EC 501: Problem Set 1, Solutions

1. (a) The equation of the line is

$$\frac{F-32}{C-0} = \frac{212-32}{100-0}.$$

This simplifies as follows:

$$\frac{F-32}{C} = \frac{180}{100} \qquad \rightarrow \qquad F - 32 = \frac{9}{5}C \qquad \rightarrow \qquad F = 32 + \frac{9}{5}C.$$

(b) $F = C$ when

$$F = 32 + \frac{9}{5}F \qquad \rightarrow \qquad \frac{4}{5}F = -32 = \qquad \rightarrow \qquad F = -40.$$

2. Differentiating $u(y) = y^{\frac{1}{2}}$:

$$u'(y) = \frac{1}{2}y^{-\frac{1}{2}} \qquad and \qquad u"(y) = -\frac{1}{4}y^{-\frac{3}{2}}.$$

Then

$$R = \frac{-yu"}{u'} = \frac{-y\left(-\frac{1}{4}y^{-\frac{3}{2}}\right)}{\frac{1}{2}y^{-\frac{1}{2}}} = \frac{1}{2}.$$

3. Differentiating implicitly:

$$4x + 6y + 6xy' + 2yy' = 0. \tag{1}$$

Substituting $x = 1, y = 2$, we get

$$4 + 12 + 6y' + 4y' = 0 \qquad \rightarrow \qquad y' = -\frac{8}{5}.$$

Differentiating (1) implicitly:

$$4 + 6y' + 6y' + 6xy" + 2y'y' + 2y'y' = 0.$$

Substituting $x = 1, y = 2$, and $y' = -\frac{8}{5}$, we get

$$4 + 12\left(-\frac{8}{5}\right) + 6y" + 2 \cdot \frac{64}{25} + 4y" = 0 \qquad \rightarrow \qquad y" = \frac{126}{125}.$$

4. Differentiating:

$$f'(x) = \frac{e^{x-3}}{2+e^{x-3}} > 0 \; for \; all \; x.$$

Thus there are no values of x for which $f'(x) < 0$.

5. Note that

$$f(1) = \frac{1-1}{1+1} = 0.$$

Now

$$f'(x) = \frac{(x^p + x^q)\left(px^{p-1} - qx^{q-1}\right) - (x^p - x^q)\left(px^{p-1} + qx^{q-1}\right)}{(x^p + x^q)^2}$$

and therefore

$$f'(1) = \frac{2(p-q) - 0}{2^2} = \frac{p-q}{2} > 0.$$

The first-order Taylor approximation is then

$$f(x) \approx f(1) + f'(1)(x-1) = \frac{p-q}{2}(x-1).$$

6. (a)

$$N(10) = \frac{1000}{1+999e^{-3.9}} = 47.12.$$

(b) We need to solve for t the following equation:

$$500 = \frac{1000}{1+999e^{-0.39t}}.$$

This yields

$$t = -\frac{1}{0.39} ln\left(\frac{1}{999}\right) = 17.71 \; days.$$

7. (a) To test for homogeneity, note that:

$$Q(tK, tL) = A\left[a\,(tK)^{-v} + b\,(tL)^{-v}\right]^{-\frac{1}{v}}.$$

This simplifies to

$$Q(tK, tL) = tA\left[aK^{-v} + bL^{-v}\right]^{-\frac{1}{v}} = tQ(K, L).$$

Thus $Q(K,L)$is homogeneous of degree 1.

(b)

$$\frac{Q}{L} = A\left[\frac{aK^{-v}+bL^{-v}}{L^{-v}}\right]^{-\frac{1}{v}} = A\left[\frac{aK^{-v}}{L^{-v}}+b\right]^{-\frac{1}{v}} = A\left[a\left(\frac{K}{L}\right)^{-v}+b\right]^{-\frac{1}{v}}.$$

8. (a) To maximize profits:

$$\frac{\partial \pi}{\partial x} = -0.2x - 0.2y + 47 = 0 \qquad (2)$$

$$\frac{\partial \pi}{\partial y} = -0.2x - 0.4y + 48 = 0 \qquad (3)$$

Subtracting (3) from (2) and simplifying yields:

$$0.2y = 1 \qquad \rightarrow \qquad y = 5.$$

Then, from (2):

$$0.2x = 46 \qquad \rightarrow \qquad x = 230.$$

To check that this yields a maximum, check the second-order conditions:

$$\frac{\partial^2 \pi}{\partial x^2} = -0.2 < 0 \qquad \surd$$

$$\frac{\partial^2 \pi}{\partial y^2} = -0.4 < 0 \qquad \surd$$

Further, note that

$$\frac{\partial^2 \pi}{\partial x \partial y} = -0.2.$$

Therefore

$$\frac{\partial^2 \pi}{\partial x^2} - \frac{\partial^2 \pi}{\partial y^2} - \left(\frac{\partial^2 \pi}{\partial x \partial y}\right)^2 = (-0.2)(-0.4) - (0.2)^2 = 0.04 > 0 \qquad \surd$$

Thus we have confirmed that the solution at $x = 230, y = 5$ does indeed yield a maximum.

(b) If $x + y = 200$, the problem becomes a constrained optimization problem:

$$Max \quad \pi(x,y) \qquad subject\ to \quad x + y = 200.$$

To solve this problem, write down the Lagrangian for it:

$$\mathcal{L} = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600 + \lambda\,(200 - x - y)$$

and consider the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = -0.2x - 0.2y + 47 - \lambda = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial y} = -0.2x - 0.4y + 48 - \lambda = 0 \tag{5}$$

Combining (4) and (5) yields:

$$x = 195, y = 5.$$

9. (a) The firm wants to maximize its profit:

$$\pi = 10Q - \frac{Q^2}{2} - Q^2 - 8Q - 5.$$

It should choose Q to satisfy

$$\frac{\partial \pi}{\partial Q} = 10 - Q - 2Q - 8 = 0 \qquad \rightarrow \qquad Q = \frac{2}{3}.$$

Checking the second-order condition:

$$\frac{\partial^2 \pi}{\partial Q^2} = -3 < 0 \qquad \surd$$

(b) The new cost curve is

$$C(Q) = Q^2 + (8 + t)Q + 5$$

and profit would be given by

$$\pi = 10Q - \frac{Q^2}{2} - Q^2 - (8 + t)Q - 5.$$

(c) Profit would now be maximized where

$$\frac{\partial \pi}{\partial Q} = 10 - Q - 2Q - 8 - t = 0 \qquad \rightarrow \qquad Q = \frac{2-t}{3}.$$

Checking the second-order condition:

$$\frac{\partial^2 \pi}{\partial Q^2} = -3 < 0 \qquad \surd$$

(d) Government's tax revenue would be given by

$$T = t\left(\frac{2-t}{3}\right) = \frac{2}{3}t - \frac{t^2}{3}.$$

This is maximized where

$$\frac{dT}{dt} = \frac{2}{3} - \frac{2t}{3} = 0 \qquad \rightarrow \qquad t = 1.$$

Checking the second-order condition:

$$\frac{\partial^2 T}{\partial t^2} = -\frac{2}{3} < 0. \qquad \surd$$

10. Since MR is just the derivative of total revenue with respect to Q, we may find TR by integrating MR with respect to Q:

$$TR = \int MR dQ = \int (25 - 2Q) dQ = 25Q - Q^2 + k.$$

But $TR(q = 0) = 0$; therefore $k = 0$ and $TR = 25Q - Q^2$.