Underidentification?

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Road map

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   (b) Cross-equation restrictions
   (c) Sequential moment conditions
3. Non-linear in parameters models
   (a) Tests of H_0 : Global underidentification
   (b) Tests of H_0 : First order underidentification
   (c) Tests of H_0 : Local underidentification
4. Fundamentally non-linear models

- x : observed random variables (cross-section, time-series or panel)
- $f(x;\alpha)$: influence functions
- α : unknown parameters
- $E\left[f\left(x;\alpha_0\right)\right]=0$: moment conditions
- α_0 : true value of the parameters

- α_0 **globally identifiable** iff $E[f(x;\alpha)]\neq 0$ for all $\alpha\neq\alpha_0$
- α_0 **locally identifiable** iff $E[f(x;\alpha^i)]\neq 0$ for any sequence α^i such that $\lim_{i\to\infty}\alpha^i=\alpha_0$

Order condition (necessary):

- If $\dim(f) > \dim(\alpha)$, then α (seemingly) overidentified
- If $\dim(f) = \dim(\alpha)$, then α (seemingly) exactly identified
- If $\dim(f) < \dim(\alpha)$, then α unequivocally underidentified

Rank condition (sufficient)

- If $D(\alpha) = E\left[\partial f(x,\alpha)/\partial \alpha'\right]$ continuous at α_0, and $rank[D(\alpha_0)] = \dim(\alpha)$, then α_0 locally identified
- If $rank[D(\alpha)]$ is constant in a neighbourhood of α_0, then this condition is also necessary
- There are non-linear models in which the rank condition fails, and yet α_0 is locally identified
- In that case, α_0 is **first-order** underidentifiable

Weak instruments literature

- Assumes that the rank condition is satisfied, but only just
- More formally $D(\alpha_0^T) = T^{-1} \cdot D$ with $rank(D) = \dim(\alpha)$
- Its objective is to compute reliable standard errors for testing hypotheses about α_0

Set estimation literature

- Assumes that α_0 is not locally identified
- Consequently, $E[f(x;\alpha)] = 0$ for some manifold of values of α that includes α_0
- Its objective is to make inferences about this manifold

Our paper

- We take an agnostic view
- In the context of local indentification, the assumption made by the set identification literature is our null
- The assumption made by the weak instrument literature is our local alternative
- The usual identification assumptions are our fixed alternative
- We test the null versus those alternatives
- The early econometric literature on linear simultaneous equations recognised that underidentification is testable, but to date such tests are uncommon in econometric practice
- And yet there are many situations of economic interest in which seemingly point identified models may be only set identified

Implementation approach

- We systematically work through the structural form
- If α_0 is not identified, then there is a manifold of α's that satisfy the original moment conditions
- We implicitly parametrise this underidentified manifold, and write all the implied moment conditions together as an extended system
- Then we simply compute the overidentification test of the extended system

- Influence functions: $f(x, \alpha) = \Psi(x)\alpha$
- $\Psi(x)$: function of data
- α only identifiable up to scale (direction)
- We need an additional normalisation to go from a direction to a particular parameter value
- But we want our inferences to be invariant to normalisation
- Order condition: $r \geq k$ (necessary)
- Rank condition: $rank\{E[\Psi(x)]\} = k$ (necessary and sufficient)

- If $r \geq k+1$ and $rank\{E[\Psi(x)]\} = k$ then α is overidentified
- But since $rank\{E[\Psi(x)]\} \leq min(r, k+1)$ this imposes testable restrictions
- The usual overidentification (J) test of Sargan (1958) and Hansen (1982) assesses precisely those restrictions

- But if $rank\,\{E[\Psi(x)]\} \leq k-1$ then α is underidentified
- Suppose $rank\,\{E[\Psi(x)]\} = k-1$
- Then, we can find a second linearly independent direction such that $E[\Psi(x)]\alpha^{*} = 0$ too
- We test for unideridentification by testing the overidentification of the extended system $E[\Psi(x)](\alpha, \alpha^{*}) = (0, 0)$
- The only tricky bit is to parametrise the nullspace of $E[\Psi(x)]$ in such a way that we don't have either redundant parameters or redundant moment conditions
- This is important to sort out in advance, since the degrees of freedom for the I test are the effective number of moment conditions minus the effective number of parameters.
- To understand better implementation, consider three examples

- $y'\alpha = u$ (structural equation)
- $E(z \cdot u) = 0$ (orthogonality conditions)
- z: r instrumental variables
- $\Psi(y, z) = zy'$
- We need to parametrise the nullspace of $\Psi(y, z)$
- Two orthogonal directions can be parametrised with $k + (k - 1)$ elements
- But since there is an extra degree of flexibility, the required number of parameters is $2k - 2$
- Therefore, the number of degrees of freedom of the I test will be $2r - 2k + 2$

- With two endogenous variables, the I test is $E[z \cdot (y_1, y_2)] = (0, 0)$
- The instrument relevance test in a model with a normalisation restriction on the first endogenous variable is $E(zy_2) = 0$
- But when $k > 1$ the I test involves parameter estimation
- We could set the top two rows of (α, α^*) to the identity matrix of order 2 (symmetric normalisation)
- Alternatively, we could impose $|\alpha| = |\alpha^*| = 1$, $\alpha' \alpha^* = 0$ and set the first entry of α^* to 0 (asymmetric normalisation)
- We can extend our approach to deal with $rank[E(zy')] < k - 1$
- Our I test is asymptotically equivalent to the LR-based tests of Koopman and Hood (1953) and Sargan (1958), and the minimum distance rank test of Cragg and Donald (1993), which looks at the reduced form regression coefficient matrix $[E(zz')]^{-1} E(zy')$

- $(y_1, y_2)'\alpha = u_1$ (first equation)
- $(y_2, y_3)'\alpha = u_2$ (second equation)
- $E[z(u_1, u_2)] = (0, 0)$ (moment conditions)
- z: r^* instrumental variables
- $vec[\Psi(y, z)] = (zy_1, zy_2|zy_3, zy_3)$ with $r = 2r^*$
- Replace the second equation by the difference between the two original equations so that $vec[\Psi^*(y, z)] = [zy_1, z(y_1 - y_2)|zy_3, 0)$
- If we mechanically duplicate the new equations we get $E[z(y_1 - y_2)] = 0$ twice
- The I test should include this moment condition only once
- Then we parametrise again the corresponding 2-dimensional null space with $2k - 2$ free elements
- Thus the number of degrees of freedom is $3r^* - 2k + 2$

Example: Panel data AR(2) model with individual effects
$(Y_{it+2} - \eta_i) + \alpha_1(Y_{it+1} - \eta_i) + \alpha_2(Y_{it} - \eta_i) = v_{it+2}$,
$(Y_{i1}, ..., Y_{iT}, \eta_i)$ is a cross-sectionally *i.i.d.* random vector, and
$E(v_{it+2}|Y_{i1}, ..., Y_{it+1}; \eta_i) = 0$

- The Arellano and Bond (1991) influence functions are $z_{it}u_{it+2}$, u_{it+2}=ΔY_{it+2}-$\alpha_1\Delta Y_{it+1}$-$\alpha_2\Delta Y_{it}$=Δv_{it+2}, z_{it}=$(Y_{i1}, Y_{i2}, \ldots, Y_{it})'$
- This is a system of equations with common coefficients and an increasing sequence of instruments, whose reduced form is non-standard
- If there are 5 or more time series observations, underidentication arises if and only if the autoregressive polynomial contains a unit root, so that ΔY_{it} is an AR(1)
- After eliminating redundancies, the I test will be based on $E[Y_{it-j}(\Delta Y_{it} - \gamma\Delta Y_{it-1})] = 0$ for $j \geq 1$ and $t \geq 2$

- Replace α by a non-linear, continuously differentiable function $\phi : \mathbb{P} \to \mathbb{R}^{k+1}$, where $\mathbb{P} \subseteq \mathbb{R}^l$
- Influence function: $f(x, \beta) = \Psi(x)\phi(\beta)$ for $\beta \in \mathbb{P}$
- Identification is only meaningful if ϕ is one-to-one, for if there are two distinct parameter values β and β^* for which $\phi(\beta) = \phi(\beta^*)$ then we know *a priori* that we cannot identify β
- We make the stronger assumption that for any two values of the parameter vector $\beta \neq \beta^*$ in $\mathbb{P}$, $\phi(\beta) \neq \mathsf{c}\phi(\beta^*)$ for some $\mathsf{c} \in \mathbb{R}$
- Define the parameter space $\mathbb{Q} \equiv \{\alpha : \alpha = \phi(\beta) \text{ for some } \beta \in \mathbb{P}\}$, and write the moment conditions as $E[\Psi(x)]\alpha = 0$ for $\alpha \in \mathbb{Q}$
- Assume $rank\{E[\Psi(x)]\} < k$, for otherwise α (and β) identified
- We can then proceed as in the linear case by defining the extendend 'linear subspace' $\mathbb{Q}^* \equiv \{\alpha : \alpha = \mathsf{c}_1\alpha_1 + \mathsf{c}_2\alpha_2, \alpha_1, \alpha_2 \in \mathbb{Q}, \mathsf{c}_1, \mathsf{c}_2 \in \mathbb{R}\}$

But unlike in the linear case, several different meaningful underidentified situations may arise

- **Global underidentification**: There are two (or more) separate values of β that satisfy the moment conditions $E[\Psi(x)]\phi(\beta) = 0$
- **Local underidentification**: There is a continuum of values of β that satisfy the moment conditions $E[\Psi(x)]\phi(\beta) = 0$
- **First-order underidentification**: There is only one β_0 that satisfies the moment conditions $E[\Psi(x)]\phi(\beta) = 0$, but $rank\{E[\Psi(x)]\partial\phi(\beta_0)/\partial\beta'\} < l$
- **Second-order underidentification**: There is only one β_0 that satisfies the moment conditions $E[\Psi(x)]\phi(\beta) = 0$, but $rank\{E[\Psi(x)]\partial\phi(\beta_0)/\partial\beta'\} < l$ **and** $rank[\partial vec\{E[\Psi(x)]\partial\phi(\beta_0)/\partial\beta'\}/\partial\beta'] < l$

Example: Euler equation with taste shifter u_t and innovations w_t

- $y_t'\delta = u_t + \gamma_1' w_t, \quad u_t = \rho u_{t-1} + \gamma_2' w_t, \ dim(y_t) = 2$
- $\{w_t\}$: vector martingale difference sequence (economic shocks)
- Valid predetermined instruments z_{t-2}: distributed lag of $\{w_{t-2}\}$
- Define $\Psi(x) = (z_{t-2}y_t' | - z_{t-2}y_{t-1}')$
- Moment conditions: $E[\Psi(x)]\phi(\delta, \rho) = 0$
- Non-linear functions: $\phi(\delta, \rho) = (\delta', \rho\delta')'$
- Parameters: δ (up to a normalisation such as $|\delta| = 1$) and ρ

Implementation

- Define ρ^* and δ^*, with $|\delta^*| = 1$
- Then, compute the usual overidentification test of the extended system $E[\Psi(x)][\phi(\delta,\rho)|\phi(\delta^*,\rho^*)] = (0,0)$ making sure that the 'linear subspace' $\mathsf{c}_1\phi(\delta,\rho) + \mathsf{c}_2\phi(\delta^*,\rho^*)$ is indeed two-dimensional
- If the model is locally but not globally identified, so there exists a vector $(\delta^*,\rho^*) \neq (\delta,\rho)$ with $|\delta| = |\delta^*| = 1$, then this I test will converge to a χ^2 with $2r^* - 4$ degrees of freedom, where r^* is the number of instruments

But other things can happen

- For instance, if $E[z_{t-1}(u_t + \gamma_1' w_t)] = 0$, then ρ is not locally identified
- The appropriate I test should jointly consider $E(z_{t-1}y_t')\delta = 0$ and $E(z_{t-2}y_t')\delta = 0$ after eliminating moment redundacies
- Similarly, if $E[z_{t-2}(y_t - \rho y_{t-1})'] = 0$, then δ is not locally identified
- The appropriate I test should assess if $E[z_{t-2}(y_t - \rho y_{t-1})'] = 0$
- It might also happen that the Jacobian matrix has reduced rank even though $E[z_{t-1}(u_t + \gamma_1' w_t)] \neq 0$ and $E(z_{t-2}(y_t - \rho y_{t-1})'] \neq 0$, in which case δ and ρ are locally identified, but not first-order identified
- Since this is a rather uninteresting situation in this example, I'll postpone the discussion of the appropriate I test

Example: Panel data AR(2) model with individual effects
$(Y_{it+2} - \eta_i) + \alpha_1(Y_{it+1} - \eta_i) + \alpha_2(Y_{it} - \eta_i) = v_{it+2}$,
$(Y_{i1}, ..., Y_{iT}, \eta_i)$ is a cross-sectionally *i.i.d.* random vector, and
$E(v_{it+2}|Y_{i1}, ..., Y_{it+1}; \eta_i) = 0$

- As we saw before, identification problems may arise if $\alpha_1 + \alpha_2 = 1$ so that the AR polynomial contains a unit root.
- If we use the Arellano and Bond (1991) linear moment conditions for estimation, then the I test checks whether $E[Y_{it-j}(\Delta Y_{it} - \gamma \Delta Y_{it-1})] = 0$ for $j \geq 1$ and $t \geq 2$
- Ahn and Schmidt (1995) proposed the additional influence functions $(Y_{it+2} - \alpha_1 Y_{it+1} - \alpha_2 Y_{it})(\Delta Y_{it+1} - \alpha_1 \Delta Y_{it} - \alpha_2 \Delta Y_{it-1})$ to obtain more efficient estimators of α_1 and α_2 in the identified case
- The question is whether these influence functions can rescue identification in the unit root case

- Given that the additional influence functions are quadratic, time series heteroskedasticity matters
- $V(v_{it}) = \sigma_t^2$: unconditional cross-sectional variance
- If σ_t^2 varies freely over time, then α_1 is globally identified by the Ahn-Schmidt moment conditions
- If σ_t^2 grows or decreases exponentially over time, then α_1 is first-order and locally identified, but not globally identified, as there is a second unit root solution that satisfies the original moment conditions
- We can easily test for the existence of two separate solutions along the lines of the previous example

- But if there is time series homoskedasticity (i.e. $\sigma_t^2 = \sigma^2 \quad \forall t$), then α_1 is locally and globally identified, but it is not first-order identified
- We can easily test for this by combining the original set of moment conditions with a new moment condition given by the expected Jacobian with respect to α_1 after imposing the unit root restriction
- Effectively, the moment conditions of such a first-order underidentification test are equivalent to the moment conditions of the global underidentification test when the two separate solutions converge

- $f(x;\alpha)$: influence functions
- α : unknown parameters
- $E\left[f\left(x;\alpha\right)\right]=0$: moment conditions

CCAPM Example

- $f(x;\alpha)=z\cdot y_1 exp(\alpha y_2)$
- y_1: excess returns on a financial asset
- y_2: consumption growth
- z: lagged instruments
- α : coefficient of relative risk aversion

- Suppose α is an unrestricted scalar parameter
- To test for global underidentification, we simply test if $E[f(x;\alpha)] = 0$ and $E[f(x;\alpha^*)] = 0$ simultaneously for $\alpha \neq \alpha^*$
- To test for first-order underidentification, we test if $E[f(x;\alpha)]{=}0$ and $E[\partial f(x;\alpha)/\partial\alpha]{=}0$ simultaneously for some α
- This second test can be regarded as a limiting case of the first one when $\alpha^* \to \alpha$
- But the really novel case is local unidentifiability, when $E[f(x;\alpha)] = 0 \ \forall\alpha$
- In this case, we can no longer construct a 'linear' nullspace $\mathbb{Q}^*$

- A locally underidentified α leads to a continuum of moment conditions
- *Et voila!* Our I test is simply the overidentification test of Carrasco and Florens (2000) applied to $E\left[f\left(x;\alpha\right)\right]=0$
- Intuitively, it is as if we chose a grid of N points α_1, α_2, ..., α_N, and simultaneously tested $E\left[f\left(x;\alpha_i\right)\right]=0$ for $i=1,2,\ldots,N$
- However, the optimal GMM weighting matrix of this discrete grid approach would become increasingly singular as N increases, and would require some sort of regularisation
- This is precisely what the Carrasco and Florens estimation procedure does in a rather elegant way
- However, their standard normal asymptotic approximation does not provide a completely reliable approximation in finite samples
- We develop a more reliable Imhof approximation to their overidentification test with the same asymptotic justification

- Suppose now that α is a vector of k parameters
- If there is full identification breakdown, a direct generalisation of the previous approach applies
- Assume instead we can separate the moment conditions in two groups such that the first group just identifies $k-1$ parameters as a function of a scalar parameter β in an implicit manner.
- Let $\varphi(\beta)$ denote this implicit manifold, so that $\alpha = [\varphi'(\beta), \beta]'$
- If we knew $\varphi(\beta)$, then we could directly apply the Carrasco-Florens procedure to $E\{f[x; \varphi(\beta), \beta]\} = 0$
- In practice, we could use a two-step procedure in which we first estimate $\varphi(\beta)$ as a function of β, plug-in the resulting estimate in f, and adjust inference appropriately
- Alternatively, we could also estimate the functional $\varphi(\beta)$ simultaneously with β

Summary

- There are many situations of economic interest in which seemingly point identified models are only set identifed
- We propose a unifying approach to underidentification tests
- We implicitly parametrise the set of identified values compatible with the structural form, and then apply an overidentification test to the extended system of moment conditions
- Our tests simplify considerably for non-linear in parameters models, in which the set of observationally equivalent structures must belong to some restricted linear subspace
- And of course, they become even simpler for linear models
- Apart from tractability, the main advantage of working with the structural form is that our underidentification tests usually have a direct interpretation in terms of the model of interest