Testing Jointly for Structural Changes in the Error Variance and Coefficients of a Linear Regression Model*

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Abstract

We provide a comprehensive treatment of the problem of testing jointly for structural change in both the regression coefficients and the variance of the errors in a single equation regression involving stationary regressors. Our framework is quite general in that we allow for general mixing-type regressors and the assumptions imposed on the errors are quite mild. The errors' distribution can be non-normal and conditional heteroskedasticity is permissable. Extensions to the case with serially correlated errors are also treated. We provide the required tools for addressing the following testing problems, among others: a) testing for given numbers of changes in regression coefficients and variance of the errors; b) testing for some unknown number of changes less than some pre-specified maximum; c) testing for changes in variance (regression coefficients) allowing for a given number of changes in regression coefficients (variance); and d) estimating the number of changes present. These testing problems are important for practical applications as witnessed by recent interests in macroeconomics and finance for which documenting structural change in the variability of shocks to simple autoregressions or vector autoregressive models has been a concern.

Keywords: Change-point; Variance shift; Conditional heteroskedasticity; Likelihood ratio tests.

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1 Introduction

Both the statistics and econometrics literature contain a vast amount of work on issues related to structural changes with unknown break dates (for an extensive review, see Perron, 2006). In the context of multiple structural changes, Bai and Perron (1998, 2003a) confronted various issues for a univariate regression: the consistency of estimates of the break dates, tests for structural change, confidence intervals for the break dates, methods to select the number of breaks and efficient algorithms to compute the estimates (see also Hawkins, 1976). Perron and Qu (2004) extended this analysis to the case where arbitrary linear restrictions are imposed on the coefficients of the model, considerably relaxing the prior assumptions. Liu, Wu and Zidek (1997) considered multiple structural changes in the context of a threshold model. Bai, Lumsdaine and Stock (1998) analyzed inference for a single break date in a multivariate time series, allowing for stationary or integrated regressors as well as trends. Bai (2000) considered inference on multiple break dates in a segmented stationary vector autoregressive (VAR) model with breaks occuring in the parameters of the conditional mean, the variance of the error term or both. Kejriwal and Perron (2006a,b) addressed issues related to testing and inference with multiple structural changes in a single equation cointegrated model. Qu and Perron (2007a) considered a multivariate system and provided methods to estimate models with structural changes in both the regression coefficients and the covariance matrix of the errors.

With respect to testing for structural change in both the coefficients and the variance of the regression error, the results are quite sparse. Qu and Perron (2007a) addressed this issue in the context of a multivariate system of equations, though their analysis is restricted to models with normally distributed errors and a prior that the breaks in the coefficients and in the variance occur at different dates. Horváth

(1993) considered a change in the mean and variance (occurring at the same time) of a sequence of independent and identically distributed (i.i.d.) random variables with moments corresponding to those of a normal distribution. Davis, Huang, and Yao (1995) extended this analysis to an autoregressive process under similar conditions.

We build on the work of Qu and Perron (2007a) to provide a comprehensive treatment of the problem of testing jointly for structural change in both the regression coefficients and the variance of the errors in a single equation system involving stationary regressors, allowing the break dates for the two components to be different or coincide. Our framework is general and allows for general mixing-type regressors and the assumptions imposed on the errors are mild. The errors' distribution can be non-normal and conditional heteroskedasticity is permissable. Extensions to the case with serially correlated errors are also treated. We provide the required tools for addressing the following testing problems, among others: a) testing for given numbers of changes in regression coefficients and variance of the errors; b) testing for some unknown number of changes less than some pre-specified maximum; c) testing for changes in variance (regression coefficients) allowing for a given number of changes in regression coefficients (variance); and d) estimating the number of changes present.

These testing problems are important for practical applications, as witnessed by recent interests in macroeconomics and finance for which documenting structural change in the variability of shocks to simple autoregressions or VAR models has been a concern (e.g., see Stock and Watson, 2002). Given the lack of proper testing procedures, a common approach is to apply standard sup-Wald type tests (e.g., see Andrews, 1993 and Bai and Perron, 1998) for changes in the mean of the absolute value of the estimated residuals, a rather ad hoc procedure. To test for a change in variance only (imposing no change in the regression coefficients), Deng and Perron (2008) extended the CUSUM of squares test of Brown, Durbin and Evans (1975) to allow general conditions on the regressors and errors (as initially suggested by Inclán and Tiao (1994) for normally distributed time series). This test is, however, adequate only if no change in coefficients is present. It is often the case that changes in both coefficients and variance occur with different break dates. A commonly used method for joint break detection is to first test for changes in the regression coefficients and, conditioning on the break dates found, then test for changes in variance. This methodology is clearly inappropriate as in the first step, the test suffers from severe size distortions (see Zhou and Perron, 2008). Also, neglecting changes in coefficients when testing for changes in variance induces both size distortions and a loss of power. Hence, a joint approach is needed. To this effect, we propose testing procedures based on quasi-likelihood ratio tests, constructed using a likelihood function appropriate for i.i.d. Gaussian errors. We then apply corrections to these likelihood ratio tests so that their limit distributions are free of nuisance parameters in the presence of non-normal distributions and conditional heteroskedasticity. We also consider extensions that allow for serial correlation.

This paper is structured as follows. Section 2 presents the models and testing problems considered. Section 3 presents the quasi-likelihood ratio tests to be used as the basis of the various testing procedures. Section 4 discusses the main assumptions imposed on the regressors and errors, derives the relevant limit distributions under the various null hypotheses and proposes corrected versions of the tests that have limit distributions free of nuisance parameters. Section 4.1 deals with the case of martingale difference errors, Section 4.2 extends the analysis to serially correlated errors, Section 4.3 covers the case with an unknown number of breaks under the alternative hypothesis and Section 4.4 discusses tests for an additional break in either the regression coefficients or the variance. Section 5 presents a specific to general method to estimate the number of breaks in each of the regression coefficients and the variance. Section 6 presents an application to illustrate the use of our tests. Section 7 provides brief concluding remarks and an appendix contains some technical derivations. Simulation results to assess the adequacy of the suggested procedures, in terms of their finite sample size and power, are presented in Zhou and Perron (2008).

2 Model and testing problems

We start with a description of the most general specification considered, where multiple breaks occur in both the coefficients of the conditional mean and the variance of the errors at possibly different times. This allows us to establish the notation used throughout the paper. The main framework of analysis is the following multiple linear regression with m breaks (or $m+1$ regimes) in the conditional mean equation:

$$y_t = x_t'\beta + z_t'\delta_j + u_t, \qquad t = T_{j-1}^c + 1, ..., T_j^c, \tag{1}$$

for $j = 1, ..., m+1$. In this model, y_t is the observed dependent variable at time t; both x_t $(p \times 1)$ and z_t $(q \times 1)$ are vectors of covariates and β and δ_j $(j = 1, ..., m+1)$ are the corresponding vectors of coefficients; u_t is the disturbance at time t. The indices $(T_1^c, ..., T_m^c)$, or the break points, are treated as unknown (the convention that $T_0^c = 0$ and $T_{m+1}^c = T$ is used). This is a partial structural change model since the parameter vector β is not subject to shifts. When $p = 0$, we obtain a pure structural change model where all the model's coefficients are subject to change. Note that using a partial structural change model where only some coefficients are allowed to change can be beneficial both in terms of obtaining more precise estimates and more powerful tests. We also allow for n breaks (or $n+1$ regimes) for the variance of the errors, occurring at unknown dates $(T_1^v, ..., T_n^v)$. Accordingly, the error term u_t has zero mean and variance σ_i^2 for $T_{i-1}^v + 1 \leq t \leq T_i^v$ $(i = 1, ..., n+1)$, where again we use the convention that $T_0^v = 0$ and $T_{n+1}^v = T$. We allow the breaks in the variance and in the coefficients to happen at different times. Hence the m-vector $(T_1^c, ..., T_m^c)$ and the n-vector $(T_1^v, ..., T_n^v)$ can be composed of distinct elements or they can overlap partially or completely. We let K denote the total number of break dates, noting that $max[m, n] \leq K \leq m + n$. When the the breaks completely coincide, $m = n = K$.

The regression (1) may be expressed in matrix form as $Y = X\beta + \bar{Z}\delta + U$, where $Y = (y_1, ..., y_T)', X = (x_1, ..., x_T)'$, $U = (u_1, ..., u_T)'$, $\delta = (\delta_1', ..., \delta_{m+1}')'$ and $\bar{Z}$ is the matrix which diagonally partitions $Z = (z_1, ..., z_T)$ at $(T_1^c, ..., T_m^c)$, i.e., $\bar{Z} = diag(Z_1, ..., Z_{m+1})$ with $Z_i = (z_{T_{i-1}^c+1}, ..., z_{T_i^c})'$. We denote the true value of a parameter with a 0 superscript. In particular, $\delta^0 = (\delta_1^{0\prime}, ..., \delta_{m+1}^{0\prime})'$ and $(T_1^{c0}, ..., T_m^{c0})$ are used to denote, respectively, the true values of the parameters δ and the true break dates in the regression coefficients. The matrix $\bar{Z}^{c0}$ diagonally partitions Z at $(T_1^{c0}, ..., T_m^{c0})$. Hence, in its most general form, the data-generating process is

$$Y = X\beta^0 + \bar{Z}^0\delta^0 + U \tag{2}$$

with $E(UU') = \Omega^0$, where the diagonal elements of Ω^0 are σ_{i0}^2 for $T_{i-1}^{v0} + 1 \leq t \leq T_i^{v0}$ $(i = 1, ..., n+1)$. We also consider cases with serial correlation in the errors for which the off-diagonal elements of Ω^0 need not be 0.

This model is a special case of the class of models considered by Qu and Perron (2007a). The method of estimation considered is quasi-maximum likelihood (QML), assuming serially uncorrelated Gaussian errors. They proved consistency of the estimates of the break fractions $(\lambda_1^0, ..., \lambda_K^0) \equiv (T_1^0/T, ..., T_K^0/T)$, where T_i^0 $(i = 1, ..., K)$ denotes an element of the union of $(T_1^{c0}, ..., T_m^{c0})$ and $(T_1^{v0}, ..., T_n^{v0})$. This proof was provided under general conditions on the regressors and errors. Substantial heterogeneity in the distributions of the regressors is allowed across regimes, though unit root processes are not permitted. The series $z_t u_t$ and u_t were assumed to be short memory processes with bounded fourth moments. Otherwise, the imposed conditions were mild and allowed for substantial conditional heteroskedasticity and autocorrelation. They also derived the limit distribution of the estimates of the break dates.

Remark 1 *Our framework is general enough to encompass models with endogenous regressors. On the one hand, instead of the regressors correlated with the errors, one can use the fitted values from a first-stage regression with instruments uncorrelated with the errors. All results in this paper go through unchanged. On the other hand, one can cast the model in terms of the limit values of the least-squares estimates, say* δ^* *instead of the true values* δ^0*. Since a change in* δ^0 *implies an equivalent change in* δ^**, one can simply tests for changes in* δ^**. The latter strategy is highly preferable. It leads to more precise estimates of the break dates and tests with higher power since using an instrumental variable procedure implies regressors with less quadratic variation, especially with weak instruments. See Perron and Yamamoto (2008) for details.*

The testing problems we consider include the following: TP-1) $H_0 : \{m = n = 0\}$ versus $H_1 : \{m = 0,\ n = n_a\}$; TP-2) $H_0 : \{m = m_a, n = 0\}$ versus $H_1 : \{m = m_a,\ n = n_a\}$; TP-3) $H_0 : \{m = 0, n = n_a\}$ versus $H_1 : \{m = m_a,\ n = n_a\}$; and TP-4) $H_0 : \{m = n = 0\}$ versus $H_1 : \{m = m_a,\ n = n_a\}$, where m_a and n_a are some positive numbers, selected a priori. We shall subsequently consider testing problems in which the alternatives specify some unknown numbers of breaks, up to some maximum. These are given by TP-5) $H_0 : \{m = n = 0\}$ versus $H_1 : \{m = 0,\ 1 \leq n \leq N\}$; TP-6) $H_0 : \{m = m_a, n = 0\}$ versus $H_1 : \{m = m_a,\ 1 \leq n \leq N\}$; TP-7) $H_0 : \{m = 0, n = n_a\}$ versus $H_1 : \{1 \leq m \leq M,\ n = n_a\}$; and TP-8) $H_0 : \{m = n = 0\}$ versus $H_1 : \{1 \leq m \leq M,\ 1 \leq n \leq N\}$. Finally, we shall also be concerned with the following testing problems: TP-9) $H_0 : \{m = m_a, n = n_a\}$ versus $H_1 : \{m = m_a + 1,\ n = n_a\}$ and TP-10) $H_0 : \{m = m_a, n = n_a\}$ versus $H_1 : \{m = m_a,\ n = n_a + 1\}$, where m_a and n_a non-negative integers. These last two tests are useful for assessing the adequacy of a model with a particular number of breaks by looking at whether including one more break is warranted. In Section 5, we also consider a sequential testing procedure that involves estimating the number and types of breaks in both the conditional mean regression and the variance of the errors.

3 The quasi-likelihood ratio tests

In this section, we consider the likelihood ratio tests obtained by assuming normally distributed and serially uncorrelated errors for the testing problems TP-1 to TP-4. In Section 4, we derive their limit distributions which, in general, are not free of nuisance parameters. We then propose modifications to the tests whose asymptotic distributions are free of nuisance parameters. Results for the testing problems TP-5 to TP-8 follow as straightforward corollaries and are discussed in Section 4.3.

Consider TP-1, where one specifies no change in the regression coefficients ($m = q = 0$) but tests for a given number n_a of changes in the variance of the errors. Under the null hypothesis, the log-likelihood function is given by

$$\log \widetilde{L}_T = -(T/2)\left(\log 2\pi + 1\right) - (T/2)\log \widetilde{\sigma}^2 \tag{3}$$

where $\widetilde{\sigma}^2 = T^{-1}\sum_{t=1}^{T}(y_t - x_t'\widetilde{\beta})^2$ and $\widetilde{\beta} = (\sum_{t=1}^{T} x_t x_t')^{-1}(\sum_{t=1}^{T} x_t y_t)$. Under the alternative hypothesis, for a given partition $\{T_1^v, ..., T_n^v\}$, the log-likelihood is

$$\log \hat{L}_T\left(T_1^v, ..., T_n^v\right) = -(T/2)\left(\log 2\pi + 1\right) - (1/2)\sum_{i=1}^{n_a+1}\left(T_i^v - T_{i-1}^v\right)\log \hat{\sigma}_i^2, \tag{4}$$

where the quasi-maximum likelihood estimates (QMLE) jointly solve the system

$$\begin{aligned}
\hat{\beta} &= \left(\sum_{i=1}^{n_a+1}\sum_{t=T_{i-1}^v+1}^{T_i^v}\frac{x_t x_t'}{\hat{\sigma}_i^2}\right)^{-1}\left(\sum_{i=1}^{n_a+1}\sum_{t=T_{i-1}^v+1}^{T_i^v}\frac{x_t y_t}{\hat{\sigma}_i^2}\right), \\
\hat{\sigma}_i^2 &= \frac{1}{T_i^v - T_{i-1}^v}\sum_{t=T_{i-1}^v+1}^{T_i^v}(y_t - x_t'\hat{\beta})^2,
\end{aligned}$$

for $i=1,...,n_a+1$. Hence, the sup-Likelihood ratio test considered here is

$$\begin{aligned}\sup LR_{1,T}\left(n_a,\varepsilon|m=n=0\right) &= \sup_{(\lambda_1^v,...,\lambda_{n_a}^v)\in\Lambda_{v,\varepsilon}} 2[\log \hat{L}_T\left(T_1^v,...,T_{n_a}^v\right)-\log \widetilde{L}_T] \\ &= 2[\log \hat{L}_T(\hat{T}_1^v,...,\hat{T}_{n_a}^v)-\log \widetilde{L}_T],\end{aligned}$$

where $(\hat{T}_1^v,...,\hat{T}_{n_a}^v)$ are the QMLE of the break dates in variance obtained by imposing no change in the coefficients, with the search restricted to the set

$$\Lambda_{v,\varepsilon}=\{(\lambda_1^v,...,\lambda_{n_a}^v):\ |\lambda_{i+1}^v-\lambda_i^v|\geq\varepsilon\ (i=1,...,n_a-1),\lambda_1^v\geq\varepsilon,\lambda_{n_a}^v\leq 1-\varepsilon\}.$$

The parameter ε acts as a truncation which imposes a minimal length for each segment and will affect the limiting distribution of the test. For the testing problem TP-2, there are m_a breaks in the regression coefficients under both the null and alternative hypotheses so that the test pertains to assessing whether there are 0 or n_a breaks in the variance. For a given partition $\{T_1^c,...,T_{m_a}^c\}$, the likelihood function under the null hypothesis is $\log \widetilde{L}_T(T_1^c,...,T_{m_a}^c)=-(T/2)\left(\log 2\pi+1\right)-(T/2)\log\widetilde{\sigma}^2$, where $\widetilde{\sigma}^2=T^{-1}\sum_{j=1}^{m_a+1}\sum_{t=T_{j-1}^c+1}^{T_j^c}(y_t-x_t'\widetilde{\beta}-z_t'\widetilde{\delta}_j)^2$, $\widetilde{\beta}=(X'M_{\bar{Z}}X)^{-1}X'M_{\bar{Z}}Y$ and $\widetilde{\delta}_j=(Z_j'Z_j)^{-1}Z_j'(Y_j-X_j\widetilde{\beta})$ with $M_{\bar{Z}}=I-\bar{Z}(\bar{Z}'\bar{Z})^{-1}\bar{Z}'$, $\bar{Z}=diag(Z_1,...,Z_{m_a+1})$, $Z_j=(z_{T_{j-1}^c+1},...,z_{T_j^c})'$, $Y_j=(y_{T_{j-1}^c+1},...,y_{T_j^c})'$ and $X_j=(x_{T_{j-1}^c+1},...,x_{T_j^c})'$. For given partitions $\{T_1^c,...,T_{m_a}^c\}$ and $\{T_1^v,...,T_{n_a}^v\}$, the log-likelihood value under the alternative hypothesis is

$$\log \hat{L}_T\left(T_1^c,...,T_{m_a}^c;T_1^v,...,T_{n_a}^v\right)=-\frac{T}{2}\left(\log 2\pi+1\right)-\frac{1}{2}\sum_{i=1}^{n_a+1}(T_i^v-T_{i-1}^v)\log\hat{\sigma}_i^2, \quad (5)$$

where the QMLE solve the following equations: $\hat{\sigma}_i^2=(T_i^v-T_{i-1}^v)^{-1}\sum_{t=T_{i-1}^v+1}^{T_i^v}(y_t-x_t'\hat{\beta}-z_t'\hat{\delta}_j)^2$ for $i=1,...,n_a+1$, and $\hat{\beta}=(X'M_{\bar{Z}_\sigma}X)^{-1}X'M_{\bar{Z}_\sigma}Y$ where $M_{\bar{Z}_\sigma}=I_T-\bar{Z}_\sigma(\bar{Z}_\sigma'\bar{Z}_\sigma)^{-1}\bar{Z}_\sigma'$ with $\bar{Z}_\sigma=diag(Z_1^\sigma,...,Z_{m_a+1}^\sigma)$, $Z_j^\sigma=(z_{T_{j-1}^c+1}^\sigma,...,z_{T_j^c}^\sigma)'$ and $z_t^\sigma=(z_t/\hat{\sigma}_i)$ for $T_{i-1}^v<t\leq T_i^v$ $(i=1,...,n_a+1)$. Using the same notation, $\hat{\delta}_j=(Z_j^{\sigma\prime}Z_j^\sigma)^{-1}Z_j^{\sigma\prime}(Y_j^\sigma-X_j^\sigma\hat{\beta})$ where $Y_j^\sigma=(y_{T_{j-1}^c+1}^\sigma,...,y_{T_j^c}^\sigma)'$ and $X_j^\sigma=(x_{T_{j-1}^c+1}^\sigma,...,x_{T_j^c}^\sigma)'$

with $x_t^\sigma = (x_t/\hat{\sigma}_i)$ and $y_t^\sigma = (y_t/\hat{\sigma}_i)$ for $T_{i-1}^v < t \leq T_i^v$. The test is then

$$\begin{aligned}\sup LR_{2,T}(m_a, n_a, \varepsilon | n = 0, m_a) &= 2[\sup_{(\lambda_1^c,\ldots,\lambda_{m_a}^c;\lambda_1^v,\ldots,\lambda_{n_a}^v)\in\Lambda_\varepsilon} \log \hat{L}_T(T_1^c, \ldots, T_{m_a}^c; T_1^v, \ldots, T_{n_a}^v) \\ &\quad - \sup_{(\lambda_1^c,\ldots,\lambda_{m_a}^c)\in\Lambda_{c,\varepsilon}} \log \widetilde{L}_T(T_1^c, \ldots, T_{m_a}^c)] \\ &= 2[\log \hat{L}_T(\widetilde{T}_1^c, \ldots, \widetilde{T}_{m_a}^c; \widetilde{T}_1^v, \ldots, \widetilde{T}_{n_a}^v) - \log \widetilde{L}_T(\hat{T}_1^c, \ldots, \hat{T}_{m_a}^c)],\end{aligned}$$

where

$$\Lambda_{c,\varepsilon} = \{(\lambda_1^c, \ldots, \lambda_m^c) : |\lambda_{j+1}^c - \lambda_j^c| \geq \varepsilon \ (j = 1, \ldots, m_a - 1), \lambda_1^c \geq \varepsilon, \lambda_{m_a}^c \leq 1 - \varepsilon\},$$

$$\begin{aligned}\Lambda_\varepsilon &= \{(\lambda_1^c, \ldots, \lambda_{m_a}^c; \lambda_1^v, \ldots, \lambda_{n_a}^v) : \text{ for } (\lambda_1, \ldots, \lambda_K) = (\lambda_1^c, \ldots, \lambda_m^c) \cup (\lambda_1^v, \ldots, \lambda_n^v) \\ &\quad |\lambda_{j+1} - \lambda_j| \geq \varepsilon \ (j = 1, \ldots, K - 1), \lambda_1 \geq \varepsilon, \lambda_K \leq 1 - \varepsilon\}.\end{aligned} \quad (6)$$

Note that we denote the estimates of the break dates in coefficients and variance by a "$\sim$" when they are obtained jointly and by a "^" when obtained separately.

Remark 2 *The set Λ_ε which defines the possible values of the break fractions in coefficients $(\lambda_1^c, \ldots, \lambda_m^c)$ and variance $(\lambda_1^v, \ldots, \lambda_n^v)$ allows them to have some (or all) common elements or not. It is important to note that each break fraction is separated by some non-zero ε. This complicates inference since many cases must be considered. To illustrate, consider the case with $m_a = n_a = 1$. We can have $K = 1$, in which case there is one break with the coefficients and the variance of the errors changing at the same date. On the other hand, if $K = 2$, the break date for the change in coefficients is different from that for the change in variance. This leads to two additional possible cases: a) $\lambda_1^c \leq \lambda_1^v - \varepsilon$ (the break in the coefficients occurs before the break in the variance) or b) $\lambda_1^c \geq \lambda_1^v + \varepsilon$ (the break in the coefficients occurs after the break in the variance). The maximized likelihood function for these two cases can be evaluated using the algorithm of Qu and Perron (2007a) since it permits* *the imposition of restrictions. For example, if $\lambda_1^c \leq \lambda_1^v - \varepsilon$, we have a two break model and the restrictions needed are that the variance of the errors in the first and second regimes is identical and the coefficients are identical in the second and third regimes. Hence, there are three maximized likelihood values to construct and the test corresponds to the maximal value over these three cases. When m_a or n_a is greater than one, more cases need to be considered, but the principle remains the same.*

For the testing problem TP-3, the null hypothesis specifies n_a breaks in variance and no break in coefficients so that, for a given partition $\{T_1^v, ..., T_n^v\}$, the likelihood function is given by $\log \widetilde{L}_T(T_1^v, ..., T_{n_a}^v) = -(T/2)(\log 2\pi + 1) - \sum_{i=1}^{n_a+1} (1/2)(T_i^v - T_{i-1}^v) \log \widetilde{\sigma}_i^2$, where $\widetilde{\sigma}_i^2 = (T_i^v - T_{i-1}^v)^{-1} \sum_{t=T_{i-1}^v+1}^{T_i^v} (y_t - x_t'\widetilde{\beta} - z_t'\widetilde{\delta})^2$ for $i = 1, ..., n_a + 1$, $(\widetilde{\beta}', \widetilde{\delta}')' = (W^{\sigma\prime} W^{\sigma})^{-1} W^{\sigma\prime} Y^{\sigma}$ and $W^{\sigma} = (w_1^{\sigma}, ..., w_T^{\sigma})'$ with $w_t^{\sigma} = (x_t^{\sigma\prime}, z_t^{\sigma\prime})'$. Under the alternative hypothesis, there are m_a breaks in coefficients and n_a breaks in variance so the likelihood function is given by (5) and the test is

$$\sup LR_{3,T}(m_a, n_a, \varepsilon | m = 0, n_a) =$$

$$2[\sup_{(\lambda_1^c, ..., \lambda_{m_a}^c; \lambda_1^v, ..., \lambda_{n_a}^v) \in \Lambda_\varepsilon} \log \hat{L}_T(T_1^c, ..., T_{m_a}^c; T_1^v, ..., T_{n_a}^v) - \sup_{(\lambda_1^v, ..., \lambda_{n_a}^v) \in \Lambda_{v,\varepsilon}} \log \widetilde{L}_T(T_1^v, ..., T_{n_a}^v)]$$
$$= 2[\log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_{m_a}^c; \widetilde{T}_1^v, ..., \widetilde{T}_{n_a}^v) - \log \widetilde{L}_T(\hat{T}_1^v, ..., \hat{T}_{n_a}^v)].$$

For the testing problem TP-4, the null hypothesis specifies no break in either coefficients or variance and the log-likelihood is given by (3). The alternative hypothesis specifies m_a breaks in coefficients and n_a breaks in the variance of the errors and the log likelihood value is given by (5). Hence, the Sup-Likelihood ratio test is

$$\begin{aligned} & \sup LR_{4,T}(m_a, n_a, \varepsilon | n = m = 0) \\ = \; & 2[\sup_{(\lambda_1^c, ..., \lambda_{m_a}^c; \lambda_1^v, ..., \lambda_{n_a}^v) \in \Lambda_\varepsilon} \log \hat{L}_T\left(T_1^c, ..., T_{m_a}^c; T_1^v, ..., T_{n_a}^v\right) - \log \widetilde{L}_T] \\ = \; & 2[\log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_{m_a}^c; \widetilde{T}_1^v, ..., \widetilde{T}_{n_a}^v) - \log \widetilde{L}_T]. \end{aligned} \quad (7)$$

4 Limiting distributions of the tests

We now consider the limit distributions of the tests. We start with the case where the errors u_t in regression (1) are martingale differences in Section 4.1 and consider extensions to serially correlated errors in Section 4.2.

4.1 The case with martingale difference errors

Since some testing problems imply breaks under the null hypothesis, we need some conditions to ensure that the estimates of the break fractions are consistent at a fast enough rate and that the estimates of the parameters are also consistent. This problem was analyzed by Qu and Perron (2007a) and we simply impose the same set of assumptions. If breaks are allowed in the regression coefficients under both the null and alternative hypotheses, we specify the following conditions:

• Assumption A1: The conditions stated in Assumptions A1-A4 and A6-A8 of Qu and Perron (2007a) are assumed to hold.

When the null hypothesis specifies no change in the regression coefficients, we shall assume, with $w_t = (x_t', z_t')'$:

• Assumption A2: $T^{-1} \sum_{t=1}^{[Ts]} w_t w_t' \rightarrow_p sQ$ uniformly in $s \in [0, 1]$, with Q some fixed positive definite matrix.

Assumption A2 rules out trending regressors and imposes the condition that the limit moment matrix of the regressors is homogeneous throughout the sample. Hence, we avoid the case where the marginal distribution of the regressors may change while the coefficients do not (see, e.g., Hansen, 2000). This follows from our basic premise that regimes are defined by changes in some coefficients. When changes in the variance of the errors are allowed under both the null and alternative hypotheses, we specify

• Assumption A3: The conditions stated in Assumption A5 of Qu and Perron (2007a) are assumed to hold with the addition that the errors $\{u_t\}$ form an array of martingale differences relative to $\mathcal{F}_t = \sigma\text{-}field\ \{..., z_{t-1}, z_t, ..., x_{t-1}, x_t, ..., u_{t-2}, u_{t-1}\}$.

When the null hypothesis imposes no change in variance, we shall need

• Assumption A4: The errors $\{u_t\}$ form an array of martingale differences relative to $\mathcal{F}_t = \sigma\text{-}field\ \{..., z_{t-1}, z_t, ..., x_{t-1}, x_t, ..., u_{t-2}, u_{t-1}\}$ and, additionally, $E\left(u_t^2\right) = \sigma_0^2$ for all t and $T^{-1/2}\sum_{t=1}^{[Ts]} z_t u_t \Rightarrow \sigma_0 Q^{1/2} W_q\left(s\right)$, where $W_q\left(s\right)$ is a q-vector of independent Wiener processes. Also, $T^{-1/2}\sum_{t=1}^{[Ts]}(u_t^2/\sigma_0^2 - 1) \Rightarrow \psi W(s)$, where $W(s)$ is a Wiener process independent of $W_q(s)$ and $\psi = \lim_{T\to\infty} var(T^{-1/2}\sum_{t=1}^{T}(u_t^2/\sigma_0^2) - 1)$.

Assumption A4 rules out instability in the error process and states that a basic functional central limit theorem holds for the weighted partial sums of the errors and their squares. Note that A4 assumes away serial correlation in the errors u_t. This will be relaxed later. The limiting distributions, under the relevant null hypotheses, of the sup-Likelihood ratio tests for testing problems TP-1 to TP-4 are stated in the following Theorem, where "$\Rightarrow$" denotes weak convergence under the Skorohod topology and "$||\cdot||$" denotes the Eucledian norm.

Theorem 1 *Under the relevant null hypotheses, we have, as $T\to\infty$, a) For TP-1, under A2 and A4:*

$$\sup LR_{1,T}\left(n_a, \varepsilon | m = n = 0\right) \Rightarrow \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_{v,\epsilon}} \frac{\psi}{2}\sum_{i=1}^{n_a}\frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)}.$$

b) For TP-2, under A1 and A4:

$$\begin{aligned}\sup LR_{2,T}\left(m_a, n_a, \varepsilon | n = 0, m_a\right) &\Rightarrow \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_{v,\epsilon}^c} \frac{\psi}{2}\sum_{i=1}^{n_a}\frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)} \\ &\leq \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_{v,\epsilon}} \frac{\psi}{2}\sum_{i=1}^{n_a}\frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)},\end{aligned}$$

where

$$\begin{aligned}\Lambda_{v,\varepsilon}^c &= \{\left(\lambda_1^v, ..., \lambda_{n_a}^v\right) :\ for\ (\lambda_1, ..., \lambda_K) = (\lambda_1^{0c}, ..., \lambda_{m_a}^{0c}) \cup (\lambda_1^v, ..., \lambda_{n_a}^v) \\ &\quad |\lambda_{j+1} - \lambda_j| \geq \varepsilon\ (j = 1, ..., K-1), \lambda_1 \geq \varepsilon, \lambda_K \leq 1 - \varepsilon\}\end{aligned}$$

$$\Lambda_{v,\varepsilon}=\{\left(\lambda_1^v,...,\lambda_{n_a}^v\right):\ \left|\lambda_{i+1}^v-\lambda_i^v\right|\geq\varepsilon\ (i=1,...,n_a-1),\lambda_1^v\geq\varepsilon,\lambda_{n_a}^v\leq 1-\varepsilon\}.$$

c) For TP-3, under A2-A3:

$$\begin{aligned}&\sup LR_{3,T}\left(m_a,n_a,\varepsilon|m=0,n_a\right)\\ \Rightarrow\ &\sup_{\left(\lambda_1^c,...,\lambda_{m_a}^c\right)\in\Lambda_{c,\epsilon}^v}\sum_{j=1}^{m_a}\frac{||\lambda_j^cW_q(\lambda_{j+1}^c)-\lambda_{j+1}^cW_q(\lambda_j^c)||^2}{\lambda_{j+1}^c\lambda_j^c(\lambda_{j+1}^c-\lambda_j^c)}\equiv H_c^*(m_a)\\ \leq\ &\sup_{\left(\lambda_1^c,...,\lambda_{m_a}^c\right)\in\Lambda_{c,\epsilon}}\sum_{j=1}^{m_a}\frac{||\lambda_j^cW_q(\lambda_{j+1}^c)-\lambda_{j+1}^cW_q(\lambda_j^c)||^2}{\lambda_{j+1}^c\lambda_j^c(\lambda_{j+1}^c-\lambda_j^c)}\equiv H_c(m_a),\end{aligned}\tag{8}$$

where

$$\begin{aligned}\Lambda_{c,\varepsilon}^v\ &=\ \{\left(\lambda_1^c,...,\lambda_{m_a}^c\right):\ \mathit{for}\ (\lambda_1,...,\lambda_K)=(\lambda_1^c,...,\lambda_{m_a}^c)\cup(\lambda_1^{0v},...,\lambda_{n_a}^{0v}),\\ &\quad|\lambda_{j+1}-\lambda_j|\geq\varepsilon\ (j=1,...,K-1),\lambda_1\geq\varepsilon,\lambda_K\leq 1-\varepsilon\},\end{aligned}$$

$$\Lambda_{c,\varepsilon}=\{\left(\lambda_1^c,...,\lambda_{m_a}^c\right):\ \left|\lambda_{j+1}^c-\lambda_j^c\right|\geq\varepsilon\ (j=1,...,m_a-1),\lambda_1^c\geq\varepsilon,\lambda_{m_a}^c\leq 1-\varepsilon\}.$$

d) For TP-4, under A2 and A4:

$$\begin{aligned}&\sup LR_{4,T}\left(m_a,n_a,\varepsilon|n=m=0\right)\\ \Rightarrow\ &\sup_{\left(\lambda_1^c,...,\lambda_{m_a}^c;\lambda_1^v,...,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon}\left[\begin{array}{c}\sum_{j=1}^{m_a}\frac{||\lambda_j^cW_q(\lambda_{j+1}^c)-\lambda_{j+1}^cW_q(\lambda_j^c)||^2}{\lambda_{j+1}^c\lambda_j^c\left(\lambda_{j+1}^c-\lambda_j^c\right)}\\ +\frac{\psi}{2}\sum_{i=1}^{n_a}\frac{(\lambda_i^vW\left(\lambda_{i+1}^v\right)-\lambda_{i+1}^vW(\lambda_i^v))^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v-\lambda_i^v\right)}\end{array}\right],\end{aligned}$$

where

$$\begin{aligned}\Lambda_\varepsilon\ &=\ \{\left(\lambda_1^c,...,\lambda_{m_a}^c,\lambda_1^v,...,\lambda_{n_a}^v\right):\ \mathit{for}\ (\lambda_1,...,\lambda_K)=(\lambda_1^c,...,\lambda_{m_a}^c)\cup(\lambda_1^v,...,\lambda_{n_a}^v),\\ &\quad|\lambda_{j+1}-\lambda_j|\geq\varepsilon\ (j=1,...,K-1),\lambda_1\geq\varepsilon,\lambda_K\leq 1-\varepsilon\}.\end{aligned}$$

Remark 3 *For the testing problems TP-2 and TP-3, the limit distributions depend on the true unknown values of the relevant break fractions corresponding to the break dates allowed under both the null and alternative hypotheses. These distributions can, however, be bounded by limit random variables which do not depend on such unknown values. This follows since* $\Lambda_{v,\varepsilon}^c\subseteq\Lambda_{v,\varepsilon}$ *and* $\Lambda_{c,\varepsilon}^v\subseteq\Lambda_{c,\varepsilon}$*. Hence, a conservative testing* *procedure is possible. As documented in Zhou and Perron (2008), the test is barely conservative if the trimming parameter ε is small, though if ε is large (e.g.,* 0.20*), the test will be somewhat undersized.*

The proof of this theorem is given in the appendix. For the testing problem TP-3, the bound is the same as the limit distribution in Bai and Perron (1998). Hence, the critical values provided by Bai and Perron (1998, 2003b) can be used. The limit distribution (for a one parameter change) for TP-1 is the same as the bounding limit distribution for TP-2, except for the scaling factor $(\psi/2)$. This quantity can nevertheless be consistently estimated. Consider the following class of estimates:

$$\hat{\psi} = T^{-1} \sum_{j=-(T-1)}^{T-1} \omega(j,m) \sum_{t=|j|+1}^{T} \hat{\eta}_t \hat{\eta}_{t-j} \tag{9}$$

where $\hat{\eta}_t = (\hat{u}_t^2/\hat{\sigma}^2) - 1$ and $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T} \hat{u}_t^2$ with $\hat{u}_t$ being the residuals under the null hypotheses. Here $w(j,m)$ is a weight function and m is some bandwidth parameter which can be selected using one of the many alternative methods that have been proposed; see e.g., Andrews (1991). The estimate $\hat{\psi}$ will be consistent under some conditions on the choice of $w(j,m)$ and the rate of increase of m as a function of T. Following Kejriwal and Perron (2006a), we use the residuals under the null hypothesis to construct $\hat{\psi}$ but the residuals under the alternative hypothesis to select the bandwidth parameter m (see also Kejriwal, 2007). Simulations have shown that using the residuals under the alternative hypothesis to both select m and construct $\hat{\psi}$ leads to tests with important size distortions. Using the residuals under the null for both leads to conservative and less powerful tests. Using the above hybrid method allows one to control the exact size in small samples, without significant loss of power (as documented in Zhou and Perron, 2008). In our empirical applications, we use the Quadratic Spectral kernel for the weight function and we adopt the method suggested by Andrews (1991) with an AR(1) approximation to select m.

Remark 4 *If the errors are i.i.d., $\psi = \mu_4/\sigma^4 - 1$, which can be consistently estimated using $\hat{\psi} = \hat{\mu}_4/\hat{\sigma}^4 - 1$, where $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2$ and $\hat{\mu}_4 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^4$ with $\hat{u}_t$ being the residuals under the null or alternative hypotheses. Also, if the errors are normally distributed, $\psi = 2$ so that no adjustment is necessary. This case was covered by Qu and Perron (2007a). Since these cases are less relevant in practical applications, we shall only consider a correction involving $\hat{\psi}$ as defined by (9). But it is useful to note that a simpler correction is available when the i.i.d. assumption is reasonable.*

We have the following statistics with nuisance parameter free limit distributions:

$$\begin{aligned}
\sup LR_{1,T}^{*} &= (2/\hat{\psi})\sup LR_{1,T} \\
&\Rightarrow \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_{v,\epsilon}} \sum_{i=1}^{n_a} \frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)} \equiv H_v(n_a) \qquad (10) \\
\sup LR_{2,T}^{*} &= (2/\hat{\psi})\sup LR_{2,T} \\
&\Rightarrow \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_{v,\epsilon}^c} \sum_{i=1}^{n_a} \frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)} \equiv H_v^{*}(n_a) \leq H_v(n_a).
\end{aligned}$$

For the testing problem TP-4, the transformation is more involved and given by

$$\sup LR_{4,T}^{*} = \sup LR_{4,T} - \frac{\hat{\psi} - 2}{\hat{\psi}} LR_v, \qquad (11)$$

where LR_v is the likelihood ratio test for no break in variance versus n_a breaks, evaluated using the estimates $\{\widetilde{T}_1^v, ..., \widetilde{T}_{n_a}^v\}$ obtained by maximizing the likelihood function that jointly allows for m_a breaks in the coefficients, i.e., $LR_v = 2[\log \hat{L}_T(\widetilde{T}_1^v, ..., \widetilde{T}_{n_a}^v) - \log \widetilde{L}_T]$ where $\log \hat{L}_T(\cdot)$ and $\log \widetilde{L}_T$ are defined by (4) and (3), respectively. Note that LR_v is not equivalent to $LR_{1,T}(n_a, \varepsilon | m = n = 0)$ which is based on the estimates of the break dates for the changes in variance assuming no break in coefficients. Since $\{\widetilde{T}_1^v/T, ..., \widetilde{T}_{n_a}^v/T\}$ are consistent estimates of the break fractions for the variance whether we have breaks in the coefficients or not, we deduce that

$$LR_v \Rightarrow (\psi/2) \sup_{\left(\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon} \sum_{i=1}^{n_a} \frac{(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right))^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)}$$

and hence,

$$\sup LR_{4,T}^{*} \Rightarrow \sup_{\left(\lambda_1^c,\ldots,\lambda_{m_a}^c;\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon} \left[\begin{array}{c} \sum_{j=1}^{m_a} \frac{||\lambda_j^c W_q(\lambda_{j+1}^c)-\lambda_{j+1}^c W_q(\lambda_j^c)||^2}{\lambda_{j+1}^c\lambda_j^c\left(\lambda_{j+1}^c-\lambda_j^c\right)} \\ +\sum_{i=1}^{n_a} \frac{(\lambda_i^v W\left(\lambda_{i+1}^v\right)-\lambda_{i+1}^v W(\lambda_i^v))^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v-\lambda_i^v\right)} \end{array} \right] \equiv H_{c,v}(m_a,n_a). \tag{12}$$

The limit distribution (12) is new. To obtain its critical values, we first simulate a dependent variable and q regressors as independent $N(0,1)$ random variables. This is done without loss of generality since the limit distribution does not depend on the distribution of the regressors and the use of normally distributed series ensures a closer correspondence to the asymptotic distribution for a given sample size, which we set to $T = 500$. The algorithm of Qu and Perron (2007a), imposing appropriate restrictions, is then used to obtain the estimates of the m_a break dates in coefficients and the n_a break dates in variance, using the trimming specified by Λ_ϵ. We then simulate a $(q+1)$ vector of independent Wiener processes, denoted $W_{q+1}(\cdot)$, as $T^{-1/2}$-scaled partial sums of independent $N(0,1)$ random vectors, again with $T = 500$. Finally, we evaluate the quantity

$$\sum_{j=1}^{m_a} \frac{||\lambda_j^c W_q\left(\lambda_{j+1}^c\right) - \lambda_{j+1}^c W_q\left(\lambda_j^c\right)||^2}{\lambda_{j+1}^c\lambda_j^c\left(\lambda_{j+1}^c-\lambda_j^c\right)} + \sum_{i=1}^{n_a} \frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v-\lambda_i^v\right)},$$

where $W_q(\cdot)$ contains the first q elements of the vector $W_{q+1}(\cdot)$ and $W(\cdot)$ is the $(q+1)^{th}$ element. This exercise is repeated 2,000 times to obtain the relevant quantiles of the distribution of the sum of the two terms. The critical values for tests of size 1%, 2.5%, 5% and 10% are presented in Table 1 for q between 1 and 5 and $\varepsilon = 0.1$, 0.15, 0.20 and 0.25. For $\varepsilon = 0.1$, $0.15, 0.2$, $m_a = 1,2$ and $n_a = 1,2$. For $\varepsilon = 0.25, m_a = 1$ and $n_a = 1$, given that $\varepsilon = 0.25$ imposes a maximal number of 2 breaks.

4.2 Extensions to serially correlated errors

We now consider the case where the errors u_t may be serially correlated. To this effect, Assumptions A3 and A4 are replaced by the following:

• Assumption A3*: The conditions stated in Assumption A5 of Qu and Perron (2007a) are assumed to hold.

And when the null hypothesis imposes no changes in variance, we shall need:

• Assumption A4*: $E(u_t^2) = \sigma_0^2$ for all t and $T^{-1/2}\sum_{t=1}^{[Ts]} z_t u_t \Rightarrow \omega Q^{1/2} W_q(s)$, where $W_q(s)$ is a q-vector of independent Wiener processes and $\omega = \lim_{T\to\infty} E(\sum_{t=1}^{T} u_t)^2$. Also, $T^{-1/2}\sum_{t=1}^{[Ts]}(u_t^2/\sigma_0^2 - 1) \Rightarrow \psi W(s)$, where $W(s)$ is a Wiener process independent of $W_q(s)$ and $\psi = \lim_{T\to\infty} var(T^{-1/2}\sum_{t=1}^{T}(u_t^2/\sigma_0^2) - 1)$.

For the testing problems TP-1 and TP-2, the same results apply as in the martingale difference case: sup $LR_{1,T}^*$ and sup $LR_{2,T}^*$ are asymptotically invariant to non-normal errors, serial correlation and conditional heteroskedasticity and the limit distribution (10) still holds. For the testing problems TP-3 and TP-4, things are more complex. Consider first TP-3. When the errors u_t are serially correlated, the limit distribution of the likelihood ratio test for changes in the coefficients depends on nuisance parameters and would be hard to implement in practice. In such a case, structural changes in the regression coefficients can still be tested for using the following sup-Wald type statistic that takes the presence of serial correlation into account: $\sup_{(\lambda_1^c,\ldots,\lambda_{m_a}^c)\in\Lambda_\varepsilon} F_{3,T}(m_a, n_a, \varepsilon | m = 0, n_a)$, where

$$F_{3,T}(m_a, n_a, \varepsilon | m = 0, n_a) = (T - (m_a+1)q - p)\,\hat{\delta}' R'(R\hat{V}(\hat{\delta})R')^{-1}R\hat{\delta}/(m_a q), \quad (13)$$

$\hat{\delta} = (\hat{\delta}_1', \ldots, \hat{\delta}_{m_a+1}')'$ are the QMLE of the coefficients that are subject to change under a given partition of the sample, R is the conventional matrix such that $(R\delta)' = (\delta_1' - \delta_2', \ldots, \delta_{m_a}' - \delta_{m_a+1}')$ and $\hat{V}(\hat{\delta})$ is an estimate of the variance covariance matrix of $\hat{\delta}$ that is robust to serial correlation and heteroskedasticity, i.e, a consistent estimate of $V(\hat{\delta}) = \text{plim}_{T\to\infty} T(\bar{Z}_\sigma^{*\prime}\bar{Z}_\sigma^*)^{-1}\Omega_{\bar{Z}_\sigma^*}(\bar{Z}_\sigma^{*\prime}\bar{Z}_\sigma^*)^{-1}$, where $\bar{Z}_\sigma^* = M_{X_\sigma}\bar{Z}_\sigma$, $\Omega_{\bar{Z}_\sigma^*} = E(\bar{Z}_\sigma^{*\prime}U_b^* U_b^{*\prime}\bar{Z}_\sigma^*)$, $U_b^* = M_{X_\sigma}U_\sigma$, $M_{X_\sigma} = I_T - X_\sigma(X_\sigma' X_\sigma)^{-1}X_\sigma'$, $X_\sigma = (x_1^\sigma, \ldots, x_T^\sigma)$, $\bar{Z}_\sigma = diag(Z_1^\sigma, \ldots, Z_{m_a+1}^\sigma)$, $Z_j^\sigma = (z_{T_{j-1}^c+1}^\sigma, \ldots, z_{T_j^c}^\sigma)'$, $U_\sigma = (u_1^\sigma, \ldots, u_T^\sigma)'$, $x_t^\sigma = (x_t^\sigma/\sigma_{i0})$,

$z_t^\sigma = (z_t/\sigma_{i0})$ and $u_t^\sigma = (u_t/\sigma_{i0})$, for $T_{i-1}^{v0} < t \leq T_i^{v0}$ $(i = 1, ..., n_a + 1)$. Under A2, A3* and additional assumptions under which a consistent estimate of $V(\hat{\delta})$ can be obtained using kernel based methods, as in Andrews (1991), the limiting distribution of $\sup F_{3,T}(m_a, n_a, \varepsilon | m = 0, n_a)$ is the same as for $\sup LR_{3,T}(m_a, n_a, \varepsilon | m = 0, n_a)$ with martingale difference errors, i.e, as stated in (8). In practice, the computation of this test could be very involved, especially if a data dependent method is used to construct $\hat{V}(\hat{\delta})$. Following Bai and Perron (1998), we suggest first finding the break points that correspond to the global maximization of the likelihood function defined by (5), then plugging these estimates into (13) to construct the test. This will not affect the consistency of the test since the break fractions are consistently estimated.

For the testing problem TP-4, things are more complex. We shall adopt a quasi-Wald testing procedure. Note first that the information matrix is block diagonal with respect to δ and σ^2, hence the test will involve one component for changes in δ and one for changes in σ^2. The first is the same as discussed above, namely $\sup F_{3,T}$ as defined by (13), except that one should use z_t, x_t and u_t in place of z_t^σ, x_t^σ and u_t^σ since the null hypothesis specifies no break in variance. Complications arise with the second component. The Wald test for equality in variance across regimes is asymptotically different from the LR test even with martingale difference errors. In fact, its limit distribution is quite complex and would necessitate additional tabulations of critical values. A simple compromise that leads to a consistent test is to sum the individuals Wald tests for each successive pair of regimes. This leads to the component

$$\sup F_T^\sigma = \hat{\psi}^{-1} \sum_{i=1}^{n_a} (\hat{\sigma}_{i+1}^2 - \hat{\sigma}_i^2)^2 (\hat{\sigma}_{i+1}^4 / (\widetilde{\lambda}_{i+1}^v - \widetilde{\lambda}_i^v) - \hat{\sigma}_i^4 / (\widetilde{\lambda}_i^v - \widetilde{\lambda}_{i-1}^v))^{-1},$$

where $\hat{\sigma}_i^2 = (\widetilde{T}_i^v - \widetilde{T}_{i-1}^v)^{-1} \sum_{t=\widetilde{T}_{i-1}^v+1}^{\widetilde{T}_i^v} \hat{u}_t^2$ and the estimates are constructed by maximizing the likelihood (5) subject to the restrictions imposed by the set Λ_ε. The test statistic suggested is then $\sup F_{4,T}(m_a, n_a, \varepsilon | m = n = 0) = \sup F_{3,T} + \sup F_T^\sigma$. It is easy to show that under A2 and A4*, the limit distribution of $\sup F_{4,T}$ is the same as that of the modified LR test with martingale difference errors, i.e., (12).

4.3 A double maximum test

The tests discussed above require the prior information on the number of breaks in the regression parameters and in the variance of the errors under the alternative hypothesis. In practice researchers may lack such information, hence the motivation for testing problems TP-5 to TP-8. Bai and Perron (1998) proposed so-called "double maximum" tests to tackle this testing problem with only breaks in the parameters. They are tests for no structural break against an unknown number of breaks, given some upper bound. Bai and Perron (1998) suggested two versions of such tests. The first is an equal-weight version, labelled $UD\max$. It can be given a Bayesian interpretation in which the prior assigns equal weight to the possible number of changes. The second test, denoted $WD\max$, applies weights to the individual tests such that the marginal p-values are equal across values of m and n. Bai and Perron (2006) showed via simulations that the two versions have similar finite sample properties. Hence, we shall only consider the $UD\max$ test because it is simpler to construct.

The Double Maximum test can play a significant role in testing for structural change and it is arguably the most useful test for trying to determine if structural changes are present. While the test for a single break is consistent against alternatives involving multiple changes, its power in finite samples can be poor. The primary advantages of the double maximum test are as follows. First, there are types of multiple structural changes that are difficult to detect with a test for a single change (for example, two breaks with the first and third regimes being the same). Second, tests for a particular number of changes may have non-monotonic power when the true number of changes is greater than specified. Third, the simulations of Bai and

Perron (2006) show, in the context of testing for changes in regression coefficients, that the power of the double maximum tests is almost as high as the highest power attainable using a test that accounts for the correct number of breaks. The tests and their limit distributions are presented in the following theorem.

Theorem 2 *Under the relevant null hypotheses, we have, as $T \to \infty$, a) For TP-5, under A2 and either A4 or A4* :*

$$UD\max LR_{1,T}^{*} = \max_{1 \le n_a \le N} \sup LR_{1,T}^{*}\left(n_a, \varepsilon | m = n = 0\right) \Rightarrow \max_{1 \le n_a \le N} H_v(n_a);$$

b) For TP-6, under A1 and either A4 or A4:*

$$\begin{aligned} UD\max LR_{2,T}^{*} &= \max_{1 \le n_a \le N} \sup LR_{2,T}^{*}\left(m_a, n_a, \varepsilon | n = 0, m_a\right) \\ &\Rightarrow \max_{1 \le n_a \le N} H_v^{*}(n_a) \le \max_{1 \le n_a \le N} H_v(n_a); \end{aligned}$$

c) For TP-7, under A2-A3:

$$\begin{aligned} UD\max LR_{3,T} &= \max_{1 \le m_a \le M} \sup LR_{3,T}\left(m_a, n_a, \varepsilon | m = 0, n_a\right) \\ &\Rightarrow \max_{1 \le m_a \le M} H_c^{*}(m_a) \le \max_{1 \le m_a \le M} H_c(m_a); \end{aligned}$$

d) For TP-8, under A2 and A4:

$$\begin{aligned} UD\max LR_{4,T}^{*} &= \max_{1 \le n_a \le N} \max_{1 \le m_a \le M} \sup LR_{4,T}^{*}\left(m_a, n_a, \varepsilon | n = m = 0\right) \\ &\Rightarrow \max_{1 \le n_a \le N} \max_{1 \le m_a \le M} H_{c,v}(m_a, n_a). \end{aligned}$$

For TP-5 to TP-7, the critical values of the limit distributions above are available from Bai and Perron (1998, 2003b) for N or M equal to 5. Note that for the testing problems TP-5 and TP-6, the results are valid whether the errors are martingale differences or serially correlated. This is not the case for TP-7 and TP-8 due to the same reasons discussed above: the likelihood ratio tests are not applicable when the errors are serially correlated. In this case, we consider the maximum of the Wald-type test and the results are presented in the following theorem.

Theorem 3 *Under the relevant null hypotheses, we have, as $T \to \infty$, a) For TP-7, under A2 and A3*:*

$$\begin{aligned} UD\max F_{3,T} &= \max_{1\le m_a \le M} \sup F_{3,T}\left(m_a, n_a, \varepsilon | m=0, n_a\right) \\ &\Rightarrow \max_{1\le m_a \le M} H_c^*(m_a) \le \max_{1\le m_a \le M} H_c(m_a); \end{aligned}$$

b) For TP-8, under A2 and A4:*

$$\begin{aligned} UD\max F_{4,T} &= \max_{1\le n_a \le N} \max_{1\le m_a \le M} \sup F_{4,T}\left(m_a, n_a, \varepsilon | n=m=0\right) \\ &\Rightarrow \max_{1\le n_a \le N} \max_{1\le m_a \le M} H_{c,v}(m_a, n_a). \end{aligned}$$

The limit distribution applicable to the testing problem TP-8 is new. We obtained its critical values using simulations as discussed above for the case of a fixed number of breaks under the alternative hypothesis. These are presented in Table 1 for $\varepsilon = 0.1$, 0.15, and 0.20, and values of M and N equal to 2.

4.4 Testing for an additional break

We now consider the testing problems TP-9 and TP-10 which look at whether including an additional break is warranted. Let $(\widetilde{T}_1^c, ..., \widetilde{T}_m^c; \widetilde{T}_1^v, ..., \widetilde{T}_n^v)$ denote the estimates of the break dates in the regression coefficients and the variance of the errors obtained jointly by maximizing the quasi-likelihood function (5) assuming m breaks in the coefficients and n breaks in the variance. For the testing problem TP-9, the issue is whether an additional break in the regression coefficients is present. Following Bai and Perron (1998) and Qu and Perron (2007a), the test is

$$\begin{aligned} \sup Seq_T\left(m+1, n | m, n\right) &= 2[\max_{1\le j \le m+1} \sup_{\tau \in \Lambda_{j,\varepsilon}^c} \log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_{j-1}^c, \tau, \widetilde{T}_j^c, ..., \widetilde{T}_m^c; \widetilde{T}_1^v, ..., \widetilde{T}_n^v) \\ &\quad - \log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_m^c; \widetilde{T}_1^v, ..., \widetilde{T}_n^v)], \end{aligned}$$

where $\Lambda_{j,\varepsilon}^c = \{\tau : \widetilde{T}_{j-1}^c + (\widetilde{T}_j^c - \widetilde{T}_{j-1}^c)\varepsilon \le \tau \le \widetilde{T}_j^c - (\widetilde{T}_j^c - \widetilde{T}_{j-1}^c)\varepsilon\}$. This amounts to performing $m+1$ tests for a single break in the regression coefficients within each of the $m+1$ regimes, defined by the partition $\{\widetilde{T}_1^c, ..., \widetilde{T}_m^c\}$. Note that the different scenarios that arise when allowing breaks in the coefficients and variance to occur at different dates imply two possible cases (since $(\widetilde{T}_1^c, ..., \widetilde{T}_m^c)$ and $(\widetilde{T}_1^v, ..., \widetilde{T}_n^v)$ can partly or completely overlap or be altogether different): 1) if the n break dates in variance are a subset of the m break dates in coefficients, then there is no variance break between $\widetilde{T}_{j-1}^c$ and $\widetilde{T}_j^c$; 2) otherwise, there is one or more variance breaks between $\widetilde{T}_{j-1}^c$ and $\widetilde{T}_j^c$. In either case, one can appeal to the results of part (c) of Theorem 1 with $m_a = 1$, since any value of n_a (the number of breaks in variance) is allowed, including 0. It is then easy to deduce that, in the case of martingale difference errors, under A1-A3 the limiting distribution of the test is given by $\lim_{T\to\infty} P(\sup Seq_T(m+1, n|m, n) \leq x) = G_{q,\varepsilon}(x)^{m+1}$, where $G_{q,\varepsilon}(x)$ is the cdf of the random variable $\sup_{\lambda\in\Lambda_{1,\varepsilon}} ||W_q(\lambda) - \lambda W_q(1)||^2/[\lambda(1-\lambda)]$, where $\Lambda_{1,\varepsilon} = \{\lambda : \varepsilon < \lambda < 1-\varepsilon\}$. The critical values can be found in Bai and Perron (1998, 2003b). With serial correlation in the errors, the principle is the same except that the statistic is based on the robust Wald test $\sup F_{3,T}$, as defined by (13), applied to a one break test for each segment. For the testing problem TP-10, similar considerations apply. Here the issue is whether an additional break in the variance is present. The test is

$$\sup Seq_T(m, n+1|m, n) = \frac{2}{\hat{\psi}}[\max_{1\leq j\leq n+1} \sup_{\tau\in\Lambda_{j,\varepsilon}^v} 2\log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_m^c; \widetilde{T}_1^v, ..., \widetilde{T}_{j-1}^v, \tau, \widetilde{T}_j^v, ..., \widetilde{T}_m^v)$$
$$-2\log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_m^c; \widetilde{T}_1^v, ..., \widetilde{T}_n^v)]$$

where $\Lambda_{j,\varepsilon}^v = \{\tau : \widetilde{T}_{j-1}^v + (\widetilde{T}_j^v - \widetilde{T}_{j-1}^v)\varepsilon \leq \tau \leq \widetilde{T}_j^v - (\widetilde{T}_j^v - \widetilde{T}_{j-1}^v)\varepsilon\}$. The correction factor $(2/\hat{\psi})$ is needed to ensure that the limit distribution of the test is free of nuisance parameters when the errors are allowed to be non-normal, serially correlated and/or conditionally heteroskedastic. One can then use part (b) of Theorem 1 with $n_a = 1$ to deduce that, under A1 and A4 or A1 and A4* applied to each segment under the null hypothesis, $\lim_{T\to\infty} P(\sup Seq_T(m, n+1|m, n) \leq x) = G_{1,\varepsilon}(x)^{n+1}$.

5 Estimating the numbers of breaks in coefficients and in variance

We now discuss a specific to general sequential procedure for estimating the number of breaks in coefficients and the variance. We start by using a modification of the sequential procedure discussed in Qu and Perron (2007a). The problem here is, however, more complex since we wish to ascertain not only to know whether some kind of break occured but also the types of breaks that occured at each break date. Hence, we need some refinements. The main difficulty is the fact that if a break occurs, it can be associated with a change in either or both the regression coefficients and the variance, and a method to decide which case is in effect needs to be applied.

The starting point is to use the $\sup LR^*_{4,T}$ with $m_a = n_a = 1$ in a sequential manner to test $H_0 : \{m = \ell, n = \ell\}$ versus $H_1 : \{m = \ell + 1, n = \ell + 1\}$. The procedure tests the null hypothesis of ℓ breaks versus the alternative of $\ell + 1$ breaks by performing a one break test within each of the $\ell + 1$ segments defined by the partition $(\hat{T}_1, ..., \hat{T}_\ell)$, the estimates of the break dates obtained by maximizing the likelihood (5) with $T^c_j = T^v_j = T_j$ for $j = 1, ..., \ell$. The statistic is then the maximal value of the tests over all $\ell + 1$ segments, denoted $\sup Seq_T(\ell + 1|\ell)$. Its limit distribution is given by $\lim_{T\to\infty} P\left(\sup Seq_T\left(\ell + 1|\ell\right) \leq x\right) = G_{q+1,\varepsilon}\left(x\right)^{\ell+1}$, where $G_{q+1,\varepsilon}\left(x\right)$ is the distribution function of $\sup_{\lambda\in\Lambda_{1,\varepsilon}} ||W_{q+1}\left(\lambda\right) - \lambda W_{q+1}\left(1\right)||^2/[\lambda\left(1 - \lambda\right)]$. Upon rejection, a model with $\ell + 1$ breaks is preferred with the additional break inserted in the segment associated with the maximal value of the likelihood function. This procedure is iterated until a non-rejection and the number of breaks selected denoted by $\bar{K}$.

The next step is to decide whether a break in coefficients, variance or both has occurred at each of the selected break dates. We perform standard hypothesis testing for the equality of the parameters across adjacent segments. Since the limiting distribution of the estimates of the parameters are the same whether using the estimates or the true values of the break dates, standard procedures can be applied. Consider first testing whether the regression coefficients are equal across the two regimes $(\hat{T}_{k-1}, \hat{T}_k)$ (regime k) and $(\hat{T}_k, \hat{T}_{k+1})$ (regime $k+1$), separated by the k^{th} break ($k = 1, ..., \bar{K}$). The null hypothesis is then $H_0 : \delta_k^0 = \delta_{k+1}^0$ and the alternative hypothesis is $H_1 : \delta_k^0 \neq \delta_{k+1}^0$. Note that since there is either a break in the regression coefficients and/or the variance of the errors, under the null hypothesis there must be a change in the variance of the errors. Hence, the test to be applied is a standard Chow-type test allowing for a change in variance across regimes (see Goldfeld and Quandt, 1978).

Consider now the testing problem $H_0 : \sigma_{k,0}^2 = \sigma_{k+1,0}^2$ versus $H_1 : \sigma_{k,0}^2 \neq \sigma_{k+1,0}^2$. The Wald test corrected for potential non-normality and conditional heteroskedasticity, is

$$W_k = \frac{(\hat{T}_k - \hat{T}_{k-1})(\hat{T}_{k+1} - \hat{T}_k)}{(\hat{T}_{k+1} - \hat{T}_{k-1})(\hat{\mu}_4 - \hat{\sigma}^4)} \left(\hat{\sigma}_{k+1}^2 - \hat{\sigma}_k^2\right)^2,$$

with $\hat{\sigma}_k^2$ and $\hat{\sigma}_{k+1}^2$ as defined in (5) (allowing the regression coefficients to be different in regimes k and $k+1$) and $\hat{\mu}_4$ a consistent estimate of $E(u_t^4)$, e.g., $\hat{\mu}_4 = (\hat{T}_{k+1} - \hat{T}_{k-1})^{-1} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_{k+1}} \hat{u}_t^4$, constructed under the alternative to maximize power.

In general, the procedure works quite well in selecting the correct number and types of breaks (see Zhou and Perron, 2008). There are cases, however, where the probability of making the correct selection is quite low. The primiary case is when both changes in coefficients and variance are not large and occur at different dates, especially when their respective break dates are far apart. The reason for this is that the $\sup Seq_T(\ell + 1|\ell)$ statistic jointly tests for a break in both regression coefficients and variance. Hence, if only one type of break occurs, the power can be quite low unless the magnitudes of the breaks are large. Unfortunately, this situation is expected to be quite common in practice (see Zhou and Perron, 2008). Hence, though this specific to general procedure is valid in large samples, it should not be applied mechanically. Care must be exercised to assess whether one is in a situation for which the procedure's finite sample properties are rather poor. An alternative approach is to use a general to specific type of procedure to determine the appropriate number and types of breaks. This involves using the battery of tests presented in this paper in a judicious way. The procedure cannot be mechanized but is likely to deliver better results. We illustrate an application of this procedure in the next section.

6 Application

Stock and Watson (2002) presented an exhaustive analysis that documented facts about potential changes in the volatility of macroeconomic time series using a two step approach. Of interest is the fact that many such series seem to exhibit a decline in volatility in the mid 80s. To illustrate our methods, we consider the consumption of non-durables time series whose graph is given in Figure 1 (the data are quarterly, covering the period 1960-2001; the data source is the DRI-McGraw Hill Basic Economics database and the data was kindly posted by Mark Watson on his web page). To achieve stationarity, we consider annual growth rates ($100\ln(x_t/x_{t-4})$, with x_t being the original series). The regression is a simple autoregression of order 4 with a fitted intercept. All tests are based on the trimming parameter $\varepsilon = 0.15$.

We first discuss how we used our testing procedures to select the number of breaks in coefficients (intercept and autoregressive parameters) and variance. With the types of breaks in this series, the sequential procedure outlined in Section 5 did not perform well. This is due to the fact that changes in both the coefficients and the variance occured at different times, a case for which the specific to general procedure performs poorly. Hence, we used a procedure more akin to a general to specific one. To start, we set an upper bound of two breaks for each of the coefficients and variance, implying a maximum of four total breaks. We later present evidence that two breaks are enough.

The first statistic used was the $UD\max$ test with $M = N = 2$, which is significant at the 5% level (value of 41.63). We then computed a wide range of tests to assess whether a model with $m = n = 1$ is adequate. First, the $\sup LR^*_{4,T}(1,1)$ is indeed significant at the 1% level (value of 30.17). We then considered the sequential test $\sup Seq_T\,(m+1, n|m, n)$ to assess whether too few coefficient breaks are included when $m = n = 1$. The test $\sup Seq_T\,(2,1|1,1)$ is insignificant (value of 11.62), indicating that an additional break in coefficients is unwarranted. The $\sup Seq_T\,(1,2|1,1)$ test is also insignificant (value of 2.29), indicating that a second break in variance is also unwarranted. The $\sup LR_{3,T}(1,1|0,1)$ test is significant at the 1% level (value of 25.55) and the $\sup LR^*_{2,T}(1,1|1,0)$ is significant at the 10% level (value of 7.44), indicating that one break in each coefficients and variance is present.

The parameter estimates are the following: the break date in coefficients is 1992:1 and in variance is 1982:1; the value of the intercept in the first regime is 0.008 and 0.005 in the second, indicating that a change in mean did not occur; the sum of the autoregressive coefficients in each regime are 0.692 and 0.826, respectively, indicating an increase in persistence in 1992:1; the standard deviation of the errors before 1982:1 is 0.010 while it is 0.006 after, indicating a 40% reduction. These results contrast with those of Stock and Watson (2002) who, using an ad hoc two step procedure, found only a change in coefficients in 1991:4 and missed the reduction in variance.

Since the statistic $UD\max LR^*_{4,T}$ tests jointly for changes in coefficients and variance of the errors, it may lack power if only changes in variance occur (especially if the number of regression coefficients allowed to change is large). In that case, an alternative strategy is possible. It involves using the $UD\max LR^*_{1,T}$ and $SupLR^*_{1,T}$ tests to assess whether changes in variance are present, assuming no change in coefficients, and then computing the statistic $\sup LR_{3,T}\,(m_a, n_a, \varepsilon|m = 0, n_a)$ and $UD\max LR_{3,T}$, where n_a is the number of changes in variance found in the first step, as well as the statistic $\sup Seq_T\,(m, n+1|m, n)$. Non-rejections with these tests are then viewed as confirmatory evidence that the results based on the $UD\max LR_{1,T}$ and $SupLR^*_{1,T}$ tests were adequate (see Zhou and Perron, 2008 for applications). A similar strategy applies when considering testing for only breaks in coefficients.

7 Conclusion

This paper provided tools for testing for multiple structural breaks in the error variance and regression coefficients in a linear regression model. An innovation to the existing literature we offered here is the non-imposition of any restrictions on the break dates, i.e., the breaks in the regression coefficients and in the variance can happen at the same or different times. We proposed statistics with asymptotic distributions that are invariant to nuisance parameters and valid in the presence of non-normal errors and conditional heteroseksaticity, as well as serial correlation. Extensive simulations of the finite sample properties presented in Zhou and Perron (2008) show that our procedures perform well in terms of size and power, though a specific to general procedure to estimate the number and types of breaks has some shorthcomings when the breaks in coefficients and variance of the errors are small and occur at different dates. To that effect, we presented an alternative general to specific procedure for identifying the number and types of breaks. Further applications are presented in Zhou and Perron (2008), illustrating the usefulness of our tests.

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Appendix

Proof of Theorem 1: Part (a) follows from Qu and Perron (2007a, Th. 5). For (b),

$$\begin{aligned}
&\sup LR_{2,T}\left(m_a, n_a, \varepsilon | n=0, m_a\right) \\
= \;& 2[\log \hat{L}_T(\widetilde{T}_1^c, ..., \widetilde{T}_{m_a}^c; \widetilde{T}_1^v, ..., \widetilde{T}_{n_a}^v) - \log \widetilde{L}_T] = T\log\widetilde{\sigma}^2 - \sum_{i=1}^{n_a+1}(\widetilde{T}_i^v - \widetilde{T}_{i-1}^v)\log\hat{\sigma}_i^2 \\
= \;& \sum_{i=1}^{n_a}[\widetilde{T}_{i+1}^v \log\widetilde{\sigma}_{1,i+1}^2 - \widetilde{T}_i^v\log\widetilde{\sigma}_{1,i}^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)\log\hat{\sigma}_{i+1}^2] + \widetilde{T}_1^v\left(\log\widetilde{\sigma}_{1,1}^2 - \log\hat{\sigma}_1^2\right),
\end{aligned}$$

where $\widetilde{\sigma}_{1,i}^2 = (\widetilde{T}_i^v)^{-1}\sum_{t=1}^{\widetilde{T}_i^v}(y_t - x_t'\widetilde{\beta} - z_t'\widetilde{\delta}_{t,j})^2$ and $\widetilde{\delta}_{t,j} = \widetilde{\delta}_j$ for $\hat{T}_{j-1}^c < t \leq \hat{T}_j^c$ (also let $\delta_{t,j}^0 = \delta_j^0$ for $T_{j-1}^{c0} < t \leq T_j^{c0}$) ($j = 1, ..., m_a+1$). Applying Taylor expansions to $\log\widetilde{\sigma}_{1,i+1}^2$, $\log\widetilde{\sigma}_{1,i}^2$ and $\log\hat{\sigma}_{i+1}^2$, around σ_0^2, we obtain $\sup LR_{2,T}\left(m_a, n_a, \varepsilon|n=0, m_a\right) = \sum_{i=1}^{n_a}(F_{1,T}^i + F_{2,T}^i) + o_p(1)$, since $\widetilde{T}_1^v\left(\log\widetilde{\sigma}_{1,1}^2 - \log\hat{\sigma}_1^2\right) = o_p(1)$, where

$$\begin{aligned}
F_{1,T}^i &= \frac{1}{\sigma_0^2}[\widetilde{T}_{i+1}^v\widetilde{\sigma}_{1,i+1}^2 - \widetilde{T}_i^v\widetilde{\sigma}_{1,i}^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)\hat{\sigma}_{i+1}^2] \\
&= \frac{1}{\sigma_0^2}\sum_{t=\widetilde{T}_i^v+1}^{\widetilde{T}_{i+1}^v}\left[(y_t - x_t'\widetilde{\beta} - z_t'\widetilde{\delta}_{t,j})^2 - (y_t - x_t'\hat{\beta} - z_t'\hat{\delta}_{t,j})^2\right]
\end{aligned}$$

$$F_{2,T}^i = -\frac{1}{2}[\widetilde{T}_{i+1}^v(\frac{\widetilde{\sigma}_{1,i+1}^2 - \sigma_0^2}{\sigma_0^2})^2 - \widetilde{T}_i^v(\frac{\widetilde{\sigma}_{1,i}^2 - \sigma_0^2}{\sigma_0^2})^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)(\frac{\hat{\sigma}_{i+1}^2 - \sigma_0^2}{\sigma_0^2})^2].$$

Now we show that $F_{1,T}^i = o_p(1)$. We can express $F_{1,T}^i$ as

$$\frac{1}{\sigma_0^2}\left[\begin{array}{c}
(U_{i+1} + X_{i+1}(\beta^0 - \widetilde{\beta}) + Z_{i+1}(\delta_{t,j}^0 - \widetilde{\delta}_{t,j}))'(U_{i+1} + X_{i+1}(\beta^0 - \widetilde{\beta}) + Z_{i+1}(\delta_{t,j}^0 - \widetilde{\delta}_{t,j})) \\
-(U_{i+1} + X_{i+1}(\beta^0 - \hat{\beta}) + Z_{i+1}(\delta_{t,j}^0 - \hat{\delta}_{t,j}))'(U_{i+1} + X_{i+1}(\beta^0 - \hat{\beta}) + Z_{i+1}(\delta_{t,j}^0 - \hat{\delta}_{t,j}))
\end{array}\right]$$

$$= \frac{1}{\sigma_0^2}\left[\begin{array}{c}
(\hat{\beta} - \widetilde{\beta})'X_{i+1}'X_{i+1}(\hat{\beta} - \widetilde{\beta}) + (\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j})'Z_{i+1}'Z_{i+1}(\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j}) \\
+(\hat{\beta} - \widetilde{\beta})'X_{i+1}'Z_{i+1}(\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j}) + 2(\beta^0 - \hat{\beta})'X_{i+1}'X_{i+1}(\hat{\beta} - \widetilde{\beta}) \\
+2(\delta_{t,j}^0 - \hat{\delta}_{t,j})'Z_{i+1}'Z_{i+1}(\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j}) + 2(\hat{\beta} - \widetilde{\beta})'X_{i+1}'Z_{i+1}(\delta_{t,j}^0 - \hat{\delta}_{t,j}) \\
+2(\beta^0 - \hat{\beta})'X_{i+1}'Z_{i+1}(\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j}) + 2(\hat{\beta} - \widetilde{\beta})'X_{i+1}'U_{i+1} + 2(\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j})'Z_{i+1}'U_{i+1}
\end{array}\right].$$

The result follows using the facts that, from A1 and A4, $X_{i+1}'X_{i+1} = O_p(T)$, $Z_{i+1}'Z_{i+1} = O_p(T)$, $X_{i+1}'Z_{i+1} = O_p(T)$, $X_{i+1}'U_{i+1} = O_p(T^{1/2})$ and $Z_{i+1}'U_{i+1} = O_p(T^{1/2})$. Also, since under the null hypothesis with A1, the estimates of the break fractions for the changes in coefficients converge to the true break fractions at a fast enough rate so that estimates of the parameters of the models are consistent and have the same limit distribution as when the break dates are known, we have $\beta^0 - \hat{\beta} = O_p(T^{-1/2})$, $\delta^0_{t,j} - \hat{\delta}_{t,j} = O_p(T^{-1/2})$, $\hat{\beta} - \widetilde{\beta} = o_p\left(T^{-1/2}\right)$ and $\hat{\delta}_{t,j} - \widetilde{\delta}_{t,j} = o_p\left(T^{-1/2}\right)$. The last two quantities are $o_p(T^{-1/2})$ since $\sqrt{T}(\hat{\beta} - \beta^0)$ and $\sqrt{T}(\widetilde{\beta} - \beta^0)$ have the same limit distribution under the null hypothesis likewise for $\sqrt{T}(\hat{\delta}_{t,j} - \delta^0_{t,j})$ and $\sqrt{T}(\widetilde{\delta}_{t,j} - \delta^0_{t,j})$. Part (b) then follows from Lemma S.1 in Qu and Perron (2007b), since we have

$$\sum_{i=1}^{n_a} F^i_{2,T} \Rightarrow \sup_{\left(\lambda^v_1,\ldots,\lambda^v_{n_a}\right)\in\Lambda^c_{v,\epsilon}} \sum_{i=1}^{n_a} \frac{\psi}{2} \frac{\left(\lambda^v_i W\left(\lambda^v_{i+1}\right) - \lambda^v_{i+1} W\left(\lambda^v_i\right)\right)^2}{\lambda^v_{i+1}\lambda^v_i\left(\lambda^v_{i+1} - \lambda^v_i\right)}.$$

The proof of part (c) is similar. We have,

$$\begin{aligned}\sup LR_{3,T}\left(m_a, n_a, \varepsilon | m = 0, n_a\right) &= 2[\log \hat{L}_T(\widetilde{T}^c_1, ..., \widetilde{T}^c_{m_a}; \widetilde{T}^v_1, ..., \widetilde{T}^v_{n_a}) - \log \widetilde{L}_T(\hat{T}^v_1, ..., \hat{T}^v_{n_a})] \\ &= \sum_{i=1}^{n_a+1} (\hat{T}^v_i - \hat{T}^v_{i-1}) \log \widetilde{\sigma}^2_i - \sum_{i=1}^{n_a+1} (\widetilde{T}^v_i - \widetilde{T}^v_{i-1}) \log \hat{\sigma}^2_i,\end{aligned}$$

where $\widetilde{\sigma}^2_i = (\hat{T}^v_i - \hat{T}^v_{i-1})^{-1} \sum_{t=\hat{T}^v_{i-1}+1}^{\hat{T}^v_i} (y_t - x'_t\widetilde{\beta} - z'_t\widetilde{\delta})^2$ and $\hat{\sigma}^2_i = (\widetilde{T}^v_i - \widetilde{T}^v_{i-1})^{-1} \sum_{t=\widetilde{T}^v_{i-1}+1}^{\widetilde{T}^v_i} (y_t - x'_t\hat{\beta} - z'_t\hat{\delta}_{t,j})^2$. Applying Taylor extensions to $\log \widetilde{\sigma}^2_i$ and $\log \hat{\sigma}^2_i$ around σ^2_0, we have

$$\begin{aligned}\sup LR_{3,T}\left(m_a, n_a, \varepsilon | m = 0, n_a\right) &= \frac{1}{\sigma^2_0} \sum_{i=1}^{n_a+1} [(\hat{T}^v_i - \hat{T}^v_{i-1})\widetilde{\sigma}^2_i - (\widetilde{T}^v_i - \widetilde{T}^v_{i-1})\hat{\sigma}^2_i] \\ -\frac{1}{2} \sum_{i=1}^{n_a+1} [(\hat{T}^v_i - \hat{T}^v_{i-1})(\frac{\widetilde{\sigma}^2_i - \sigma^2_0}{\sigma^2_0})^2 &- (\widetilde{T}^v_i - \widetilde{T}^v_{i-1})(\frac{\hat{\sigma}^2_i - \sigma^2_0}{\sigma^2_0})^2] + o_p(1).\end{aligned}$$

Using arguments similar to those in part (b), under A1 and A2, the second term can be shown to be $o_p\left(1\right)$. For the first term,

$$\frac{1}{\sigma^2_0} \sum_{i=1}^{n_a+1} [(\hat{T}^v_i - \hat{T}^v_{i-1})\widetilde{\sigma}^2_i - (\widetilde{T}^v_i - \widetilde{T}^v_{i-1})\hat{\sigma}^2_i] = \frac{1}{\sigma^2_0} \sum_{j=1}^{m_a} [\widetilde{T}^c_{j+1}\bar{\sigma}^2_{1,j+1} - \widetilde{T}^c_j\bar{\sigma}^2_{1,j} - (\widetilde{T}^c_{j+1} - \widetilde{T}^c_j)\bar{\sigma}^2_{j+1}] + o_p(1),$$

where $\bar{\sigma}^2_{1,j} = (\widetilde{T}^c_j)^{-1} \sum_{t=1}^{\widetilde{T}^c_j} (y_t - x'_t\widetilde{\beta} - z'_t\widetilde{\delta})^2$ and $\bar{\sigma}^2_j = (\widetilde{T}^c_j - \widetilde{T}^c_{j-1})^{-1} \sum_{t=\widetilde{T}^c_{j-1}+1}^{\widetilde{T}^c_j} (y_t - x'_t\hat{\beta} - z'_t\hat{\delta}_{t,j})^2$. Following the proof of Theorem 5 in Qu and Perron (2007b), we have

$$\begin{aligned}&\frac{1}{\sigma^2_0} \sum_{j=1}^{m_a} [\widetilde{T}^c_j\bar{\sigma}^2_{1,j+1} - \widetilde{T}^c_j\bar{\sigma}^2_{1,j} - (\widetilde{T}^c_{j+1} - \widetilde{T}^c_j)\bar{\sigma}^2_{j+1}] \\ &\Rightarrow \sup_{\left(\lambda^c_1,\ldots,\lambda^c_{m_a}\right)\in\Lambda^v_{c,\epsilon}} \sum_{j=1}^{m_a} \frac{||\lambda^c_j W_q(\lambda^c_{j+1}) - \lambda^c_{j+1} W_q(\lambda^c_j)||^2}{\lambda^c_{j+1}\lambda^c_j(\lambda^c_{j+1} - \lambda^c_j)}.\end{aligned}$$

For part (d),

$$
\begin{aligned}
&\sup LR_{4,T}\left(m_a, n_a, \varepsilon | n=m=0\right) \\
=\;& 2[\sup_{\left(\lambda_1^c,\ldots,\lambda_{m_a}^c;\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon} \log \hat{L}_T\left(T_1^c,\ldots,T_{m_a}^c;T_1^v,\ldots,T_{n_a}^v\right) - \log \widetilde{L}_T] \\
=\;& 2[\log \hat{L}_T(\widetilde{T}_1^c,\ldots,\widetilde{T}_{m_a}^c;\widetilde{T}_1^v,\ldots,\widetilde{T}_{n_a}^v) - \log \widetilde{L}_T] = T\log\widetilde{\sigma}^2 - \sum_{i=1}^{n_a+1}(\widetilde{T}_i^v - \widetilde{T}_{i-1}^v)\log\hat{\sigma}_i^2 \\
=\;& \sum_{i=1}^{n_a}[\widetilde{T}_{i+1}^v \log\widetilde{\sigma}_{1,i+1}^2 - \widetilde{T}_i^v\log\widetilde{\sigma}_{1,i}^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)\log\hat{\sigma}_{i+1}^2] + \widetilde{T}_1^v(\log\widetilde{\sigma}_{1,1}^2 - \log\hat{\sigma}_1^2),
\end{aligned}
$$

where $\widetilde{\sigma}_{1,i}^2 = (\widetilde{T}_i^v)^{-1}\sum_{t=1}^{\widetilde{T}_i^v}(y_t - x_t'\widetilde{\beta})^2$ and $\hat{\sigma}_i^2$ is evaluated under the relevant break dates. Applying Taylor expansions to $\log\widetilde{\sigma}_{1,i+1}^2$, $\log\widetilde{\sigma}_{1,i}^2$ and $\log\hat{\sigma}_{i+1}^2$, we obtain after some algebra, $\sup LR_{4,T}\left(m_a, n_a, \varepsilon | n=m=0\right) = \sum_{i=1}^{n_a}\left(F_{1,T}^i + F_{2,T}^i\right) + o_p(1)$, where

$$
\begin{aligned}
\sum_{i=1}^{n_a} F_{1,T}^i &= \sum_{i=1}^{n_a}\frac{1}{\sigma_0^2}\left[\widetilde{T}_{i+1}^v\widetilde{\sigma}_{1,i+1}^2 - \widetilde{T}_i^v\widetilde{\sigma}_{1,i}^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)\hat{\sigma}_{i+1}^2\right] \\
&= \frac{1}{\sigma_0^2}\sum_{j=1}^{m_a}[\widetilde{T}_j^c\bar{\sigma}_{1,j+1}^2 - \widetilde{T}_j^c\bar{\sigma}_{1,j}^2 - (\widetilde{T}_{j+1}^c - \widetilde{T}_j^c)\bar{\sigma}_{j+1}^2],
\end{aligned}
$$

$$
F_{2,T}^i = -\frac{1}{2}[\widetilde{T}_{i+1}^v(\frac{\widetilde{\sigma}_{1,i+1}^2 - \sigma_0^2}{\sigma_0^2})^2 - \widetilde{T}_i^v(\frac{\widetilde{\sigma}_{1,i}^2 - \sigma_0^2}{\sigma_0^2})^2 - (\widetilde{T}_{i+1}^v - \widetilde{T}_i^v)(\frac{\hat{\sigma}_{i+1}^2 - \sigma_0^2}{\sigma_0^2})^2],
$$

with $\bar{\sigma}_{1,j}^2 = (\widetilde{T}_j^c)^{-1}\sum_{t=1}^{\widetilde{T}_j^c}(y_t - x_t'\widetilde{\beta})^2$ and $\bar{\sigma}_j^2 = (\widetilde{T}_j^c - \widetilde{T}_{j-1}^c)^{-1}\sum_{t=\widetilde{T}_{j-1}^c+1}^{\widetilde{T}_j^c}(y_t - x_t'\hat{\beta})^2$. >From the proof of part (c), we have

$$
\sum_{i=1}^{n_a} F_{1,T}^i \Rightarrow \sup_{\left(\lambda_1^c,\ldots,\lambda_{m_a}^c;\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon}\sum_{j=1}^{m_a}||\lambda_j^c W_q(\lambda_{j+1}^c) - \lambda_{j+1}^c W_q(\lambda_j^c)||^2/[\lambda_{j+1}^c\lambda_j^c(\lambda_{j+1}^c - \lambda_j^c)],
$$

and from that of part (b),

$$
\sum_{i=1}^{n_a} F_{2,T}^i \Rightarrow \sup_{\left(\lambda_1^c,\ldots,\lambda_{m_a}^c;\lambda_1^v,\ldots,\lambda_{n_a}^v\right)\in\Lambda_\varepsilon}\sum_{i=1}^{n_a}\frac{\psi}{2}\frac{\left(\lambda_i^v W\left(\lambda_{i+1}^v\right) - \lambda_{i+1}^v W\left(\lambda_i^v\right)\right)^2}{\lambda_{i+1}^v\lambda_i^v\left(\lambda_{i+1}^v - \lambda_i^v\right)}
$$

under assumptions A2 and A4. Hence, the result stated in the Theorem follows.

Table 1: Asympotic critical values of the sup $LR^*_{4,T}$ test (the entries are quantiles x such that $P\left(H_{c,v}(m_a, n_a) \leq x\right) = \alpha$)

		$\varepsilon = 0.10$				$\varepsilon = 0.15$				$\varepsilon = 0.20$			$\varepsilon = 0.25$	$UDmaxLR^*_4$		
		$n_a = 1$		$n_a = 2$		$n_a = 1$		$n_a = 2$		$n_a = 1$		$n_a = 2$	$n_a = 1$	$M = N = 2$		
q	α	$m_a = 1$	$m_a = 2$	$m_a = 1$	$m_a = 2$	$m_a = 1$	$m_a = 2$	$m_a = 1$	$m_a = 2$	$m_a = 1$	$m_a = 2$	$m_a = 1$	$m_a = 1$	$\varepsilon = 0.10$	$\varepsilon = 0.15$	$\varepsilon = 0.20$
1	.90	12.11	17.58	18.21	22.92	11.53	15.91	16.15	20.04	10.83	14.14	14.55	10.08	22.92	20.04	17.14
	.95	14.09	19.91	20.49	25.37	13.54	18.33	18.38	22.54	12.94	16.54	16.44	12.04	25.37	22.54	19.47
	.975	16.59	22.06	22.82	27.66	15.47	20.50	20.88	24.79	14.82	18.54	18.80	13.61	27.66	24.79	22.26
	.99	19.97	24.79	25.48	30.12	18.86	22.83	23.96	27.40	17.21	21.60	22.23	16.45	30.12	27.40	24.41
2	.90	14.94	22.52	20.44	27.36	14.05	20.71	18.56	24.59	13.07	18.32	16.34	12.19	27.36	24.59	20.83
	.95	17.34	24.72	22.72	29.90	16.56	23.42	20.59	27.06	15.24	21.17	18.81	14.02	29.89	27.06	23.29
	.975	19.04	26.84	25.06	32.91	18.41	25.59	22.66	29.55	17.62	23.42	20.65	15.84	32.91	29.55	26.15
	.99	20.82	29.94	27.49	34.72	19.99	27.87	25.13	32.06	18.99	26.15	23.39	18.33	34.72	32.06	29.34
3	.90	16.76	26.61	22.62	31.99	16.10	24.40	20.98	28.35	15.35	22.55	18.97	14.60	31.99	28.35	25.17
	.95	18.79	28.99	25.14	34.36	17.96	27.03	23.01	31.19	17.17	25.01	20.75	16.26	34.36	31.19	27.32
	.975	20.36	30.63	26.96	36.29	19.73	29.61	25.01	33.42	18.67	26.81	22.39	17.86	36.29	33.42	29.33
	.99	22.28	33.93	29.51	39.14	21.99	31.31	27.81	36.13	20.24	29.07	24.60	19.87	39.14	36.13	31.96
4	.90	19.31	30.63	25.07	36.07	18.31	28.14	22.66	31.94	17.37	26.05	20.57	16.41	36.07	31.94	28.33
	.95	21.54	33.71	27.34	38.91	20.49	30.84	24.81	34.34	19.43	28.48	22.84	18.75	38.91	34.34	30.91
	.975	23.81	36.50	29.78	41.25	22.52	33.50	26.84	37.31	21.54	30.88	25.11	20.55	41.25	37.01	33.20
	.99	26.37	39.79	31.87	44.50	24.84	37.10	29.50	41.07	24.31	34.31	26.88	22.69	44.51	41.07	36.08
5	.90	21.35	34.69	26.76	39.78	20.22	32.18	24.40	35.76	19.37	29.92	22.09	18.06	39.78	35.76	31.82
	.95	23.74	37.53	29.34	43.03	22.38	34.62	26.44	38.19	21.57	32.15	24.42	20.36	43.03	38.19	34.21
	.975	26.51	39.75	32.16	45.89	24.32	37.32	28.98	41.09	23.53	34.38	26.70	22.54	45.89	41.09	36.71
	.99	29.23	43.38	35.04	49.63	28.65	40.70	32.53	44.94	27.10	38.11	30.04	24.86	49.63	44.94	40.19

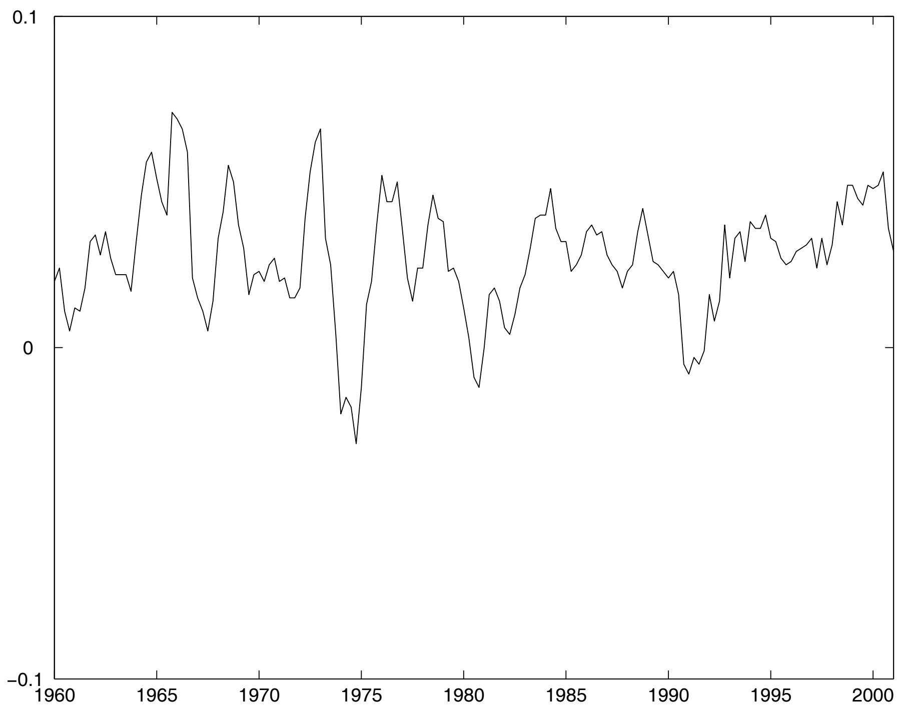

Figure 1: Consumption of non-durables; quarterly series of annual growth rates, 1960-2001