A Non-local Perspective on the Power Properties of the CUSUM and CUSUM of Squares Tests for Structural Change*

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Abstract

We consider the power properties of the CUSUM and CUSUM of squares tests in the presence of a one-time change in the parameters of a linear regression model. A result due to Ploberger and Krämer (1990) is that the CUSUM of squares test has only trivial asymptotic local power in this case, while the CUSUM test has non-trivial local asymptotic power unless the change is orthogonal to the mean regressor. The main theme of the paper is that such conclusions obtained from a local asymptotic framework are not reliable guides to what happens in finite samples. The approach we take is to derive expansions of the test statistics that retain terms related to the magnitude of the change under the alternative hypothesis. This enables us to analyze what happens for non-local to zero breaks. Our theoretical results are able to explain how the power function of the tests can be drastically different depending on whether one deals with a static regression with uncorrelated errors, a static regression with correlated errors, a dynamic regression with lagged dependent variables, or whether a correction for non-Normality is applied in the case of the CUSUM of squares. We discuss in which cases the tests are subject to a non-monotonic power function that goes to zero as the magnitude of the change increases, and uncover some curious properties. All theoretical results are verified to yield good guides to the finite sample power through simulation experiments. We finally highlight the practical importance of our results.

Keywords: Change-point, Mean shift, Local asymptotic power, Recursive residuals, Dynamic models.

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1 Introduction

A statistic which has played an important role in theory and applications related to structural change is the CUSUM test proposed by Brown, Durbin and Evans (1975) [1]. This test is based on the maximum of partial sums of recursive residuals. More precisely, for a linear regression with k regressors

$$y_t = x_t'\beta + u_t,$$

it is defined by

$$CUSUM^{rec} = \max_{k+1<r\leq T} \left| \frac{\sum_{t=k+1}^{r} \widetilde{u}_t}{\hat{\sigma}\sqrt{T-k}} \right| / (1 + 2\frac{r-k}{T-k}),$$

where throughout we shall use the estimate $\hat{\sigma}^2 = \sum_{t=k+1}^{T}(\widetilde{u}_t - \overline{\widetilde{u}})^2$ with $\overline{\widetilde{u}} = (T-k)^{-1}\sum_{t=k+1}^{T}\widetilde{u}_t$, as suggested by Harvey (1975), and $\widetilde{u}_t$ are the recursive residuals defined by

$$\begin{aligned} \widetilde{u}_t &= (y_t - x_t'\hat{\beta}_{t-1})/f_t, \\ f_t &= (1 + x_t'(X_{t-1}'X_{t-1})^{-1}x_t)^{1/2}, \end{aligned}$$

where X_{t-1} contains the observations on the regressors up to time $t-1$ and $\hat{\beta}_{t-1}$ is the OLS estimate of β using data up to time $t-1$ [2]. The limit distribution of the CUSUM test can be expressed in terms of the maximum of a weighted Wiener process, i.e.,

$$CUSUM^{rec} \Rightarrow \sup_{0\leq r\leq 1} \left| \frac{W(r)}{1+2r} \right|,$$

where $\Rightarrow$ denotes weak convergence under the Skorohod topology and $W(r)$ is a unit Wiener process defined on $[0,1]$, see Sen (1982). Also, it was shown by Krämer, Ploberger and Alt (1988) that the limit distribution remains valid even if lagged dependent variables are present as regressors. Ploberger and Krämer (1992) considered a version of the CUSUM using OLS residuals instead of recursive residuals defined by

$$CUSUM^{ols} = \max_{1<r\leq T} \left| \frac{1}{\hat{\sigma}\sqrt{T}} \sum_{t=1}^{[Tr]} \hat{u}_t \right|,$$

with $\hat{u}_t$ the OLS residuals and $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2$. Its limit distribution is given by $\sup_{0\leq r\leq 1}|BB(r)|$ where $BB(r) = W(r) - rW(1)$, a Brownian Bridge process (see also Krämer, Ploberger and Schluter, 1991). Their simulations showed the OLS-based CUSUM

[1] The literature on testing for structural change is extensive and the reader is referred to Perron (2006) for a comprehensive review.

[2] For a review of the use of recursive methods in the analysis of structural change, see Dufour (1982).

test to have higher power except for shifts that occur very early in the sample, the standard CUSUM tests having small power for late shifts. An alternative, also suggested by Brown, Durbin and Evans (1975), is the CUSUM of squares test (CUSQ). It takes the form:

$$CUSQ = \max_{k+1<r\leq T} \sqrt{T} \left| S_T^{(r)} - \frac{r-k}{T-k} \right|, \tag{1}$$

where

$$S_T^{(r)} = \left(\sum_{t=k+1}^{r} \widetilde{u}_t^2 \right) / \left(\sum_{t=k+1}^{T} \widetilde{u}_t^2 \right).$$

The limit distribution of CUSQ is derived by Ploberger and Krämer (1986) for the case of martingale difference errors. They show that it depends on the distribution of the errors, though a studentized version has the same limit distribution as in the Normal case. McCabe and Harrison (1980) suggested the use of CUSQ using least-squares instead of recursive residuals. Deng and Perron (2005) consider the limit distribution of the CUSQ allowing mixing conditions on the regressors and the errors, in particular serial correlation and conditional heteroskedasticity. They suggest a modification with a limit distribution free of nuisance parameters, namely the supremum of a Brownian Bridge process.

Another variant using partial sums is the fluctuations test of Ploberger, Krämer and Kontrus (1989) which looks at the maximum difference between the OLS estimate of β using the full sample and the OLS estimates using subsets of the sample from the first observation to some date t, ranging from $t = k$ to T. A similar test for a change in the slope of a linear trend function is analyzed in Chu and White (1992). Also, Chu, Hornik and Kuan (1995) looked at the maximum of moving sums of recursive and least-squares residuals. Kuan and Hornik (1995) provide a unified perspective on generalized fluctuations tests.

Ploberger and Krämer (1990) considered the local power functions of the CUSUM and CUSQ tests. The former has non-trivial local asymptotic power unless the mean regressor is orthogonal to all structural changes. On the other hand, the latter has only trivial local power (i.e., power equal to size) for local changes that specify a one-time change in the coefficients (see also Deshayes and Picard, 1986; Ploberger, 1989, also shows that the CUSQ has non-trivial local power against heteroskedasticity). This local analysis strongly suggests that to test for the presence of structural change in the regression coefficients, the CUSUM test should be preferred, and the use of OLS versus recursive residuals should be based on whether power is to be maximized for early or late shifts.

The main thrust of this paper is that such a conclusion based on a local asymptotic framework is a highly unreliable guide to what happens in practice for shifts that are not "very small". We analyze the power function of the tests for shifts that are not local to zero and reach conclusions that can be drastically different depending on the type of change and the presence of serial correlation and how one corrects the statistics to eliminate its effect. Our analysis considers the CUSUM and CUSQ tests with recursive (labeled $CUSUM^{rec}$ and $CUSQ^{rec}$) or OLS residuals (labeled $CUSUM^{ols}$ and $CUSQ^{ols}$) in the static case (i.e., no lagged dependent variable nor serial correlation in the errors) or dynamic models, with lagged dependent variables or with a non-parametric correction. Our theoretical analysis is based on expansions of the tests that keep terms that are of higher orders in the magnitude of the change, along with selected simulations to illustrate the implications in finite samples.

If we are dealing with a static regression with serially uncorrelated errors, the guidelines offered by the local asymptotic analysis of Ploberger and Krämer (1990) apply and the CUSUM test has indeed better power unless the change is orthogonal to the mean regressor. Our theoretical analysis also allows us to show that the CUSQ test constructed with OLS residuals has zero power when the break occurs at mid-sample. When the errors in the regression are serially correlated and commonly used non-parametric corrections are applied, both tests have power that can decrease to zero as the magnitude of the change increases. Partial conclusions to that effect have been documented for a simple mean shift model by Vogelsang (1999) and Crainiceanu and Vogelsang (2001) for the CUSUM with OLS residuals in the context of a regression where the errors are assumed to be potentially serially correlated so that a correction is applied (see also Hashimzade and Vogelsang, 2006). Our results extend their analysis to the CUSUM with recursive residuals and the CUSQ test. But more importantly, we show the mean change to be a knife-edge case in the sense that for structural changes involving regressors other than a constant this feature is not present.

When dealing with a dynamic regression with lagged dependent variables, things are very different from what is suggested by a local asymptotic power analysis. In the case of a change in mean, the non-monotonic power of the CUSUM test still applies. This, in particular, leads to the curious feature that the mere inclusion of an irrelevant lagged dependent variable as regressor is enough to subject the test to this problem. Furthermore, the magnitude of the change at which the reversal in power occurs is, in general, quite small, and well within what can be expected in practice. On the other hand, the power function of the CUSQ test is not subject to this problem. Indeed, for large breaks, the CUSQ is superior to the CUSUM, except for some specific exceptions, provided the errors are Normally distributed [3]. With

[3] Garbade (1977) presented simulation results for the static case in which the CUSUM of squares dominates the CUSUM, but these are due to the highly specialized nature of the experimental design in which the

non-Normal errors, it is necessary to apply a non-parametric correction to the CUSQ to have the desired size and, in the context of a dynamic model, this correction destroys the power of the CUSQ test in many cases of interest.

The paper is organized as follows. Section 2 considers the static regression. Section 3 analyzes the case when a correction for potential serial correlation in the errors is applied. Section 4 discusses the case of a dynamic regression. Section 5 offers concluding comments and summarizes the practical implications of our results. An appendix contains the proofs of the theoretical results stated in the text.

2 Static regression

We have a scalar dependent variable y_t $(t = 1, ..., T)$ and a k dimensional vector of regressors x_t. The data are assumed to be generated by

$$y_t = x_t'\beta_t + u_t, \tag{2}$$

$$\beta_t = \beta + \delta 1_{(t>[T\lambda])}, \tag{3}$$

for some $0 < \lambda < 1$, with $1_{(A)}$ the indicator function that takes value 1 if event A holds and zero otherwise. We impose the following assumptions of the errors and the regressors.

- Assumption A1. The disturbances u_t are stationary and ergodic, with $E(u_t|U_t) = 0$, $E(u_t^2|U_t) = \sigma_u^2$, and $E(u_t^4) < \infty$, where U_t is the σ field generated by $\{x_{t-s}, u_{t-s-1}|s \geq 0\}$.

- Assumption A2. The regressors x_t satisfy $p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} x_t = c$, some constant $k \times 1$ vector, $p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{[Tr]} x_t x_t' = R(r)$ and $p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T} (x_t x_t')^2 = \Gamma$, where $R(r)$ and Γ are nonsingular constant $k \times k$ matrices (as a matter of notation, $R(1) \equiv R$).

Note that the limit distribution of the CUSQ test depends on the exact distribution of the errors. The form of the statistic given by (1) will have a limit distribution invariant to nuisance parameters with Normal errors that are uncorrelated and conditionally homoskedastic. The limit distribution of the CUSUM does not depend on the Normality assumption, though it depends on the presence of correlation in the errors. In this section, we first consider the limit of the standard CUSUM and CUSQ tests as defined in the introduction. In Section 2.1, we consider the properties of a modification of the CUSQ test that incorporates a correction

regressor is a Normal random variable with mean 0, making the problem akin to that of a change in variance.

for potential non-Normality and conditional heteroskedasticity. In Section 3, we consider the properties of both tests when a correction for serial correlation (and non-Normality for the CUSQ) is applied.

We first consider simple simulations. For that purpose, we follow the framework used in Krämer, Ploberger and Alt (1998). The vector of regressors is $x_t = \{x_{1t}, x_{2t}\} = \{1, (-1)^t\}'$ and the change in the coefficients is specified by $\delta = b[\cos(\psi), \sin(\psi)]'$. Here ψ is the angle between δ and the mean regressor $x_{1t} = \{1\}$. When $\psi = 0$, only the intercept changes and when $\psi = 90°$ the shift occurs only for the regressor $x_{2t} = \{(-1)^t\}$. The break is assumed to occur at date $[\lambda T]$ and we consider $\lambda = 0.3$, 0.5 and 0.7. The sample size is set to $T = 120$, the errors u_t are *i.i.d.* $N(0, 1)$, and we consider three types of breaks with $\psi = 0$, 45° and 90°. The nominal size of the test is 5% and we use the asymptotic critical values to evaluate the power. The results are presented in Table 1 for the CUSUM and Table 2 for the CUSQ.

The results show some interesting features. First, as expected from the local asymptotic power analysis of Ploberger and Krämer (1990), the power of the CUSUM is superior to that of the CUSQ when the shift is small. But the power of the CUSQ test increases rapidly as the magnitude of the change increases. Also, its power function shows interesting differences when using OLS or recursive residuals. For an early break, the power with OLS residuals is higher than with recursive residuals, and vice versa for a late break. More interestingly, with OLS residuals the power is virtually zero when a break occurs at mid-sample, while it remains high with recursive residuals. As documented in Ploberger and Krämer (1992), the power of the CUSUM test is higher with recursive residuals when the break occurs early in the sample, and vice versa, the power is higher with OLS residuals when the break occurs late in the sample. The following Theorem provides theoretical explanations for these findings.

Theorem 1 *Under the assumption that the data are generated by (2) and (3) satisfying A1 and A2, we have, if the statistics are constructed from regression (2):*

$$(a)\ T^{-1/2}CUSQ^{ols} \xrightarrow{p} \frac{|\,(2\lambda-1)\,\lambda\,(1-\lambda)\,(\delta' R\delta)\,|}{\sigma_u^2 + \lambda\,(1-\lambda)\,(\delta' R\delta)};$$

$$(b)\ T^{-1/2}CUSQ^{rec} \xrightarrow{p} \frac{\lambda^2(1-\lambda)\delta' R\delta}{\sigma_u^2 + \lambda\,(1-\lambda)\,\delta' R\delta};$$

$$(c)\ T^{-1/2}CUSUM^{ols} \xrightarrow{p} \frac{|c'\delta|\lambda(1-\lambda)}{\sqrt{\sigma_u^2 + \lambda\,(1-\lambda)\,\delta' R\delta}};$$

$$(d)\ T^{-1/2}CUSUM^{rec} \xrightarrow{p} \frac{-|c'\delta|\lambda\log(\lambda)/3}{\sqrt{\sigma_u^2 + \lambda\,(1-\lambda)\,\delta' R\delta - (c'\delta\lambda\log(\lambda))^2}}.$$

The results first imply that the power function of the test is monotonic in the sense that, as $||\delta||$ gets large the limit is non-decreasing (this hold generally for the CUSQ tests; for the CUSUM it holds for the cases considered in the simulations but may not be true in general; however, the limit never tends to zero as $||\delta||$ increases). The result of part (a) shows that the power problem of $CUSQ^{ols}$ when the break is near mid-point is not specific to the simulation design used but holds with more general regressors and types of changes. It also allows us to get an idea about the region where zero power can be expected. With $||\delta||$ large, the limit is approximately $|2\lambda - 1|$. Hence, zero power will occur if $T^{1/2}|2\lambda - 1|$ is less than the critical value used. With a 5% test and $T = 100$, this occurs when $0.4 \precsim \lambda \precsim 0.6$. The intuition for this result is that least-squares allocates the sum of squared residuals in an inverse proportion to the length of the regime (pre and post break), which is why the test has power. With a break at mid-sample, the sum of squared residuals is equal for both segments and the test accordingly takes a zero value.

The result in part (b) shows that the power problem at mid-sample is not present when using recursive residuals. The CUSQ with OLS residuals will be more powerful than the CUSQ with recursive residuals when $|2\lambda - 1| > \lambda$, i.e, when $\lambda < 1/3$ so that the break occurs early in the sample. Also, for large breaks, the power of the CUSQ with recursive residuals will be maximized the later the break is. For the CUSUM tests, when the structural break is orthogonal to the mean regressor, i.e., $c'\delta = 0$, both CUSUM tests have a limit of 0. This is a well-known result previously obtained in a local asymptotic framework.

2.1 The CUSQ test with non-Normal errors.

It is well known that the limit distribution of the CUSQ test depends on the distribution of the errors. As shown in Ploberger and Krämer (1986) (see also, Deng and Perron, 2005), if the errors are uncorrelated and conditionally homoskedastic,

$$CUSQ \Rightarrow \frac{\gamma^{1/2}}{\sigma^2} \sup_{r \in [0,1]} |BB(r)|,$$

where $BB(r)$ is a Brownian Bridge process and $\gamma = E(u_t^2 - \sigma^2)^2 = \mu_4 - \sigma^4$, where $\mu_4 = E(u_t^4)$. This result holds whether OLS or recursive residuals are used (Deng and Perron, 2005). Hence, to account for non-Normal errors, one needs to use the following modification

$$CUSQ_M = \frac{\hat{\sigma}^2}{\hat{\gamma}^{1/2}} CUSQ,$$

where $\hat{\gamma} = \hat{\mu}_4 - \hat{\sigma}^4$, with $\hat{\mu}_4 = T^{-1}\sum_{t=1}^{T}\check{u}_t^4$, and $\check{u}_t$ denoting either the OLS or recursive residuals. An asymptotically equivalent representation is given by

$$CUSQ_M = \frac{\max_{1\leq r\leq T}\left|T^{-1/2}\left[\sum_{s=1}^{r}\check{u}_s^2 - \frac{[Tr]}{T}\sum_{s=1}^{T}\check{u}_s^2\right]\right|}{\hat{\gamma}^{1/2}}, \tag{4}$$

where it is understood, here and throughout the text, that in the case of recursive residuals, $u_s = 0$ for $s = 1, ..., k$. Table 3 presents the finite sample size and power of this modification using the same data-generating process as above. The results show that the behavior of the test is qualitatively the same as in the case with Normal errors, though some power loss can be observed. There is, however, one interesting difference in that for $\lambda = 0.5$, the $CUSQ^{ols}$ no longer has zero power when $\psi = 0$ or $90°$ and the power increases as the magnitude of the change increases. This is explained by the following Theorem.

Theorem 2 *Under the assumptions that the data are generated by (2) and (3) satisfying A1 and A2, we have if the statistics are constructed from regression (2):*

$$(a)\ T^{-1/2}CUSQ_M^{ols} = \frac{|\,(2\lambda-1)\,\lambda\,(1-\lambda)\,(\delta' R\delta)\,|}{d(\delta,\lambda)^{1/2}} + o_p(1),$$
$$(b)\ T^{-1/2}CUSQ_M^{rec} = \frac{\lambda^2(1-\lambda)\delta' R\delta}{d(\delta,\lambda)^{1/2}} + o_p(1),$$

where

$$\begin{aligned} d(\delta,\lambda) &= \lambda\,(1-\lambda)\left(3\lambda^2 - 3\lambda + 1\right) g\,(\delta) - (1-\lambda)^2\,\lambda^2\,(\delta' R\delta)^2 + o\left(||\delta||^4\right), \\ g\,(\delta) &= p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\left(\delta' x_t x_t'\delta\right)^2. \end{aligned}$$

This Theorem shows that as $||\delta||$ increases, the order of magnitude of the numerator and denominator are the same. Hence, we should not expect a decrease in power resulting from the application of the correction for non-Normality; in particular it does not induce a non-monotonic power function. Further, in the case of a mean change, namely, $x_t = 1$, it can be shown that $d(\delta,\lambda) = (2\lambda-1)^2\,(1-\lambda)\,\lambda\delta^4 + o\,(||\delta||^4)$. Therefore, the factor $(2\lambda - 1)$ in the numerator and the denominator cancels out for $CUSQ^{ols}$. The same result holds when $x_t = (-1)^t$. This observation clearly explains our simulation findings that the power of $CUSQ^{ols}$ in those two cases is monotonically increasing, unlike the behavior under Normality.

3 Tests that account for serial correlation in the errors

We now relax the assumption on the errors while maintaining that the regression on which the tests are based is the static one, namely (2). The limit distribution of the tests can be obtained under various conditions on the errors and the regressors. For the CUSUM test the reader is referred to Tang and MacNeill (1993) and for the CUSQ test to Deng and Perron (2005). The limit distribution of the CUSUM test with recursive residuals is

$$CUSUM^{rec} \Rightarrow (\frac{h(0)}{\sigma^2})^{1/2} \sup_{0\leq r\leq 1} \left| \frac{W(r)}{1+2r} \right|,$$

where $h(0) = \lim_{T\to\infty} var(T^{-1/2}\sum_{t=1}^{T} u_t)$, which is equivalent to (2π times) the spectral density at frequency zero of u_t when the latter is a stationary process (with OLS residuals, the term $[W(r)/(1+2r)]$ is replaced by $BB(r)$, a Brownian Bridge process). For the CUSQ, we have, with either OLS or recursive residuals,

$$CUSQ \Rightarrow (\frac{\varphi^{1/2}}{\sigma^2}) \sup_{r\in[0,1]} |BB(r)|,$$

where $\varphi = \lim_{T\to\infty} var(T^{-1/2}\sum_{t=1}^{T} \xi_t)$ with $\xi_t = u_t^2 - \sigma^2$. We study the properties of modifications that have the same limit distribution as in the case with no serial correlation (and Normal errors for the CUSQ test). The modified CUSUM test is

$$CUSUM_S = \left(\frac{\hat{\sigma}^2}{\hat{h}(0)}\right)^{1/2} CUSUM,$$

where $\hat{h}(0)$ is, under the null hypothesis of no structural change, a consistent estimate of $h(0)$. We consider an estimate based on a weighted sum of autocovariances defined by:

$$\hat{\omega}_{WS}^2 = \hat{\gamma}_0 + 2\sum_{j=1}^{T-1} k(j,m)\,\hat{\gamma}_j,$$

with $\hat{\gamma}_j = T^{-1}\sum_{t=j+1}^{T} \check{u}_t\check{u}_{t-j}$, where $\check{u}_t$ are either the OLS or recursive residuals from regression (2), and $k(j,m)$ is some kernel function and m the bandwidth. Hence, the version of the CUSUM test analyzed is $CUSUM_S = \left(\hat{\sigma}^2/\hat{\omega}_{WS}^2\right)^{1/2} CUSUM$. The modification for the CUSQ test suggested by Deng and Perron (2005) is

$$CUSQ_S = \frac{\sup_{0\leq r\leq 1} \left| T^{-1/2}\left[\sum_{s=1}^{[Tr]} \check{u}_s^2 - \frac{[Tr]}{T}\sum_{s=1}^{T} \check{u}_s^2\right]\right|}{\hat{\varphi}^{1/2}},$$

where

$$\hat{\varphi} = T^{-1} \sum_{j=-(T-1)}^{(T-1)} k(j,m) \sum_{t=|j|+1}^{T} \check{\eta}_t \check{\eta}_{t-j},$$

with $\check{\eta}_t = \check{u}_t^2 - \hat{\sigma}^2$, $\hat{\sigma}^2 = T^{-1} \sum_{t=1}^{T} \hat{u}_t^2$ and $\check{u}_t^2$ denoting either the OLS or recursive residuals.

Tables 4 and 5 present the simulation results when these transformed statistics are applied to the same data-generating processes used in Section 2. For simplicity, we use the Bartlett kernel $k(j,m) = 1 - j/(m+1)$ if $j \leq m$ and 0 otherwise. A popular data-dependent method to determine the bandwidth is that proposed by Andrews (1991). Here, we use the $AR(1)$ approximation, so that $m \propto (C(\delta)T)^{1/3}$where $C(\delta) = 4\hat{\rho}(\delta)^2 / (1 - \hat{\rho}(\delta)^2)^2$ with $\hat{\rho}(\delta)$ the OLS estimate from a regression of $\check{u}_t$ on $\check{u}_{t-1}$[4]. The results are now quite different. For the case of a change in mean, the power of the tests initially increases as the magnitude of the change increases but rapidly decreases to zero as this change gets larger. This phenomenon is referred to as non-monotonic power and was analyzed by Perron (1991) for Gardner's (1969) structural change test and the analysis was generalized by Vogelsang (1999) and Crainiceanu and Vogelsang (2001) in studies which included the CUSUM with OLS residuals (see also Hashimzade and Vogelsang, 2006). Our results show this phenomenon to extend to the CUSUM with recursive residuals and to the CUSQ with either OLS or recursive residuals. However, the simulations also show that this reversal of power does not occur when the angle of the change (relative to the mean regressor) is 45° or 90° (for the CUSUM, the power of the test can increase when the angle is 90° and the power of the CUSQ is again close to zero with OLS residuals and a change at mid-sample). This non-monotonic power is therefore DGP specific and the question is then whether the behavior with a change in mean is a knife-edge case or is expected to hold more generally.

The following Theorem proved in the appendix provides an answer to this question. For the sake of generality, we consider a general kernel function $k(j,m)$ satisfying the regularity conditions stated in Andrews (1991), in particular the fact that $\sum_{j=1}^{T-1} |k(j,m)| = O(m)$. When Andrews' (1991) data dependent method to select m is used, we have $m = c(\hat{\alpha}T)^{1/\vartheta}$ where ϑ depends on the kernel (e.g., $\vartheta = 3$ for the Bartlett kernel and $\vartheta = 5$ for the Quadratic Spectral and others). The value of $\hat{\alpha}$ is obtained as described above for the simulations.

Theorem 3 *Let the data be generated by (2) and (3) satisfying A1 and A2, and the statistics*

[4] We also considered an autoregressive spectral density estimate for $\hat{h}(0)$ based on the following $AR(p)$ approximation estimated by OLS: $\check{u}_t = \sum_{j=1}^{p} \hat{d}_j \check{u}_{t-j} + \hat{v}_t$. The estimate is then $\hat{\omega}_{AR}^2 = \hat{\sigma}_v^2/(1 - \sum_{j=1}^{p} \hat{d}_j)^2$, where $\hat{\sigma}_v^2 = T^{-1} \sum_{t=p+1}^{T} \hat{v}_t^2$. The results are qualitatively similar and, hence, not reported.

constructed from regression (2): i) for the CUSUM test, if the following condition is satisfied:

$$\delta' \left(R_1 - R_0\right) \delta \to 0 \;\; as \; |\delta| \to \infty, \tag{5}$$

where $R_j = p\lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} x_t x'_{t-j}$, *we have*

$$(a)\; T^{-1/2+1/\vartheta} CUSUM_S^{ols} = \frac{|c'\delta|\lambda(1-\lambda)}{O(||\delta||^{5/3})} + o_p(1),$$

$$(b)\; T^{-1/2+1/\vartheta} CUSUM_S^{rec} = \frac{-|c'\delta| \log\left(\lambda\right)/3}{O(||\delta||^{5/3})} + o_p(1),$$

and if condition (5) is not satisfied,

$$(c)\; T^{-1/2+1/\vartheta} CUSUM_S^{ols} = \frac{|c'\delta|\lambda(1-\lambda)}{O(||\delta||)} + o_p(1),$$

$$(d)\; T^{-1/2+1/\vartheta} CUSUM_S^{rec} = \frac{-|c'\delta| \log\left(\lambda\right)}{O(||\delta||)} + o_p(1);$$

ii) for the CUSQ test, if the following condition is satisfied

$$\frac{1}{T} \sum_{t=2}^{T} \left(A_t A_{t-1} - A_t^2\right) \xrightarrow{p} 0, \tag{6}$$

where $A_t = \delta' x_t x'_t \delta$, *then,*

$$(e)\; T^{-1/2+1/\vartheta} CUSQ_S^{ols} = \frac{|\left(2\lambda - 1\right) \lambda \left(1-\lambda\right) \left(\delta' R \delta\right)|}{O(\|\delta\|^{2+4/3})} + o_p(1),$$

$$(f)\; T^{-1/2+1/\vartheta} CUSQ_S^{rec} = \frac{\lambda^2 (1-\lambda) \delta' R \delta}{O(\|\delta\|^{2+4/3})} + o_p(1),$$

and, otherwise,

$$(g)\; T^{-1/2+1/\vartheta} CUSQ_S^{ols} = \frac{|\left(2\lambda - 1\right) \lambda \left(1-\lambda\right) \left(\delta' R \delta\right)|}{O(\|\delta\|^{2})} + o_p(1),$$

$$(h)\; T^{-1/2+1/\vartheta} CUSQ_S^{rec} = \frac{\lambda^2 (1-\lambda) \delta' R \delta}{O(\|\delta\|^{2})} + o_p(1).$$

This Theorem is quite informative. First, consider the case with a given fixed change δ. The rate of divergence of the tests is $T^{1/2-1/\vartheta}$, slower than the $T^{1/2}$ rate that applies in the static case with no correction for serial correlation. This explains the substantially lower power of the test for relatively small changes (compare Table 4 with Table 1 and Table

5 with Table 2, for small shifts). The main result of interest, however, is that the limit of the statistics will be 0 as the magnitude of the change increases when condition (5) is satisfied for the CUSUM and (6) for the CUSQ.[5] Otherwise, the limits of the statistics are bounded above zero and the limit of the power as the change increases depends on a variety of factors. The intuition for this results is that a change in mean is akin to a permanent change in the level of the series and, as shown in Perron (1990), induces a bias of $\hat{\rho}(\delta)$ towards one. The larger the change the faster $\hat{\rho}(\delta)$ approaches one. Since the bandwidth is proportional to $C(\delta) = 4\hat{\rho}(\delta)^2 / (1 - \hat{\rho}(\delta)^2)^2$, this effect makes the bandwidth an increasing function of the change. A higher bandwidth then translates into summing more covariance terms that are themselves proportional to the change in mean so that the estimates $\hat{\omega}^2_{WS}$ or $\hat{\varphi}$ are an increasing function of $|\delta|$ and the value of the statistic eventually goes to zero as the magnitude of the change increases. For other types of changes, which do not imply a permanent effect on the level of the series, the bias of $\hat{\rho}(\delta)$ towards one is no present and, hence, the bandwidth is not an increasing function of the magnitude of the change.

Now, for the CUSUM, condition (5) is obviously satisfied if the change affects only the constant regressor, since then $\delta'(R_1 - R_0)\delta$ is exactly zero. In general, however, $R_1 \neq R_0$ and, unless some fortuitous cancellation occurs, condition (5) will not be satisfied. The same applies for the CUSQ test since condition (6) will hold when the change affects only the constant, but will otherwise not hold generally. Hence, the phenomenon of non-monotonic power, in the context studied here, can be seen as a knife edge case that does not generalize to cases involving changes affecting regressors other than a simple constant, though this later case is one that is practically important.

4 Dynamic Regressions

As discussed in the previous section, a non-parametric correction for possible serial correlation in the errors can lead to a substantial loss of power in the case of a change in mean (or the constant). It is, however, often the case that one can eliminate serial correlation in the residuals by introducing an appropriate number of lags of the dependent variable (and possibly the regressors) as additional regressors. We consider, in this section, the simple case where a lagged dependent variable is included as a regressor so that the data-generating

[5]Note that in the simulations, when the data dependent bandwidth m is larger than T, we truncate m at T, with the implication that it is then possible to have power to go back up again when the break magnitude increases further. This occurred only when $\psi = 90°$.

process and the regression estimated is

$$y_t = \alpha y_{t-1} + \beta_t x_t' + u_t, \tag{7}$$

and, again, $\beta_t = \beta + \delta 1_{(t>[T\lambda])}$. Note that no change in the coefficient of the lagged dependent variable is permitted and that, in general, this is a different model than the static one defined by (2). This dynamic model can be rewritten as

$$y_t = z_t' \left(\phi + \gamma 1_{(t>[T\lambda])}\right) + u_t, \tag{8}$$

where $z_t' = (y_{t-1}, x_t')$, $\phi = (\alpha, \beta)'$ and $\gamma = (0, \delta)'$. We first consider the basic form of the CUSQ test appropriate with uncorrelated and conditionally homoskedastic Normal errors, and return in Section 4.3 to the case where one would correct for potential non-Normality.

We again first present simple simulations. The setup is basically the same as in the previous sections, except that now the data are generated by (8). All other specifications remain the same and we consider the following values for the autoregressive parameter, $\alpha = -0.5$, 0.0, 0.5 and 0.8. The results are presented in Table 6 for the CUSUM test and in Table 7 for the CUSQ test. They show yet again a different picture of the qualitative properties of the tests.

Consider first the CUSUM test. The results now show that, unless the value of the autoregressive parameter is negative, the CUSUM test with either OLS or recursive residuals now exhibits non-monotonic power for both the case of a change in the constant regressor and that of a change at 45° relative to the constant (again the power is virtually zero if the change is orthogonal to the constant regressor). The case with $\alpha = 0$ and $\psi = 0$ is especially interesting when one compares the results to those in Table 1 corresponding to the static regression. They show that introducing an irrelevant lagged dependent variable is enough to drastically change the properties of the test and make its power function eventually decrease to 0 as the magnitude of the change increases!

As shown in Table 7, the behavior of the CUSQ test is very different. Here, the power is monotonic for all values of the autoregressive parameter. Also, for the CUSQ with OLS residuals and for a break occurring at mid-sample, the power is close to zero only when the change is at 45° relative to the constant. Here, the case with $\alpha = 0$ is especially interesting. As seen in Table 2, with *i.i.d.* Normal errors, the power of the CUSQ test with OLS residuals is 0 in large samples for all types of changes. What Table 7 shows is that introducing an irrelevant lagged dependent variable is enough to make the power of the test increase as the magnitude of the change increases, unless the change is at 45° relative to the constant!

The questions to be addressed are: a) is the non-monotonic power function of the CUSUM test now a generalized phenomenon that applies to all types of changes? Does the CUSUM test have a non-monotonic power function for all types of changes, except when the change is at mid-sample and at 45° relative to the constant? Is this latter case where power is zero a knife-edge case? The theoretical analysis below will attempt to shed light on these questions. We start with a Theorem that presents general results.

Theorem 4 *Suppose the data are generated by (7) and (8) with the regressors x_t and errors satisfying A1 and A2. As a matter of notation, let $S_1 = \sum_{j=1}^{\infty} \alpha^{j-1} R_j$ and $S_2 = (1-\alpha^2)^{-1} \sum_{j=-\infty}^{\infty} \alpha^{|j|} R_j$. Consider first the case with recursive residuals. For the CUSUM test, we have,*

$$T^{-1/2} CUSUM^{rec} = \frac{\sup_{r\in[0,1]} \left| N(\lambda,\delta,r)/(1+2r) \right|}{\sqrt{D(\lambda,\delta,1) - N(\lambda,\delta,1)^2}} + o_p(1),$$

with

$$\begin{aligned} N(\lambda,\delta,r) &= c'\delta(n-\lambda)1_{(r>\lambda)} - \int_\lambda^r \frac{s-\lambda}{s} ds\,(c'\delta) \\ &\quad -\lambda \int_\lambda^r \frac{s-\lambda}{s} A(s)^{11} \left(\frac{c'\delta}{1-\alpha} - \frac{s-\lambda}{s} c' R^{-1} S_1 \delta \right) ds\,(\delta' S_1 \delta), \end{aligned}$$

$$\begin{aligned} D(\lambda,\delta,r) &= \left(\int_\lambda^r (s-\lambda)^2 \Lambda(s,\delta)\, ds - 2\int_\lambda^r (\lambda^2/s^2)(s-\lambda) A(s)^{11} ds \right) (\delta' S_1 \delta)^2 \\ &\quad + \left(\lambda - r^{-1}\lambda^2\right)(\delta' R \delta), \end{aligned}$$

where

$$\begin{aligned} A(s)^{11} &= \left(A_{11}(s) - (1/s)(s-\lambda)^2 1_{(s>\lambda)} \delta' S_1 R^{-1} S_1 \delta \right)^{-1}, \\ A_{11}(s) &= (s-\lambda) 1_{(s>\lambda)} (\delta' S_2 \delta), \\ \Lambda(s,\delta) &= r^{-2}\lambda^2 \left(A(s)^{11}\right)^2 A_{11} + [s^{-4}(s+\lambda)\lambda^2(\lambda-s)] \left(A(s)^{11}\right)^2 \left(\delta' S_1 R^{-1} S_1 \delta\right) \\ &\quad + 2s^{-3}\lambda^2 A(s)^{11} \end{aligned}$$

and $A_{11} = (1-\lambda)(\delta' S_2 \delta)$. For the CUSUM of squares,

$$T^{-1/2} CUSQ^{rec} = \sup_{r\in[0,1]} \frac{|D(\lambda,\delta,r) - rD(\lambda,\delta,1)|}{D(\lambda,\delta,1)} + o_p(1).$$

When the statistics are constructed with OLS residuals, we have,

$$T^{-1/2} CUSUM^{ols} = \frac{\sup_{r\in[0,1]} |B(\lambda,\alpha,r,\delta)|}{\sqrt{(1-\lambda)\lambda[\delta' R\delta - (1-\lambda)\lambda A(1)^{11} (\delta' S_1 \delta)^2]}} + o_p(1),$$

where

$$
\begin{aligned}
B(\lambda, \alpha, r, \delta) &= (r-\lambda)\, 1_{(r>\lambda)} c'\delta - (1-\lambda)[rc'\delta - (1-\lambda)\,\lambda r A^{11} c' R^{-1} S_1 \delta\,(\delta' S_1 \delta) \\
&\quad + \lambda A^{11} c'\delta (r-\lambda) 1_{(r>\lambda)} \delta' S_1 \delta/(1-\alpha)]
\end{aligned}
$$

and

$$
T^{-1/2} CUSQ^{ols} = \sup_{r\in[0,1]} \frac{|\Upsilon(\lambda,\alpha,r) - r\Upsilon(\lambda,\alpha,1)|}{(1-\lambda)\,\lambda[\delta' R\delta - (1-\lambda)\,\lambda A(1)^{11}\,(\delta' S_1\delta)^2]} + o_p(1),
$$

with $\Upsilon(\lambda,\alpha,r)$ *as defined in (A.4) in the appendix.*

The results presented in this Theorem are difficult to analyze in full generality. In particular, for the case of a change in mean, $\delta' R\delta - (1-\lambda)\,\lambda A(1)^{11}\,(\delta' S_1\delta)^2 = 0$, so that we need to consider lower order terms. In the next subsections, we provide explicit formulae for important cases.

4.1 Specialized results for the case of a change in mean

In the following Proposition, we establish explicit results for the special case of a change in mean whose magnitude is given by the scalar δ.

Proposition 1 *Suppose the data are generated by (7) and (8) with the regressors and errors satisfying A1 and A2. Suppose also that only the coefficient associated with the constant regressor is subject to change, then we have*

$$
\begin{aligned}
T^{-1/2} CUSUM^{ols} &= \frac{O_p\left(|\delta|^{-1}\right) + O_p\left(|\delta|/T\right)}{\sqrt{O_p(1) + O_p\left(|\delta|^2/T\right)}}, \\
T^{-1/2} CUSUM^{rec} &= O_p\left(|\delta|^{-1}\right), \\
T^{-1/2} CUSQ^{ols} &= \frac{(2\lambda-1)\, O\left(|\delta|^{-2}\right) + O_p\left(|\delta|^2/T\right)}{O(1) + O_p\left(|\delta|^2/T\right)}, \\
T^{-1/2} CUSQ^{rec} &= \left[\frac{2\lambda(1-\lambda)}{(1+\alpha) + 2(1-\lambda)}\right] + o_p(1), \qquad (9)
\end{aligned}
$$

where for the last case the result holds as $|\delta| \to \infty$.

The results of this Proposition are quite interesting. First, with a change in mean, the recursive CUSUM test converges to zero as the magnitude of the change increases and, accordingly, the power also goes to zero for any significance level. This theoretical result, which holds for any value of α, is corroborated by the simulation results in Table 7. Note, however, that since the sample size is rather small ($T = 120$), the quality of our asymptotic approximation need not be accurate when the autoregressive parameter is large, e.g., $\alpha = 0.8$.

Consider now the CUSUM^{ols}. What is of interest here is the fact that when $|\delta|$ is large, such that $|\delta|^2/T$ is large, $CUSUM^{ols} = O_p(1)$. So in an asymptotic setting where $|\delta|^2$ would increase faster than T, the test does not converge to zero but it is inconsistent since it does not diverge. In finite samples, the power will depend, in particular, on whether this limit value exceeds the critical value used. Hence, the power can be zero or one depending on the size of the test. Our simulations show that, for large $|\delta|$, the statistic exceeds the 5% critical value for $T = 120$ when α is large, e.g., 0.8. With values of α below 0.5, the power quickly reaches zero as $|\delta|$ increases. Our theoretical result supports the simulation finding that with a change in mean in a static regression with uncorrelated errors, the mere introduction of an irrelevant lagged dependent variable is enough to drastically alter the power properties of the CUSUM test and reduce its power.

The behavior of the CUSQ^{ols} test is rather different from the CUSUM^{ols}. Consider first the case with $\lambda \neq 0.5$. The first thing to note is that for large $|\delta|$, $T^{-1/2}CUSQ^{ols} = O_p(1)$ so that the test diverges and is consistent, which explains the high power in the simulation results when $|\delta|$ is large. For a fixed T, increasing $|\delta|$ implies that $T^{-1/2}CUSQ^{ols} = O_p(1)$ and the power will depend on whether this quantity is above the critical value. For T large enough, it will always be. Our simulations show that it is the case with $T = 120$ and a 5% size, at least for the design considered. A curious feature can be obtained by considering the case where $|\delta|^2/\sqrt{T} = O(1)$. In this case, $CUSQ^{ols} = O_p(1)$. This suggest that for large breaks, it is possible to observe non-monotonic power with respect to T.

A more curious feature is the behavior of the CUSQ^{ols} when $\lambda = 0.5$. In this case, for a fixed δ, the limit value of the test is zero, as $T \to \infty$, and it is inconsistent. Further unreported simulations showed that the power of the CUSQ^{ols} can indeed decrease as the sample size increases for a given fixed break but the rate of decrease is slow and depends on the value of the autoregressive parameter. For example, consider the case with $\delta = 20$ and $\lambda = 0.5$. With $\alpha = 0$, the power is 1 when $T = 120$, with $T = 5,000$ the power is 0.75 and with $T = 8,000$ it is 0.58. When $\alpha = 0.2$, the power is 1 when $T = 120$, with $T = 5,000$ the power is 0.90 and with $T = 8,000$ it is 0.74. When $\alpha = 0.8$, the power is one whether $T = 120$ or $T = 8,000$. While inconsistency is, in general, a rather poor feature of any test statistic, it does not imply that the test has no power. This is especially the case here. Indeed, for any fixed sample size, the CUSQ^{ols} is bounded as the magnitude of the change increases and, hence, the power goes to one when the sample size is large enough. This is consistent with the results reported in Table 7, where it is shown that with $T = 120$, the change in mean need not be large for power to attain a value close to one. This is indeed a curious feature that makes a comparison between the CUSQ^{ols} and the CUSUM^{ols} difficult. Denote the power function of a test as a function of T and δ by $P(T, \delta)$. Then, with the limits taken sequentially, we have $\lim_{T\to\infty} \lim_{\delta\to\infty} P(T, \delta)$ is 1 for the CUSQ^{ols} but not necessarily so for the CUSUM^{ols} (it can be 0 or 1 depending on the data generating process). On the other hand, for any fixed δ, $\lim_{T\to\infty} P(T, \delta)$ is 1 for the CUSUM^{ols} and 0 for the CUSQ^{ols}. Hence, for any break magnitude, the CUSUM^{ols} will eventually have greater power than the CUSQ^{ols} as the sample size increases; and for any fixed sample, the CUSQ^{ols} will dominate the CUSUM^{ols} for large breaks. What the simulation results show is that for average to large breaks and common sample sizes, the latter effect dominates. Our theoretical results confirm the curious feature that including such an irrelevant lagged dependent variable can improve the power of the CUSQ^{ols} when the change occurs at mid-sample (recall that in this case with OLS residuals its power in the static regression is zero).

For the case of CUSQ^{rec}, $T^{-1/2}CUSQ^{rec} = O_p(1)$ so that the test is consistent. It is also bounded as $|\delta|$ increases. Figure 1 presents graphs of (9) as a function of α for the values of λ used in the simulations. The results show that the limit value is above the 5% critical value for all value of α unless the break fraction is early, in which case it is below for positive values of α. The results are in general agreement with the simulations, though with a sample at only $T = 120$, the last feature is not observed. Our theoretical results also show that the conclusions from the simulation evidence can be sensitive to the size of the test used.

Some intuition for the power differences of the CUSUM and CUSQ tests can be stated as follows. In a dynamic model, as the magnitude of the break gets large the estimate of the autoregressive coefficient converges to one and, accordingly to avoid a fitted drift, the estimate of the constant goes to zero (see Perron, 1990). Hence, the fitted residuals become the first difference of the series plus an outlier for the change. What the CUSUM does is to cumulate these first differences so that they effectively cancel out. On the other hand, the CUSQ cumulates their squares, avoiding the cancellation effect. Hence, with a similar denominator the CUSUM has a smaller numerator and power is accordingly low.

4.2 Specialized results for the case of a change at 45° angle from the mean regressor

The next Proposition presents explicit theoretical results for the case where the regressors are as in the simulation design, i.e., with $x_t = (1, (-1)^t)$, and the change in the coefficients associated with each regressor is of equal magnitude, so that the change is at a 45° angle from the mean regressor. We start with the case where the statistics are constructed using OLS residuals.

Proposition 2 *Suppose the data are generated by (7) and (8) with the regressors specified by $x_t = (1, (-1)^t)$ and the errors satisfying A2. Suppose also that $\delta_1 = \delta_2$, then, when the statistics are constructed with OLS residuals,*

$$\begin{aligned} T^{-1/2}CUSUM^{ols} &= (1-\alpha)\sqrt{\lambda(1-\lambda)/[2(1+\alpha^2)]} + o_p(1), \\ T^{-1/2}CUSQ^{ols} &= |2\lambda - 1| + o_p(1), \end{aligned} \tag{10}$$

where the results hold as $|\delta| \to \infty$.

Again the results of this Proposition show interesting features. Consider first that pertaining to the CUSQ test. Since the limit function has the factor $(2\lambda - 1)$ for all values of r, the value of the test will be zero irrespective of the magnitude of the change, and power is accordingly zero when the break occurs at mid-sample. Hence, this problem which held for all types of change in the case of a static regression, is seen now to be a knife-edge problem in the case of a dynamic regression. It does not hold with other types of changes such as a change in mean (see Proposition 1 above) and changes orthogonal to the mean regressor. This result accords with the simulations reported in Table 7. Furthermore, the limit function is invariant to the true AR coefficient α. Therefore, we expect the power to be flat as a function of this parameter, which is indeed observed in our simulations.

In the case of the CUSUM test, the approximation is invariant to the magnitude of the change δ. To get a better grasp of this problem, Figure 2 presents graphs of (10) as a function of α for the values of λ used in the simulations. The results show that the limit value will be below the 5% critical value when α is larger than some value which depends on the position of the break. The results are in general agreement with the simulations, though with a sample at only $T = 120$, a zero power for large breaks occurs only when $\alpha > 0.5$ for all values of λ. The reversal is, however, shown to be more rapid the earlier the break is. Our theoretical results also show that the conclusions from the simulation evidence can be sensitive to the size of the test used. This shows that in the dynamic case with a lagged dependent variable the change in mean is no longer a knife-edge case as it was in the static regression with a non-parametric correction for serial correlation.

Consider now the case where the statistics are constructed with recursive residuals, whose results are presented in the following Proposition.

Proposition 3 *Suppose the data are generated by (7) and (8) with the regressors specified by $x_t = (1, (-1)^t)$ and the errors satisfying A2. Suppose also that $\delta_1 = \delta_2$, then, when the statistics are constructed with recursive residuals, we have,*

$$T^{-1/2}CUSUM^{rec} = \frac{(1-\alpha)\lambda|\log(\lambda)|/3}{\sqrt{2\lambda(1-\lambda)(1+\alpha^2)-(1-\alpha)^2\lambda^2\log(\lambda)^2}} + o_p(1), \tag{11}$$

$$T^{-1/2}CUSQ^{rec} = \lambda + o_p(1),$$

where the results hold as $|\delta| \to \infty$.

Note that in both cases, the approximations are invariant to the magnitude of the change δ. To get a better grasp of this problem, Figure 3 presents a graph of (11) for the CUSUM^{rec} as a function of α for the values of λ used in the simulations. The general conclusions are similar to the OLS case, namely that zero power will occur for values of α above some threshold, which is lower the earlier the break. This feature is consistent with the simulations reported in Table 6. Note again that using a different size of the test would result in zero power for values of α above a lower threshold for a given break date. For the CUSQ^{rec} test, the limit function is again invariant to the AR coefficient and is equal to the break fraction. This implies that power is higher the later the break. This is consistent with our simulation findings in Table 7. Further, the 5% critical value divided by the square root of 120 is equal to 0.18. Hence, this explains the fact that power is one throughout for the simulations reported in Table 7 given our choices of break fractions. The power would be zero for very early shifts, e.g., λ smaller than 0.18 at the 5% level.

4.3 The CUSQ test in dynamic models with a correction for non-Normality

If one suspects that the errors from the dynamic regression are not Normally distributed, then the CUSQ$_M$ defined by (4), which involves a correction factor, should be used. In this section, we address the issue of the effect of such a correction on the power of the test in the context of the dynamic regression model. We shall restrict the analysis to simulations as they will clearly illustrate the problems that arise.

The simulation design is exactly the one used previously as described at the beginning of Section 4. All we do is to use the CUSQ$_M$ applied to the OLS or recursive residuals instead of the original CUSQ test. The results are presented in Table 8. They show that the power of the test decreases substantially. With OLS residuals, the power is now very small with breaks at mid-samples. For other break dates, the power decreases as the magnitude of the break increases, except for the case with a break at 45° from the mean regressor. With recursive residuals, a similar pattern emerges except that the decrease in power is not as severe for a change orthogonal to the mean regressor when the coefficient on the lagged dependent variable is large. These results suggest the importance of testing for Normality in the residuals and to not apply the correction if the Normality hypothesis is not rejected.

5 Conclusions

We have documented a number of features about the power function of the CUSUM and CUSUM of squares tests in the context of an alternative which specifies a one-time change in the regression coefficients. The local asymptotic analysis of Ploberger and Krämer (1990) states that the CUSQ has only trivial power in such cases and the CUSUM has non-trivial local power unless the change is orthogonal to the mean regressor. With respect to the latter feature, we have shown that it extends to breaks that are non-local to zero. The CUSUM has indeed trivial power for all cases considered when the change is orthogonal to the mean regressor. The fact that the CUSQ is expected to have lower power than the CUSUM for changes not orthogonal to the mean regressor was shown to hold also for large breaks in the case of a static regression with uncorrelated errors. Beyond that case, our results show the relative properties of the CUSUM and CUSQ to be very different from what is suggested by such a local asymptotic analysis. Indeed, our analysis casts doubts about the usefulness of local asymptotic frameworks in the context of structural change models more generally.

First, if one suspects the errors to be correlated and a correction to eliminate the effect of nuisance parameters is applied, the impact on the power of the CUSQ and CUSUM depends on the type of change present. A change in the mean regressor will induce a non-monotonic power function, such that the power goes to zero as the magnitude of the break increases, for both the CUSUM and CUSQ, whether using OLS or recursive residuals. We have shown, however, that such non-monotonic power functions are peculiar to the mean change case and do not hold generally. Hence, for changes that do not affect only the mean regressor, the CUSUM is superior to the CUSQ test unless the change is orthogonal to the mean regressor.

When dealing with a dynamic regression, our analysis has uncovered a number of interesting features. For the mean change case, the power of the CUSUM with either OLS or recursive residuals goes to zero as the magnitude of the change increases in most cases. On the other hand, the power of the CUSQ goes to one with larger breaks, in most cases. Indeed, for most cases of practical interest the power of the CUSQ is higher than that of the CUSUM. With other types of changes, the CUSUM test need not go to zero but power can still be zero depending on the break date and the extent of the dynamics. The CUSQ in such cases has higher power than the CUSUM. It is also clearly superior when the change is orthogonal to the mean regressor. However, these nice features for the CUSQ hold only if the errors can safely be assumed to be Normal. The properties of the CUSQ when corrected for non-Normality deteriorate, except in some special cases.

The practical implications are as follows. If one is dealing with a static regression with no correlation in the errors, the CUSUM is clearly the preferred test. If serial correlation is present and accounted for via a non-parametric correction using the residuals, both tests have bad properties for the important case of a change in mean. In this situation, it is better to account for correlation by introducing dynamics in the regression (e.g., via lagged dependent variables). The CUSQ test is then preferable for most types of changes, provided the errors can be assumed to be Normally distributed. With non-Normal errors, the CUSQ test loses its superiority and both tests have poor properties, except in special cases.

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Appendix

Proof of Theorem 1(a): We have

$$T^{-1/2}CUSQ^{ols} = \frac{\sup_{r\in[0,1]}\left|T^{-1}\left[\sum_{s=1}^{[Tr]}\hat{u}_s^2 - \frac{[Tr]}{T}\sum_{s=1}^{T}\hat{u}_s^2\right]\right|}{T^{-1}\sum_{s=1}^{T}\hat{u}_s^2} + o_p(1). \tag{A.1}$$

The OLS estimate of β satisfies

$$\begin{aligned}
\hat{\beta} &= \left(\sum_{s=1}^{T} x_s x_s'\right)^{-1}\sum_{s=1}^{T} x_s y_s, \\
&= \left(\sum_{s=1}^{T} x_s x_s'\right)^{-1}\sum_{s=1}^{T}\left(x_s x_s'\beta + x_s x_s'\delta 1_{(s>[T\lambda])} + x_s u_s\right), \\
&= \beta + \left(\sum_{s=1}^{T} x_s x_s'\right)^{-1}\sum_{s=1}^{T}\left[x_s x_s'\delta 1_{(s>[T\lambda])} + x_s u_s\right]
\end{aligned}$$

and the OLS residuals are such that

$$\hat{u}_s = u_s + x_s'\delta 1_{(s>[T\lambda])} - x_s'\left(\sum_{t=1}^{T} x_t x_t'\right)^{-1}\left[\sum_{t=1}^{T} x_t x_t'\delta 1_{(t>[T\lambda])} + \sum_{t=1}^{T} x_t u_t\right].$$

We have

$$T^{-1}\sum_{s=1}^{T} x_s'\delta 1_{(s>[T\lambda])}\delta' 1_{(s>[T\lambda])} x_s \xrightarrow{p} (1-\lambda)\left(\delta' R\delta\right),$$

$$T^{-1}\sum_{t=1}^{T} x_t x_t'\delta 1_{(t>[T\lambda])} \xrightarrow{p} \int_0^1 R\delta 1_{(s>\lambda)} ds = (1-\lambda) R\delta.$$

Hence,

$$\begin{aligned}
&T^{-1}\sum_{s=1}^{T}\left[x_s'(T^{-1}\sum_{t=1}^{T} x_t x_t')^{-1} T^{-1}\sum_{t=1}^{T} x_t x_t'\delta 1_{(t>[T\lambda])}\right]^2 \\
&\xrightarrow{p} \int_0^1 tr\left(R^{-1}(1-\lambda)R\delta(1-\lambda)\delta' R\right) dz = (1-\lambda)^2\left(\delta' R\delta\right).
\end{aligned}$$

Furthermore,

$$T^{-1}\sum_{s=1}^{T}\left[x_s'(T^{-1}\sum_{t=1}^{T} x_t x_t')^{-1} T^{-1}\sum_{j=1}^{T} x_j u_j\right]^2 \xrightarrow{p} 0,$$

$$T^{-1}\sum_{s=1}^{T} x_s'\delta 1_{(s>[T\lambda])}(\sum_{j=1}^{T} x_j u_j)'(\sum_{t=1}^{T} x_t x_t')^{-1} x_s \xrightarrow{p} 0,$$

and

$$2T^{-1}\sum_{s=1}^{T}x_s'\delta 1_{(s>[T\lambda])}(T^{-1}\sum_{t=1}^{T}x_t x_t'\delta 1_{(t>[T\lambda])})'(T^{-1}\sum_{j=1}^{T}x_j x_j')^{-1}x_s' \xrightarrow{p} 2\left(1-\lambda\right)^2\left(\delta' R\delta\right).$$

Collecting results,

$$T^{-1}\sum_{s=1}^{T}\hat{u}_s^2 \xrightarrow{p} \sigma_u^2+\left[\left(1-\lambda\right)-\left(1-\lambda\right)^2\right]\left(\delta' R\delta\right)=\sigma_u^2+\lambda\left(1-\lambda\right)\left(\delta' R\delta\right).$$

Consider the scaled numerator of (A.1). We have,

$$T^{-1}\sum_{s=1}^{[Tr]}x_s'\delta 1_{(s>[T\lambda])}\delta' 1_{(s>[T\lambda])}x_s \xrightarrow{p} \left(\delta' R\delta\right)\int_0^r 1_{(v>\lambda)}dv,$$

$$T^{-1}\sum_{s=1}^{[Tr]}\left[x_s'(T^{-1}\sum_{s=1}^{T}x_s x_s')^{-1}T^{-1}\sum_{s=1}^{T}x_s x_s'\delta 1_{(s>[T\lambda])}\right]^2 \xrightarrow{p} r\left(1-\lambda\right)^2\left(\delta' R\delta\right),$$

$$2T^{-1}\sum_{s=1}^{[Tr]}x_s'\delta 1_{(s>[T\lambda])}(T^{-1}\sum_{t=1}^{T}x_t x_t'\delta 1_{(t>[T\lambda])})'(T^{-1}\sum_{j=1}^{T}x_j x_j')^{-1}x_s' \xrightarrow{p} 2\left(1-\lambda\right)\left(\delta' R\delta\right)\int_0^r 1_{(v>\lambda)}dv.$$

Collecting terms,

$$T^{-1/2}CUSQ^{ols} \xrightarrow{p} \frac{\sup_{r\in[0,1]}\left|(2\lambda-1)\left[\int_0^r 1_{(v>\lambda)}dv+r\left(\lambda-1\right)\right]\left(\delta' R\delta\right)\right|}{\sigma_u^2+\lambda\left(1-\lambda\right)\left(\delta' R\delta\right)},$$

and the result follows.

Proof of Theorem 1(b): The proof is analogous to the OLS case, though the derivations are more complex. Note that we repeatedly use the fact that $\text{plim}_{t\to\infty} f_t = 1$, with f_t the standard deviation of the recursive residual at time t. We start by writing the recursive residuals as:

$$\widetilde{u}_s=\frac{u_s}{f_s}+\frac{1}{f_s}x_s'\delta 1_{(s>[T\lambda])}-\frac{1}{f_s}\left\{x_s'(\sum_{t=1}^{s-1}x_t x_t')^{-1}\left[\sum_{i=1}^{s-1}x_i x_i'\delta 1_{(i>[T\lambda])}+\sum_{i=1}^{s-1}x_i u_i\right]\right\}.$$

Now,

$$T^{-1}\sum_{s=1}^{T}\frac{1}{f_s^2}x_s'\delta 1_{(s>[T\lambda])}\delta' 1_{(s>[T\lambda])}x_s \xrightarrow{p} \left(1-\lambda\right)\left(\delta' R\delta\right),$$

$$\begin{aligned}
&T^{-1}\sum_{s=1}^{T}\left[\frac{1}{f_s}x_s'(T^{-1}\sum_{t=1}^{s-1}x_t x_t')^{-1}T^{-1}\sum_{i=1}^{s-1}x_i x_i'\delta 1_{(i>[T\lambda])}\right]^2\\
&\xrightarrow{p}\int_0^1 tr\left(R\left(r\right)^{-1}\int_0^r R\delta 1_{(v>\lambda)}dv\left(\int_0^r R\delta 1_{(v>\lambda)}dv\right)' R\left(r\right)^{-1}R\right)dr,\\
=\;&\int_0^1\left(\frac{1}{r}\int_0^r 1_{(v>\lambda)}dv\right)^2 dr\left(\delta' R\delta\right).
\end{aligned}$$

Also,

$$T^{-1}\sum_{s=1}^{T}x_s'(T^{-1}\sum_{t=1}^{s-1}x_tx_t')^{-1}T^{-1}\sum_{j=1}^{s-1}x_ju_j\xrightarrow{p}0$$

and

$$\begin{aligned}&2T^{-1}\sum_{s=1}^{T}x_s'\delta 1_{(s>[T\lambda])}(T^{-1}\sum_{t=1}^{s-1}x_tx_t'\delta 1_{(t>[T\lambda])})'(T^{-1}\sum_{j=1}^{s-1}x_jx_j')^{-1}x_s'\\&\xrightarrow{p}2\int_0^1 tr(\delta 1_{(r>\lambda)}\int_0^r\delta' R1_{(v>\lambda)}dvR\left(r\right)^{-1}R)dr=2\int_0^1(\frac{1}{r}\int_0^r 1_{(v>\lambda)}dv)dr\left(\delta' R\delta\right).\end{aligned}$$

Collecting results, the limit of the denominator is

$$\begin{aligned}&T^{-1}\sum_{s=1}^{T}\widetilde{u}_s^2\xrightarrow{p}\sigma_u^2+\left(1-\lambda\right)\left(\delta' R\delta\right)+\int_0^1(\int_0^r 1_{(v>\lambda)}dv)^2\delta'(RR\left(r\right)^{-1}RR\left(r\right)^{-1}R)\delta dr\\&-2\int_0^1(\int_0^r 1_{(v>\lambda)}dv)\delta'(RR\left(r\right)^{-1}R)\delta dr=\sigma_u^2+\lambda(1-\lambda)\delta' R\delta,\end{aligned}$$

since $\int_0^1(1_{(r>\lambda)}-(1/r)\int_0^r 1_{(v>\lambda)}dv)^2dr=\lambda\left(1-\lambda\right)$. Consider now the scaled numerator. We have,

$$T^{-1}\sum_{s=1}^{[Tr]}(1/f_s^2)x_s'\delta 1_{(s>[T\lambda])}\delta' 1_{(s>[T\lambda])}x_s\xrightarrow{p}\left(\delta' R\delta\right)\int_0^r 1_{(v>\lambda)}ds,$$

$$T^{-1}\sum_{s=1}^{[Tr]}\left[x_s'(T^{-1}\sum_{s=1}^{s-1}x_sx_s')^{-1}T^{-1}\sum_{s=1}^{s-1}x_sx_s'\delta 1_{(s>[T\lambda])}\right]^2\xrightarrow{p}\int_0^r\left(\frac{1}{s}\int_0^s 1_{(v>\lambda)}dv\right)^2ds\left(\delta' R\delta\right),$$

$$2T^{-1}\sum_{s=1}^{[Tr]}x_s'\delta 1_{(s>[T\lambda])}(T^{-1}\sum_{t=1}^{T}x_tx_t'\delta 1_{(t>[T\lambda])})'(T^{-1}\sum_{j=1}^{T}x_jx_j')^{-1}x_s'\xrightarrow{p}2\int_0^r(\frac{1}{s}\int_0^s 1_{(v>\lambda)}dv)ds\left(\delta' R\delta\right).$$

Collecting results, we have,

$$T^{-1/2}CUSQ^{rec}\xrightarrow{p}\frac{\sup_{r\in[0,1]}\left|\delta' B\left(r,\lambda\right)\delta\right|}{\sigma_u^2+\delta' A\left(\lambda\right)\delta},$$

where

$$\begin{aligned}B\left(r,\lambda\right)&=R\left[\int_0^r(1_{(s>\lambda)}-s^{-1}\int_0^s 1_{(v>\lambda)}dv)^2ds-r\int_0^1(1_{(r>\lambda)}-r^{-1}\int_0^r 1_{(v>\lambda)}dv)^2dr\right],\\&=R\left[r^{-1}\lambda\left(r-\lambda\right)1_{(r>\lambda)}-r\lambda\left(1-\lambda\right)\right],\end{aligned}$$

The numerator $\left|\delta' B\left(r,\lambda\right)\delta\right|$ achieves a maximum at $r=\lambda$, in which case it is equal to $\lambda^2\left(1-\lambda\right)\delta' R\delta$, and the result follows.

Proof of Theorem 1(c,d): With recursive residuals, we have, following Ploberger and Krämer (1990),

$$T^{-1}\sum_{t=1}^{[Tr]}\widetilde{u}_t \xrightarrow{p} c'\int_0^r \delta 1_{(v>\lambda)}dv - c'\int_0^r\left(\frac{1}{v}\int_0^v \delta 1_{(w>\lambda)}dw\right)dv$$
$$= c'\int_0^r \delta\left(1_{(v>\lambda)} - \frac{1}{v}\int_0^v 1_{(w>\lambda)}dw\right)dv = c'\delta\lambda\left[\log(r)-\log(\lambda)\right]1_{(r>\lambda)}.$$

Hence, the limit of $T^{-1}\sum_{t=1}^{[Tr]}\widetilde{u}_t/(1-2r)$ is maximized when $r=1$, in which case it reduces to $-c'\delta\lambda\log(\lambda)$. It is straightforward to show that, with OLS residuals

$$T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t \xrightarrow{p} c'\delta\left[(r-\lambda)1_{(r>\lambda)} - r(1-\lambda)\right],$$

whose absolute value is maximized at $r=\lambda$. Using the results in parts (a) and (b), we have in the case of OLS residuals $T^{-1}\sum_{s=1}^{T}\hat{u}_s^2 \xrightarrow{p} \sigma_u^2 + \lambda(1-\lambda)(\delta' R\delta)$ and with recursive residuals

$$T^{-1}\sum_{s=1}^{T}(\widetilde{u}_s - \overline{\widetilde{u}})^2 \xrightarrow{p} \sigma_u^2 + \lambda(1-\lambda)\delta' R\delta - (c'\delta\lambda\log(\lambda))^2,$$

and the results follow.

Proof of Theorem 2: Since $\hat{\psi} = T^{-1}\sum_{t=1}^{T}\breve{u}_t^4 - (T^{-1}\sum_{t=1}^{T}\breve{u}_t^2)^2$, we only need to consider the limit of $T^{-1}\sum_{t=1}^{T}\breve{u}_t^4$. The results are the same with OLS or recursive residuals. Keeping only the terms of highest order in δ, we have

$$\begin{aligned} T^{-1}\sum_{t=1}^{T}\breve{u}_t^4 &= \left[(1-\lambda)-4(1-\lambda)^2+6(1-\lambda)^3-3(1-\lambda)^4\right]g(\delta)+o_p\left(\|\delta\|^4\right), \\ &= \lambda(1-\lambda)\left(3\lambda^2-3\lambda+1\right)g(\delta)+o_p\left(\|\delta\|^4\right), \end{aligned}$$

where $g(\delta) = p\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}(\delta' x_t x_t'\delta)^2$. Hence, using the limit of $T^{-1}\sum_{t=1}^{T}\breve{u}_t^2$ derived in Theorem 1(a,b), we have,

$$T^{-1}\sum_{t=1}^{T}\breve{u}_t^4 - (T^{-1}\sum_{t=1}^{T}\breve{u}_t^2)^2 = \lambda(1-\lambda)\left(3\lambda^2-3\lambda+1\right)g(\delta) - (1-\lambda)^2\lambda^2(\delta' R\delta)^2 + o_p\left(\|\delta\|^4\right).$$

Proof of Theorem 3: We first consider the CUSUM test with recursive residuals (the same result holds if OLS residuals are used; see Crainiceanu and Vogelsang, 2001, for the simple mean shift case; the extension to more general regressors is straightforward). It is easy to verify that,

$$\hat{\gamma}_j = T^{-1}\sum_{t=j+1}^{T}\widetilde{u}_t\widetilde{u}_{t-j} \xrightarrow{p} \gamma_j + \lambda(1-\lambda)\delta' R_j\delta, \text{ uniformly in } j, \tag{A.2}$$

with $R_j = p\lim_{T\to\infty} T^{-1}\sum_{t=j+1}^{T} x_t x_{t-j}'$ and $\gamma_j = p\lim_{T\to\infty} T^{-1}\sum_{t=j+1}^{T} u_t u_{t-j}$. Hence,

$$\hat{\rho}(\delta) = \frac{T^{-1}\sum_{t=2}^{T}\widetilde{u}_t\widetilde{u}_{t-1}}{T^{-1}\sum_{t=2}^{T}\widetilde{u}_{t-1}^2} \xrightarrow{p} \frac{\gamma_1 + \lambda(1-\lambda)\delta' R_1\delta}{\sigma_u^2 + \lambda(1-\lambda)\delta' R_0\delta}.$$

Using (A.2) and the fact that $\sum_{j=1}^{T-1} k(j,m) = O(m)$

$$\begin{aligned}
\hat{\omega}_{WS}^2 &= \hat{\gamma}_0 + 2\sum_{j=1}^{T-1} k(j,m)\,\hat{\gamma}_j, \\
&= (\gamma_0 + 2\sum_{j=1}^{T-1} k(j,m)\,\gamma_j) + \lambda(1-\lambda)(\delta' R_0\delta + 2\sum_{j=1}^{T-1} k(j,m)\,\delta' R_j\delta) + o_p(1), \\
&= h(0) + \lambda(1-\lambda)(\delta' R_0\delta + 2\sum_{j=1}^{T-1} k(j,m)\,\delta' R_j\delta) + o_p(1), \qquad \text{(A.3)} \\
&\leq h(0) + \lambda(1-\lambda)(\delta' R_0\delta + 2\sum_{j=1}^{T-1} k(j,m)\,\delta' R_0\delta) + o_p(1), \\
&\leq h(0) + \lambda(1-\lambda)O(m)\delta' R_0\delta + o_p(||\delta||^2).
\end{aligned}$$

If condition (5) holds, then $\hat{\rho}(\delta) \xrightarrow{p} 1$ as $||\delta||$ increases such that $C(\delta) = O_p(||\delta||^4)$, $m = O_p(||\delta||^{4/\vartheta} T^{1/\vartheta})$ and $\hat{\omega}_{WS}^2 = O_p(||\delta||^{4/\vartheta+2} T^{1/\vartheta})$. If condition (5) does not hold, $\hat{\rho}(\delta) \xrightarrow{p} \kappa < 1$, say, as $||\delta||$ increases such that $C(\delta) = O_p(1)$, $m = O_p(T^{1/\vartheta})$ and $\hat{\omega}_{WS}^2 = O_p(||\delta||^2 T^{1/\vartheta})$, which completes the proof. For the CUSQ test, the estimate of the autocovariance at delay one is

$$\begin{aligned}
T^{-1}\sum_{t=2}^{T}\left(\breve{u}_t^2 - \hat{\sigma}^2\right)\left(\breve{u}_{t-1}^2 - \hat{\sigma}^2\right) &= T^{-1}\sum_{t=2}^{T}\left(u_t^2 - \sigma^2 + A_t^*\right)\left(u_{t-1}^2 - \sigma^2 + A_{t-1}^*\right), \\
&= T^{-1}\sum_{t=2}^{T}\left(u_t^2 - \sigma^2\right)\left(u_{t-1}^2 - \sigma^2\right) + T^{-1}\sum_{t=2}^{T} A_t^*\left(u_{t-1}^2 - \sigma^2\right) \\
&\quad + T^{-1}\sum_{t=2}^{T} A_{t-1}^*\left(u_t^2 - \sigma^2\right) + T^{-1}\sum_{t=2}^{T} A_t^* A_{t-1}^*, \\
&= T^{-1}\sum_{t=2}^{T} A_t^* A_{t-1}^* + o_p\left(\|\delta\|^4\right),
\end{aligned}$$

where

$$\begin{aligned}
A_t^* &= \left[1_{(t>[T\lambda])} + (1-\lambda)^2 + 2(1-\lambda)\right]\delta' x_t x_t'\delta - 2\left[1_{(t>[T\lambda])} - (1-\lambda)\right] u_t x_t'\delta, \\
&= \left[1_{(t>[T\lambda])} + (1-\lambda)^2 + 2(1-\lambda)\right]\delta' x_t x_t'\delta + o_p\left(\|\delta\|^2\right).
\end{aligned}$$

If condition (6) holds, $T^{-1}\sum_{t=2}^{T}\left(A_t^* A_{t-1}^* - A_t^{*2}\right) \xrightarrow{p} 0$, and $\hat{\rho}(\delta) \xrightarrow{p} 1$ as $||\delta||$ increases such that $C(\delta) = O_p(||\delta||^8)$ and $m = O_p(||\delta||^{8/\vartheta}\, T^{1/\vartheta})$. Using similar arguments as for the CUSUM test, we have $\hat{\omega}_{WS}^2 = O_p(T^{1/\vartheta}\,\|\delta\|^{4+8/\vartheta})$ which yields the stated result. When (6) does not hold, $\hat{\rho}(\delta) \xrightarrow{p} \kappa < 1$, say, as $||\delta||$ increases such that $C(\delta) = O_p(1)$ and $m = O_p(T^{1/\vartheta})$, which yields $\hat{\omega}_{WS}^2 = O_p\left(T^{1/\vartheta}\,\|\delta\|^4\right)$, using similar arguments as for the CUSUM.

Proof of Theorem 4: Without loss of generality, we set $\beta = 0$. Consider first the case with recursive residuals, it can be shown that

$$\begin{aligned}
&T^{-1}\sum_{t=1}^{[Tr]}\widetilde{u}_t \xrightarrow{p} c'\delta\,(r-\lambda)\,1_{(r>\lambda)} - \int_\lambda^r \frac{s-\lambda}{s}ds\,(c'\delta) \\
&-\lambda\int_\lambda^r \frac{s-\lambda}{s}A(s)^{11}\left(\frac{c'\delta}{1-\alpha} - \frac{s-\lambda}{s}c'R^{-1}S_1\delta\right)ds\,(\delta' S_1\delta) \equiv N(\lambda,\delta,r)
\end{aligned}$$

and, ignoring terms of lower order than $||\delta||^2$,

$$\begin{aligned}
T^{-1}\sum_{t=1}^{[Tr]}\widetilde{u}_t^2 &= (r-\lambda)\,1_{(r>\lambda)}\delta' R\delta \\
&\quad + \left(\int_\lambda^r (s-\lambda)^2\,\Lambda(s,\delta)\,ds - 2\int_\lambda^r (s-\lambda)\,\lambda^2 s^{-2} A(s)^{11}\,ds\right)(\delta' S_1\delta)^2 \\
&\quad + \left(\int_\lambda^r \left(\frac{s-\lambda}{s}\right)^2 ds - 2\int_\lambda^r \frac{s-\lambda}{s}ds\right)(\delta' R\delta) + o_p(||\delta||^2), \\
&= \left(\int_\lambda^r (s-\lambda)^2\,\Lambda(s,\delta)\,ds - 2\int_\lambda^r (s-\lambda)\,\lambda^2 s^{-2} A(s)^{11}\,ds\right)(\delta' S_1\delta)^2 \\
&\quad +(\lambda - r^{-1}\lambda^2)\,(\delta' R\delta) + o_p(||\delta||^2) \equiv D(\lambda,\delta,r) + o_p(||\delta||^2).
\end{aligned}$$

Combining these results with the fact that, for the CUSQ test, $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 - (T^{-1}\sum_{t=1}^{T}\hat{u}_t)^2$, the result stated in the Theorem follows. For the case with OLS residuals, we have,

$$\begin{aligned}
&T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t \xrightarrow{p} (r-\lambda)\,1_{(r>\lambda)}c'\delta - (1-\lambda)\left[rc'\delta - \lambda(1-\lambda)\,rA^{11}c'R^{-1}S_1\delta\,(\delta' S_1\delta)\right. \\
&\left.+\lambda A^{11}\frac{c'\delta\,(r-\lambda)\,1_{(r>\lambda)}}{1-\alpha}\delta' S_1\delta\right] \equiv B(\lambda,\alpha,r,\delta).
\end{aligned}$$

Also,

$$\begin{aligned}
T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t^2 &= \left[(r-\lambda)\,1_{(r>\lambda)}\,(2\lambda-1) + (1-\lambda)^2 r\right]\delta' R\delta \qquad \text{(A.4)} \\
&\quad + (1-\lambda)^2\lambda^2\left(A(1)^{11}\right)^2 A_{11}(r)\,(\delta' S_1\delta)^2 \\
&\quad +\lambda^2(1-\lambda)^3\left[r - 2(r-\lambda)\,1_{(r>\lambda)}\right]\left(A(1)^{11}\right)^2\left(\delta' S_1 R^{-1} S_1\delta\right)(\delta' S_1\delta)^2 \\
&\quad +2\lambda(1-\lambda)\left[(r-\lambda)\,1_{(r>\lambda)}\,(1-2\lambda) - (1-\lambda)^2 r\right] A(1)^{11}\,(\delta' S_1\delta)^2 + o_p(||\delta||^2),
\end{aligned}$$

and the result follows. Note that with $r=1$, the expression simplifies to

$$T^{-1}\sum_{s=1}^{T}\hat{u}_s^2=(1-\lambda)\lambda\left[\delta' R\delta-(1-\lambda)\lambda A(1)^{11}\left(\delta' S_1\delta\right)^2\right]+o_p(||\delta||^2),$$

and the result follows.

Proof of Proposition 1: As a matter of notation, $o_p^*(1)$ denotes a term that is $o_p(1)$ uniformly in T and $|\delta|$. We first consider tests based on OLS residuals. We have

$$\hat{u}_t=y_t-\hat{\alpha}y_{t-1}-\hat{c},$$

where

$$\hat{\alpha}=\alpha+\delta\left[T^{-1}\sum_{t=[T\lambda]+1}^{T}y_{t-1}-T^{-1}(1-\lambda)\sum_{t=1}^{T}y_{t-1}\right]/\left[T^{-1}\sum_{t=1}^{T}y_{t-1}^2-\left(T^{-1}\sum_{t=1}^{T}y_{t-1}\right)^2\right].$$

Using the fact that $y_t=\alpha y_{t-1}+\delta 1(t>[T\lambda])+u_t$ (setting $\beta=0$ without loss of generality), we have $T^{-1}\sum_{t=1}^{[Tr]}y_{t-1}=o_p(1)$ for $r<\lambda$ and, for $r>\lambda$,

$$T^{-1}\sum_{t=1}^{[Tr]}y_{t-1}=\frac{\delta(r-\lambda)}{1-\alpha}+O_p\left(1/\sqrt{T}\right).$$

Also,

$$T^{-1}\sum_{t=1}^{T}y_{t-1}^2=\frac{1}{1-\alpha^2}\left[\sigma_u^2+\delta^2\frac{(1-\lambda)(1+\alpha)}{1-\alpha}+O_p\left(|\delta|/\sqrt{T}\right)\right],$$

which implies that,

$$\delta\left[T^{-1}\sum_{t=[T\lambda]+1}^{T}y_{t-1}-T^{-1}(1-\lambda)\sum_{t=1}^{T}y_{t-1}\right]=\frac{\delta^2\lambda(1-\lambda)}{1-\alpha}+O_p\left(|\delta|/\sqrt{T}\right)$$

and

$$T^{-1}\sum_{t=1}^{T}y_{t-1}^2-\left(T^{-1}\sum_{t=1}^{T}y_{t-1}\right)^2=\frac{\lambda(1-\lambda)\delta^2}{(1-\alpha)^2}+\frac{1}{1-\alpha^2}\left[\sigma_u^2+O_p\left(|\delta|/\sqrt{T}\right)\right].$$

Hence,

$$\begin{aligned}
\hat{\alpha}&=\alpha+\frac{\frac{\delta^2\lambda(1-\lambda)}{1-\alpha}+O_p\left(|\delta|/\sqrt{T}\right)}{\frac{\lambda(1-\lambda)\delta^2}{(1-\alpha)^2}+\frac{1}{1-\alpha^2}\left[\sigma_u^2+O_p\left(|\delta|/\sqrt{T}\right)\right]}\\
&=1+\frac{O_p\left(1/[\sqrt{T}|\delta|]\right)+O_p\left(1/\delta^2\right)}{O_p(1)+O_p\left(1/[\sqrt{T}|\delta|]\right)+O_p\left(1/\delta^2\right)}\\
&=1+O_p\left(1/[\sqrt{T}|\delta|]\right)+O_p\left(1/\delta^2\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\hat{c} &= \frac{1}{T-1}\sum_{t=2}^{T} y_t - \hat{\alpha}\frac{1}{T-1}\sum_{t=1}^{T-1} y_t \\
&= (1-\hat{\alpha})\frac{1}{T-1}\sum_{t=2}^{T-1} y_t + \hat{\alpha}\frac{1}{T-1}y_1 + \frac{1}{T-1}y_T \\
&= \left[O_p\left(1/[\sqrt{T}|\delta|]\right) + O_p\left(1/\delta^2\right)\right]\left[\frac{\delta(1-\lambda)}{1-\alpha} + O_p\left(|\delta|/\sqrt{T}\right)\right] + O_p\left(T^{-1}\right) + O\left(|\delta|/T\right) \\
&= \frac{(1-\lambda)}{1-\alpha}\left[O_p\left(1/\sqrt{T}\right) + O_p\left(1/|\delta|\right)\right] + O_p\left(T^{-1}\right) + O_p\left(|\delta|/T\right) + O_p\left(1/[\sqrt{T}|\delta|]\right).
\end{aligned}$$

Combining these results, we have

$$\begin{aligned}
T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t &= T^{-1}\sum_{t=1}^{[Tr]}(y_t - \hat{\alpha}y_{t-1} - \hat{c}) \\
&= T^{-1}\sum_{t=1}^{[Tr]}(y_t - y_{t-1} - (\hat{\alpha}-1)y_{t-1} - \hat{c}) \\
&= T^{-1}y_{[Tr]} - T^{-1}y_0 - (\hat{\alpha}-1)T^{-1}\sum_{t=1}^{[Tr]} y_{t-1} - T^{-1}[Tr]\hat{c}.
\end{aligned}$$

Hence, for $r < \lambda$,

$$T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t = rO_p\left(|\delta|/T\right) + o_p^*(1)$$

and for $r > \lambda$,

$$T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t = rO_p\left(|\delta|/T\right) + \left[O_p\left(1/[\sqrt{T}|\delta|]\right) + O_p\left(1/\delta^2\right)\right]\left[\frac{\delta(r-\lambda)}{1-\alpha} + O_p\left(|\delta|/\sqrt{T}\right)\right],$$

so that $\max_r |T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t| = O_p\left(|\delta|/T\right) + O_p\left(1/|\delta|\right) + o_p^*(1)$. Using the results above, one can also show that

$$\hat{\sigma}^2 = T^{-1}\sum_{t=1}^{T}\hat{u}_t^2 = \frac{2\sigma_u^2}{1+\alpha} + O_p\left(|\delta|^2/T\right) + o_p^*(1) \tag{A.5}$$

and the result for the $CUSUM^{ols}$ follows. It also follows that

$$T^{-1}\sum_{t=1}^{[Tr]}\hat{u}_t^2 - rT^{-1}\sum_{t=1}^{T}\hat{u}_t^2 = (2\lambda-1)\,O\left(|\delta|^{-2}\right) + O_p\left(|\delta|^2/T\right) + o_p^*(1),$$

which yields the stated result for the $CUSQ^{ols}$ using (A.5). For the recursive tests, we can show that, as $T \to \infty$, followed by $\delta \to \infty$,

$$T^{-1} \sum_{t=1}^{[Tr]} \widetilde{u}_t^2 = \left[r\sigma_u^2 + (r - \lambda) 1_{(r>\lambda)} \frac{2}{1+\alpha} \sigma_u^2 \right] + o_p(1).$$

The result for CUSQ^{rec} follows easily. Consider now the CUSUM^{rec} test. For the numerator, it can be shown to be $O_p(|\delta|)$ as $T \to \infty$. The result follows showing that the denominator is $O_p(|\delta|^2)$.

Proof of Proposition 2: Again we specify $\beta = 0$, without loss of generality, so that the data generating process is

$$y_t = \alpha y_{t-1} + \left(1 + (-1)^t\right) \delta 1_{(t>[T\lambda])} + u_t.$$

With $z_t' = (y_{t-1}, 1, (-1)^t)$, for this special case, we have

$$T^{-1} \sum_{t=1}^{[Tn]} z_t \xrightarrow{p} \begin{pmatrix} \frac{\delta(n-\lambda)1_{(n>\lambda)}}{1-\alpha} \\ n \\ 0 \end{pmatrix},$$

$$T^{-1} \sum_{t=1}^{[Tn]} z_t z_t' \xrightarrow{p} \begin{pmatrix} \frac{1}{1-\alpha^2}\left[2(n-\lambda)1_{(n>\lambda)}\delta^2 + n\sigma_u^2 + \frac{4\alpha^2}{1-\alpha^2}(n-\lambda)1_{(n>\lambda)}\delta^2\right] & \frac{(n-\lambda)1_{(n>\lambda)}\delta}{1-\alpha} & -\frac{(n-\lambda)1_{(n>\lambda)}\delta}{1+\alpha} \\ \frac{(n-\lambda)1_{(n>\lambda)}\delta}{1-\alpha} & n & 0 \\ -\frac{(n-\lambda)1_{(n>\lambda)}\delta}{1+\alpha} & 0 & n \end{pmatrix},$$

$$T^{-1} \sum_{t=[T\lambda]}^{T} z_t z_t' \xrightarrow{p} (1-\lambda) \begin{pmatrix} \frac{1}{1-\alpha^2}\left[2\delta^2 + \sigma_u^2 + \frac{4\alpha^2}{1-\alpha^2}\delta^2\right] & \frac{\delta}{1-\alpha} & -\frac{\delta}{1+\alpha} \\ \frac{\delta}{1-\alpha} & 1 & 0 \\ -\frac{\delta}{1+\alpha} & 0 & 1 \end{pmatrix},$$

$$T^{-1} \sum_{t=[T\lambda]}^{T} z_t z_t' \gamma \xrightarrow{p} (1-\lambda)\delta \begin{pmatrix} \frac{\delta}{1-\alpha} - \frac{\delta}{1+\alpha} \\ 1 \\ 1 \end{pmatrix}$$

and

$$T^{-1} \sum_{t=[T\lambda]}^{[Tn]} z_t z_t' \xrightarrow{p} (n-\lambda) 1_{(n>\lambda)} \begin{pmatrix} \frac{1}{1-\alpha^2}\left[2\delta^2 + \sigma_u^2 + \frac{4\alpha^2}{1-\alpha^2}\delta^2\right] & \frac{\delta}{1-\alpha} & -\frac{\delta}{1+\alpha} \\ \frac{\delta}{1-\alpha} & 1 & 0 \\ -\frac{\delta}{1+\alpha} & 0 & 1 \end{pmatrix}.$$

Collecting these results, we have, using Theorem 3,

$$\sup_r B(\lambda, \delta, \alpha, r) = \frac{(1-\lambda)\lambda(1-\alpha)}{1+\alpha^2}\delta,$$

and the denominator converges to $(2\lambda(1-\lambda)/(1+\alpha^2))^{1/2}\delta$. Combining these two facts, we obtain the result in Proposition 2. For the CUSQ test, the result follows from tedious algebra and is omitted.

Proof of Proposition 3: Consider first the CUSUM test. In this special case, we have, using Theorem 3 where we only retain the highest order terms in δ, $\sup_r |N(\lambda, \delta, r)| = [(1-\alpha)/(1+\alpha^2)]\lambda|\log(\lambda)|\delta + o_p(|\delta|)$ and $D(\lambda, \delta, r) = 2\lambda(r-\lambda)1_{(r>\lambda)}\delta^2/[(1+\alpha^2)r] + o_p(|\delta|^2)$. Setting $r = 1$ gives $D(\lambda, \delta, 1) = 2\lambda(1-\lambda)\delta^2/(1+\alpha^2) + o_p(|\delta|^2)$. Again, these yield the desired result for the CUSUM test. With the expression for $D(\lambda, \delta, r)$, the result for CUSQ test also follows straightforwardly.

Table 1: Power of the Cusum Test; Static Regression with *i.i.d.* Normal errors

		a) Using recursive residuals								b) Using OLS residuals							
λ	$\psi\backslash b$	0	.5	1	2	5	10	20	30	0	.5	1	2	5	10	20	30
.3	0°	.04	.34	.91	1.0	1.0	1.0	1.0	1.0	.07	.44	.97	1.0	1.0	1.0	1.0	1.0
	45°	.05	.17	.58	1.0	1.0	1.0	1.0	1.0	.06	.26	.67	1.0	1.0	1.0	1.0	1.0
	90°	.05	.04	.03	.01	.00	.00	.00	.00	.06	.06	.04	.01	.00	.00	.00	.00
.5	0°	.06	.24	.85	1.0	1.0	1.0	1.0	1.0	.07	.65	1.0	1.0	1.0	1.0	1.0	1.0
	45°	.05	.16	.43	.97	1.0	1.0	1.0	1.0	.07	.38	.89	1.0	1.0	1.0	1.0	1.0
	90°	.06	.04	.03	.00	.00	.00	.00	.00	.07	.07	.03	.01	.00	.00	.00	.00
.7	0°	.04	.11	.43	.98	1.0	1.0	1.0	1.0	.06	.54	.97	1.0	1.0	1.0	1.0	1.0
	45°	.05	.08	.18	.65	1.0	1.0	1.0	1.0	.07	.31	.81	1.0	1.0	1.0	1.0	1.0
	90°	.05	.04	.03	.01	.00	.00	.00	.00	.08	.05	.04	.01	.00	.00	.00	.00

Table 2: Power of the Cusum of Squares Test; Static Regression with *i.i.d.* Normal errors

		a) Using recursive Residuals								b) Using OLS Residuals							
λ	$\psi\backslash b$	0	.5	1	2	5	10	20	30	0	.5	1	2	5	10	20	30
.3	0°	.04	.04	.05	.11	1.0	1.0	1.0	1.0	.05	.06	.16	.86	1.0	1.0	1.0	1.0
	45°	.05	.04	.04	.11	1.0	1.0	1.0	1.0	.04	.06	.17	.86	1.0	1.0	1.0	1.0
	90°	.04	.05	.05	.10	1.0	1.0	1.0	1.0	.06	.15	.84	1.0	1.0	1.0	1.0	1.0
.5	0°	.05	.03	.08	.91	1.0	1.0	1.0	1.0	.05	.04	.02	.00	.00	.00	.00	.00
	45°	.04	.03	.09	.91	1.0	1.0	1.0	1.0	.04	.02	.01	.00	.00	.00	.00	.00
	90°	.04	.03	.09	.92	1.0	1.0	1.0	1.0	.04	.04	.02	.00	.00	.00	.00	.00
.7	0°	.05	.05	.21	.99	1.0	1.0	1.0	1.0	.05	.04	.05	.52	1.0	1.0	1.0	1.0
	45°	.03	.04	.20	.99	1.0	1.0	1.0	1.0	.04	.04	.04	.50	1.0	1.0	1.0	1.0
	90°	.05	.04	.21	.99	1.0	1.0	1.0	1.0	.04	.03	.05	.54	1.0	1.0	1.0	1.0

Table 3: Power of the Cusum of Squares Test;
static regression with a correction for non Normal errors

		a) Using recursive residuals								b) Using OLS residuals							
λ	$\psi\backslash b$	0	.5	1	2	5	10	20	30	0	.5	1	2	5	10	20	30
.3	0°	.04	.04	.04	.15	1.0	1.0	1.0	1.0	.04	.03	.09	.77	1.0	1.0	1.0	1.0
	45°	.04	.04	.04	.05	.53	1.0	1.0	1.0	.03	.04	.08	.65	1.0	1.0	1.0	1.0
	90°	.04	.04	.04	.13	1.0	1.0	1.0	1.0	.03	.04	.09	.76	1.0	1.0	1.0	1.0
.5	0°	.03	.03	.09	.94	1.0	1.0	1.0	1.0	.03	.02	.01	.01	.00	.02	.13	.60
	45°	.03	.03	.06	.85	1.0	1.0	1.0	1.0	.03	.04	.01	.00	.00	.00	.00	.00
	90°	.03	.03	.09	.94	1.0	1.0	1.0	1.0	.03	.02	.02	.01	.01	.02	.12	.60
.7	0°	.03	.03	.20	1.0	1.0	1.0	1.0	1.0	.03	.03	.11	.77	1.0	1.0	1.0	1.0
	45°	.05	.02	.19	.99	1.0	1.0	1.0	1.0	.04	.03	.07	.64	1.0	1.0	1.0	1.0
	90°	.05	.03	.22	.99	1.0	1.0	1.0	1.0	.03	.03	.10	.78	1.0	1.0	1.0	1.0

Table 4: Power of the CUSUM test;
static regression with a correction for serial correlation

		a) Using recursive residuals								b) Using OLS residuals							
λ	$\psi\backslash b$	0	.5	1	2	5	10	20	30	0	.5	1	2	5	10	20	30
.3	0°	.03	.26	.67	.17	.00	.00	.00	.00	.06	.47	.98	1.0	.11	.00	.00	.00
	45°	.04	.16	.53	.98	1.0	1.0	1.0	1.0	.08	.25	.79	1.0	1.0	1.0	1.0	1.0
	90°	.03	.04	.04	.02	.02	.32	.59	.72	.07	.07	.06	.06	.15	.86	.93	.95
.5	0°	.04	.14	.38	.01	.00	.00	.00	.00	.55	.67	1.0	1.0	.14	.00	.00	.03
	45°	.04	.09	.40	.93	1.0	1.0	1.0	1.0	.08	.39	.94	1.0	1.0	1.0	1.0	1.0
	90°	.03	.04	.04	.02	.01	.13	.30	.46	.07	.07	.07	.06	.27	.94	.96	.94
.7	0°	.04	.07	.13	.00	.00	.00	.00	.00	.06	.54	.99	1.0	.08	.00	.00	.00
	45°	.03	.06	.13	.52	.98	1.0	1.0	1.0	.08	.32	.86	1.0	1.0	1.0	1.0	1.0
	90°	.03	.04	.04	.02	.00	.04	.04	.02	.07	.07	.07	.06	.17	.89	.89	.87

Table 5: Power of the CUSUM of Squares test;
static regression with a correction for serial correlation

		a) Using recursive residuals								b) using OLS residuals							
λ	b	0	.5	1	2	5	10	20	30	0	.5	1	2	5	10	20	30
.3	0°	.04	.04	.00	.00	.00	.00	.00	.00	.03	.04	.05	.01	.00	.00	.00	.00
	45°	.05	.03	.04	.09	.99	1.0	1.0	1.0	.04	.03	.09	.61	1.0	1.0	1.0	1.0
	90°	.04	.04	.04	.07	.01	.00	.00	.05	.04	.04	.07	.68	.96	.00	.00	.00
.5	0°	.03	.02	.04	.00	.00	.00	.00	.00	.03	.01	.01	.00	.00	.00	.00	.00
	45°	.04	.04	.07	.85	1.0	1.0	1.0	1.0	.03	.03	.01	.00	.00	.00	.00	.00
	90°	.04	.03	.07	.86	.56	.00	.25	.88	.03	.02	.01	.00	.00	.00	.00	.00
.7	0°	.04	.04	.14	.48	.00	.00	.00	.00	.03	.03	.06	.01	.00	.00	.00	.00
	45°	.04	.03	.21	.99	1.0	1.0	1.0	1.0	.03	.03	.05	.55	1.0	1.0	1.0	1.0
	90°	.04	.04	.18	.98	.20	.00	.00	.00	.04	.04	.08	.75	1.0	.02	.00	.00

Table 6: Power of the Cusum Test in a Dynamic Regression

		a) Using recursive residuals												b) Using OLS residuals											
		$\alpha = -.5$			$\alpha = 0$			$\alpha = .5$			$\alpha = .8$			$\alpha = -.5$			$\alpha = 0$			$\alpha = .5$			$\alpha = .8$		
λ	$b \backslash \psi$	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°
.3	0	.03	.04	.03	.03	.04	.04	.05	.04	.04	.04	.06	.04	.04	.04	.03	.04	.04	.04	.03	.04	.03	.05	.04	.04
	.5	.27	.16	.04	.28	.18	.04	.27	.15	.04	.24	.10	.04	.36	.20	.05	.37	.19	.04	.34	.16	.03	.26	.14	.04
	1	.90	.60	.03	.83	.55	.02	.78	.49	.03	.71	.38	.02	.97	.72	.05	.93	.68	.04	.92	.62	.03	.82	.48	.02
	2	1.0	1.0	.02	1.0	1.0	.02	1.0	.95	.01	1.0	.86	.00	1.0	1.0	.06	1.0	1.0	.06	1.0	1.0	.02	1.0	.97	.00
	5	1.0	1.0	.01	1.0	1.0	.02	1.0	1.0	.00	1.0	.97	.00	1.0	1.0	.07	1.0	1.0	.15	1.0	1.0	.05	1.0	1.0	.00
	10	1.0	1.0	.01	1.0	1.0	.03	.95	1.0	.03	1.0	.12	.00	1.0	1.0	.03	1.0	1.0	.21	1.0	1.0	.42	1.0	1.0	.00
	20	1.0	1.0	.00	.72	1.0	.00	.32	1.0	.14	.96	.00	.00	1.0	1.0	.00	1.0	1.0	.12	1.0	1.0	.80	1.0	.01	.00
	30	1.0	1.0	.00	.21	1.0	.00	.06	1.0	.13	.54	.00	.20	1.0	1.0	.00	.15	1.0	.03	1.0	1.0	.70	1.0	.00	.05
	200	1.0	1.0	.00	.00	1.0	.00	.00	1.0	.00	1.0	.00	.00	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	1.0	.00	.66
.5	0	.04	.03	.04	.03	.05	.05	.05	.03	.05	.05	.05	.05	.04	.04	.04	.04	.03	.04	.04	.03	.03	.03	.05	.05
	.5	.24	.12	.04	.22	.11	.04	.22	.09	.03	.21	.10	.02	.58	.34	.05	.60	.34	.05	.56	.31	.03	.51	.26	.03
	1	.82	.46	.03	.78	.43	.02	.74	.39	.02	.67	.32	.02	.99	.91	.05	.99	.87	.04	1.0	.86	.02	.99	.83	.01
	2	1.0	.99	.01	1.0	.97	.00	1.0	.88	.00	.99	.82	.00	1.0	1.0	.06	1.0	1.0	.06	1.0	1.0	.02	1.0	1.0	.00
	5	1.0	1.0	.00	1.0	1.0	.00	1.0	1.0	.00	1.0	.95	.00	1.0	1.0	.06	1.0	1.0	.17	1.0	1.0	.05	1.0	1.0	.00
	10	1.0	1.0	.00	1.0	1.0	.00	.82	.99	.00	.98	.02	.00	1.0	1.0	.03	1.0	1.0	.23	1.0	1.0	.58	1.0	1.0	.00
	20	1.0	1.0	.00	.42	1.0	.00	.07	.91	.00	.54	.00	.00	1.0	1.0	.00	1.0	1.0	.09	1.0	1.0	.87	1.0	.46	.00
	30	1.0	1.0	.00	.03	1.0	.00	.00	.80	.00	.02	.00	.00	1.0	1.0	.00	.86	1.0	.02	1.0	1.0	.72	1.0	.00	.08
	200	1.0	1.0	.00	.00	1.0	.00	.00	.76	.00	1.0	.00	.00	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	1.0	.00	.68
.7	0	.04	.03	.05	.04	.04	.04	.04	.04	.05	.03	.04	.03	.04	.04	.03	.02	.05	.03	.03	.03	.04	.03	.04	.03
	.5	.10	.05	.04	.09	.06	.03	.09	.06	.03	.09	.05	.04	.43	.25	.04	.43	.25	.04	.46	.24	.03	.46	.24	.03
	1	.42	.18	.02	.37	.17	.02	.37	.16	.02	.38	.15	.01	.98	.80	.03	.98	.80	.03	.97	.76	.02	.97	.75	.02
	2	.96	.73	.01	.92	.64	.00	.88	.50	.00	.91	.48	.00	1.0	1.0	.05	1.0	1.0	.05	1.0	1.0	.01	1.0	1.0	.01
	5	1.0	1.0	.01	1.0	1.0	.00	.97	.82	.00	1.0	.76	.00	1.0	1.0	.05	1.0	1.0	.13	1.0	1.0	.02	1.0	1.0	.00
	10	1.0	1.0	.00	.94	1.0	.00	.57	.50	.00	.88	.02	.00	1.0	1.0	.02	1.0	1.0	.15	1.0	1.0	.38	1.0	1.0	.00
	20	1.0	1.0	.00	.18	1.0	.00	.01	.01	.00	.14	.00	.00	1.0	1.0	.00	1.0	1.0	.04	1.0	1.0	.85	1.0	.87	.00
	30	1.0	1.0	.00	.00	1.0	.00	.00	.00	.00	.00	.00	.00	1.0	1.0	.00	.97	1.0	.00	1.0	1.0	.60	1.0	.00	.02
	200	.43	1.0	.00	.00	1.0	.00	.00	.00	.00	1.0	.00	.00	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	1.0	.00	1.0

Table 7: Power of the Cusum of Squares Test in a Dynamic Regression

		a) Using recursive residuals												b) Using OLS residuals											
		$\alpha=-.5$			$\alpha=0$			$\alpha=.5$			$\alpha=.8$			$\alpha=-.5$			$\alpha=0$			$\alpha=.5$			$\alpha=.8$		
λ	$b\backslash\psi$	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°
.3	0	.04	.03	.03	.03	.04	.03	.03	.04	.03	.04	.03	.03	.05	.05	.05	.05	.05	.05	.07	.05	.06	.05	.05	.05
	.5	.03	.03	.03	.03	.04	.04	.04	.03	.04	.04	.02	.03	.07	.08	.06	.09	.08	.08	.06	.08	.06	.08	.09	.08
	1	.04	.03	.03	.04	.03	.03	.04	.03	.03	.04	.04	.04	.18	.20	.13	.17	.21	.17	.17	.19	.20	.15	.19	.21
	2	.08	.09	.08	.07	.10	.08	.13	.09	.09	.19	.10	.08	.71	.86	.25	.54	.90	.51	.48	.89	.83	.57	.82	.88
	5	.79	.99	.72	.65	1.0	.69	.94	.99	.95	1.0	.99	.99	.97	1.0	.63	.78	1.0	.79	.92	1.0	1.0	1.0	1.0	1.0
	10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.95	1.0	1.0	.98	1.0	.98	1.0	1.0	1.0	1.0	1.0	1.0
	20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	200	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
.5	0	.04	.03	.04	.03	.03	.03	.03	.04	.03	.04	.04	.04	.05	.06	.05	.05	.05	.05	.05	.05	.07	.05	.04	.06
	.5	.03	.03	.04	.03	.03	.03	.02	.04	.04	.03	.04	.03	.05	.05	.04	.04	.04	.04	.05	.05	.05	.04	.05	.06
	1	.12	.10	.06	.09	.12	.09	.06	.11	.12	.07	.09	.10	.02	.03	.04	.03	.03	.04	.03	.04	.03	.05	.04	.02
	2	.78	.88	.20	.62	.91	.59	.47	.88	.88	.49	.86	.90	.01	.01	.07	.03	.01	.02	.04	.01	.13	.11	.01	.00
	5	1.0	1.0	.86	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.04	.00	.27	.14	.00	.15	.38	.00	.00	.93	.00	.00
	10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.38	.00	.99	.70	.00	.70	.98	.00	.14	1.0	.00	.00
	20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99	.00	1.0	1.0	.00	1.0	1.0	.00	.98	1.0	.00	.00
	30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	1.0	.00	1.0	1.0	.00	.79
	200	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.00	1.0	1.0	.00	1.0	1.0	.00	1.0	1.0	.00	1.0
.7	0	.02	.04	.03	.03	.04	.04	.04	.03	.03	.04	.04	.04	.04	.05	.06	.07	.05	.04	.05	.05	.04	.05	.05	.05
	.5	.05	.05	.04	.04	.05	.05	.04	.05	.05	.05	.04	.05	.06	.03	.03	.04	.04	.04	.03	.04	.03	.03	.02	.04
	1	.25	.25	.15	.20	.25	.17	.22	.30	.25	.17	.25	.28	.02	.03	.04	.02	.02	.03	.02	.03	.04	.02	.04	.03
	2	.98	.99	.62	.94	1.0	.94	.88	.99	.99	.92	.99	1.0	.13	.23	.03	.05	.37	.06	.02	.27	.23	.02	.19	.40
	5	1.0	1.0	.99	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.40	1.0	.12	.09	1.0	.10	.06	1.0	.98	.02	1.0	1.0
	10	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.51	1.0	.99	.59	1.0	.57	.94	1.0	.91	1.0	1.0	1.0
	20	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.99	1.0	1.0	1.0
	30	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
	200	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 8: Power of the Cusum of Squares Test in a Dynamic Regression: correcting for non-Normality

		a) Using recursive residuals												b) Using OLS residuals											
		$\alpha=-.5$			$\alpha=0$			$\alpha=.5$			$\alpha=.8$			$\alpha=-.5$			$\alpha=0$			$\alpha=.5$			$\alpha=.8$		
λ	$b\backslash\psi$	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°	0°	45°	90°
.3	0	.05	.04	.06	.04	.05	.05	.05	.06	.05	.05	.04	.03	.05	.04	.07	.05	.05	.06	.06	.06	.05	.05	.05	.05
	.5	.04	.04	.05	.04	.05	.05	.04	.06	.04	.05	.04	.05	.07	.08	.07	.06	.09	.09	.07	.08	.06	.07	.07	.09
	1	.06	.06	.06	.05	.06	.05	.05	.06	.06	.06	.06	.06	.19	.19	.09	.13	.22	.14	.09	.21	.16	.07	.18	.19
	2	.14	.08	.12	.11	.05	.14	.16	.07	.15	.23	.08	.16	.54	.85	.14	.23	.92	.26	.13	.82	.56	.13	.70	.81
	5	.49	.27	.13	.30	.18	.29	.56	.18	.96	.94	.09	1.0	.59	1.0	.16	.21	1.0	.20	.18	1.0	.58	.53	1.0	.95
	10	.29	.96	.00	.02	.80	.02	.51	.86	.98	.95	.36	1.0	.34	1.0	.02	.02	1.0	.01	.01	1.0	.35	.90	1.0	.83
	20	.00	1.0	.00	.00	1.0	0.0	.00	1.0	.79	.00	.99	1.0	.01	1.0	.00	.00	1.0	.00	.00	1.0	.01	1.0	1.0	.84
	30	.00	1.0	.00	.00	1.0	0.0	.00	1.0	.01	.00	1.0	1.0	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	1.0	1.0	.83
.5	0	.05	.05	.05	.05	.04	.05	.05	.04	.06	.04	.05	.04	.06	.05	.06	.06	.05	.05	.06	.05	.05	.05	.05	.06
	.5	.03	.03	.02	.04	.03	.03	.04	.02	.04	.03	.02	.02	.05	.05	.04	.06	.05	.04	.07	.04	.05	.04	.04	.05
	1	.03	.04	.02	.04	.05	.05	.03	.05	.05	.02	.04	.06	.03	.03	.05	.05	.03	.04	.05	.04	.03	.05	.05	.04
	2	.72	.66	.08	.46	.76	.45	.26	.70	.83	.29	.57	.88	.03	.03	.06	.04	.01	.05	.06	.03	.03	.06	.04	.02
	5	1.0	1.0	.02	.93	1.0	.95	.45	1.0	1.0	.76	1.0	1.0	.09	.02	.04	.05	.00	.05	.04	.02	.12	.07	.05	.11
	10	1.0	1.0	.00	.02	1.0	.02	.00	1.0	1.0	.00	1.0	1.0	.05	.01	.00	.00	.00	.00	.00	.01	.05	.00	.22	.28
	20	.01	1.0	.00	.00	1.0	.00	.00	1.0	.99	.00	1.0	1.0	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	.84	.18
	30	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	.00	1.0	1.0	.00	.00	.00	.00	.00	.00	.00	.00	.00	.00	1.0	.05
0.7	0	.07	.04	.04	.04	.04	.05	.05	.04	.05	.05	.05	.04	.08	.05	.05	.05	.05	.05	.05	.05	.06	.05	.05	.05
	.5	.02	.03	.03	.03	.03	.03	.02	.03	.03	.02	.02	.03	.03	.03	.03	.03	.04	.03	.03	.03	.04	.03	.03	.03
	1	.13	.11	.07	.12	.13	.09	.08	.12	.15	.10	.12	.15	.03	.02	.03	.02	.02	.02	.04	.02	.03	.04	.03	.04
	2	.97	.96	.40	.90	.99	.89	.81	.97	.98	.87	.95	.99	.13	.27	.03	.05	.34	.04	.05	.26	.11	.03	.16	.30
	5	1.0	1.0	.54	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	.17	1.0	.02	.04	1.0	.02	.01	1.0	.16	.08	1.0	.70
	10	1.0	1.0	.00	.96	1.0	.96	.37	1.0	1.0	1.0	1.0	1.0	.06	1.0	.00	.00	1.0	.00	.00	1.0	.06	.25	1.0	.46
	20	.96	1.0	.00	.00	1.0	.00	.00	1.0	1.0	.56	1.0	1.0	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	.35	1.0	.39
	30	.08	1.0	.00	.00	1.0	.00	.00	1.0	1.0	.44	1.0	1.0	.00	1.0	.00	.00	1.0	.00	.00	1.0	.00	.31	1.0	.34

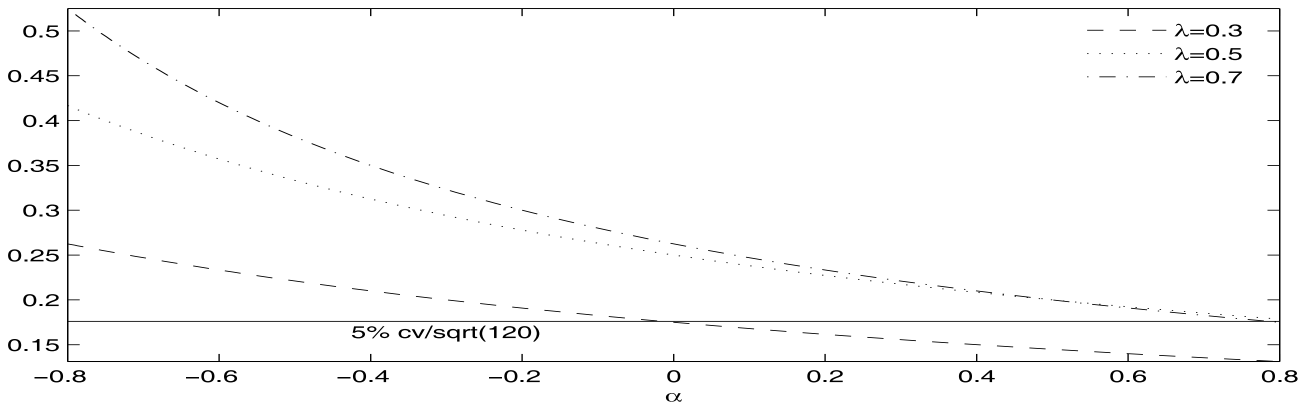

Figure 1: Limit of $T^{-1/2}CUSQ^{rec}$, recursive residuals from a dynamic regression, 0^o.

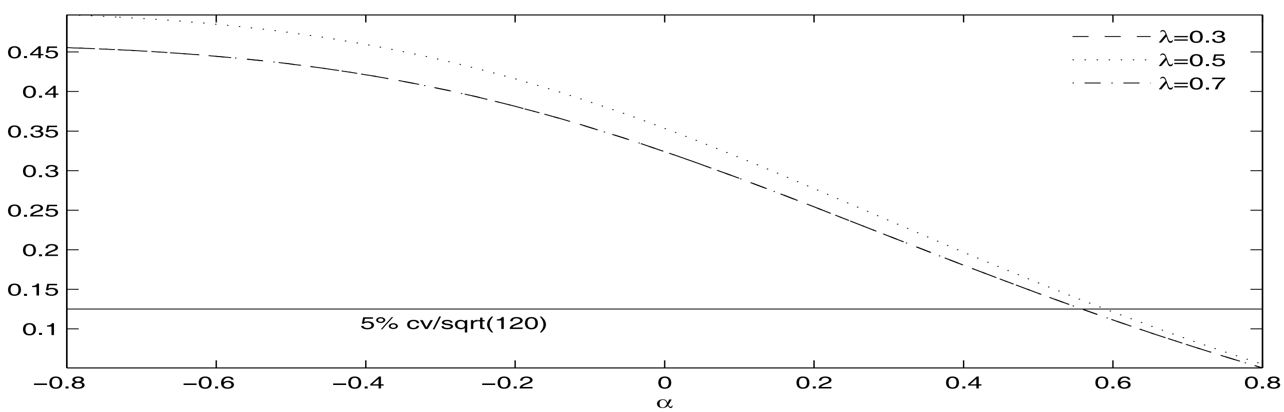

Figure 2: Limit of $T^{-1/2}CUSUM^{ols}$, OLS residuals from a dynamic regression, 45^o.

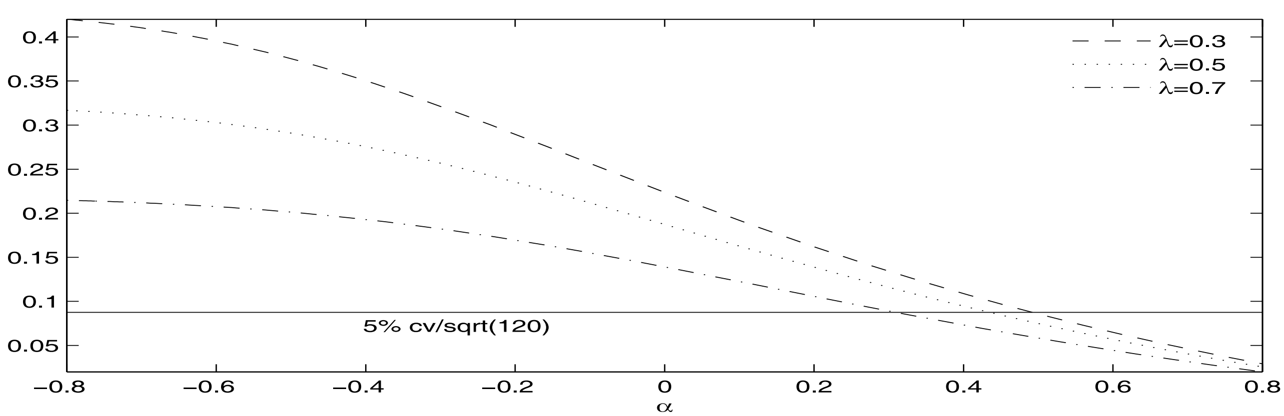

Figure 3: Limit of $T^{-1/2}CUSUM^{rec}$, recursive residuals from a dynamic regression, 45^o.