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A SIMPLE DERIVATION OF EXACT RELIABILITY FORMULAS FOR LINEAR AND CIRCULAR CONSECUTIVE-k-of-n : F SYSTEMS

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Abstract

Exact reliability formulas for linear and circular consecutive-k-of-n : F systems are derived in the case of equal component reliabilities.

SYSTEM OF COMPONENTS; EQUAL COMPONENT RELIABILITIES

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1. Introduction

Consider a system of n components arranged in a line or circle, and suppose that each component fails independently with probability $q = 1 - p$. We say the system fails if there is a run of at least k consecutive failures. The failure probability of these systems has been studied in many places; see Derman et al. [1], Shanthikumar [5], Lambris and Papastavridis [2], Papastavridis and Hadzichristos [3], and the references therein. In Section 2 we derive formulas for the system failure probability for both the linear and circular models. The formulas and their derivation are shorter and simpler than those given elsewhere.

2. Main results

Theorem 1. *For $n \geqq k$, the failure probability for the consecutive-k-of-n : F system is given by*

(a)

$$\sum_{m=1}^{(n+1)/(k+1)} (-1)^{m+1} \left[\binom{n-mk}{m} + \frac{1}{p} \binom{n-mk}{m-1} \right] (q^k p)^m$$

for the linear system, and

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(b)

$$q^n + \sum_{m=1}^{[n/(k+1)]} (-1)^{m+1} \left[\binom{n-mk}{m} + k\binom{n-mk-1}{m-1} \right] (q^k p)^m$$

for the circular system, where $\binom{i}{j} = 0$ *for* $i < j$, *and* $\binom{0}{0} = 1$.

To prove the theorem we need the following lemma.

Lemma 1. *For an n-component system, the number of ways to choose m non-overlapping runs of* $k+1$ *consecutive components is*:

(a)

$$\binom{n-mk}{m}$$

for the linear system, and

(b)

$$\binom{n-mk}{m} + k\binom{n-mk-1}{m-1}$$

for the circular system.

Proof of (a). For each run, imagine that the $k+1$ components are compressed into a single 'marked' component. We then have m marked components out of a total of $n - mk$ components. The number of ways to arrange them is $\binom{n-mk}{m}$.

Proof of (b). Cut the circle at an arbitrary point between two components. Call this point the *breakpoint*. Then

1. the number of ways where no run crosses the breakpoint is the same as that for the linear system, or $\binom{n-mk}{m}$ ways, and

2. the number of ways where a run crosses the breakpoint and contains i components, $1 \leqq i \leqq k$, to the left of the breakpoint and $k+1-i$ to the right, is the same as the number of ways to choose $m-1$ runs in a linear system with $n-k-1$ components. Summing over i, the total number of ways in this case is

$$k\binom{n-k-1-(m-1)k}{m-1} = k\binom{n-mk-1}{m-1}.$$

Adding the counts for the two cases, (b) is obtained.

Proof of Theorem 1. For either system, number the components and order them from left to right (linear system) or clockwise (circular system) and for $i = 1, \cdots, n$ define the following events:

$$A_i = \left\{ \begin{array}{l} \text{a run of } k \text{ failed components followed by one} \\ \text{working component ends with component } i \end{array} \right\}.$$

Also let

$$A_{n+1} = \{\text{a run of } k \text{ failed components ends with component } n\}.$$

Note that

$$\{\text{linear system fails}\} = \bigcup_{i=1}^{n+1} A_i$$

and

$$\{\text{circular system fails}\} = \left(\bigcup_{i=1}^{n} A_i\right) \cup \{\text{all components fail}\}.$$

Using inclusion–exclusion we have

$$P\left(\bigcup_i A_i\right) = \sum_m (-1)^{m+1} \sum_{i_1<i_2<\cdots<i_m} P(A_{i_1}A_{i_2}\cdots A_{i_m}).$$

To prove (b) note that for the circular system

$$P(A_{i_1}A_{i_2}\cdots A_{i_m}) = \begin{cases} (q^k p)^m & \text{if none of the runs overlap} \\ 0 & \text{otherwise} \end{cases}$$

and the number of times $(q^k p)^m$ appears in the sum is the same as the number of ways to choose m non-overlapping runs of $k+1$ components, which was calculated in Lemma 1(b). Adding the probability that all components fail and noting that there is space for at most $[n/(k+1)]$ non-overlapping runs, (b) is obtained.

For the linear system, with $i_1 < i_2 < \cdots < i_m$

$$P(A_{i_1}A_{i_2}\cdots A_{i_m}) = \begin{cases} (q^k p)^m & \text{if } i_m < n+1 \text{ and none of the runs overlap} \\ (q^k p)^m / p & \text{if } i_m = n+1 \text{ and none of the runs overlap} \\ 0 & \text{otherwise} \end{cases}$$

and the number of times $(q^k p)^m$ appears in the sum is the same as the number of ways to choose m non-overlapping runs of $k+1$ components in an n-component system, and the number of times $(q^k p)^m/p$ appears (the case where the last k components fail) is the same as the number of ways to choose $m-1$ of these runs in an $n-k$ component system. By Lemma 1(a) the former is $\binom{n-mk}{m}$ and the latter is

$$\binom{n-k-(m-1)k}{m-1} = \binom{n-mk}{m-1}.$$

Combining and noting that not more than $[(n+1)/(k+1)]$ of the events can occur together, (a) is obtained.

Remarks

1. By letting A_i be the event that a run of k consecutive failures ends with component i, an expression for the system failure probability can be obtained (using inclusion–exclusion), though the combinatorics become complicated due to overlapping runs. Defining A_i as in the proof of Theorem 1 eliminates the possibility of overlaps and greatly simplifies the derivation.

2. For the case where the components are not identical, see Peköz and Ross [4] for a method of approximating the failure probability.

References

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