

Funcitia: A Tale of Fourier Transform

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1 Neighbors

There was a quiet small town named Funcitia. Naturally, the people living there were called funcitians. Sinc was an ordinary guy in this town. At least the neighbors called him Sinc. He, as you can imagine, had a round head, no neck, no waist, looked like a salt shaker. The robe had a long train on the ground as a wedding dress, like the ripples of water. When he walked on the street, sometimes people would look at him for a while and finally recognized him. "Hey, aren't you the sine function divided by the argument itself? I've heard about you." Then he would happily answer, "just call me Sinc." So people called him Sinc, although nobody knew the meaning of the letter C.

Sinc loved plants. He had some flowers in his garden. He wouldn't let some random person enter to mess his pretty garden, so he built a gate. The gate had no door on it. It was actually an opening on the wall. When Sinc came home, he couldn't get in directly, because the train of the robe was too wide. Once he stopped in front of the gate, then turned around changing the appearance into a rectangle. ... OK, *Sinc could transform into a rectangle*. The transform was done simply by turning around. He didn't have to think about some ugly integrals. This was embedded in his gene. With the appearance of a rectangle, he could easily go through the gate. If he would like, he could turn around back into the original shape. In fact he comfortably stayed in either form. His turning was not a secret at all, because everyone familiar with Sinc knew that his full name was Sincus Rectanguli.

Actually everyone in this town, that is, every funcitian could perform the turning move. This move was called Fourier Turning by people out of the town. Ironically, it was also well known that Mr. Fourier himself couldn't do this. One of Sinc's best friends was Lambda. From one side he was a triangle Λ , and from the other side he looked similar to Sinc, except that he was a little slimmer than Sinc. Perhaps they were good buddies because of similar looks. Lambda could go through Sinc's gate, if he faced the gate with the triangle side. There were strange rumors saying that Lambda was the son of Sinc. One rumor was that Sinc changed into the rectangle form and created Lambda by the magic of "convolution". Another rumor was that Sinc could perform the trick of "multiplication", then came Lambda. Nobody knew which was more likely to be true. However, the seniors in the town claimed that both were false, because either convolution or multiplication was the magic only in the age of mythology. If anyone still knew how to use these magics, the whole world would be messed up.

Among the seniors, we should definitely start from the mayor, whose name was Gauss. He was fat, wearing a loose robe. Nobody in the town was elder than him, or had more prestige than him. He seemed to know everything. Every time he began to talk, all the other funcitians listened

carefully. There was a bell tower on top of the highest building. The bell was believed to be a statue of Mayor Gauss. Sinc respected Gauss, and at the same time, kept him at a respectful distance. Sinc was happy about his gate, because no matter how Gauss manipulated his fatty body, he couldn't get through. Therefore Gauss didn't frequently visit Sinc, giving criticisms.

Gauss's turning skill was lame. His turning could only make himself from short and fat into a little taller and thinner. The robe was always there in both shapes. Somebody said that this robe was the symbol of senior which he would not give up. By the way, people in Functia also became fatter if they ate too much. But this was never a problem, even for girls, because when they turned around, they became even slimmer than previously. Hence people did not care about losing weight. Gauss once published a "Heisenberg's Losing Weight Useless Principle", proving that even you lost some weight, once you turned around you would surly get the it back. Actually everybody knew this fact, Gauss published this only to refute the saying that he was too fat.

Gauss also had a buddy, called Sech, who also always wore a robe. He was thinner than Gauss, but the robe laying on the ground was thicker. People saw Sech retaining the same look all the day. In fact few functians knew that he was also unchanged after turning, which was similar to Gauss. No need to say, because of the robe of senior, he could not enter Sinc's garden. Somebody thought Sech was comparable to Gauss, and was also qualified to be the Mayor. But because he was humble, or maybe Gauss was his friend, he never revealed such an idea. People saw him going to the beach surfing very often. When he slid on the surface of the sea, he looked quite like a soliton wave.

There were many other functians. One lady called Bessel had similar shape to Sinc. Gauss said, "her name is not proper. She is the first order Bessel function divided by the argument. Just like Sinc should not be called sine, she shouldn't be called Bessel." Anyway Bessel was a bright name. She turned into an exact half circle with the diameter on the ground when she performed the turning move. Some said that this shape was perfect, but some thought it looked like a tortoise. But anyway she could go to Sinc's garden, where she often visited.

Another guy was called Trapez. He looked ordinary, wearing rippled clothes. After turning around, he transformed into a trapezoid. He was famous for his profession, a traveler. He brought lots of photos and shared them with functians. His stories about the outside world were very popular in the town.

Although there were two classes, seniors and citizens, all functians lived harmonic lives. Until one day, the peace was broken.

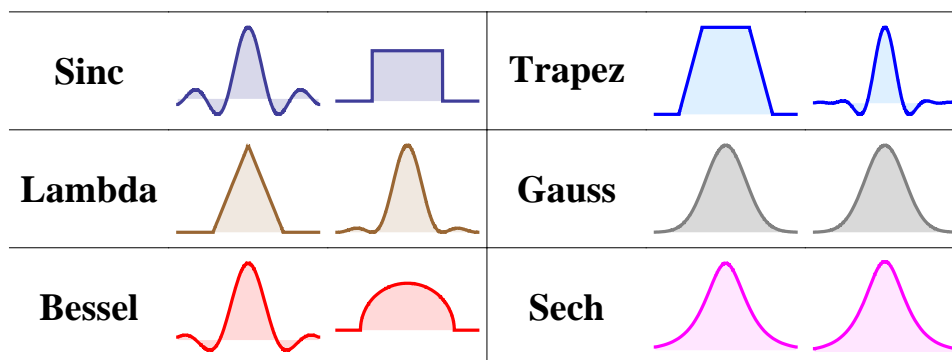


Figure 1: All the characters mentioned so far

2 Invasion

One day, people in Functia were gathering in the square chatting with each other when the traveler Trapez came back. Functians were all excited to see him. He was coming with the shape of trapezoid, otherwise if he turned to the other side, it would be hard to distinguish him from Sinc looking from a distance. However, when he walked closer, functians got a feeling of terror. This was no longer the normal Trapez. He was *injured*. His original healthy and continuous body changed into a bunch of vertical spikes forming a trapezoid. What's more, people from the other side surprisingly found that, his shape was no longer like Sinc consisting of a body with surrounding clothes on the ground. He became a long line of many identical bodies, which was so long that no one could see the end. This phenomenon was completely new to the functians, because all the people in this town had finite size. What was this endless line of bodies? He was simply not a creature on this planet *Integrable*, but an alien. Trapez himself didn't know why he became like this. He only knew that he was walking in a forest, when suddenly someone pushed him down from the back, then he transformed into spikes.

Whenever weird things happened, find Gauss. In no time Gauss hurried to the square from the library, carrying his corpulent body. He examined the injure of Trapez, then slowly said, "Hmm...you are sampled.."

The functians didn't know the word *sample*. But they didn't mind hearing a new word, because the mayor, being a philosopher as well, often used uncommon words to express the common meanings. However, this time was different because after saying that Gauss started contemplating, which was not consistent with his flaunting character. Then Gauss walked to Sinc, said something and went away. Gauss asked Sinc to his home. But Sinc wouldn't like to go alone. He was simply not comfortable talking with a senior like Gauss. Finally he got his girl friend Bessel to go together, which made him to feel better.

Because of Gauss's long robe, he had no door for his home, I should say, no wall at all. His home was only a clear area, otherwise he couldn't enter. So this home was very open for the guests. On the grass in Gauss's home, Gauss said to Sinc, "Troubles come..." Then he took a gilt-edged huge book from the shelf.

The citizens were still talking about Trapez's injury in the square when somebody ran into the town, shouting that a monster was coming. Sech climbed onto the bell tower with some other guys to check. The thing moving towards Functia looked like a fence, which was a line of innumerable identical spikes extending to far away in both sides. The whole fence was moving slowly to here. After Sech describing the monster to the crowd, Trapez recalled that he saw similar scene after being pushed down. It was extremely dangerous.

There was a functian called Laplace, who always felt responsible to protect others. Having heard the coming of an enemy, he walked out without doubt. He was the strongest man in the town, having a sharp head and exponentially decaying arms, which were much stronger than the squared-exponentially decaying robe of Gauss. His profile was referred as Laplace distribution by the probabilistists, hence the name. When he turned around, his head became round, and the arms got stronger to the polynomial decay. Whenever threaten came, Laplace always stood in front, which relieved all the other functians. But now what surprised Laplace was that, the monster knew *magic*. As he got closed to the monster, he was turned into the spiky shape, too. Only the lengths of the spikes remained the original shape. He, following Trapez, was also sampled. The strongest man couldn't stop it. Were all the functians destined to be overridden by this legion of spikes?

Some citizens proposed that maybe they could find out the monster's appearance of the other side. Any other shape would be better than this spiky fence. However, as Sech found out from the tower, the monster had the exact same shape even if you looked from the other side. He was a turning-invariant person, just like Gauss and Sech! No weakness, invulnerable.

Meanwhile, Gauss was in his home talking with Sinc and Bessel. Gauss hold the book and said, "in no time a monster called Comb will come. His formal name is Dirac Array. He is derived from Dirac Delta, that is, he is the result of Dirac Delta copying himself many times... Dirac Delta committed crimes a long time ago. He transformed others into his shape by the magic of multiplication..."

"Wait a second! You mean the trick of multiplication really exists?" Asked Sinc.

Gauss answered, "It truly exists. But because of the devastating power, the abolishment of this magic became a moral obligation and was accepted by people unconditionally...Although the power of Dirac Delta's magic is large, the weakness of himself is also obvious. Once he turns around, or people go around to the other side he becomes a flat shape of a constant function. Then the multiplication magic has no effect to anything at all. For this reason, during that crime, Dirac Delta could only sneak attack... Maybe later he merged with the victims who were transformed into the same shape as him, and became a comb. Now even he turns around, nobody can escape any longer..."

Hearing this, Sinc was so desperate. Noises came from the square, when Sech described the monster to the crowd. Gauss continued, "Some people call Comb using multiplication by the term *sample*. Do you know, Sinc, that someone calls you by sampling function?"

"Ahh... I have some relation with that monster?"

"Indeed, because if you perform the multiplication magic, you can get rid of him... Look at yourself and your clothes on the ground. At some certain locations there is nothing. If we adjust the locations of the zeros such that they are well aligned with the spikes of Comb, you can get him by multiplication. Only one spike will be left, which locates at your head. The result is that Comb will be reduced from the Dirac Array back to Dirac Delta. Then it becomes easy to avoid him..." Although Gauss's words were too technical, at this crucial stage Sinc and Bessel had no other choice but bearing the tedious and accurate language.

Sinc was very upset about this mission. He never thought he was THE ONE in such a war. Bessel also thought that was too dangerous for Sinc. If the alignment was not exact, it wouldn't work. Then Comb would be likely to beat Sinc. It was even better to run away than taking the risk.

Looking at Sinc, Bessel said to Gauss, "Let me do it. My shape is similar to Sinc."

Gauss said, "I'm surprised of your courage. But sadly, Miss Bessel, your zeros are not equally spaced, which can't beat the hateful monster."

Sinc was still wondering, "Who on earth is able to do that long lost magic?" "You can't? Don't be hiding now. Aren't there lots of people saying that..." "The rumors are not true. Don't believe them. So what to do now? Is it recorded in the book?"

Gauss blinked his eyes. "But I really know that trick... I can teach you..." The fat mayor really was something.

The mayor taught Sinc the spell and the moves. Then together with Bessel, he carefully measured and adjusted Sinc's clothes, that is, the locations of zeros. When the error was exactly zero, Bessel gave him the deepest blessing.

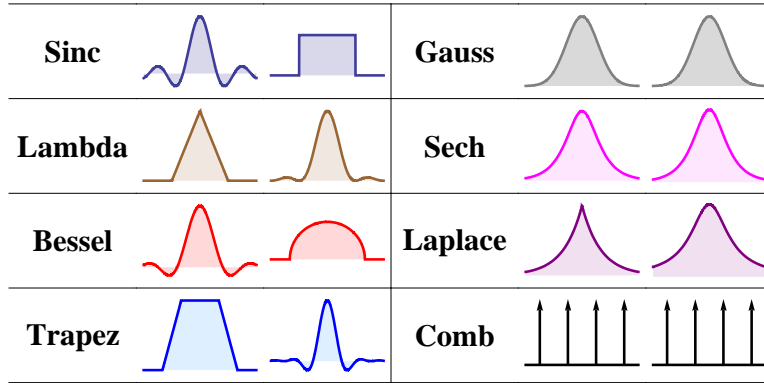


Figure 2: All the characters

3 Future

Functia was facing the greatest ever menace. Under the instruction of Gauss, Sinc took the mission to tame Comb, the legion of spikes. When they were about to leave Gauss's home, they saw Sinc's friend Lambda on the path outside, who was also leaving in haste. In fact, Lambda followed them to here and heard everything about the multiplication trick. He noticed that his zeros were exactly as same as Sinc's, so he adjusted the size by himself, willing to take the risk for his friend. When Sinc found that Lambda's clothes was unusually trimmed like himself, he got a bad feeling. It was already too late to chase Lambda, because he ran away swiftly. After all, Lambda was thinner than Sinc, hence running faster. It was said that when he ran fast, people on the other side (the triangle side) could see the phase was changing rapidly.

All the Functians were watching Lambda running to the monster Comb, who was approaching this town. He was aiming during running, and preparing for the spell at the same time. Comb was also ready to perform the magic. At the moment they met, "MULTIPLICARE!" Both launched the same trick simultaneously.

As a result, they were multiplied. Because both tricks worked, they merged into one body. Only the product functian was left. The product was nothing but *Dirac Delta*. Only one spike by itself. Lambda successfully reduced the array into only one spike. Delta looked at himself, sighing. All the work in these years were in vain now. Moreover, he saw someone else doing the multiplication trick, which irritated him. Becoming angry and even crazy, he rushed to the town madly.

This was certainly an unwise action. Because although his head was sharp, the attacking area was too small. And this was not a sneak attack any more. It was Sinc waiting for him in the path. Delta rushed to him. Sinc turned around into the rectangle form and dodged Delta. The even better thing was that, Sinc faced the flat side of Delta now. "MULTIPLICARE!" He launched the magic immediately. After a spark, Delta transformed into a rectangle, as same as Sinc.

Losing the spike, Delta was like a lion without teeth. The citizens surrounded and subdued him in one shot. He even didn't get the opportunity to say the spell again.

The enemy was defeated, but our friend Lambda merged with Comb becoming the new Dirac Delta. Trapez the traveler and Laplace the fighter were injured. How to heal them all? Fortunately Gauss came to help. Following his directions, Sinc turned to the rectangle form, adjusted the size and performed the multiplication trick to the side of Trapez consisting of a line of many identical

bodies. The magic filtered all the replicas leaving only the original body. In this way Trapez approximately recovered the old appearance. The only pity was that his previous infinitely long train of the clothes was cut. On the other side, the trapezoid was no longer in a perfect shape, as if a previously trimmed and neat suit gets wrinkled. Then Sinc did the same thing to Laplace, recovering him approximately, too.

Finally, Sinc still wanted to do something to the rectangulized Dirac Delta, who was the result of merging between evil Comb and friendly Lambda. Gauss suggested to convolve two rectangles to restore the triangular profile of Lambda. Sinc said, “well, I also hope I can do it. You must know the convolution magic, right? Would you teach me?” Gauss laughed, “You are a hero in this town. How is it possible that I don’t want to teach you? But you’ve already learned it. You need only *turn around, multiply and turn back*. This is the so called convolution trick!” “Really?” In this way Sinc saw the triangle head of Lambda again. With the sequence of manipulations the mind of Dirac Delta was fading out and Lambda regained this body. Till now, every problem got solved. Long live the friendship!

In the next morning the new hero Sinc was invited to Gauss’s home again. Gauss told him that he was considering retirement and asked Sinc to be the mayor. Gauss was long regarded as the best function because of many merits, like turning-invariance (some one considered it bad) and the maximum entropy fixing the variance. Just like the perfect circle is worthy to be the chief in *Flatland*, Gauss is always worthy to be the mayor of Functia. Who could accept the fact that he was retiring? Looking outside of his home, Gauss speaked to Sinc slowly,

“A long time ago I was as same as the others, having my own defects and peculiarities, for example, some discontinuous and undifferentiable points. Everyone is like that, which is not a big deal. Later I found an old book which recorded the multiplication magic and the method to use it to do convolution. Although I knew this was a dangerous move, I was so curious that I applied it to myself. I practiced the convolution to myself by a mirror tons of times. As a result, I gradually turned into this particular shape. Then I got respects and became the mayor. After all I am not born to be special... But do you know how the future will turn to be?”

“*Future?*” Of course Sinc knew this word, but he never considered this question. Because the life in Functia was so relaxed, nobody needed to think of any change in the future.

“In the future, some day, Comb the monster will invade again. At that time every continuous function will be sampled and become a spiky shape. On the other side it will be an infinitely long line of bodies. Furthermore, this long line will be sampled again such that in the original side the spiky shape will be copied infinitely many times. Thus in either side, the shape will be both spiky and periodic. And people only need to care about our appearance in one period... ”

Although Sinc didn’t understand all of the words, but he couldn’t stand any more. “Is this the future of us functions? All of us are going to be like Comb?”

“But as long as you are here, we have the hope... You are a key. If you perform the multiplication and convolution tricks, our original forms can be restored. Many guys can do similar things, like Trapez, Bessel, Butterworth, even me. But nobody gets your talent. You are the *ideal filter*... So you can take care of all the people in Functia through the dark age of the discretized future. When that trend comes, my Gaussian shape will be entirely useless. For this reason we need you to be the mayor...”

Sinc never thought his squared silly body or rippled clothes was so significant. “I can save all the sampled functions?” But thinking of the horrible monster Comb and Gauss’s words “*some day, all the continuous function will be sampled*”, Sinc felt that maintaining or restoring current

peaceful life was the most important thing for everyone. In fact he still hoped Gauss to be the mayor, because he was already the symbol of this happy life. However Sinc himself being the mayor, was the omen of the coming disaster and savior.

In that afternoon all the functionaries voted. Sinc became the mayor, who was the only candidate.