Appendix

Competition Policy as Strategic Trade with Differentiated Products

Martino De Stefano and Marc Rysman

In this appendix we will repeatedly make use of the definition of dominant diagonal matrix and its properties.

Definition 1 *An $n \times n$ matrix $A= [a_{ij}]$ is said to have a dominant diagonal (d.d.) if there exist positive numbers $d_1, d_2, ..., d_n$ such that:*

$$d_i|a_{ii}| > \sum_{j \neq i} d_j|a_{ij}| \qquad for\ i = 1, 2, .., n.$$

Remark 1 *Assume that an $n \times n$ matrix $A= [a_{ij}]$ has positive diagonal elements and non-positive off-diagonal elements. A sufficient condition for $\mathbf{A}$ to have d.d. is that $\mathbf{Ae} > \mathbf{0}$, where $\mathbf{e}$ is a vector of ones. That is, a sufficient condition is that its row sums are all positive.*

Lemma 7 *Assume that an $n \times n$ matrix $\mathbf{B}$ is such that $\mathbf{B} \geq \mathbf{0}$. If the matrix $(\mathbf{I} - \mathbf{B})$ has the d.d. property, then it is definite positive and it has a nonnegative inverse, which can be computed as*

$$(\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + ...$$

where the series is converging. Given two matrices $\mathbf{B}_1 \geq \mathbf{B}_2$, then $(\mathbf{I} - \mathbf{B}_2)^{-1} \geq (\mathbf{I} - \mathbf{B}_1)^{-1}$.

Lemma 8 *A d.d. matrix $\mathbf{B}$ with positive diagonal elements and negative off diagonal elements, has a nonnegative inverse $\mathbf{B}^{-1} \geq \mathbf{0}$.*

Before proceeding, we discus a few issues of notation. We write market structure G^c in matrix notation as $\mathbf{\Omega}_c$, where $\mathbf{\Omega}_c$ is $n^c \times n^c$ and element $\omega_{ij} = 1$ if i and j are sold by the same firm, and $\omega_{ij} = 0$ otherwise. We define $\mathbf{\Omega} = \begin{bmatrix} \mathbf{\Omega}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}_f \end{bmatrix}$. Throughout this paper, we use a bold $\mathbf{1}$ to represent an $n \times n$ matrix of ones and $\mathbf{1}^{cs}$ to represent an $n^c \times n^s$ matrix of ones, where c and s are either h or f.

5.1 Solution of the Third Stage Bertrand Game

The demand functions are:

$$q_i = a_i - \beta p_i + \sigma \sum_{j \neq i} p_j$$

Where the parameters are given by:

$$a_i = \frac{\alpha_i[\gamma(n-2)+1] - \gamma \sum_{j \neq i} \alpha_j}{(1-\gamma)[\gamma(n-1)+1]}; \quad \beta = \frac{\gamma(n-2)+1}{(1-\gamma)[\gamma(n-1)+1]}; \quad \sigma = \frac{\gamma}{(1-\gamma)[\gamma(n-1)+1]}.$$

and $0 \leq \gamma < 1$. In the last period, exporters take ownership structure and taxes as given and maximize profit independently. Our assumptions assure that the objective functions are concave. For each product i, the first order condition is:

$$a_i - 2(\beta+\sigma)p_i + 2\sigma \sum_{j \in \mathcal{F}_i} p_j + \sigma \sum_{j \notin \mathcal{F}_i} p_j + (\beta+\sigma)t_i - \sigma \sum_{j \in \mathcal{F}_i} t_j = 0$$

Where $\mathcal{F}_i$ is the set of products produced by the same firm producing product i (including product i) and t_i is the tax rate for product i. Note that tax rates are assumed to be product specific. Assuming an interior equilibrium, the n first order conditions can be written as:

$$[2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]\, \mathbf{p}(G^h, \mathbf{t}^h, G^f, , \mathbf{t}^f) = \mathbf{a} + [(\beta+\sigma)\mathbf{I} - \sigma\mathbf{\Omega}]\, \mathbf{t}$$

Where $\mathbf{p} = \begin{bmatrix} \mathbf{p}^h \\ \mathbf{p}^f \end{bmatrix}$ is the $n \times 1$ vector of prices and $\mathbf{a} = \begin{bmatrix} \mathbf{a}_h \\ \mathbf{a}_f \end{bmatrix}$ is the $n \times 1$ vector of intercepts. To show that the matrix $[2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]$ is invertible, we will prove that it has the d.d. property. Note that all diagonal elements are positive (they are 2β) and all off-diagonal elements are negative (they are either $-\sigma$ or -2σ). To prove the d.d. property it is sufficient to show that:

$$[2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]\, \mathbf{e} > \mathbf{0} \Longrightarrow 2(\beta+\sigma)\mathbf{e} - \sigma\mathbf{\Omega}\mathbf{e} - n\sigma\mathbf{e} > \mathbf{0}$$

where $\mathbf{e}$ is an $n \times 1$ vector of ones. As $\mathbf{\Omega}\mathbf{e} \leq n\mathbf{e}$, it is sufficient to show that:

$$2(\beta+\sigma)\mathbf{e} - 2n\sigma\mathbf{e} > \mathbf{0} \Longleftrightarrow 2(\beta+\sigma) > 2n\sigma \Longleftrightarrow \beta > (n-1)\sigma$$

Using the definition of β and σ:

$$\beta > (n-1)\sigma \Longleftrightarrow 1 > \gamma$$

Which is true by assumption. The equilibrium prices are given by:

$$\mathbf{p}(G^h, \mathbf{t}^h, G^f, , \mathbf{t}^f) = [2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]^{-1}[\mathbf{a} + ((\beta+\sigma)\mathbf{I} - \sigma\mathbf{\Omega})\,\mathbf{t}] \tag{4}$$

5.2 Proof of Lemma (1)

When countries do not impose taxes, equilibrium prices are given by:

$$\mathbf{p}(G^h, \mathbf{t}^h, G^f, , \mathbf{t}^f) = [2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]^{-1}\,\mathbf{a}$$

Note that a less competitive market structure corresponds to a "more positive" $\mathbf{\Omega}$ matrix, some zeros are replaced by ones. To prove the Lemma, just use Lemma 10 and the fact that the matrix $[2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}+\mathbf{1})]$ has the d.d. property.

5.3 Proof of Lemma (2)

From Lemma (1) we know that, given the foreign market structure, the highest equilibrium home country prices are obtained when the home country assigns all goods to the same firm. A home country monopolist solves:

$$\max_{\mathbf{p}^h} \Pi^h\left(\mathbf{p}^h, \mathbf{p}^f(G^f)\right) = \mathbf{p}^{h\prime}\mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(G^f)\right)$$

Where $\mathbf{p}^f(G^f)$ is the equilibrium foreign prices when the home country chooses a monopoly and the foreign market structure is G^f. The vector of first-order conditions is:

$$\mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(G^f)\right) + \frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(G^f)\right)}{\partial \mathbf{p}^h}\mathbf{p}^h = \mathbf{0}. \tag{5}$$

In contrast, the first-order condition for (1) is:

$$\mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right) + \left(\frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right)}{\partial \mathbf{p}^h} + \frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right)}{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}\frac{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}{\partial \mathbf{p}^h}\right)' \mathbf{p}^h = \mathbf{0} \tag{6}$$

Let $\mathbf{p}^h(G^f)$ be the equilibrium home prices when the home country chooses a monopoly and the foreign market structure is G^f. Note that $\mathbf{p}^f(G^f) = \mathbf{p}^f\left(\mathbf{p}^h(G^f), G^f\right)$. Hence, using (5):

$$\begin{aligned}
&\mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right) + \left(\frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right)}{\partial \mathbf{p}^h} + \frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right)}{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}\frac{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}{\partial \mathbf{p}^h}\right)' \mathbf{p}^h\Bigg|_{\mathbf{p}^h=\mathbf{p}^h(G^f)} \\
= \quad & \frac{\partial \mathbf{q}^h\left(\mathbf{p}^h, \mathbf{p}^f(\mathbf{p}^h, G^f)\right)}{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}\frac{\partial \mathbf{p}^f(\mathbf{p}^h, G^f)}{\partial \mathbf{p}^h}' \mathbf{p}^h\Bigg|_{\mathbf{p}^h=\mathbf{p}^h(G^f)} > \mathbf{0}
\end{aligned}$$

Where the inequality comes from the fact that $\frac{\partial \mathbf{q}^h\left(\mathbf{p}^h,\mathbf{p}^f(\mathbf{p}^h,G^f)\right)}{\partial \mathbf{p}^f(\mathbf{p}^h,G^f)} > \mathbf{0}$ and $\frac{\partial \mathbf{p}^f(\mathbf{p}^h,G^f)}{\partial \mathbf{p}^h} > \mathbf{0}$. Therefore, the vector of prices that solves equation 5 must be lower (element by element) than the vector that solves equation 6.

5.4 Proof of Theorem (1)

Assume that the home country is not exporting through a single exporter: $\Omega_h < e_h e_h^{'}$. Assume that the home country is thinking of using a national champion policy. Define the matrix $\mathbf{\Omega}_h(x)$ whose generic element (i,j) is one if the corresponding element of $\mathbf{\Omega}_h$ is one and it is x if the corresponding element of $\mathbf{\Omega}_h$ is zero. This implies that $\mathbf{\Omega}_h(0) = \mathbf{\Omega}_h$ and $\mathbf{\Omega}_h(1) = \mathbf{e}_h\mathbf{e}_h^{'}$ (this is the national champion policy). Define $\mathbf{\Omega}(x)$ as $\mathbf{\Omega}$ where $\mathbf{\Omega}_h$ has been replaced by $\mathbf{\Omega}_h(x)$ and $\mathbf{p}(x) = [2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega}(x) + \mathbf{e}\mathbf{e}^{'})]^{-1}\mathbf{a}$. $\mathbf{p}_h(x)$ and $\mathbf{p}_f(x)$ are the corresponding elements of $\mathbf{p}(x)$

Consider the following function of x:

$$\Pi_h(x) = \mathbf{p}_h'(x)[\mathbf{a}_h - (\beta+\sigma)\mathbf{p}_h(x) + \sigma\mathbf{e}_h\mathbf{e}_h'\mathbf{p}_h(x) + \sigma\mathbf{e}_h\mathbf{e}_f'\mathbf{p}_f(x)]$$

With this formulation, the home country will decide to export with a single exporter if $\Pi_h(1) > \Pi_h(0)$. The derivative of the previous expression with respect to x is:

$$\frac{\partial \Pi_h(x)}{\partial x} = [\mathbf{a}_h - 2(\beta+\sigma)\mathbf{p}_h(x) + 2\sigma\mathbf{e}_h\mathbf{e}_h'\mathbf{p}_h(x) + \sigma\mathbf{e}_h\mathbf{e}_f'\mathbf{p}_h(x)]'\frac{\partial \mathbf{p}_h(x)}{\partial x} + \sigma\mathbf{p}_h(x)\mathbf{e}_h\mathbf{e}_f'\frac{\partial \mathbf{p}_f(x)}{\partial x}$$

Consider the definition of $\mathbf{p}(x)$:

$$\begin{bmatrix} 2(\beta+\sigma)\mathbf{I}_h - \sigma(\mathbf{\Omega}_h(x) + \mathbf{e}_h\mathbf{e}_h^{'}) & -\sigma\mathbf{e}_h\mathbf{e}_f^{'} \\ -\sigma\mathbf{e}_f\mathbf{e}_h^{'} & 2(\beta+\sigma)\mathbf{I}_f - \sigma(\mathbf{\Omega}_f(x) + \mathbf{e}_f\mathbf{e}_f^{'}) \end{bmatrix} \begin{bmatrix} \mathbf{p}_h(x) \\ \mathbf{p}_f(x) \end{bmatrix} = \begin{bmatrix} \mathbf{a}_h \\ \mathbf{a}_f \end{bmatrix}$$

From the first set of equations:

$$\mathbf{a}_h - 2(\beta+\sigma)\mathbf{p}_h + 2\sigma\mathbf{e}_h\mathbf{e}_h'\mathbf{p}_h(x) + \sigma\mathbf{e}_h\mathbf{e}_f^{'}\mathbf{p}_f(x) = \sigma\left[\mathbf{e}_h\mathbf{e}_h^{'} - \mathbf{\Omega}_h(x)\right]\mathbf{p}_h(x)$$

Substituting in the previous derivative:

$$\frac{\partial \Pi_h(x)}{\partial x} = \sigma\mathbf{p}_h'(x)[\mathbf{e}_h\mathbf{e}_h' - \mathbf{\Omega}_h(x)]\frac{\partial \mathbf{p}_h(x)}{\partial x} + \sigma\mathbf{p}_h'(x)\mathbf{e}_h\mathbf{e}_f'\frac{\partial \mathbf{p}_f(x)}{\partial x}$$

To sign this expression notice that:

(1) $[\mathbf{e}_h\mathbf{e}_h' - \mathbf{\Omega}_h(x)] \geq 0$. This comes directly from the fact that $\mathbf{\Omega}_h(x) < \mathbf{e}_h\mathbf{e}_h'$ if $x < 1$.

(2) $\frac{\partial \mathbf{p}_h(x)}{\partial x} > 0$ and $\frac{\partial \mathbf{p}_f(x)}{\partial x} > 0$. This is because $\mathbf{p} = [2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega} + \mathbf{e}\mathbf{e}')]^{-1}\mathbf{a}$ and a more positive Ω (an increase in x) will make the inverse matrix multiplying $\mathbf{a}$ more positive (element by element).

(3) $\mathbf{p}_h(x) > 0$

The previous three conditions imply that $\frac{\partial \Pi_h(x)}{\partial x} > 0$, so that $\Pi_h(1) - \Pi_h(0) = \int_0^1 \frac{\partial \Pi_h(x)}{\partial x} dx > 0$.

5.5 Proof of Lemma (4)

The equilibrium prices arising when the countries choose partitions G^h and G^f are given by:

$$\begin{bmatrix} \mathbf{p}^h(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \\ \mathbf{p}^f(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \end{bmatrix} = [2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega} + \mathbf{1})]^{-1} \left\{ \mathbf{a} + ((\beta+\sigma)\mathbf{I} - \sigma\mathbf{\Omega}) \begin{bmatrix} \mathbf{t}^h \\ \mathbf{t}^f \end{bmatrix} \right\}$$

To prove the theorem, it is sufficient to show that the $n \times n$ matrix multiplying the vector of taxes is full rank. We previously proved that the matrix $[2(\beta+\sigma)\mathbf{I} - \sigma(\mathbf{\Omega} + \mathbf{1})]$ is full rank. We will prove that the matrix $((\beta+\sigma)\mathbf{I} - \sigma\mathbf{\Omega})$ is full rank as well. We show that it has the d.d. property. It has positive diagonal elements and negative off-diagonal elements. It is sufficient to show that:

$$((\beta+\sigma)\mathbf{I} - \sigma\mathbf{\Omega})\mathbf{e} > \mathbf{0}$$

Where $\mathbf{e}$ is a $n \times 1$ vector of ones. Given that $\mathbf{\Omega}\mathbf{e} < n\mathbf{e}$, using the definition of β and σ,it is sufficient to show that:

$$(\beta+\sigma) - n\sigma > 0 \Longleftrightarrow \gamma < 1$$

Which is true by assumption. Now, just note that the product of full rank matrices is full rank.

5.6 Proof of Lemma (5)

5.6.1 Step 1

We first will show how the equilibrium prices, $\mathbf{p}^h(G^h, G^f)$ and $\mathbf{p}^f(G^h, G^f)$, change when the market structure changes.

We use the fact that the home and foreign governments will picks taxes $\mathbf{t}^h$ and $\mathbf{t}^f$ such that (see Lemma 3):

$$\begin{aligned} \mathbf{p}^h(G^h,\mathbf{t}^h,G^f,\mathbf{t}^f) &= \mathbf{p}^h_{ST}(G^f,\mathbf{t}^f) \\ \mathbf{p}^f(G^h,\mathbf{t}^h,G^f,\mathbf{t}^f) &= \mathbf{p}^f_{ST}(G^h,\mathbf{t}^h) \end{aligned} \tag{7}$$

The equilibrium prices are such that (see equation 4):

$$\begin{bmatrix} \mathbf{A}_h & -\sigma\mathbf{1}^{hf} \\ -\sigma\mathbf{1}^{fh} & \mathbf{A}_f \end{bmatrix} \begin{bmatrix} \mathbf{p}^h \\ \mathbf{p}^f \end{bmatrix} = \begin{bmatrix} \mathbf{a}_h + \left[(\beta+\sigma)\mathbf{I}_h - \sigma\mathbf{\Omega}_h\right]\mathbf{t}^h \\ \mathbf{a}_f + \left[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{\Omega}_f\right]\mathbf{t}^f \end{bmatrix} \tag{8}$$

Where we defined:

$$\begin{aligned} \mathbf{A}_h &= 2(\beta+\sigma)\mathbf{I}_h - \sigma(\mathbf{\Omega}_h + \mathbf{1}^{hh}) \\ \mathbf{A}_f &= 2(\beta+\sigma)\mathbf{I}_f - \sigma(\mathbf{\Omega}_f + \mathbf{1}^{ff}) \end{aligned}$$

It is easy to solve the Stackelberg problems. The home country Stackelberg problem is:

$$\begin{aligned} &\max_{\mathbf{p}_h} \mathbf{p}^{h\prime}\left[\mathbf{a}_h - (\beta+\sigma)\mathbf{p}^h + \sigma\mathbf{1}^{hh}\mathbf{p}^h + \sigma\mathbf{1}^{hf}\mathbf{p}^f(\mathbf{p}^h,G^f,\mathbf{t}^f)\right] \\ &\text{subject to } \mathbf{p}^f(\mathbf{p}^h,G^f,\mathbf{t}^f) = \sigma\mathbf{A}_f^{-1}\mathbf{1}^{fh}\mathbf{p}_h + \mathbf{A}_f^{-1}\left[\mathbf{a}_f + ((\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{\Omega}_f)\mathbf{t}^f\right] \end{aligned}$$

It is easy to show that the Stackelberg prices are such that:

$$\begin{bmatrix} \mathbf{B}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_f \end{bmatrix} \begin{bmatrix} \mathbf{p}^h_{ST}(G^f,\mathbf{t}^f) \\ \mathbf{p}^f_{ST}(G^h,\mathbf{t}^h) \end{bmatrix} = \begin{bmatrix} \mathbf{a}_h \\ \mathbf{a}_f \end{bmatrix} + \sigma \begin{bmatrix} \mathbf{0} & \mathbf{1}^{hf}\mathbf{A}_f^{-1} \\ \mathbf{1}^{fh}\mathbf{A}_h^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a}_h + [(\beta+\sigma)\mathbf{I}_h - \sigma\mathbf{\Omega}_h]\mathbf{t}^h \\ \mathbf{a}_f + [(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{\Omega}_f]\mathbf{t}^f \end{bmatrix} \tag{9}$$

Where we defined:

$$\begin{aligned} \mathbf{B}_h &= 2(\beta+\sigma)\mathbf{I}_h - 2\sigma\left(\mathbf{1}^{hh} + \sigma(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh})\right) \\ \mathbf{B}_f &= 2(\beta+\sigma)\mathbf{I}_f - 2\sigma\left(\mathbf{1}^{ff} + \sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{aligned}$$

Equations 8 and 9 have to be both verified at the equilibrium. They represent $2n$ equations in $2n$ unknowns, prices and taxes. It a matter of algebraic manipulation to derive equilibrium prices:

$$\begin{bmatrix} \mathbf{p}^h(G^h,G^f) \\ \mathbf{p}^f(G^h,G^f) \end{bmatrix} =$$

$$\left\{2(\beta+\sigma)\mathbf{I}-\sigma\left[\begin{array}{cc} 2\left(\mathbf{1}^{hh}+\frac{1}{2}\sigma\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{array}\right]\right\}^{-1}\left[\begin{array}{c}\mathbf{a}_h \\ \mathbf{a}_f\end{array}\right]$$

Where the matrix in curled parentheses can be proved to have the d.d. property (the proof is reported at the end of the appendix). We want to show that equilibrium prices increase if market structure becomes less competitive. Assume that the home market becomes less competitive. Some zeros in the matrix $\mathbf{\Omega}_h$ are replaced by ones. Given that $\mathbf{A}_h$ has the d.d. property, Lemma 10 implies that $\mathbf{A}_h^{-1}$ is positive and it becomes more positive as home market structure becomes less competitive. This implies that $\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)$ becomes more positive. Applying Lemma 10 again to the matrix in curled parentheses, we can show that its inverse is positive and it becomes more positive as the home market structure becomes less competitive. This proves that a less competitive home market structure increases equilibrium prices.

5.6.2 Step 2

Using 8, we can write the equilibrium taxes as:

$$\mathbf{t}^h(G^h,G^f)=-\left[(\beta+\sigma)\mathbf{I}_h-\sigma\mathbf{\Omega}_h\right]^{-1}\left[\mathbf{a}_h-\mathbf{A}_h\mathbf{p}^h(G^h,G^f)+\sigma\mathbf{1}^{hf}\mathbf{p}^f(G^h,G^f)\right]$$
$$\mathbf{t}^f(G^h,G^f)=-\left[(\beta+\sigma)\mathbf{I}_f-\sigma\mathbf{\Omega}_f\right]^{-1}\left[\mathbf{a}_f-\mathbf{A}_f\mathbf{p}^f(G^h,G^f)+\sigma\mathbf{1}^{fh}\mathbf{p}^h(G^h,G^f)\right]$$

Consider that the equilibrium prices have to satisfy the condition:

$$\left\{2(\beta+\sigma)\mathbf{I}-\sigma\left[\begin{array}{cc} 2\left(\mathbf{1}^{hh}+\frac{1}{2}\sigma\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{array}\right]\right\}\left[\begin{array}{c}\mathbf{p}^h(G^h,G^f) \\ \mathbf{p}^f(G^h,G^f)\end{array}\right]$$
$$=\left[\begin{array}{c}\mathbf{a}_h \\ \mathbf{a}_f\end{array}\right] \tag{10}$$

From the second set of equations, we can derive:

$$2(\beta+\sigma)\mathbf{p}^f(G^h,G^f)-2\sigma\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right)\mathbf{p}^f(G^h,G^f)-\sigma\mathbf{1}^{fh}\mathbf{p}^h(G^h,G^f)=\mathbf{a}_f$$
$$\Longrightarrow\quad 2(\beta+\sigma)\mathbf{p}^f(G^h,G^f)-2\sigma\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right)\mathbf{p}^f(G^h,G^f)=\mathbf{a}_f+\sigma\mathbf{1}^{fh}\mathbf{p}^h(G^h,G^f)$$

Replace in the expression for $\mathbf{t}^f(G^h, G^f)$:

$$\mathbf{t}^f(G^h, G^f) = \sigma\left[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f\right]^{-1}\left[\mathbf{1}^{ff} - \boldsymbol{\Omega}_f + \sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right]\mathbf{p}^f(G^h, G^f) > \mathbf{0} \tag{11}$$

Where we used the fact that $\mathbf{A}_f = 2(\beta+\sigma)\mathbf{I}_f - \sigma(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})$. Notice that a country will always tax (it will never subsidize) its goods. We noted before that if a country, in particular country h, chooses a less competitive market structure, given the other country's market structure, the equilibrium prices increase, in particular $\mathbf{p}^f(G^h, G^f)$ increases. Besides $\mathbf{A}_h^{-1}$ becomes more positive for less competitive home market structures. Given that $\left[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f\right]^{-1} > \mathbf{0}$ and $\left[\mathbf{1}^{ff} - \boldsymbol{\Omega}_f + \sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right] > \mathbf{0}$, it is clear that if country h makes its market structure less competitive, the equilibrium foreign taxes $\mathbf{t}^f(G^h, G^f)$ increase:

$$\frac{\partial \mathbf{t}^f(G^h, G^f)}{\partial G^h} > \mathbf{0}$$

5.7 Proof of Theorem (3)

Here we analytically prove that $\left(\frac{\partial \Pi^h}{\partial \mathbf{p}^f}\frac{\partial \mathbf{p}^f}{\partial \mathbf{t}^f}\right) > \mathbf{0}$. It is easy to derive that:

- $\frac{\partial \Pi^h}{\partial \mathbf{p}^f} = \sigma\mathbf{p}^h\mathbf{1}^{hf}$.
- $\frac{\partial \mathbf{p}^f}{\partial \mathbf{t}^f} = \left[2(\beta+\sigma)\mathbf{I}_f - \sigma(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}\left[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f\right]$.

We prove that $\frac{\partial \mathbf{p}^f}{\partial \mathbf{t}^f} \geq \mathbf{0}$. The following are easy algebraic manipulations:

$$\begin{aligned}
& \left[2(\beta+\sigma)\mathbf{I}_f - \sigma(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}\left[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f\right] = \\
= \; & \frac{1}{2(\beta+\sigma)}\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}(\beta+\sigma)\left[\mathbf{I}_f - \frac{\sigma}{(\beta+\sigma)}\boldsymbol{\Omega}_f\right] = \\
= \; & \frac{1}{2}\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}\left[\mathbf{I}_f - \frac{\sigma}{(\beta+\sigma)}\boldsymbol{\Omega}_f\right] = \\
= \; & \frac{1}{2}\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff}) + \frac{\sigma}{2(\beta+\sigma)}(\mathbf{1}^{ff} - \boldsymbol{\Omega}_f)\right] = \\
= \; & \frac{1}{2}\mathbf{I}_f + \frac{\sigma}{4(\beta+\sigma)}\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]^{-1}(\mathbf{1}^{ff} - \boldsymbol{\Omega}_f)
\end{aligned}$$

Now, just observe that the matrix $\left[\mathbf{I}_f - \frac{\sigma}{2(\beta+\sigma)}(\boldsymbol{\Omega}_f + \mathbf{1}^{ff})\right]$ has positive diagonal elements, negative off-diagonal elements, and a d.d.. Its inverse is positive. Besides $(\mathbf{1}^{ff} - \boldsymbol{\Omega}_f) \geq \mathbf{0}$.

5.8 Proof of Theorem (4)

The profit to the exporter for each product in the country c is:

$$(p_i^c - t_i^c)q_i^c = (p_i^c - t_i^c)\left(a_i - \beta p_i + \sigma\sum_{j\neq i} p_j\right) \tag{12}$$

Each exporter chooses prices by taking into account the cross effect among all goods she owns. The first order condition for good i is:

$$q_i^c - (\beta+\sigma)(p_i^c - t_i^c) + \sigma\sum_{j\in\mathcal{F}_i}(p_j^c - t_j^c) = 0$$

where $\mathcal{F}_i$ is the set of products produced by the same firm producing product i (including product i). Using the first order conditions corresponding to all goods in $\mathcal{F}_i$, it is possible to derive:

$$(p_i^c - t_i^c) = \frac{1}{\beta+\sigma}\left[q_i^c + \frac{\sigma}{\beta-(n^p-1)\sigma}\sum_{j\in\mathcal{F}_i} q_j^c\right]$$

Replacing in 12:

$$(p_i^c - t_i^c)q_i^c = \frac{1}{\beta+\sigma}\left[q_i^c + \frac{\sigma}{\beta-(n_p-1)\sigma}\sum_{j\in\mathcal{F}_i} q_j^c\right] q_i^c$$

Where n^p is the number of goods produced by the firm that produces product i. To prove the theorem, it is sufficient to show that a merger among home firms increases the quantity produced of all home goods (note that $\beta - (n^p-1)\sigma > 0$). We will prove that $\frac{d\mathbf{q}^h}{dG^h} > \mathbf{0}$. From 10 we can derive:

$$\left[(\beta+\sigma)\mathbf{I}_h - \sigma\mathbf{1}^{hh} - \sigma^2\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right]\mathbf{p}^h = \mathbf{a}_h - (\beta+\sigma)\mathbf{p}^h + \sigma\mathbf{1}^{hh}\mathbf{p}^h + \sigma\mathbf{1}^{hf}\mathbf{p}^f = \mathbf{q}^h \tag{13}$$

It is sufficient to show that the quantity on the left increases if home market structure becomes less competitive. Consider the Stackelber leader price for the home country (see 9):

$$2\left[(\beta+\sigma)\mathbf{I}_h - \sigma\mathbf{1}^{hh} - \sigma^2(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh})\right]\mathbf{p}^h = \mathbf{a}_h + \mathbf{1}^{hf}\mathbf{A}_f^{-1}\left\{\mathbf{a}_f + [(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f]\mathbf{t}^f\right\}$$

It is sufficient to show that $\mathbf{A}_f^{-1}[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f]\mathbf{t}^f$ increases as home market structure becomes less competitive. Note that $\mathbf{A}_f^{-1}[(\beta+\sigma)\mathbf{I}_f - \sigma\boldsymbol{\Omega}_f]$ does not depend on home market structure and that it is a positive matrix (see proof of theorem 3). It is now sufficient to note that $\mathbf{t}^f$ increases if the home market structure becomes less competitive (see Lemma 5).

5.9 Proof of Theorem (5)

From previous results, we know that a less competitive home market structure determines an increase of all prices and an increase in the quantity of all home goods (see previous theorem). This implies that the quantity of at least one foreign product decreases (prices and quantities cannot increase all at the same time). We will now show that the quantity of all foreign goods change in the same direction as the home market structure becomes less competitive. Use the equivalent of 13:

$$\mathbf{q}^f = \left[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{1}^{ff} - \sigma^2\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right]\mathbf{p}^f = \mathbf{a}_f - (\beta+\sigma)\mathbf{p}^f + \sigma\mathbf{1}^{ff}\mathbf{p}^f + \sigma\mathbf{1}^{fh}\mathbf{p}^h$$

After some manipulations:

$$2\mathbf{q}^f = 2\left[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{1}^{ff} - \sigma^2\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right]\mathbf{p}^f = \mathbf{a}_f + \sigma\mathbf{1}^{fh}\mathbf{p}^h - \sigma^2\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\mathbf{p}^f$$

Differentiate the previous expression, and use the fact that $\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right) = \left(\mathbf{e}_h'\mathbf{A}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f\mathbf{e}_f'$ and $\mathbf{1}^{fh} = \mathbf{e}_f\mathbf{e}_h'$, where $\mathbf{e}_f$ and $\mathbf{e}_h$ are $n^f \times 1$ and $n^h \times 1$ vectors of ones.:

$$\begin{aligned}
2\frac{d\mathbf{q}^f}{dG^h} &= \sigma\mathbf{e}_f\mathbf{e}_h'\frac{d\mathbf{p}^h}{dG^h} - \sigma^2\left(\mathbf{e}_h'\mathbf{A}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f\mathbf{e}_f'\frac{d\mathbf{p}^f}{dG^h} - \sigma^2\mathbf{e}_f\mathbf{e}_f'\mathbf{p}^f\frac{d\left(\mathbf{e}_h'\mathbf{A}_h^{-1}\mathbf{e}_h\right)}{dG^h} \\
&= \left(\sigma\mathbf{e}_h'\frac{d\mathbf{p}^h}{dG^f} - \sigma^2\left(\mathbf{e}_h'\mathbf{A}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f'\frac{d\mathbf{p}^f}{dG^f} - \sigma^2\mathbf{e}_f'\mathbf{p}^f\frac{d\left(\mathbf{e}_h'\mathbf{A}_h^{-1}\mathbf{e}_h\right)}{dG^h}\right)\mathbf{e}_f
\end{aligned}$$

where the quantity in parentheses is a scalar. Given that the quantity of at least one foreign product decreases as the home market structure becomes less competitive, this quantity must be negative.

5.10 Solution of the Third Stage Cournot Game

The inverse demand functions are:

$$p_i = \alpha_i - q_i - \gamma\sum_{j\neq i} q_j$$

In the third period, firms choose quantities, given market structure and taxes. For each product i, the first order condition is:

$$\alpha_i - t_i - 2(1-\gamma)q_i - 2\gamma\sum_{j\in\mathcal{F}_i} q_j - \gamma\sum_{j\notin\mathcal{F}_i} q_j = 0$$

In matrix formula:

$$\boldsymbol{\alpha} - \mathbf{t} - [2(1-\gamma)\mathbf{I} + \gamma\boldsymbol{\Omega} + \gamma\mathbf{1}]\mathbf{q} = \mathbf{0}$$

Define:

$$\begin{aligned}
\boldsymbol{\Delta}_h &= 2(1-\gamma)\mathbf{I}_h + \gamma\boldsymbol{\Omega}_h + \gamma\mathbf{1}^{hh} \\
\boldsymbol{\Delta}_f &= 2(1-\gamma)\mathbf{I}_f + \gamma\boldsymbol{\Omega}_f + \gamma\mathbf{1}^{ff}
\end{aligned}$$

The first order conditions can be written:

$$\begin{bmatrix} \boldsymbol{\Delta}_h & \gamma\mathbf{1}^{hf} \\ \gamma\mathbf{1}^{fh} & \boldsymbol{\Delta}_f \end{bmatrix} \begin{bmatrix} \mathbf{q}^h(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \\ \mathbf{q}^f(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_h \\ \boldsymbol{\alpha}_f \end{bmatrix} - \begin{bmatrix} \mathbf{t}_h \\ \mathbf{t}_f \end{bmatrix} \tag{14}$$

It is now easy to show that for $\gamma \in [0,1)$ the matrix $2(1-\gamma)\mathbf{I} + \gamma\boldsymbol{\Omega} + \gamma\mathbf{1}$ is definite positive, hence non singular, so that:

$$\begin{bmatrix} \mathbf{q}^h(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \\ \mathbf{q}^f(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Delta}_h & \gamma\mathbf{1}^{hf} \\ \gamma\mathbf{1}^{fh} & \boldsymbol{\Delta}_f \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \boldsymbol{\alpha}_h \\ \boldsymbol{\alpha}_f \end{bmatrix} - \begin{bmatrix} \mathbf{t}_h \\ \mathbf{t}_f \end{bmatrix} \right\}$$

The fact that the matrix multiplying the tax vector has rank n, proves the equivalent of Lemma (4) for the case of quantity competition.

5.11 Proof of Lemma (6)

5.11.1 Step 1

We first will show how the equilibrium quantities, $\mathbf{q}^h(G^h, G^f)$ and $\mathbf{q}^f(G^h, G^f)$, change when the market structure changes.

We use the fact that the home and foreign governments will picks taxes $\mathbf{t}^h$ and $\mathbf{t}^f$ such that:

$$\begin{aligned}
\mathbf{q}^h(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) &= \mathbf{q}^h_{ST}(G^f, \mathbf{t}^f) \\
\mathbf{q}^f(G^h, \mathbf{t}^h, G^f, \mathbf{t}^f) &= \mathbf{q}^f_{ST}(G^h, \mathbf{t}^h)
\end{aligned}$$

It easy to solve the Stackelberg problem:

$$\max_{\mathbf{q}^h} \mathbf{q}^{h\prime}[\boldsymbol{\alpha}_h - (1-\gamma)\mathbf{q}^h - \gamma\mathbf{1}^{hh}\mathbf{q}^h - \gamma\mathbf{1}^{hf}\mathbf{q}^f(\mathbf{q}^h, \boldsymbol{\Omega}_f, \mathbf{t}^f)]$$

$$\text{subject to: } \mathbf{q}^f(\mathbf{q}^h, \boldsymbol{\Omega}_f, \mathbf{t}^f) = \boldsymbol{\Delta}_f^{-1}(\boldsymbol{\alpha}_f - \mathbf{t}^f) - \gamma\boldsymbol{\Delta}_f^{-1}\mathbf{1}^{fh}\mathbf{q}^h$$

Where the constraint can be derived from (14). The Stackelberg quantities are:

$$\begin{bmatrix} \mathbf{q}^h_{ST}(\boldsymbol{\Omega}_f, \mathbf{t}^f) \\ \mathbf{q}^f_{ST}(\boldsymbol{\Omega}_h, \mathbf{t}^h) \end{bmatrix} = \begin{bmatrix} \mathbf{L}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_f \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \boldsymbol{\alpha}_h \\ \boldsymbol{\alpha}_f \end{bmatrix} - \gamma \begin{bmatrix} \mathbf{0} & \mathbf{1}^{hf}\boldsymbol{\Delta}_f^{-1} \\ \mathbf{1}^{fh}\boldsymbol{\Delta}_h^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\alpha}_h - \mathbf{t}^h \\ \boldsymbol{\alpha}_f - \mathbf{t}^f \end{bmatrix} \right\} \tag{15}$$

Where we defined:

$$\begin{aligned} \mathbf{L}_h &= 2(1-\gamma)\mathbf{I}_h + 2\gamma\left(1 - \gamma\left(\mathbf{e}_f'\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\right)\mathbf{1}^{hh} \\ \mathbf{L}_f &= 2(1-\gamma)\mathbf{I}_f + 2\gamma\left(1 - \gamma\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\right)\mathbf{1}^{fh} \end{aligned}$$

and $\mathbf{e}_f$ and $\mathbf{e}_h$ are $n^f \times 1$ and $n^h \times 1$ vectors of ones. It is possible to show that $\mathbf{L}_h$ and $\mathbf{L}_f$ are definite positive and, hence, invertible.

Equations 15 and 14 have both to be verified at the equilibrium. They represent $2n$ equations in $2n$ unknowns, quantities and subsidies. It a matter of algebraic manipulation to derive equilibrium quantities:

$$\begin{bmatrix} \mathbf{q}^h(G^h, G^f) \\ \mathbf{q}^f(G^h, G^f) \end{bmatrix} = \begin{bmatrix} \mathbf{W}_h & \gamma\mathbf{1}^{hf} \\ \gamma\mathbf{1}^{fh} & \mathbf{W}_f \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\alpha}_h \\ \boldsymbol{\alpha}_f \end{bmatrix} \tag{16}$$

Where:

$$\begin{aligned} \mathbf{W}_h &= 2(1-\gamma)\mathbf{I}_h + 2\gamma\mathbf{1}^{hh} - \gamma^2\left(\mathbf{e}_f'\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\mathbf{1}^{hh} \\ \mathbf{W}_f &= 2(1-\gamma)\mathbf{I}_f + 2\gamma\mathbf{1}^{ff} - \gamma^2\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{1}^{ff} \end{aligned}$$

Note that the equilibrium quantities depend on $\boldsymbol{\Omega}_h$ only through the scalar $\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)$. The link between market structure and $\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)$ is given by the following Lemma (whose proof is omitted for sake of brevity).

Lemma 9 $\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) = \mathbf{e}_h'\left[2(1-\gamma)\mathbf{I}_h + \gamma\boldsymbol{\Omega}_h + \gamma\mathbf{1}^{hh}\right]^{-1}\mathbf{e}_h$ *decreases if the market structure is made less competitive.*

Differentiate 16 with respect to $\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)$ (treating it as a continuous variable):

$$\begin{bmatrix} \mathbf{W}_h & \gamma\mathbf{1}^{hf} \\ \gamma\mathbf{1}^{fh} & \mathbf{W}_f \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{q}^h(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} \\ \frac{\partial \mathbf{q}^f(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \gamma^2\mathbf{1}^{ff}\mathbf{q}^f(G^h, G^f) \end{bmatrix} \tag{17}$$

Solving the system of equations 17:

$$\begin{aligned}
\frac{\partial \mathbf{q}^h(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} &= -\frac{n^f}{2(1-\gamma)+\gamma\left(2-\gamma\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)n^f}\frac{\gamma^3\mathbf{1}^{hf}\mathbf{q}^f(G^h,G^f)}{2(1-\gamma)+\gamma\left(2-\gamma\mathbf{e}_f'\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f-\gamma\mathbf{e}_f'\mathbf{W}_f^{-1}\mathbf{e}_f\right)n^h}\\
\frac{\partial \mathbf{q}^f(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} &= \frac{\gamma^2\mathbf{1}^{ff}\mathbf{q}^f(G^h,G^f)}{2(1-\gamma)+\gamma\left(2-\gamma\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h-\gamma\mathbf{e}_h'\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f}
\end{aligned}$$

Note that:

$$\left(2-\gamma\mathbf{e}_f'\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f-\gamma\mathbf{e}_f'\mathbf{W}_f^{-1}\mathbf{e}_f\right)=\left(1-\gamma\mathbf{e}_f'\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)+\left(1-\gamma\mathbf{e}_f'\mathbf{W}_f^{-1}\mathbf{e}_f\right)$$

To sign $\frac{\partial \mathbf{q}^h(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)}$, note that the expression in the first round parentheses on the right is positive (proof is omitted):

$$\left(1-\gamma\mathbf{e}_c'\boldsymbol{\Delta}_c^{-1}\mathbf{e}_c\right)>0$$

for $c=h$ or $c=f$, and that:

$$\left(1-\gamma\mathbf{e}_f'\mathbf{W}_f^{-1}\mathbf{e}_f\right)=\frac{2(1-\gamma)+\gamma\left(1-\gamma\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)n^f}{2(1-\gamma)+\gamma\left(2-\gamma\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)n^f}>0$$

Hence $\frac{\partial \mathbf{q}^h(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)}<0$. Similarly we can sign $\frac{\partial \mathbf{q}^f(G^h,G^f)}{\partial(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)}>0$.

5.11.2 Step 2

From 14 we can derive that equilibrium taxes must satisfy:

$$\begin{aligned}
\mathbf{t}^h(G^h,G^f) &= \left[\boldsymbol{\alpha}_h-\gamma\mathbf{1}^{hf}\mathbf{q}^f(G^h,G^f)-\boldsymbol{\Delta}_h\mathbf{q}^h(G^h,G^f)\right]\\
\mathbf{t}^f(G^h,G^f) &= \left[\boldsymbol{\alpha}_f-\gamma\mathbf{1}^{fh}\mathbf{q}^h(G^h,G^f)-\boldsymbol{\Delta}_f\mathbf{q}^f(G^h,G^f)\right]
\end{aligned}$$

From 16 we can derive that:

$$\boldsymbol{\alpha}_f-\gamma\mathbf{1}^{fh}\mathbf{q}^h(G^h,G^f)=\mathbf{W}_f\mathbf{q}^f(G^h,G^f)$$

And replacing in the expression for $\mathbf{t}^f(G^h,G^f)$:

$$\begin{aligned}
\mathbf{t}^f(G^h,G^f) &= \left[\mathbf{W}_f-\boldsymbol{\Delta}_f\right]\mathbf{q}^f(G^h,G^f)\\
&= -\gamma[\gamma\left(\mathbf{e}_h'\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{1}^{ff}+\boldsymbol{\Omega}_f-\mathbf{1}^{ff}]\mathbf{q}^f(G^h,G^f)
\end{aligned}$$

If the foreign country market structure is of the national champion type ($\boldsymbol{\Omega}_f = \mathbf{1}^{ff}$), then $\mathbf{t}^f(G^h, G^f) = -\gamma^2 \left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) \mathbf{1}^{ff}\mathbf{q}_f(\boldsymbol{\Omega}_h, \boldsymbol{\Omega}_f) < \mathbf{0}$, so that country f subsidizes its firms. If market structure for the foreign country is not of the national champion type, the subsidies could be negative (taxes) because a country does not want to create competition between its firms. It is easy to prove that, given any foreign country's market structure, a move to monopoly ($\boldsymbol{\Omega}_f = \mathbf{1}^{ff}$) always increases foreign subsidies.

Note that the foreign taxes depend on $\boldsymbol{\Omega}_h$ only through the scalar $\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)$ (see 16).

$$\begin{aligned}
\frac{1}{\gamma}\frac{\partial \mathbf{t}^f(G^h, G^f)}{\partial(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} &= -\gamma\mathbf{1}^{ff}\mathbf{q}^f(G^h, G^f) - \left[\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{1}^{ff} + \boldsymbol{\Omega}_f - \mathbf{1}^{ff}\right]\frac{\partial \mathbf{q}^f(G^h, G^f)}{\partial(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} \\
&= -\gamma\left\{\mathbf{I}_f + \gamma\frac{\left[\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{1}^{ff} + \boldsymbol{\Omega}_f - \mathbf{1}^{ff}\right]}{2(1-\gamma) + \gamma\left(2 - \gamma\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f}\right\}\mathbf{1}^{ff}\mathbf{q}_f(G^h, G^f) \\
&= -\gamma\left\{\mathbf{e}_f + \gamma\frac{\left[\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f\mathbf{e}_f^{'} - \left(\mathbf{e}_f\mathbf{e}_f^{'} - \boldsymbol{\Omega}_f\right)\right]}{2(1-\gamma) + \gamma\left(2 - \gamma\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f}\mathbf{e}_f\right\}\mathbf{e}_f^{'}\mathbf{q}_f(G^h, G^f) \\
&= -\gamma\left\{\mathbf{e}_f + \gamma\frac{\left[n^f\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f - n^f\mathbf{e}_f + \boldsymbol{\Omega}_f\mathbf{e}_f\right]}{2(1-\gamma) + \gamma\left(2 - \gamma\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f}\right\}\mathbf{e}_f^{'}\mathbf{q}_f(G^h, G^f)
\end{aligned}$$

Where we used the fact that $\mathbf{1}^{ff} = \mathbf{e}_f\mathbf{e}_f$. The $i's$ element of this vector has opposite sign of:

$$1 + \gamma\frac{\left[n^f\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) - n^f + n^p\right]}{2(1-\gamma) + \gamma\left(2 - \gamma\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f}$$

Where n^p is the number of products owned by the firm owning good i. This quantity is positive if:

$$\begin{aligned}
& 2(1-\gamma) + \gamma\left(2 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f + \gamma\left[n^f\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) - n^f + n^p\right] = \\
= \;\; & 2(1-\gamma) + \gamma\left(1 - \gamma\mathbf{e}_h^{'}\mathbf{W}_h^{-1}\mathbf{e}_h\right)n^f + \gamma n^p > 0
\end{aligned}$$

We noted before that $\left(1 - \gamma\mathbf{e}_f^{'}\mathbf{W}_f^{-1}\mathbf{e}_f\right) > 0$. This proves that:

$$\frac{\partial \mathbf{t}^f(G^h, G^f)}{\partial(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h)} < \mathbf{0}$$

If country h chooses a less competitive market structure, $\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)$ decreases. A less competitive home market structure induces lower foreign equilibrium subsidies (higher taxes) in the second stage of the game.

5.12 Proof of Theorem (6)

We want to prove that $\left(\frac{\partial \Pi^h}{\partial \mathbf{q}^f}\frac{\partial \mathbf{q}^f}{\partial \mathbf{t}^f}\right) > \mathbf{0}$. It is easy to show that:

- $\frac{\partial \Pi^h}{\partial \mathbf{q}^f} = -\gamma \mathbf{q}^h(G^h, G^f)\mathbf{1}^{hf} < \mathbf{0}$.
- $\frac{\partial \mathbf{q}^f}{\partial \mathbf{t}^f} = -\boldsymbol{\Delta}_f^{-1}$.

So that:

$$\left(\frac{\partial \Pi^h}{\partial \mathbf{q}^f}\frac{\partial \mathbf{q}^f}{\partial \mathbf{t}^f}\right) = \gamma \mathbf{q}^h(G^h, G^f)\mathbf{1}^{hf}\boldsymbol{\Delta}_f^{-1}$$

It is sufficient to show that $\mathbf{1}^{hf}\boldsymbol{\Delta}_f^{-1} > \mathbf{0}$. Remember that:

$$\boldsymbol{\Delta}_f = 2(1-\gamma)\mathbf{I}_f + \gamma\boldsymbol{\Omega}_f + \gamma\mathbf{1}^{ff}$$

It is sufficient to show that $\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f > \mathbf{0}$. Define $\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f = \mathbf{x} \Longrightarrow \boldsymbol{\Delta}_f\mathbf{x} = \mathbf{e}_f$. Row i of the previous equation is: $2(1-\gamma)x_i + \gamma\sum_{j\in\mathcal{F}_i} x_j + \gamma\sum_{j=1}^{n^f} x_j = 1$, where $\mathcal{F}_i$ is the set of products produced by the same firm producing product i (including product i). First, note that it must be that $\sum_{j=1}^{n^f} x_j > 0$. If not, there would be a product i for which $\sum_{j\in\mathcal{F}_i} x_j < 0$ and $x_i < 0$, so that $2(1-\gamma)x_i + \gamma\sum_{j\in\mathcal{F}_i} x_j + \gamma\sum_{j=1}^{n^f} x_j = 1$ could not be one. Besides it is clear that, for two goods i and k belonging to the same group, it must be that $x_i = x_k$ (just subtract the lines corresponding to the two products, to obtain $2(1-\gamma)x_i = 2(1-\gamma)x_k$). We can rewrite the equation for a given group p as: $[2(1-\gamma) + \gamma n^p]x_p + \gamma\sum_{j=1}^{n^f} x_j = 1$, where n^p is the number of goods in product p and x_p is the common value of x for all goods in group p. Consider two groups, p and q, and subtract the two equations. We have: $[2(1-\gamma) + \gamma n_p]x_p = [2(1-\gamma) + \gamma n_q]x_q$ so that all x_i must have the same sign and as their sum must be positive they must all be positive.

5.13 Proof of Theorem (7)

The net profit for each product in the country c is:

$$(p_i^c - t_i^c)q_i^c = \left(\alpha_i - q_i - \gamma\sum_{j\neq i} q_j - t_i^h\right) q_i^h \tag{18}$$

The exporter chooses quantity by taking into account the cross effect among all goods she owns. From the first order condition we can easily derive:

$$(p_i^c - t_i^c) = (1-\gamma) q_i^c + \gamma \sum_{j \in \mathcal{F}_i} q_j^c$$

where $\mathcal{F}_i$ is the set of products produced by the same firm producing product i (including product i). Replacing in 18:

$$(p_i^c - t_i^c) q_i^c = \left[(1-\gamma) q_i^c + \gamma \sum_{j \in \mathcal{F}_i} q_j^c \right] q_i^c$$

From the proof of Lemma 6 we know that a merger in country c increases the quantity of all country c's goods and decreases the quantity of all country $-c$'s goods. Given the previous expression (remember that $(1-\gamma) > 0$), this proves the theorem.

6 Omitted Claims

Claim 1 *The matrix:*

$$\begin{aligned} & \left\{ 2(\beta+\sigma) \begin{bmatrix} \mathbf{I}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_f \end{bmatrix} - \sigma \begin{bmatrix} 2\left(\mathbf{1}^{hh} + \frac{1}{2}\sigma \mathbf{1}^{hf} \mathbf{A}_f^{-1} \mathbf{1}^{fh}\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff} + \frac{1}{2}\sigma \mathbf{1}^{fh} \mathbf{A}_h^{-1} \mathbf{1}^{hf}\right) \end{bmatrix} \right\} \\ = & \left\{ 2(\beta+\sigma) \begin{bmatrix} \mathbf{I}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_f \end{bmatrix} - \sigma \begin{bmatrix} 2\mathbf{e}_h \mathbf{e}_h + \sigma \left(\mathbf{e}_f' \mathbf{A}_f^{-1} \mathbf{e}_f\right) \mathbf{e}_h \mathbf{e}_h & \mathbf{e}_h \mathbf{e}_f \\ \mathbf{e}_f \mathbf{e}_h & 2\mathbf{e}_f \mathbf{e}_f + \sigma \left(\mathbf{e}_h' \mathbf{A}_h^{-1} \mathbf{e}_h\right) \mathbf{e}_f \mathbf{e}_f' \end{bmatrix} \right\} \end{aligned}$$

has the d.d. property.

First we will show that:

$$\begin{aligned} \left(\mathbf{e}_f' \mathbf{A}_f^{-1} \mathbf{e}_f\right) &\leq \frac{n^f}{2[\beta - (n^f - 1)\sigma]} \\ \left(\mathbf{e}_h' \mathbf{A}_h^{-1} \mathbf{e}_h\right) &\leq \frac{n^h}{2[\beta - (n^h - 1)\sigma]} \end{aligned}$$

We will prove the first inequality. The matrix $\mathbf{A}_f$ has the d.d. property (see paper). The matrix $\mathbf{A}_f^{-1} = \left[2(\beta+\sigma)\mathbf{I}_f - \sigma(\mathbf{\Omega}_f + \mathbf{e}_f \mathbf{e}_f')\right]^{-1}$ is positive and is "maximized" (element by element) when

$\mathbf{\Omega}_f = \mathbf{e}_f\mathbf{e}_f^{'}$. So that:

$$\left[2(\beta+\sigma)\mathbf{I}_f - \sigma(\mathbf{\Omega}_f + \mathbf{e}_f\mathbf{e}_f^{'})\right]^{-1} \leq \left[2(\beta+\sigma)\mathbf{I}_f - 2\sigma\mathbf{e}_f\mathbf{e}_f^{'})\right]^{-1} = \frac{1}{2(\beta+\sigma)}\left[\mathbf{I}_f + \frac{\sigma}{(\beta+\sigma)-n^f\sigma}\mathbf{e}_f\mathbf{e}_f^{'}\right]$$

Hence:

$$\mathbf{e}_f^{'}\mathbf{A}_f^{-1}\mathbf{e}_f = \mathbf{e}_f^{'}\left[2(\beta+\sigma)\mathbf{I}_f - \sigma(\mathbf{\Omega}_f + \mathbf{e}_f\mathbf{e}_f^{'})\right]^{-1}\mathbf{e}_f \leq \frac{1}{2(\beta+\sigma)}\mathbf{e}_f^{'}\left[\mathbf{I}_f + \frac{\sigma}{\beta-(n^f-1)\sigma}\mathbf{e}_f\mathbf{e}_f^{'}\right]\mathbf{e}_f =$$

$$= \frac{1}{2(\beta+\sigma)}\left\{n_f + \frac{\sigma}{\beta-(n_f-1)\sigma}n_f^2\right\} = \frac{n^f}{2[\beta-(n^f-1)\sigma]}$$

Now consider the matrix of the claim. It has clearly negative off diagonal elements. To prove the dominant diagonal property it is sufficient to show that:

$$\left\{2(\beta+\sigma)\begin{bmatrix}\mathbf{I}_h & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_f\end{bmatrix} - \sigma\begin{bmatrix}2\mathbf{e}_h\mathbf{e}_h^{'} + \sigma\left(\mathbf{e}_f^{'}\mathbf{A}_f^{-1}\mathbf{e}_f\right)\mathbf{e}_h\mathbf{e}_h^{'} & \mathbf{e}_h\mathbf{e}_f^{'}\\ \mathbf{e}_f\mathbf{e}_h^{'} & 2\mathbf{e}_f\mathbf{e}_f^{'} + \sigma\left(\mathbf{e}_h^{'}\mathbf{A}_h^{-1}\mathbf{e}_h\right)\mathbf{e}_f\mathbf{e}_f^{'}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{e}_h\\ \mathbf{e}_f\end{bmatrix} > 0$$

Given that $\left(\mathbf{e}_f^{'}\mathbf{A}_f^{-1}\mathbf{e}_f\right) \leq \frac{n^f}{2[\beta-(n^f-1)\sigma]}$ and $\left(\mathbf{e}_h^{'}\mathbf{A}_h^{-1}\mathbf{e}_h\right) \leq \frac{n^h}{2[\beta-(n^h-1)\sigma]}$, to prove the claim is sufficient to show that:

$$\begin{aligned}2(\beta+\sigma) &> 2\sigma n^h + \frac{\sigma^2 n^h n^f}{2[\beta-(n^f-1)\sigma]} + \sigma n^f\\ 2(\beta+\sigma) &> 2\sigma n_f + \frac{\sigma^2 n^h n^f}{2[\beta-(n^h-1)\sigma]} + \sigma n^h\end{aligned}$$

We will prove that the first inequality is true. Replacing the expression for β and σ:

$$2[\gamma(n-1)+1] - 2\gamma n^h - \gamma n^f > \frac{\gamma^2 n^h n^f}{2[\gamma(n^h-1)+1]} \Longleftrightarrow$$

$$2[\gamma n_f + 2(1-\gamma)][\gamma n_h + (1-\gamma)] - \gamma^2 n_h n_f > 0$$

Clearly true because $[\gamma n_f + 2(1-\gamma)] > \gamma n_f$ and $[\gamma n_h + (1-\gamma)] > \gamma n_h$.

Claim 2 *The matrix*

$$[2(1-\gamma)\mathbf{I} + \gamma\mathbf{\Omega} + \gamma\mathbf{1}] = \left[2(1-\gamma)\mathbf{I} + \gamma\mathbf{\Omega} + \gamma\mathbf{e}\mathbf{e}^{'}\right]$$

is definite positive.

Notice that for any vector $\mathbf{z}$:

$$\mathbf{z}^{'}[2(1-\gamma)\mathbf{I}+\gamma\mathbf{\Omega}+\gamma\mathbf{e}\mathbf{e}^{'}]\mathbf{z}=2(1-\gamma)\mathbf{z}^{'}\mathbf{z}+\gamma\mathbf{z}^{'}\mathbf{\Omega}\mathbf{z}+\gamma(\mathbf{z}^{'}\mathbf{e})(\mathbf{z}^{'}\mathbf{e})^{'}$$

The first and last addend are non negative and their sum is zero iff $\mathbf{z}=0$. Besides $\mathbf{z}^{'}\mathbf{\Omega}\mathbf{z}\geq 0$. This matrix is only semidefinite positive if $\gamma=1$. In this case there is not a specific solution if a firm has more then one product. We exclude this possibility.

Claim 3 $\left(\mathbf{e}_h^{'}\mathbf{\Delta}_h^{-1}\mathbf{e}_h\right)=\mathbf{e}_h^{'}\left[2(1-\gamma)\mathbf{I}_h+\gamma\mathbf{\Omega}_h+\gamma\mathbf{e}_h\mathbf{e}_h\right]^{-1}\mathbf{e}_h$ *decreases if the market structure is made less competitive. And* $\left(1-\gamma\left(\mathbf{e}_h^{'}\mathbf{\Delta}_h^{-1}\mathbf{e}_h\right)\right)>0$.

Step 1: Calculate $[2(1-\gamma)\mathbf{I}_h+\gamma\mathbf{\Omega}_h]^{-1}$.

Assume that country h has m firms $p=1,2,....,m$, each controlling n_p goods. We can always arrange them so that Ω_h is block diagonal:

$$[2(1-\gamma)\mathbf{I}_h+\gamma\mathbf{\Omega}_h]=\begin{bmatrix} 2(1-\gamma)\mathbf{I}_1+\gamma\mathbf{e}_1\mathbf{e}_1^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2(1-\gamma)\mathbf{I}_2+\gamma\mathbf{e}_2\mathbf{e}_2^{'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2(1-\gamma)\mathbf{I}_m+\gamma\mathbf{e}_m\mathbf{e}_m^{'} \end{bmatrix}$$

Where $\mathbf{e}_i$ is a $n_i\times 1$ vector of ones. Notice that:

$$[2(1-\gamma)\mathbf{I}_i+\gamma\mathbf{e}_i\mathbf{e}_i^{'}]^{-1}=\frac{1}{2(1-\gamma)}\left[\mathbf{I}_i-\frac{\gamma}{2(1-\gamma)+\gamma n_i}\mathbf{e}_i\mathbf{e}_i^{'}\right]$$

Where n_i is the size of group i. From that:

$$[2(1-\gamma)\mathbf{I}_h+\gamma\mathbf{\Omega}_h]^{-1}=$$

$$=\frac{1}{2(1-\gamma)}\begin{bmatrix} \mathbf{I}_1-\frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1\mathbf{e}_1^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2-\frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2\mathbf{e}_2^{'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_m-\frac{\gamma}{2(1-\gamma)+\gamma n_m}\mathbf{e}_m\mathbf{e}_m^{'} \end{bmatrix}$$

Step 2: Calculate $[2(1-\gamma)\mathbf{I}_h+\gamma\mathbf{\Omega}_h+\gamma\mathbf{e}_h\mathbf{e}_h^{'}]^{-1}$.

To do that we use the following “diagonal plus theorem”: let $\mathbf{M} = \mathbf{D} + a\mathbf{x}\mathbf{y}^T$, where $\mathbf{D}$ is a diagonal invertible matrix, $\mathbf{x}$ and $\mathbf{y}$ are vectors and a is a scalar; then $\mathbf{M}^{-1} = \mathbf{D}^{-1} + b\mathbf{x}^*\mathbf{y}^{*T}$, where $b = -a\left(1 + a\sum \frac{x_i y_i}{d_{ii}}\right)^{-1}$ and $x_i^{*T} = x_i/d_{ii}$, $y_i^{*T} = y_i/d_{ii}$.

Define $\mathbf{Z}_h = [2(1-\gamma)\mathbf{I}_h + \gamma\boldsymbol{\Omega}_h]^{-1}$. Then:

$$\left[2(1-\gamma)\mathbf{I}_h + \gamma\boldsymbol{\Omega}_h + \gamma\mathbf{e}_h\mathbf{e}_h^{'}\right]^{-1} = \left[\mathbf{Z}_h^{-1} + \gamma\mathbf{Z}_h^{-1}\mathbf{Z}_h\mathbf{e}_h\mathbf{e}_h^{'}\right]^{-1} = \left[\mathbf{I}_h + \gamma\mathbf{Z}_h\mathbf{e}_h\mathbf{e}_h^{'}\right]^{-1}\mathbf{Z}_h$$

Applying the“diagonal plus theorem”:

$$\left\{\mathbf{I}_h + \gamma\mathbf{Z}_h\mathbf{e}_h\mathbf{e}_h^{'}\right\}^{-1} = \left\{\mathbf{I}_h + \begin{bmatrix} \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{\gamma}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix} \mathbf{e}_h^{'}\right\}^{-1} =$$

$$= \left\{\mathbf{I}_h - \left(\frac{1}{1 + \sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}\right) \begin{bmatrix} \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{\gamma}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix} \mathbf{e}_h^{'}\right\}$$

So that:

$$\left[2(1-\gamma)\mathbf{I}_h + \gamma\boldsymbol{\Omega}_h + \gamma\mathbf{e}_h\mathbf{e}_h^{'}\right]^{-1} = [\mathbf{I}_h + \gamma\mathbf{Z}_h\mathbf{e}_h\mathbf{e}_h^{'}]^{-1}\mathbf{Z}_h =$$

$$= \frac{1}{2(1-\gamma)}\left\{\mathbf{I}_h - \left(\frac{1}{1 + \sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}\right) \begin{bmatrix} \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{\gamma}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix} \mathbf{e}_h^{'}\right\} *$$

$$* \begin{bmatrix} \mathbf{I}_1 - \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1\mathbf{e}_1^{'} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 - \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2\mathbf{e}_2^{'} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_m - \frac{\gamma}{2(1-\gamma)+\gamma n_m}\mathbf{e}_m\mathbf{e}_m^{'} \end{bmatrix}$$

Premoltiplying by $\mathbf{e}_h'$ and postmoltiplying by $\mathbf{e}_h$:

$$\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h = \frac{1}{2(1-\gamma)}\left\{\mathbf{e}_h^{'} - \left(\frac{1}{1+\sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}\right)\mathbf{e}_h^{'}\begin{bmatrix} \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{\gamma}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix}\mathbf{e}_h^{'}\right\}\begin{bmatrix} \frac{2(1-\gamma)}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{2(1-\gamma)}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{2(1-\gamma)}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix} =$$

$$= \left\{\mathbf{e}_h^{'} - \left(\frac{1}{1+\sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}\right)\mathbf{e}_h^{'}\begin{bmatrix} \frac{\gamma}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{\gamma}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{\gamma}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix}\mathbf{e}_h^{'}\right\}\begin{bmatrix} \frac{1}{2(1-\gamma)+\gamma n_1}\mathbf{e}_1 \\ \frac{1}{2(1-\gamma)+\gamma n_2}\mathbf{e}_2 \\ . \\ \frac{1}{2(1-\gamma)+\gamma n_p}\mathbf{e}_p \end{bmatrix} =$$

$$\left(\sum_p \frac{n_p}{2(1-\gamma)+\gamma n_p}\right) - \left(\frac{\sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}{1+\sum\limits_p \frac{n_p\gamma}{2(1-\gamma)+\gamma n_p}}\right)\left(\sum_p \frac{n_p}{2(1-\gamma)+\gamma n_p}\right) =$$

$$= \sum_p \frac{n_p}{2(1-\gamma)+\gamma n_p}\frac{1}{1+\gamma\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}} = \frac{\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}}{1+\gamma\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}}$$

Proving that $\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) = \frac{\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}}{1+\gamma\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}}$ is decreasing if the market structure becomes less competitive is equivalent to prove that $\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}$ decreases as the market structure is becomes competitive. Assume that groups 1 and 2 are put together. We want to show that:

$$\frac{n_1+n_2}{2(1-\gamma)+\gamma(n_1+n_2)} < \frac{n_1}{2(1-\gamma)+\gamma n_1} + \frac{n_2}{2(1-\gamma)+\gamma n_2}$$

$$\Longleftrightarrow 4\gamma^2 n_1 n_2 - 4\gamma n_1 n_2 - \gamma^2 n_1 n_2^2 - \gamma^2 n_1^2 n_2 < 0 \Longleftrightarrow 4\gamma - 4 - \gamma n_2 - \gamma n_1 < 0$$

$$\Longleftrightarrow 4 - 4\gamma + \gamma n_2 + \gamma n_1 > 0 \Longleftrightarrow 4(1-\gamma) + \gamma n > 0$$

True because $0 \le \gamma < 1$.

From the proof we even derive that:

$$1 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) = \frac{1}{1+\gamma\sum\limits_p \frac{n_p}{2(1-\gamma)+\gamma n_p}} > 0$$

Claim 4 *The matrix:*

$$\left[2(1-\gamma)\mathbf{I}_h + 2\gamma\left(1-\gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\right)\mathbf{1}^{hh}\right] = \left[2(1-\gamma)\mathbf{I}_h + 2\gamma\left(1-\gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\right)\mathbf{e}_h\mathbf{e}_h^{'}\right]$$

is definite positive.

For any vector $\mathbf{z} \neq \mathbf{0}$:

$$\mathbf{z}'\left[2(1-\gamma)\mathbf{I}_f + 2\gamma\left(1-\gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\right)\mathbf{e}_h\mathbf{e}_h^{'}\right]\mathbf{z} = 2(1-\gamma)\mathbf{z}'\mathbf{z} + 2\gamma\left(1-\gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\right)(\mathbf{z}'\mathbf{e}_f)(\mathbf{z}'\mathbf{e}_f)' > 0$$

As, from the previous claim, $1-\gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right) > 0$

7 Existance of Interior Equilibrium

Here we show that if products are symmetric ($\alpha_i = \alpha,$ for every $i => a_i = a,$ for every i), the solution is interior. By continuity, the solution is interior if α_i are similar but not identical. However, numerical simulations show that if α_i are sufficiently different across products, the first order condition does result in negative outputs or prices for products with low α_i. In the paper, we assume that the α_i are close, so that the equilibrium is interior.

7.1 Bertrand Case

As shown in the paper's appendix (see proof of Lemma 5), equilibrium prices are:

$$\begin{bmatrix} \mathbf{p}^h \\ \mathbf{p}^f \end{bmatrix} = \left\{2(\beta+\sigma)\mathbf{I} - \sigma\begin{bmatrix} 2\left(\mathbf{1}^{hh} + \frac{1}{2}\sigma\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff} + \frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{bmatrix}\right\}^{-1}\begin{bmatrix} \mathbf{a}_h \\ \mathbf{a}_f \end{bmatrix}$$

We use the following Lemma:

Lemma 10 *Assume that an* $n \times n$ *matrix* $\mathbf{B}$ *is such that* $\mathbf{B} \geq \mathbf{0}$*. If the matrix* $(\mathbf{I} - \mathbf{B})$ *has the d.d. property, then it is definite positive and it has a nonnegative inverse, which can be computed as*

$$(\mathbf{I} - \mathbf{B})^{-1} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + ...$$

where the series is converging. Given two matrices $\mathbf{B}_1 \geq \mathbf{B}_2$*, then* $(\mathbf{I} - \mathbf{B}_2)^{-1} \geq (\mathbf{I} - \mathbf{B}_1)^{-1}$*.*

In the paper's appendix we showed that:

$$\begin{bmatrix} 2\left(\mathbf{1}^{hh}+\frac{1}{2}\sigma\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{bmatrix} > 0$$

Thus, applying the previous Lemma:

$$\left\{2(\beta+\sigma)\mathbf{I}-\sigma\begin{bmatrix} 2\left(\mathbf{1}^{hh}+\frac{1}{2}\sigma\left(\mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{1}^{fh}\right)\right) & \mathbf{1}^{hf} \\ \mathbf{1}^{fh} & 2\left(\mathbf{1}^{ff}+\frac{1}{2}\sigma\left(\mathbf{1}^{fh}\mathbf{A}_h^{-1}\mathbf{1}^{hf}\right)\right) \end{bmatrix}\right\}^{-1} < \tfrac{1}{2(\beta+\sigma)}\mathbf{I}$$

Which implies that:

$$\begin{bmatrix} \mathbf{p}^h \\ \mathbf{p}^f \end{bmatrix} > \tfrac{a}{2(\beta+\sigma)}\mathbf{e} > \mathbf{0}$$

Thus equilibrium prices are positive.

We now show that the equilibrium quantities are positive. In the paper's appendix, we showed that (see proof of theorem 4):

$$2\mathbf{q}^h = \mathbf{a}_h + \mathbf{1}^{hf}\mathbf{A}_f^{-1}\left\{\mathbf{a}_f + [(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{\Omega}_f]\mathbf{t}^f\right\}$$

At the equilibrium, the foreign country will choose the national champion policy; thus $\mathbf{\Omega}_f = \mathbf{e}_f\mathbf{e}_f^{'}$ and $\mathbf{A}_f^{-1} = \frac{1}{2}[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{e}_f\mathbf{e}_f^{'}]^{-1}$. This implies that:

$$\begin{aligned} 2\mathbf{q}^h &= \mathbf{a}_h + \mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{a}_f + \mathbf{1}^{hf}\frac{1}{2}[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{e}_f\mathbf{e}_f^{'}]^{-1}[(\beta+\sigma)\mathbf{I}_f - \sigma\mathbf{e}_f\mathbf{e}_f^{'}]\mathbf{t}^f \\ &= \mathbf{a}_h + \mathbf{1}^{hf}\mathbf{A}_f^{-1}\mathbf{a}_f + \mathbf{1}^{hf}\frac{1}{2}\mathbf{t}^f \end{aligned}$$

As shown in appendix, $\mathbf{A}_f^{-1}$ is positive, and all taxes are positive (see equation 12 in appendix). All quantities are thus positive.

7.2 Cournot Case

In the paper's appendix, we showed (see proof of Lemma 6) that the equilibrium quantities are such that:

$$\begin{bmatrix} \mathbf{W}_h & \gamma \mathbf{1}^{hf} \\ \gamma \mathbf{1}^{fh} & \mathbf{W}_f \end{bmatrix} \begin{bmatrix} \mathbf{q}^h \\ \mathbf{q}^f \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}_h \\ \boldsymbol{\alpha}_f \end{bmatrix}$$

Where:

$$\begin{aligned} \mathbf{W}_h &= 2(1-\gamma)\mathbf{I}_h + 2\gamma\mathbf{1}^{hh} - \gamma^2\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)\mathbf{1}^{hh} \\ \mathbf{W}_f &= 2(1-\gamma)\mathbf{I}_f + 2\gamma\mathbf{1}^{ff} - \gamma^2\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)\mathbf{1}^{ff} \end{aligned}$$

with:

$$\left(1 - \gamma\mathbf{e}_c^{'}\boldsymbol{\Delta}_c^{-1}\mathbf{e}_c\right) > 0$$

As products are symmetric, it is easy to show that all home products' taxes and quantities are the same (say, q_h and t_h). Similarely, all foreign products' taxes and quantities are the same (say, q_f and t_f). From the previous equations, we can derive that:

$$\begin{aligned} \left\{2(1-\gamma) + \gamma[2 - \gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h\right\} q_h + \gamma n_f q_f &= \alpha \\ \left\{2(1-\gamma) + \gamma[2 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)]n_f\right\} q_f + \gamma n_h q_h &= \alpha \end{aligned}$$

Notice that $\left\{2(1-\gamma) + \gamma[2 - \gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h\right\} > 0$ and $\left\{2(1-\gamma) + \gamma[2 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)]n_f\right\} > 0$.[10] Equating the two equations we can derive that:

$$\left\{2(1-\gamma) + \gamma[1 - \gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h\right\} q_h = \left\{2(1-\gamma) + \gamma[1 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)]n_f\right\} q_f$$

Notice that $\left\{2(1-\gamma) + \gamma[1 - \gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h\right\} > 0$ and $\left\{2(1-\gamma) + \gamma[1 - \gamma\left(\mathbf{e}_h^{'}\boldsymbol{\Delta}_h^{-1}\mathbf{e}_h\right)]n_f\right\} > 0$. Thus, q_f and q_h must have the same sign. As $\left\{2(1-\gamma) + \gamma[2 - \gamma\left(\mathbf{e}_f^{'}\boldsymbol{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h\right\} q_h + \gamma n_f q_f = \alpha > 0$, they have to be both positive.

We now show that equilibrium prices must be positive. From the first order condition:

$$\alpha - t_i - 2(1-\gamma)q_i - 2\gamma\sum_{j\in\mathcal{F}_i} q_j - \gamma\sum_{j\notin\mathcal{F}_i} q_j = 0$$

Which implies that:

$$p_i = \alpha - q_i - \gamma\sum_{j\neq i} q_j = t_i + (1-\gamma)q_i + \gamma\sum_{j\in\mathcal{F}_i} q_j$$

---

[10] This comes from $2 - \gamma\left(\mathbf{e}_c^{'}\boldsymbol{\Delta}_c^{-1}\mathbf{e}_c\right) > 0$ for $c = h, f$.

In the paper's appendix we showed (see proof of Lemma 6) that, at the equilibrium:

$$\mathbf{t}^h = -\gamma^2 \left(\mathbf{e}_f^{'}\mathbf{\Delta}_f^{-1}\mathbf{e}_f\right) \mathbf{1}^{hh}\mathbf{q}^h => t^h = -\gamma^2 \left(\mathbf{e}_f^{'}\mathbf{\Delta}_f^{-1}\mathbf{e}_f\right) n_h q_h$$

So that:

$$p_h = -\gamma^2 \left(\mathbf{e}_f^{'}\mathbf{\Delta}_f^{-1}\mathbf{e}_f\right) n_h q_h + (1-\gamma)q_h + \gamma n_h q_h = \gamma[1-\gamma \left(\mathbf{e}_f^{'}\mathbf{\Delta}_f^{-1}\mathbf{e}_f\right)]n_h q_h + (1-\gamma)q_h$$

Which is positive because $1-\gamma \left(\mathbf{e}_f^{'}\mathbf{\Delta}_f^{-1}\mathbf{e}_f\right) > 0$, as shown in appendix.