Intuitive and noncompetitive equilibria in weakly efficient auctions with entry costs

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Abstract

I study weakly efficient auctions with entry costs, under the IPV assumption, following Tan and Yilankaya [5]. First, I generalize their Proposition 4 to what I call *(generalized) intuitive equilibrium.* By such I prove that if bidders' valuation distributions are ordered in a (weak) first order domination ranking, then there exists an equilibrium in cutoff strategies where cutoffs are (weakly) increasingly ordered with respect to the domination ranking. Stronger bidders are thus ex ante more likely to participate. A second result states a necessary and sufficient condition for the existence of a noncompetitive cutoff equilibrium, in which only one bidder (if any) takes part in the auction. Neither the uniform distribution nor any distribution first order stochastically dominated by the uniform may ever satisfy that condition. If both intuitive and nonintuitive equilibria exist, I conjecture that intuitive equilibria tend to yield higher ex ante efficiency, while nonintuitive ones might yield higher expected revenues.

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1 Introduction

Tan and Yilankaya [7] give an interesting characterization of equilibria in weakly efficient auctions (in the sense of Armstrong [1][1]) with entry costs.[2] Among other issues, they study asymmetric bidders, which is my main concern in this note. They show the existence of what they call *intuitive equilibrium* (their Proposition 4). Assume that bidders can be split into two groups (strong bidders and weak ones), with respective valuation distributions F_s and F_w. F_s first order dominates F_w. Then there exists a cutoff equilibrium in which strong bidders are more likely to participate in the auction than weak bidders are.

Tan and Yilankaya focus on two groups only. In this note, I generalize their intuitive equilibrium notion to the case where there is an arbitrary number of groups of bidders that are ordered by first order stochastic dominance (of valuation distributions). I show existence of what I call a *(generalized) intuitive equilibrium* where stronger bidders have lower cutoffs.

In a second result I present, a noncompetitive outcome can arise in equilibrium. If one bidder is "strong enough", then there exists an equilibrium in which the other bidders never participate in the auction.

The next section presents the model and some background concepts. The main results are explained in the second section. A third section digresses on multiple equilibria, efficiency and revenue. There is an Appendix containing a long proof.

2 Cutoff equilibria

An indivisible object is to be auctioned off by a mechanism that is efficient among the final bidders (i.e. weakly efficient).[3] There are N potential, risk-neutral bidders. There is no reservation price. Participating in the auction has a known cost c ($0 < c < v^*$) for each final participant[4]. Each bidder learns his valuation before deciding whether to participate or not in the auction. Bidder i's valuation v_i is drawn from a distribution F_i with full support on $[0, v^*]$. I apply the IPV assumption.

Since the mechanism is weakly efficient, the only relevant component of bidder i's strategy is the

[1] That is, it is efficient among final participants.

[2] Their work extends the one of Campbell [2].

[3] See Menezes and Monteiro [5] and Levin and Smith [3] for an introduction on auctions with entry cost. I rather follow the former paper in my work. Unlike the latter model, the former one considers that bidders learn their own valuations before taking the participation decision.

[4] As in Tan and Yilankaya [7], we assume that entry costs are equal among bidders.

participation decision. It is easy to show that the participation decision of bidder i is determined by a cutoff (or threshold) value θ_i such that he participates iff $v_i > \theta_i$. In the indifference case $v_i = \theta_i$, I assume that there is no entry. Knowing others' cutoff strategies, if bidder i participates in the auction, her expected profits would be equal to

$$\int_0^{v_i} \Pr\left(\varepsilon > \max_{j\neq i}[I\{v_j > \theta_j\} \cdot v_j]\right) d\varepsilon - c = \int_0^{v_i} \prod_{j\neq i} F_j\left(\max\{\theta_j, \varepsilon\}\right) d\varepsilon - c$$

under the IPV assumption. Then, her best response $\theta_i^{\#}(\theta_{-i})$ to θ_{-i}, others' cutoff strategies, is characterized by

$$\int_0^{\theta_i^{\#}(\theta_{-i})} \prod_{j\neq i} F_j\left(\max\{\theta_j, \varepsilon\}\right) d\varepsilon - c = (\leq)0$$

The inequality only applies when the upper bound of the integral is v^*, as in further equations where the symbol appears in parentheses. $\theta_i^{\#}(\cdot)$ is single-valued, continuous and strictly decreasing (except in the corner case $\theta_i^{\#}(\theta_{-i}) = v^*$), and belongs to $[c, v^*]$. Observe that $\theta_i^{\#}(\theta_{-i}) = c \Leftrightarrow \theta_{-i} = (v^*, ..., v^*)$. A cutoff (or threshold) equilibrium is a vector $\Theta^* = (\theta_1^*, ..., \theta_N^*) \in [c, v^*]^N$ such that

$$\int_0^{\theta_i^*} \prod_{j\neq i} F_j\left(\max\{\theta_j^*, \varepsilon\}\right) d\varepsilon - c = (\leq)0, \forall i \in \{1, ..., N\}$$

Existence of cutoff equilibrium is guaranteed by Brower's fixed-point Theorem.

3 Main results

Tan and Yilankaya [5] consider a setting where there are two groups of potential bidders, one 'strong' and one 'weak', with the property that the distribution function for any bidder in the strong group, F_s, first order stochastically dominates the distribution function for any bidder in the weak group, F_w. They show that there exists at least one equilibrium, which they term an *intuitive equilibrium*, in which all the strong bidders have identical cutoff θ_s^*, all weak bidders have identical cutoff θ_w^*, and $\theta_s^* \leq \theta_w^*$. My first result extends this result to any ordering of the distribution functions among all bidders.

Proposition 1 *The (generalized) intuitive equilibrium. If bidders can be ordered in a first order stochastic dominance ranking, such that*

$$F_1(v) \leq F_2(v) \leq ... \leq F_N(v),\ \forall v \in [0, v^*]$$

, then the auction has at least one cutoff equilibrium $\Theta^* = (\theta_1^*, \theta_2^*, ..., \theta_N^*)$ *such that*

$$\theta_1^* \leq \theta_2^* \leq ... \leq \theta_N^*$$

Proof. See the appendix. ■

In such an equilibrium, a stronger bidder is ex ante more likely to participate in the auction than a weaker bidder. This fact is what makes this equilibrium intuitive, as Tan and Yilankaya point out.

Remark 1 *If, additionally,* $F_i(v) = F_{i+1}(v) = ... = F_{i+k}(v),\ \forall v \in [0, v^*]$*, for some* $k \in \mathbb{N}$ *and some* $i \in \{1, ..., N-k\}$*, then there is an intuitive cutoff equilibrium satisfying* $\theta_i^* = \theta_{i+1}^* = ... = \theta_{i+k}^*$.

This natural remark indicates that intuitive equilibria are not at odds with the idea of symmetric cutoffs for symmetric agents. In a certain sense, a truly intuitive equilibrium should be required to meet this property.

I now show that if one bidder's distribution function is "strong enough", then there is a threshold equilibrium in which the other bidders never participate.

Proposition 2 *If (and only if) there exists some potential bidder* k *and some number* $e \in [c, v^*]$ *such that*

$$E_{F_k}(v \mid v \geq e) = \frac{v^* - c}{1 - F_k(e)}$$

then the auction has a cutoff equilibrium in which the rest of bidders never participate and bidder k *participates if and only if his valuation is greater than* c.

Proof.

$$\exists e\in [c,v*]: E_{F_k}\left(v \,|\, v \geq e\right) = \frac{v*-c}{1-F_k(e)}$$

$$\Longleftrightarrow \exists e \geq c: \frac{\int_e^{v*} x dF_k(x)}{1-F_k(e)} = \frac{v*-c}{1-F_k(e)}$$

$$\Longleftrightarrow \exists e \geq c: v* - \int_e^{v*} x dF_k(x) = c$$

$$\Longleftrightarrow \exists e \geq c: v* - \left[F_k(v*)v* - F_k(e)e - \int_e^{v*} F_k(x)dx\right] = c$$

$$\Longleftrightarrow \exists e \geq c: F_k(e)e + \int_e^{v*} F_k(x)dx = c$$

$$\Longleftrightarrow F_k(c)c + \int_c^{v*} F_k(x)dx =$$

$$= \int_0^{v*} F_k(\max\{c,x\})dx \leq c$$

To see that this implies the result, consider a bidder $i \neq k$ and suppose no bidders other than i or k ever participate, and c is k's cutoff. The left-hand side of the last equation gives i's gross expected payoff to participating if his type is $v*$, as the reader can see from Section 2. Hence his best response is a threshold strategy of $v*$ (so never participating as well). If all bidders except for k never participate, k will participate in the auction whenever his valuation is higher than c. Thus, his optimal threshold strategy would be precisely c. ■

This result establishes than noncompetitive results may arise, which is bad news for the seller. The following remark enumerate contexts in which a noncompetitive equilibrium cannot ever arise.

Remark 2 *Neither the uniform distribution nor any other distribution first order stochastically dominated by it can satisfy the condition of Proposition 2.*

No matter how high entry costs are, if bidders are "weak enough", every bidder will participate in the auction with positive ex ante probability. However, noncompetitive outcomes may not be unlikely. If a distribution function satisfies the condition of Proposition 2, then any other distribution that (first order) stochastically dominates the former satisfies the same condition. An immediate consequence is that if bidders are ordered according to first order stochastic domination, and there is a noncompetitive equilibrium in which the only possible final participant is bidder k, then for each $k' < k$ there exists a noncompetitive equilibrium by which only bidder k' can possibly participate.[5] Additionally, for any distribution $F(x)$ with full support on $[0,v^*]$ there exists a positive number $\tilde{n}$ such that for any $n > \tilde{n}$, $F(x)^n$ meets the condition of Proposition 2. The latter observation may be particularly interesting in auctions with multiple units.

4 Multiplicity, efficiency and revenue

A final part of this note is devoted to the analysis of multiple equilibria. Tan and Yilankaya [7] show that even if at least one intuitive equilibrium exists, there might be nonintuitive equilibria, where the stronger bidders are indeed less likely to participate than weaker bidders. If one agrees that an intuitive equilibrium has a stronger predictive power, a set of natural questions arise. Is an intuitive equilibrium good in terms of ex ante efficiency, as compared to a nonintuitive one? And, is it good for the seller?

This section shall not provide definite answers to these questions, yet some guidelines are indicated. In order to illustrate these ideas, I restrict attention to the case in which there is only one strong potential bidder (which I call bidder G) and only one weak potential bidder (called F). Each bidder's valuation distribution is represented by the bidder's identifying letter, and G first order stochastically dominates F. The following graph illustrates each bidder's best cutoff response.

[5] In that case, the only noncompetitive *and* intuitive equilibrium is the one in which only the strongest bidder can possibly participate.

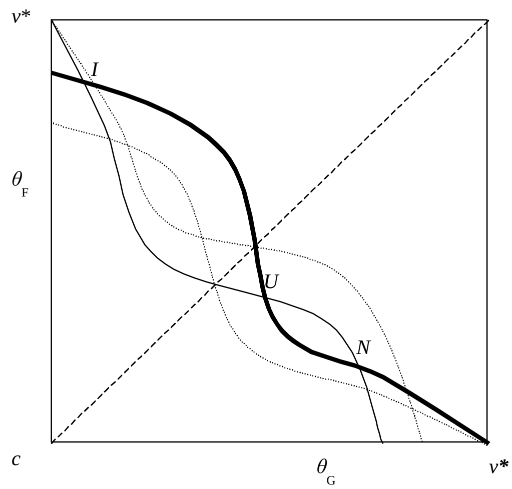

Figure 1: Multiple equilibria

The thick line represents bidder F's best response $\theta_F^{\#}(\cdot)$. The normal-sized line is $\theta_G^{\#}(\cdot)$. The dashed line is the 45-degree line. The dotted lines mirror the best responses with respect to the 45-degree line. Notice that $\theta_F^{\#}(\cdot) \geq \theta_G^{\#}(\cdot)$.

In this example, there are three cutoff equilibria: a stable intuitive equilibrium $\Theta^I = (\theta_F^I, \theta_G^I)$,[6] a stable nonintuitive equilibrium Θ^N and an unstable nonintuitive equilibrium Θ^U. The dotted lines illustrate that $\theta_G^I < \theta_F^N < \theta_G^N < \theta_F^I$. In fact, for any nonintuitive equilibrium there exists one intuitive equilibrium such that these inequalities are met.

First, total ex ante efficiency is analyzed. Consider any pair of cutoff strategies (θ_F, θ_G). The first component of efficiency is the winner's expected valuation. This is the expectation of the maximum valuation among the bidders finally participating, or $E \max_{i \in \{F,G\}} I\{v_i > \theta_i\} \cdot v_i$. This random variable follows the following distribution function H:

$$H(v; \theta_F, \theta_G) = F(\max\{\theta_F, v\}) \cdot G(\max\{\theta_G, v\})$$

The second component is the (expected) waste given by participation. All together, total ex ante efficiency is accounted as

$$TEE(\theta_F, \theta_G) = \int_0^{v^*} [1 - H(v; \theta_F, \theta_G)]dv - c[2 - F(\theta_F) - G(\theta_G)]$$

[6] In what follows, the first coordinate always refers to bidder F, and the second one refers to bidder G.

The partial derivative of TEE with respect to θ_F is $f(\theta_F)\left[c - \int_0^{\theta_F} G(\max\{\theta_G, v\})dv\right]$. An analogous expression follows when differentiating with respect to θ_G. It can be seen that only equilibrium cutoffs can maximize TEE, as pointed out by Stegeman [6] and Lu [4]. Moreover, any unstable equilibrium such as Θ^U is suboptimal. If one increases θ_G along the $\theta_F^{\#}(\cdot)$ line, the total derivative is

$$\frac{dTEE}{d\theta_G} = \frac{\partial TEE}{\partial \theta_G} + \frac{d\theta_F^{\#}(\theta_G)}{d\theta_G}\frac{\partial TEE}{\partial \theta_F} = \frac{\partial TEE}{\partial \theta_G}$$

since $\frac{\partial TEE}{\partial \theta_F} = 0$, due to $c = \int_0^{\theta_F^{\#}(\theta_G)} G(\max\{\theta_G, v\})dv$. Since Θ^U is unstable, the $\theta_F^{\#}(\cdot)$ line is below the $\theta_G^{\#}(\cdot)$ line, thus $\frac{\partial TEE}{\partial \theta_G} > 0$. This shows that any of the adjacent stable equilibria is more efficient that the unstable equilibrium.

Therefore, it is just worth it to compare Θ^I to Θ^N. A clean conclusion is not reached in this respect. However, if G reverse-hazard-rate dominates F, that is, $G(\cdot)/F(\cdot)$ is increasing, then $H(v; \theta_G^N, \theta_F^N)$ first order stochastically dominates $H(v; \theta_F^N, \theta_G^N)$. If $c[F(\theta_F^N)+G(\theta_G^N)-F(\theta_G^N)-G(\theta_F^N)]$ is low enough, then $TEE(\theta_G^N, \theta_F^N) \geq TEE(\theta_F^N, \theta_G^N)$. But (θ_G^N, θ_F^N) is above the $\theta_G^{\#}(\cdot)$ line, thus TEE is increased by moving towards $(\theta_G^N, \theta_G^{\#}(\theta_G^N))$. From there, TEE is increased again if one moves the coordinates along the $\theta_G^{\#}(\cdot)$ line towards (θ_F^I, θ_G^I). Hence $TEE(\theta_F^I, \theta_G^I) > TEE(\theta_F^N, \theta_G^N)$ provided reverse-hazard-rate domination and $c[F(\theta_F^N) + G(\theta_G^N) - F(\theta_G^N) - G(\theta_F^N)]$ low enough.

Secondly, I provide some insight into expected revenues. Given the cutoffs (θ_F, θ_G), bidder F's ex ante expected payoff is computed as

$$\begin{aligned}
\pi_F(\theta_F, \theta_G) &= E\left[I\{v_F > \theta_F\}\left(\int_0^{v_F} G(\max\{\theta_G, v\})dv - c\right)\right] \\
&= \int_0^{v^*} [1 - F(\max\{\theta_F, v\})]G(\max\{\theta_G, v\})dv - c[1 - F(\theta_F)]
\end{aligned}$$

An analogous calculation follows for π_G. Ex ante expected revenues are then

$$\begin{aligned}
R(\theta_F, \theta_G) &= TEE(\theta_F, \theta_G) - \pi_F(\theta_F, \theta_G) - \pi_G(\theta_F, \theta_G) \\
&= \int_0^{v^*} [1 - F(\max\{\theta_F, v\})][1 - G(\max\{\theta_G, v\})]dv
\end{aligned}$$

Denote the interior of the integral as $h(v; \theta_F, \theta_G)$. Now, suppose that G hazard-rate dominates F, that is, $[1-G(\cdot)]/[1-F(\cdot)]$ is increasing. Additionally, suppose that the ex ante probability of having both bidders participating at the auction is higher under the nonintuitive equilibrium Θ^N than under the intuitive one. It is readily seen that, under these conditions, $h(\cdot; \theta_F^N, \theta_G^N) > h(\cdot; \theta_F^I, \theta_G^I)$ almost everywhere, thus $R(\theta_F^N, \theta_G^N) > R(\theta_F^I, \theta_G^I)$.

While none of these observations are conclusive, it can be conjectured that intuitive equilibria are good in terms of efficiency, but they are bad for the seller as compared to nonintuitive ones. Intuitive equilibria give higher chances of participation to the bidders with higher expected valuation. However, nonintuitive equilibria are more "moderate" (the difference between highest and lowest cutoffs are lower) that intuitive counterparts. Hence, the seller might expect a higher probability of having a competitive auction in a nonintuitive equilibrium.

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5 Appendix

Proof. Proposition 1.

I assume, without loss of generality, that bidders are numbered according to the first order stochastic dominance ranking. Now, if $N = 2$, the proof is done in the following way: Define θ_1^+ as $F_2(\theta_1^+) \cdot \theta_1^+ \equiv c$, and θ_2^+ as $F_1(\theta_2^+) \cdot \theta_2^+ \equiv c$. Notice that, because of the first order domination, $\theta_1^+ \leq \theta_2^+$. It is easy to see that $\theta_1^+ = \theta_1^{\#}(\theta_1^+)$, $\theta_2^+ = \theta_2^{\#}(\theta_2^+)$, where # indicates "best cutoff response". Since $\theta_1^{\#}(\cdot)$ is strictly decreasing, it is defined under $\left[c, \theta_1^+\right] \times \left[\theta_1^+, v*\right]$. By the same reason, $\theta_2^{\#}(\cdot)$ is defined under $\left[c, \theta_2^+\right] \times \left[\theta_2^+, v*\right]$. Best responses are then closed under $\left[c, \theta_1^+\right] \times \left[\theta_2^+, v*\right]$, so we must find at least one equilibrium in this set. This equilibrium accomplishes with what is stated in the Proposition.

I focus now on the cases where $N > 2$. First of all, define an ordered (k)-pseudo-equilibrium

$$\Theta^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N) = \left(\theta_1^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N), ..., \theta_k^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N)\right)$$

as a k-first-players cutoff equilibrium that would appear if bidders $k+1, k+2, ..., N$ held their threshold strategy constant at the values above indicated. Define a continuous ordered (k)-pseudo-equilibrium line as a function that returns for each possible value $x \in [c, v*]$ and some fixed vector $(\theta_{k+2}, ..., \theta_N)$ an ordered (k)-pseudo-equilibrium

$$\Theta^k(x, \theta_{k+2}, ..., \theta_N) = \left(\theta_1^k(x, \theta_{k+2}, ..., \theta_N), ..., \theta_k^k(x, \theta_{k+2}, ..., \theta_N)\right)$$

in such a way that the resulting line is continuous and defined on $[c, v*]^N$. Pseudo-equilibria always exist in the same way as an equilibrium does. A continuous pseudo-equilibrium path can always be found as optimal threshold strategies are continuous and hence a little change in one of the parameters change the function values smoothly. Then the following claims follow:

Claim 1 *If for any parameters $(\theta_{k+2}, ..., \theta_L) \in [c, v*]^{N-k-1}$ there is a continuous k-pseudo-equilibrium line such that for any $x \in [c, v*]$*

$$\theta_1^k(x, \theta_{k+2}, ..., \theta_N) \leq ... \leq \theta_k^k(x, \theta_{k+2}, ..., \theta_N) \tag{1}$$

then for the previously fixed parameters $(\theta_{k+3}, ..., \theta_N)$ and whenever $k+2 \leq N$ there exists a continuous $(k+1)$-pseudo-equilibrium line such that for any $y \in [c, v*]$

$$\theta_1^{k+1}(y,\theta_{k+3},...,\theta_N) \leq ... \leq \theta_k^{k+1}(y,\theta_{k+3},...,\theta_N) \leq \theta_{k+1}^{k+1}(y,\theta_{k+3},...,\theta_N) \qquad (2)$$

Proof of the Claim: Each (k)-pseudo-equilibrium meets

$$\int_0^{\theta_i^k(x,\theta_{k+2},...,\theta_N)} \prod_{\substack{j\neq i\\ j\leq k}} F_j\left(\max\{\theta_j^k(x,\theta_{k+2},...,\theta_N),\varepsilon\}\right)\cdot$$

$$\cdot F_{k+1}\left(\max\{x,\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c = (\leq)0, \quad \forall i\in\{1,...,k\}$$

We define the function $\pi_{k+1}(x)$ as equal to

$$\int_0^x \prod_{j\leq k} F_j\left(\max\{\theta_j^k(x,\theta_{k+2},...,\theta_N),\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c$$

If this function equals zero in x (or is non-positive for $x = v^*$), then we have a $(k+1)$-pseudo-equilibrium $\left(\theta_1^k(x,\theta_{k+2},...,\theta_N),...,\theta_k^k(x,\theta_{k+2},...,\theta_N),x\right)$ for the vector $(\theta_{k+2},...,\theta_N)$.

Define $\omega_k(\theta_{k+2},...,\theta_N)$ as the (unique) value satisfying

$$\int_0^{\omega_k(\theta_{k+2},...,\theta_N)} \prod_{\substack{j\leq k+1\\ j\neq k}} F_j\left(\max\{\omega_k(\theta_{k+2},...,\theta_N),\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c = (\leq)0$$

Given (1), it is seen that $\theta_k^k(\omega_k(\theta_{k+2},...,\theta_N),\theta_{k+2},...,\theta_N) \geq \omega_k(\theta_{k+2},...,\theta_N)$. On the other hand, it is obvious that $\theta_k^k(v^*,\theta_{k+2},...,\theta_N) \leq v^*$. By continuity of this function, I deduce that

$$H_k \equiv \left\{x\in[\omega_k(\theta_{k+2},...,\theta_N),v^*] : \theta_k^k(x,\theta_{k+2},...,\theta_N) = x\right\} \neq \varnothing$$

Denote h_k as the supremum of H_k. Then, by first order dominance between bidders k and $k+1$, $\pi_{k+1}(h_k) \leq 0$. Hence, if $\pi_{k+1}(v^*) > 0$, by continuity of $\pi_{k+1}(x)$ there exists some point $x^*\in[h_k,v^*]$ that yields the following $(k+1)$-pseudo-equilibrium

$$\Theta^{k+1}(\theta_{k+2},...,\theta_N) = \left(\theta_1^k(x^*,\theta_{k+2},...,\theta_N),...,\theta_k^k(x^*,\theta_{k+2},...,\theta_N),x^*\right)$$

Notice that this pseudo-equilibrium meets (2). The reason is that for any $x^* \in [h_k, v^*]$, we have for sure $\theta_k^k(x, \theta_{k+2}, ..., \theta_N) \leq x$, due to the definition of h_k and the fact that $\theta_k^k(v^*, \theta_{k+2}, ..., \theta_N) \leq v^*$.

On the other hand, if $\pi_{k+1}(v^*) \leq 0$ we already have a $(k+1)$-pseudo-equilibrium

$$\Theta^{k+1}(\theta_{k+2}, ..., \theta_N) = \left(\theta_1^k(v^*, \theta_{k+2}, ..., \theta_N), ..., \theta_k^k(v^*, \theta_{k+2}, ..., \theta_N), v^*\right)$$

clearly satisfying (2). Finally, it only remains to generalize (2) to all possible values of θ_{k+2}, and to state that the $(k+1)$-pseudo-equilibrium line obtained is also continuous, which is always possible as argued before. □

Claim 2 *For any $(\theta_4, ..., \theta_N) \in [c, v^*]^{N-3}$, whenever these parameters make sense, and in any case when $N = 3$, there exists a continuous (2)-pseudo-equilibrium line such that for any $x \in [c, v^*]$ (just ignore "$\theta_4, ..., \theta_N$" in the expression below if $N = 3$)*

$$\theta_1^2(x, \theta_4, ..., \theta_N) \leq \theta_2^2(x, \theta_4, ..., \theta_N)$$

Proof of the claim: The proof is almost identical to the case we studied where $N = 2$, and it is redundant to show it here. □

By Claims 1 and 2, we see by an induction argument that we can find a continuous $(N-1)$-pseudo-equilibrium line $\Theta^{N-1}(x)$ that meets the condition

$$\theta_1^{N-1}(x) \leq ... \leq \theta_{N-1}^{N-1}(x), \quad \forall x \in [c, v^*]$$

It remains to check that this line intersects with N^{th}-bidder's optimal threshold strategy in a point that meets $\theta_1^* \leq ... \leq \theta_N^*$. This is done by means of mimicking the proof of Claim 1. ■