Easy Collusion Mechanisms in Auctions with Entry

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Abstract

I study simultaneous, weakly efficient auctions with entry costs, under the IPV assumption. I construct two simple pre-play communication equilibria that yield collusive results. In the first one, bidders publicly arrange objects from the most preferred to the least preferred. In the second one, bidders just name their most-preferred object. Asymptotic efficiency is reached in the first case. In the second, a large number of bidders operates against seller's interest. I also analyze what sellers can do concerning these communication equilibria by properly setting reservation prices.

Keywords: Auctions; Pre-play communication; Collusion; Entry costs.

JEL codes: D44, D82, C72.

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1 Introduction

Recent literature in Auction Theory has been largely devoted to the study of collusive equilibria from the bidders' point of view. Some of recent results on it apply to infinitely repeated auctions (Fudenberg, Levine and Maskin [10]; Aoyagi [2]; Blume and Heidhues [4]; Skrzypacz and Hopenhayn [17]; Fabra [9]).[1] Results found this way rely on two facts: that auctions are taken in different periods of time, and that there are infinitely many auctions. But, can we construct a model of coordination among bidders when auctions are taken in the same period of time and they are not to be infinitely many? Literature has vastly analyzed the one-object case. A good contribution is McAfee and McMillan [14]. A cartel of bidders can proceed to a knockout auction of the object, such that the highest-valuation bidder economically compensates the other ones in order to be the only participant in the auction. The collusion mechanism with more than one object has been sometimes thought as a replication of this one-object mechanism.[2] But a couple of things have been generally overlooked: that this requires direct coordination between bidders, which is frequently illegal and hard to manage; and that simple communication instead of direct coordination is also possible.

[1]Fudenberg, Levine and Maskin [10] prove that bidders-side efficiency can be reached for high discount factors. Aoyagi [2] designs a collusive mechanism in which bidders reveal information about private valuations (or signals) to a third party (called the center), who indicates who is to win the auction. Since this induces bidders to overstate their valuations (or to misreport their signals), the winner of the object is "punished" in subsequent auctions, so that incentive compatibility is reached. Blume and Heidhues [4] design a tacit collusion scheme based on bid-rotation and bounded bids. Skrzypacz and Hopenhayn [17] also rely on bid-rotation and underbidding based on the history of winners' identities. They find an asymptotically efficient collusive result when the number of bidders grows without bound. Fabra [9] analyzes the possibilities for collusion in repeated auctions under discriminatory and uniform formats, and concludes that the latter facilitates collusion more than the former.

[2]Yet the literature reports more complex collusive schemes in simultaneous auctions. One example is Grimm, Riedel and Wolfstetter [11] concerning the 3G frequency auctions in Germany. Another example is Brusco and Lopomo [5] for a series of simultaneous English auctions like the FCC spectrum auctions in the United States. The information arising during the auction process is useful to coordinate bidders. In this respect, see also Albano, Germano and Lovo [1], and Engelbrecht-Wiggans and Kahn [8], who analyze the SPaR (Stake, Protect and Revenge) strategies for a series of simultaneous English auctions.

Besides, the literature on collusion in auctions has seldom taken into account that the very existence of entry costs could allow bidders to coordinate among themselves. I found two recent exceptions. One is Tan and Yilankaya [18], where they analyze efficient collusion in a one-object second-price auction with entry costs. They observe that such collusion equilibrium is not ratifiable, in the sense that a high-valuation type could veto the collusion mechanism, hence sending a credible message about his strength that (once it modifies other bidders' beliefs) increases his expected profits with respect to the ones obtained through the collusion mechanism. The other one is Campbell [6], a remarkable paper that exploits entry costs in a series of simultaneous second-price auctions with two bidders. He finds that, if pre-play communication is possible, bidders can engage in truthful revelation of information regarding their valuations. Concretely, bidders publicly order objects from the most preferred to the least preferred. This partial revelation of information makes it possible for the bidders to obtain bidders-side efficiency with probability close to 1 when the number of objects goes large. Campbell's results are important but difficult to generalize to a more-than-two-bidder setup, as indicated in Miralles [16].

Indeed, the main purpose of this paper is to extend Campbell's results, in several ways. First, I extend his results to an arbitrary number of potential bidders, provided that we have a sufficiently high number of objects to be sold. Second, I find a simpler information revelation mechanism, namely the most-preferred-object revelation, also yielding collusive results. Third, I analyze robustness of these equilibria under relaxations on the main assumptions regarding entry costs and reservation prices. Fourth, I calculate optimal reservation prices when the existence of these communication equilibria is taken into account by the sellers. Finally, I observe that optimal reservation price setters would be ex ante better-off if these communication equilibria were not played.

I have chosen to analyze weakly efficient auctions (like the second-price one, as in Campbell [6]) for two reasons. First, a weakly efficient auction is by itself appealing and easier in dealing with calculations. Second, and most importantly, Celik and Yilankaya [7] have proved that, under the IPV assumption, the optimal (revenue-maximizing) auction with entry costs and symmetric bidders is weakly efficient, with properly induced cut-off (critical participation decision) values. So this is a good benchmark to start studying pre-play communication among bidders.

I now introduce the base problem I deal with in this paper. Consider a series of weakly efficient, separate, simultaneous auctions of N indivisible objects (assume we have N separate sellers), where there are L potential, risk-neutral bidders. There is no reservation price. Participating in each auction has a known cost c ($0 < c < v^*$) for each final participant. Each bidder learns his valuation before taking any decision. Each i^{th}-bidder's valuation of any object h (denoted v_{ih}) is known by other bidders to follow an ex ante distribution F over the support $[0, v^*]$. I assume full support on this interval for easiness of calculations, while noticing that skipping this assumption would not alter the main conclusions. Bidders are a priori symmetric among themselves and with respect to objects. Valuations are independent among bidders (IPV assumption) and among objects.

Prior to the auction stage of the game, cheap-talk is undertaken among bidders i. I could have defined a message space, say, $M_i \equiv [0, 1]$, and then I could have tried to find equilibria of interest, so that messages are interpreted only in the equilibrium. Through this paper, though, I proceed in a different way, which makes little difference at the end. I understand that the message space has an interpretation from the very beginning of the game. In this paper, I consider and study two possibilities. In a first one, communication could be constrained to the revelation of ranking profiles $r_i^N \equiv (r_{i1}^N, ..., r_{iN}^N)$, where it is understood by any bidder that $h \neq k \Leftrightarrow r_{ih}^N \neq r_{ik}^N$, that $r_{ih}^N > r_{ik}^N$ means $v_{ih} \geq v_{ik}$ and that $r_{ih}^N < r_{ik}^N$ means $v_{ih} \leq v_{ik}$. In a second one, communication is constrained to the revelation of the most-preferred object $n_i^N \in \{1, ..., N\}$. Needless to say, there are infinitely many other possibilities regarding the message space and the meaning each message could have in equilibrium, but I restrict attention to these two cases.

The message this paper communicates is that very simple pre-play communication mechanisms yield asymptotically good collusive results through truthful information revelation, so that no complicated strategies, involving signalling during the auction process, punishments, threats, third party's collaboration, enforcement... are needed. The following results can be generalized to setups other than auction contexts, like for instance local markets with entry costs, imperfect information and Bertrand competition among entrants. It is also seen and argued that the level of common knowledge that resulting equilibria require is quite mild. Namely, equilibria are sustained if (1) everybody knows that everybody is telling the truth in the equilibrium, and (2) everybody knows that everybody knows that everybody is telling the truth in the equilibrium. Two main Propositions are presented in this paper, one concerning the case where the number of objects grows without bound and another in which the number of potential bidders grows without bound. In both cases, expected profits for any seller go to zero if there is no reservation price. In the first case, an asymptotically efficient result is reached.

All this suggests how important communication and information sharing can be with respect to the performance of market mechanisms. Skrzypacz and Hopenhayn [17] correctly point out that direct, cartel-organizing communication among bidders is usually a difficult, and illegal, task. But this overlooks the fact that other kinds of communication are easy to establish and difficult to illegalize. Can a bidder (or firm) make everybody else know that he prefers some specific object (or market or geographical area) to any of the other ones? In my opinion, there are plenty of cheap, difficult-to-prosecute ways he can send a signal about that.

The paper is organized as follows. Section 2 introduces preliminary results regarding weakly efficient auctions with entry costs. These results are not only very useful in further sections of the paper, but are also very useful for other models of entry cost in auctions. Section 3 presents the main results, summarized in the two aforementioned Propositions. Section 4 analyzes the extent to which such Propositions can be generalized under relaxations of the underlying assumptions. In some cases, consequent optimal seller's reservation price is calculated, which serves to evaluate the extent to which collusion is successful. Section 5 concludes. An appendix containing long proofs is included.

2 Preliminary results: a weakly efficient auction with entry costs and asymmetric bidders

Before presenting the main results, some insight into an efficient auction with entry costs is necessary, in order to gain more understanding.[3] We need to know what is going to happen in each separate auction after the communication stage of the game is played. After publicly sharing information, bidders update beliefs about others' valuations for each object, so that for each auction it is not necessarily the case that bidders are still symmetric. That is the reason why I now focus on a unique auction with asymmetric bidders.

An indivisible object is to be sold by a mechanism that is *efficient among the final bidders* (or *weakly efficient*, following Armstrong [3]). There are L potential, risk-neutral bidders. There is no reservation price. Participating in the auction has a known cost c ($0 < c < v^*$) for each final participant. Each bidder learns his valuation before deciding whether to participate or not in the auction. Each i^{th}-bidder's valuation v_i is believed by other bidders to follow a distribution F_i over the support $[0, v^*]$. We assume full support under this interval for easiness of calculations, though noting that a relaxation of this assumption would not alter the main results. Once again, we apply the IPV assumption, so valuations are independent among bidders.

Since the mechanism is weakly efficient, the only relevant component of bidder i's strategy is the participation decision. He comes to know his valuation before taking this decision, so this strategy is closely related to a threshold (cut-off) value θ_i. Bidder i participates in the auction if and only if his valuation v_i is above this threshold value (in the limit case $v_i = \theta_i$, we simplify by assuming that bidder withdraws from the auction). Knowing others' cut-off strategies, if bidder i participates in the auction, his expected profits would be equal to

$$\int_0^{v_i} \Pr\left(\varepsilon > \max_{j \neq i}[I\{v_j > \theta_j\} \cdot v_j]\right) d\varepsilon - c = \int_0^{v_i} \prod_{j \neq i} F_j\left(\max\{\theta_j, \varepsilon\}\right) d\varepsilon - c$$

under the IPV assumption. Then, his best response $\theta_i^{\#}(\theta_{-i})$ to θ_{-i}, others' cut-off strategies, is characterized by

[3] See Menezes and Monteiro [15] and Levin and Smith [13]. I rather follow the former paper in my work, since, unlike the latter model, the former one considers that bidders learn their own valuations before taking the participation decision.

$$\int_0^{\theta_i^{\#}(\theta_{-i})} \prod_{j \neq i} F_j \left(\max\{\theta_j, \varepsilon\} \right) d\varepsilon - c = (\leq) 0$$

The less-or-equal inequality only applies when the upper bound of the integral equals v^*, as in further equations where the symbol appears in brackets. Observe that $\theta_i^{\#}(\cdot)$ is single-valued, continuous, strictly decreasing (except in the corner case $\theta_i^{\#}(\theta_{-i}) = v^*$) and belongs to $[c, v^*]$. Observe as well that $\theta_i^{\#}(\theta_{-i}) = c \Leftrightarrow \theta_{-i} = (v^*, ..., v^*)$. A cut-off equilibrium is a vector $\Theta^* = (\theta_1^*, ..., \theta_L^*) \in [c, v^*]^L$ such that

$$\int_0^{\theta_i^*} \prod_{j \neq i} F_j \left(\max\{\theta_j^*, \varepsilon\} \right) d\varepsilon - c = (\leq) 0, \forall i \in \{1, ..., L\}$$

Existence of a cut-off equilibrium is guaranteed by a simple use of Brower's fix point Theorem. I now present two contributions in this topic of Auction Theory that are very useful in the derivation of my main results. While they are presented as Lemmas, they are important by themselves. Lemma 1 is an extension of some previous work by Tan and Yilankaya [19]. Their paper about second-price sealed-bid auctions with entry costs and private values reveals, among other results, that if we have two groups of potential bidders, namely the "strong" ones and the "weak" ones, characterized by distribution functions F_s and F_w such that the former stochastically dominates the latter, then there exists at least one cut-off equilibrium in which $\theta_s^* \leq \theta_w^*$. They call it *intuitive equilibrium*. My first Lemma extends this result to a point that if bidders can be (weakly) ordered from the "strongest" to the "weakest", then a related cut-off ordering is possible in equilibrium. This Lemma is used in the proof of Proposition 1. The second Lemma I exhibit suggests the following idea: with entry costs, if one bidder's distribution function is "strong enough", then there exists a threshold equilibrium in which the other bidders never take part in the (weakly efficient) auction. This Lemma is very useful to derive Proposition 2.

Lemma 1 *The* general intuitive equilibrium. *In the auction considered here, if bidders can be ordered in a first order stochastic dominance ranking, such that*

$$F_1(v) \leq F_2(v) \leq ... \leq F_L(v), \forall v \in [0, v^*]$$

, *then this game has at least one cut-off equilibrium* $\Theta^* = (\theta_1^*, \theta_2^*, ..., \theta_L^*)$ *that meets*

$$\theta_1^* \leq \theta_2^* \leq ... \leq \theta_L^*$$

Proof. See the appendix. ■

Lemma 2 *In the auction we are considering here, if there exists some potential bidder* k *and some number* $e \geq c$ *such that*

$$E_{F_k}(v \mid v \geq e) = \frac{v^* - c}{1 - F_k(e)}$$

the auction has a cut-off equilibrium in which the rest of bidders never participate and bidder k *participates if and only if his value is greater than* c*.*

Proof.

$$E_{F_k}(v \mid v \geq e) = \frac{v^* - c}{1 - F_k(e)} \Leftrightarrow \frac{\int_e^{v^*} x f_k(x)dx}{1 - F_k(e)} = \frac{v^* - c}{1 - F_k(e)}$$
$$\Longleftrightarrow v^* - \int_e^{v^*} x f_k(x)dx = c$$
$$\Longleftrightarrow v^* - \left[F_k(v^*)v^* - F_k(e)e - \int_e^{v^*} F_k(x)dx \right] = c$$
$$\Leftrightarrow F_k(e)e + \int_e^{v^*} F_k(x)dx = c$$
$$\Longrightarrow F_k(c)c + \int_c^{v^*} F_k(x)dx = \int_0^{v^*} F_k(\max\{c, x\})dx \leq c$$

This implies that, if bidder k plays a threshold strategy c, and if the rest of bidders but one of them play a threshold strategy of v^* (hence they do not participate in any case), the remaining bidder would obtain non-positive expected profits in case of taking part in the auction. This is seen in the last inequality, where the left-hand side is the expected profit of a participating bidder i (different from k) even if his valuation is equal to v^*. His best response would be a threshold strategy of v^* (so never participating as well). If all bidders except for k never participate, k will participate in the auction whenever his valuation is higher than c. Thus, his optimal threshold strategy would be precisely c. This completes the equilibrium. ■

Some points should be noticed here, with respect to Lemma 2. First, the reader can check that neither the uniform distribution nor any other distribution stochastically dominated by it can satisfy the condition of this Lemma. Second, if a distribution function satisfies this condition, any distribution function stochastically dominating the former one satisfies the same condition.

With these Lemmas at hand, I turn to the main part of this research, which is devoted to the analysis of pre-play communication in weakly efficient auctions with entry.

3 Main results

3.1 Asymptotic efficient collusion when $N \to \infty$

After having seeing some properties of weakly efficient auctions, it is time to focus on the communication part of the game. The previous section has provided us with some tools that are going to be useful in proving the main results of the present section. First, I present a communication game where bidders are allowed to publicly reveal their ranking valuations. The result I obtain is very interesting. Asymptotically, truthful rankings are revealed. As a result, bidders obtain an efficient allocation for them, in the sense that the bidder that gives the highest value to some object always keeps it at zero price, provided this valuation is higher than the entry cost.

This result is not at odds with what other researchers have found. In another setup, Jackson and Sonnenschein [12] recently prove that prior to the resolution of an $N-$replica of some allocation problem, agents can reveal some close-to-the-truth information about their preferences, hence allowing for a more efficient allocation. The condition for this to happen is that the allocation the social planner will apply with the information at hand has to be Pareto-efficient. In the limit, as N becomes unboundedly large, the revealed information tends to become absolutely true.

There is an intuitive explanation for the result I find. Asymptotically, revealing ranking valuations is equivalent to revealing the valuations themselves. Therefore, this induces cut-off equilibria for each object in which only the bidder revealing the highest valuation for the object takes part in the auction, as long as this valuation is higher than c. Imagine that bidder i has to reveal valuations for two objects 1 and 2, where $v_1 > v_2$. For each object, he has to allocate different values that are either $\bar{v}$ or v, where $\bar{v} > v$, knowing that the other bidders will believe him. The higher the value he attaches to an object, the more likely he will win the object for free. Given that, he prefers to give the highest probability of winning to the object he really appreciates more, by a simple convex combination argument. Translating this argument to every possible pair of objects, it is seen that the bidder tends to reveal the truth. We see it more formally. I now define the communication game that is to be played, and I state and proof a Proposition about the result explained above.

Definition 1 *A ranking revelation game $\Gamma\{F, c, L, N\}$ with L potential bidders, N objects, initial distribution F and entry cost c is defined as:*

1) A strategy for bidder i in this game is a message function $(r_{i1}^N, ..., r_{iN}^N) \equiv r_i^N : [0, v^*]^N \rightarrow \wp\{1, ..., N\}$, where $\wp\{1, ..., N\}$ is the family of all permutations on $\{1, ..., N\}$ (and thus it has $N!$ elements), a belief function $B_i : [\wp\{1, ..., N\}]^{L-1} \rightarrow [\mathbf{F}^N]^{L-1}$, where $\mathbf{F}$ is the set of all distribution functions over the initial (full) support, and a cut-off strategy profile
$\left\{\theta_{ih}^{\#N}\right\}_{h\in\{1,...,N\}} : [\mathbf{F}^N]^{L-1} \times [[c, v^*]^N]^{L-1} \rightarrow [c, v^*]^N.$

2) The timing of the game is as follows: first, Nature assigns object valuations to any potential bidder; second, potential bidders engage in public communication and update beliefs using the information they receive; third, they decide their object-specific participation (cut-off) strategies, and a separate efficient auction with no reservation price takes place among the remaining bidders for each object.

3) Payoffs regarding each object are distributed as follows: any seller obtains a revenue equal to the second highest valuation among the final participants[4] *(and thus he obtains zero if there are less than two of them). For any object, any final participant pays an entry cost c, obtains the object if his valuation is the highest one among participants (with an arbitrary tie-breaking rule), and in this case obtains a gross pay-off equal to valuation minus seller's revenue. If a bidder does not take part in the auction, he does not pay any entry cost and his payoff concerning the object is equal*

[4]By the Revenue Equivalence, any weakly efficient auction yields the same revenue to the seller as the second-price auction.

to zero.

A truthful ranking revelation equilibrium in this game is a Perfect Bayesian Equilibrium in which the equilibrium messages satisfy $v_{ih} > v_{ik} \Rightarrow r^N_{ih} > r^N_{jk}$, $\forall i \in \{1, ..., L\}$, $\forall h, k \in \{1, ..., N\}$*, and beliefs are consistent with the messages that are sent.*

Campbell [6] successfully solves this game for the case of $L = 2$. He obtains that, in this case, for any number of objects, any original distribution function, and any entry cost, there exists a truthful ranking revelation equilibrium. Moreover, under this family of equilibria, he proves asymptotic efficiency, in the sense that, when N is high enough, with probability arbitrarily close to one the bidder with the highest valuation for an object keeps it for free (as long as this highest valuation is higher than the entry cost). This result is a remarkable theoretical contribution. Nevertheless, its drawback is that it is not easily generalizable to an arbitrary number of bidders. Campbell's proof critically relies on the following monotonicity property: the "stronger" (meaning higher ranking value) some bidder claims to be with respect to some object, the higher the other bidder's equilibrium cut-off strategy will become. With three bidders, and given that bidders are strategic substitutes in cut-off strategies, if bidder 1 claims to be stronger and bidder 2 reacts increasing his cut-off strategy in equilibrium, it may well be that bidder 3 finally reduces his cut-off equilibrium strategy, given that bidder 2 has increased it. So the monotonicity property may not hold with more than two bidders. Miralles [16] analyzes this problem and obtains sufficient conditions to guarantee monotonicity and therefore to find a truthful ranking revelation equilibrium. Nevertheless, conditions found in that research are too restrictive for any practical purpose.

That is the reason why, with more than two bidders and concerning ranking revelation strategies, it is worth it to focus on asymptotic cases, where the distorting effect coming from strategic substitutability in cut-off strategies becomes negligible. I now state Proposition 1, whose proof can be found in the appendix. Lemma 1 is very useful in this proof.

Proposition 1 *Consider a game* $\Gamma\{F, c, L, N\}$*, fixing* L *constant. In this context,* $\exists \tilde{N} : \forall N \geq \tilde{N}$ *there exists a truthful ranking revelation equilibrium. Denote* X_{LNO} *as the* *event "valuations are such that with L potential bidders and $N \geq \tilde{N}$ objects, this truthful ranking revelation equilibrium a) gives zero profits to any seller, and b) is interim-efficient among bidders, for a fixed set O of objects", defined on the event-space $[0, v^*]^{NL}$. Then the following is true:*

$$\forall \varepsilon > 0, \exists \bar{N} \geq \tilde{N} : \forall N \geq \bar{N}, \Pr\left(X_{LNO}\right) > 1 - \varepsilon$$

Proof. See the appendix. ■

3.2 A very poor seller's outcome as $L \to \infty$

We now focus on another asymptotic result, this time from the point of view of the number of bidders. Here, we are going to use Lemma 2. If the number of objects is high enough, the most-preferred object valuation distribution is so strong that it induces an equilibrium in which only one bidder (if any) takes finally part in the auction. This justifies a communication game in which bidders are only allowed to reveal what their most-preferred objects are. There is an equilibrium in which there is truthful revelation of this information. A consequent asymptotic result is that any seller's expected profits collapses to zero for L big enough.

This result contributes to the recent literature contradicting the "intuitive" idea that seller's expected profits are increasing in the number of potential bidders. As a matter of fact, with entry costs, seller's expected profits could decline to zero, if there is no reservation price. This is illustrated in Levin and Smith [13], for the case where bidders take the participation decision before learning their respective valuations, and in Menezes and Monteiro [15], for the case where bidders learn their own valuations before taking the participation decision.

Definition 2 *A most-preferred-object revelation game $\Gamma'\{F, c, L, N\}$ with L potential bidders, N objects, initial distribution F and entry cost c is a modification of the ranking revelation game, in which the message space has N elements instead of $N!$, and each message may denote in equilibrium what object the sender prefers among all N objects. A strategy for bidder i in this game is a message function $n_i^N : [0, v^*]^N \to \{1, ..., N\}$, a belief function $B_i : \{1, ..., N\}^{L-1} \to \left[\mathbf{F}^N\right]^{L-1}$, and a cut-off strategy profile $\left\{\theta_{ih}^{\#N}\right\}_{h \in \{1,...,N\}} : \left[\mathbf{F}^N\right]^{L-1} \times \left[[c, v^*]^N\right]^{L-1} \to [c, v^*]^N$.*

A truthful most-preferred-object revelation equilibrium in this game is a Perfect Bayesian Equilibrium in which the equilibrium messages are of the kind $n_i^N \in \underset{h \in \{1,...,N\}}{\arg\max} v_{ih}^N$, *for any* $i \in \{1, ..., L\}$, *and beliefs are consistent with the messages that are sent.*

Proposition 2 *Consider a game* $\Gamma'\{F, c, L, N\}$. *In this context,* $\exists \bar{N} : \forall L, \forall N \geq \bar{N}$ *there exists a truthful most-preferred-object revelation equilibrium. Denote* Y_{LNh} *as the event "valuations are such that with L potential bidders and N objects,* $h^{th}-$*object seller obtains zero profits under this equilibrium", defined on the event-space* $[0, v^*]^{NL}$. *Then, fixing* $N \geq \bar{N}$, *the following is true:*

$$\Pr(Y_{LNh}) \geq 1 - \left(\frac{N-1}{N}\right)^L$$

Proof. See the appendix. ■

It is obvious that this probability goes to one as L goes large. In fact, for L large enough, the probability that some seller gets exactly zero profits in the context of this communication equilibrium is higher than the same probability when no pre-play communication is undertaken (if we assume symmetric cut-off equilibrium). With no communication, the symmetric cut-off equilibrium value θ_L is characterized in any auction by the equation $F(\theta_L)^{L-1} \cdot \theta_L = c$. The probability of zero profits in this context is equal to $F^{(L-1)}(\theta_L) \equiv L \cdot F(\theta_L)^{L-1} - (L-1)F(\theta_L)^L$. Using the characterization of the equilibrium, this probability is equal to $\frac{c}{\theta_L}[L - (L-1)F(\theta_L)]$. Therefore, when $L \to \infty$, this probability tends to $\frac{c}{v^*} < 1$, leading to the aforementioned conclusion.

I illustrate the scope of this Proposition with an easy example. Let all valuations for all objects be independently drawn from a uniform distribution function between 0 and 1. Let $c = \frac{1}{10}$, i.e. a ten percent of the maximum valuation. Then, if $N \geq 10$, there exists a truthful most-preferred-object revelation equilibrium. Assume $N = 10$. Then, if $L = 3$, a seller has at least a 27.1% probability of getting zero profits. For $L \geq 22$ (2.2 bidders per object) the probability that h^{th}-object seller obtains zero profits is not lower than 90%, whereas for $L \geq 29$ the probability that this seller obtains zero profits is not lower than 95%. But if we set $N = 20$, and we want the probability that h^{th}-object seller obtain zero profits to be not lower than 90%, then we must have at least 45 potential bidders (2.25 bidders per object). If $c = \frac{1}{20}$, i.e. a five percent of the maximum valuation, then there exists a truthful most-preferred-object revelation equilibrium if $N \geq 20$. Holding $N = 20$, again we need 45 bidders to guarantee that he probability that h^{th}-object seller obtains zero profits is going to be not lower than 90%. All this points out several issues. First, the minimum L for which such equilibrium exists is relatively small. Second, if the number of bidders is high enough, it is relatively likely that a seller obtains no profits. Third, once the critical number of objects is reached, entry costs and distribution functions have an increasingly unimportant influence on the probability that a specific seller obtains zero profits. Finally, a decrease in the number of objects (sellers) may negatively affect any seller's expected profits, if he knows that the number of sellers is anyway high enough to support a truthful revelation equilibrium as the one depicted above. While the latter point could sound rather astonishing, it is not so if we think of it from a "wild-life-documentary" point of view: if a potential prey knows that some one is going to be hunted, it prefers that there be many potential prey around him, so that the probability to become the *chosen* prey diminishes.

Proposition 2 is asymptotically (for L large) robust to an enlargement of the strategy space in the sense of $n_i^N : [0, v*]^N \to \{0, 1, ..., N\}$, where "0" could represent a meaningless message in equilibrium. If no message is sent, others' beliefs about bidder's valuations are not updated. For each object, bidder's expected profits negatively depend on the probability that somebody else declares that this object is his preferred one. Since this probability tends to one as L goes large, "0"–signal sender's expected profits collapses to zero, whereas other-message sender's overall expected profits do not tend to zero if some meaningful (in equilibrium) message is sent.

4 Remarks and discussion

4.1 Sensitivity to entry cost variability

So far we have assumed that entry costs are constant among bidders and among objects. But this could be criticized as an unrealistic assumption. Although for a priori similar objects, similar entry costs could be expected, I proceed to relax this assumption. I obtain that the nature of the entry cost variability is important in determining if the main Propositions hold true.

Remark 1 *Propositions 1 and 2 are robust to entry costs variability among bidders, as long as there exist* $\underline{c} > 0, \bar{c} < v*$: $\bar{c} \geq c_i \geq \underline{c}, \forall i \in \{1, ..., L\}$.

Proof. Regarding Proposition 1, we just need to redefine W_h^N in part 2) of the proof (see the appendix), by $W_h^N \equiv \underset{i \in \{1,...,L\}}{\arg\max} \left[F^{-1}\left(\frac{r_{ih}^N}{N}\right) - c_i \right]$. The proof still follows. It is only required to add a proper subscript to c whenever necessary. With respect to Proposition 2, we redefine $\bar{N}$ as the minimum natural number such that

$$\int_0^{v*} F(\max\{\max_{i \in \{1,...,L\}} c_i, \quad x\})^{\bar{N}} dx \leq \min_{i \in \{1,...,L\}} c_i$$

, so that still only one bidder (if any) takes part in the auction for some object, provided it is most-preferred by some bidder. With that in mind, the rest of the proof follows the same way, just adding proper subscripts whenever necessary to the entry cost notation. ■

Remark 2 *Propositions 1 and 2 are NOT NECESSARILY robust to entry cost variability among objects.*

Proof. Regarding Proposition 1, step 4) of the proof fails. If $v_{i1} > v_{i2}$ but $v_{i1} - c_1 < v_{i2} - c_2$, when $N \to \infty$ bidder i will optimally attach the highest ranking value not to the most valuable good but to the one that is most valuable in net terms, i.e. after substracting entry costs. Incentive compatibility is thus violated. Regarding Proposition 2), step 4) of its proof also fails in the same terms. ■

Admittedly, the last remark could reduce the impact of pre-play communication in auctions with entry. Fortunately for the bidders, a slight modification in the designed game can induce a similar family of communication equilibria with equal consequences. Suppose that prior to the game, it is known that the entry cost for each object and each bidder is extracted from a distribution function D with support on $[0, v*]$. Define $w_{ih} \equiv v_{ih} - c_{ih}$, i.e. the "net valuation" that bidder i has for object h. Then the following remark summarizes an interesting result, which constitutes a generalization of Proposition 1.

Remark 3 *If entry costs are publicly revealed once Nature chooses them, and before any decision is taken, then* $\exists \tilde{N} : \forall N \geq \tilde{N}$ *there exists a truthful "net valuations"* *ranking revelation equilibrium. The rest of Proposition 1 follows with this modification.*

Proof. By properly combining F and D, net valuations are extracted from a new distribution function Q supported on $[-v*, v*]$. Let $a_i^N = (a_{i1}^N, ..., a_{iN}^N)$ be the ranking profile with respect to net valuations. Then, once we learn c_{ih}, it is readily seen that $Q^{-1}\left(\frac{a_{ih}^N}{N}\right) + c_{ih} \xrightarrow{prob_{N\to\infty}} v_{ih}$. With this information in mind, we can redefine W_h^N in part 2) of the proof (see the appendix), by $W_h^N \equiv \underset{i\in\{1,...,L\}}{\arg\max}\left(\frac{a_{ih}^N}{N}\right)$. From then on, the proof follows as in the main result. With a probability asymptotically close to one, a bidder with the highest reported "net valuation" ranking position obtains the object for free, if the "net valuation" is positive. Given that, we go to steps 4) and 5) of the main proof (see the appendix) and we check that a bidder is interested in revealing a higher "net valuation" for an object that he prefers in net terms to other one, hence asymptotically meeting the incentive compatibility. ■

With this modification of the game, truthful ranking revelation holds as an asymptotically optimal strategy for any bidder in equilibrium. It can be seen, therefore, that the asymptotic existence of this equilibrium cannot be avoided by the sellers setting entry fees and (or) reservation prices (in this case, "net valuations" are equal to $v_{ih} - c_{ih} - p_{ih}$, where the last element is the reservation price that bidder i has to pay for object h), just taking into account that in this latter case profits may not fall to zero. Notice as well that this generalization of Proposition 1 also works with bidder-specific distribution functions D_i and F_i, as long as this functions only vary with respect to bidders and not with respect to objects. These points are very important since they stress how robust this communication equilibrium is.

4.2 Seller's strategy: choosing reservation prices

As we have seen, Proposition 2 is weak against object-specific variation in entry cost. It can be seen that it is also not robust against object-varying reservation prices. But if it were not that vulnerable, another additional obstacle arises. Suppose for illustration that reservation prices are set identical among objects. An increase in the common reservation price may increase the critical number of objects $\bar{N}$ in Proposition 2. In some cases, for any N we can find a (common) high enough reservation price such that the equilibrium depicted in Lemma 2 does not exist. If this is the case, $L \to \infty$ is no longer bad news for the seller but rather very good ones. He would optimally set a reservation price slightly below $v^* - c$. By doing so, he both breaks any possible most-preferred-object revelation equilibrium and maximizes revenue given no communication among bidders. What I next do is to report a general case in which this does not happen, that is, a case where an increase of a (equal among objects) reservation price does not destroy previously existent most-preferred-object revelation equilibria. In that case, optimal reservation price is calculated for the asymptotic case where $L \to \infty$. I conclude that in this case any seller is ex ante worse-off than in a game without pre-play communication.

Remark 4 *Assume that reservation prices p_r are equal among objects. Let N_{p_r} be the critical number of objects such that for any $N \geq N_{p_r}$ there exists a most-preferred-object revelation equilibrium, given the reservation price p_r. Then, defining $N(p_r)$ as*

$$\int_{c+p_r}^{v^*} F(x)^{N(p_r)} dx + F(c+p_r)^{N(p_r)} \cdot c - c = 0 \qquad (3)$$

, we obtain that N_{p_r} is never increasing in p_r if $N(0)\frac{f(x)}{F(x)} \leq \frac{1}{c}$, $\forall x \in [c, v^*]$. Consequently, if $N \geq N_0$, and if $\left[1 - F(p_r + c)^N\right] \cdot p_r$ is concave with respect to p_r, optimal (common) reservation price, when seller anticipates the existence of this communication equilibrium and $L \to \infty$, is calculated as

$$J_F^N(\tilde{p}_r^* + c) = c$$

, where $J_F^N(x) \equiv x + \left[\frac{d \log\left[1 - F(x)^N\right]}{dx}\right]^{-1}$ is the "N-virtual valuation" function.

Corollary 1 *Asymptotically, the seller of any object is ex ante worse-off in this case than when there is no pre-play communication.*

Proof. The left-hand side of Equation (3) equals the expected profits of a participating bidder with valuation v^* if some other bidder has distribution function equal to $F(x)^{N(p_r)}$ and cut-off strategy equal to $c + p_r$, and if the rest of bidders do not take part in the auction, given the reservation price p_r. If the equality holds, there exists an equilibrium in which the only final participant (if any) is the one with distribution function $F(x)^{N(p_r)}$, following Lemma 2. Therefore, N_{p_r} is the minimum natural number not lower than $N(p_r)$. By differentiating Equation (3), we see that $\frac{dN(p_r)}{dp_r} \leq 0$ if and only if $N(p_r)\frac{f(c+p_r)}{F(c+p_r)} \leq \frac{1}{c}$. If this is accomplished at $p_r = 0$, the condition stated in the Remark becomes sufficient.

To calculate optimal (common) reservation prices, we first realize that for any object asymptotic expected revenue under this communication equilibrium is arbitrarily close to $\left[1 - F(p_r + c)^N\right] \cdot p_r$ as L goes large. The optimal reservation price is derived through first-order differentiation. To prove the Corollary, I follow Celik and Yilankaya [7] to conclude that the best reservation price when there is no pre-play communication and there are entry costs is arbitrarily close to $v^* - c$ as L goes large. In fact, in this way the seller obtains the maximum possible revenue, which is clearly above what he obtains when a most-preferred-object revelation equilibrium exists and is anticipated. ■

It is not very clear why different sellers should set equal reservation prices. This is admittedly a huge problem for this equilibrium to exist in a more generic context. On the other hand, sellers are ex ante identical (facing the same problem) and it is not clear if an asymmetric reservation prices equilibrium exists. If there is no communication among bidders, the optimal reservation price is identical among sellers.

This Remark deserves very serious and critical discussion. We are facing a problem of multiplicity of equilibria. Since any communication game has many equilibria in which messages are meaningless, it could be suggested that the seller could set an optimal reservation price that ignores the existence of a most-preferred-object revelation equilibrium. By doing so, he would set a reservation price arbitrarily close to $v^* - c$ as L goes large. With probability close to one as $L \to \infty$, there would exist a bidder whose valuation is arbitrarily close to v^*. This bidder correctly deduces that with probability close to one no other bidder has a valuation high enough to make it worth it to take part in the auction if he participates, so he optimally decides to participate. Therefore, with probability close to one, seller sells the object and obtains the maximum possible revenue.

The timing of the game becomes crucial in this discussion. If we changed the order of the stages in this game, selection of equilibria would not be such an important issue. Consider the following modification of the game. Bidders know that auctions are going to be weakly efficient, but they do not know the reservation prices during the communication stage. The information hence revealed is shared in such a way that it is concealed to the sellers. After communication among bidders is undertaken, sellers announce reservation prices. After that, bidders take their participation decisions, and auctions are played simultaneously. Then, if the condition of Remark 4 holds, there exists an equilibrium in which communication is truthful, and sellers will optimally set a reservation price according to the formula in the same Remark.

Having seen the vulnerabilities of Proposition 2, we now focus on the equilibrium suggested in the above generalization of Proposition 1. Define

$$\bar{F}_{NL}(x) \equiv E\left[F^{\left(\max\limits_{i\in\{1,\ldots,L\}} r_{ih}^{N}\right)}(x) \;\middle|\; L, N\right]$$

where $F^{(h)}$ is the $(N+1-h)^{th}-$order statistic distribution function. This is the ex ante distribution function of the highest valuation for some arbitrary object. Each seller is newly allowed to set a reservation price $p_r \in [0, v^{*}-c]$. He sets it before the communication stage of the game takes place. In the context of the "net valuations" ranking revelation equilibrium, any seller expects with a probability close to one that only one bidder (if any) is going to take part in the auction. Ex ante expected revenue is then arbitrarily close to

$$\left[1-\bar{F}_{NL}\left(p_r+c\right)\right]\cdot p_r$$

This kind of equilibrium exists for N high enough, so we can concentrate on the limit case where N goes to infinity. The following Lemma is going to be useful, in order to calculate the asymptotically optimal reservation price:

Lemma 3 *For a fixed L and for any $x \in [0, v^{*}]$, we have $\bar{F}_{NL}(x) \geq F(x)^{L}$, $\forall N$, and $\lim\limits_{N\to\infty} \bar{F}_{NL}(x) = F(x)^{L}$.*

Proof. See the appendix. ■

With this, we can compute expected revenue in the limit, and calculate the optimal reservation price. The result is as follows:

Proposition 3 *Let $p_r^*(L, N, F)$ be the optimal (profit-maximizing) reservation price with L potential bidders, N objects and common original distribution function F, under the communication equilibrium depicted in Proposition 1. Let $\left[1 - F\left(p_r + c\right)^L\right] \cdot p_r$ be concave with respect to p_r, and define the "L-virtual valuation" function*

$$J_F^L(x) \equiv x + \left[\frac{d \log\left[1 - F(x)^L\right]}{dx}\right]^{-1}$$

Then, $\bar{p}_r^* \equiv \lim_{N \to \infty} p_r^*(L, N, F)$ satisfies

$$J_F^L\left(\bar{p}_r^* + c\right) = c$$

Corollary 2 *Seller's ex ante expected profits are not higher in this setup than in a game without pre-play communication and with optimal (revenue-maximizing) reservation price.*

Proof. With probability arbitrarily close to one when $N \to \infty$, seller's ex ante expected profits are equal to $\left[1 - F\left(p_r + c\right)^L\right] \cdot p_r$. The optimality condition follows then from the usual first-derivative maximization method.

The Corollary is proved in the following way. Notice that in the communication game the seller is only facing the highest-valuation bidder, who is never stronger than a $F(x)^L-$distribution bidder. So he cannot optimally do better than when he only faces one bidder with a distribution equal to $F(x)^L$. Meanwhile, in the no-communication game the seller is facing the highest-valuation bidder, whose distribution function is $F(x)^L$ from the seller's point of view, *plus the rest of bidders*, so he optimally cannot do worse than when facing a unique bidder with distribution function equal to $F(x)^L$. ■

So we have seen that, under the equilibrium communication strategies depicted in Proposition 1, the seller becomes ex ante worse-off, even when acting optimally.

This result is quite robust and is only vulnerable to non-simultaneity of the auctions process. In this case, the seller of some object would act optimally if he reveals some information about the auction's outcome. For instance, revealing the winner's identities could be enough to break up the communication equilibrium. By doing so, he lets potential bidders update beliefs about others' valuations. These forecasted updates may have influence on the communication-stage strategies, hence bringing instability to the proposed equilibrium. If instead no information is transferred to the bidders during the process, we see that an infinitely repeated (weakly efficient) auction process has a simple efficient collusive result via pre-play communication, provided there are entry costs and bidders know all their valuations prior to the starting of the auction series, and under the IPV assumption.

4.3 A local markets interpretation

Having seen that sellers may do something concerning the possibility of collusion in an auction context, it is worth it to have a look at another scenario where anti-collusion policies are difficult to implement. Instead of the base context I am analyzing, one could imagine a set of N separate markets of the same good and a number L of firms. $v_{ih} \in [0, v^*]$ could be regarded as the h^{th}-market value for a firm i if it is the only firm entering this market. This value is private information, and the only that is known is the initial distribution function F, common to all bidders and objects. If more than one firm enters this market, then the market works by Bertrand competition. Entering a market h costs c_{ih} for firm i. This cost is publicly known. Pre-play communication is undertaken among firms before any one takes any entry decision.

This new context can be treated in a way very similar to the auction model I was analyzing before. Equivalent Propositions can be found, and as a result consumers suffer from either an excess in the number of markets or an excess in the number of firms. Interestingly, this time the consumer cannot himself negotiate market prices, and therefore collusion through pre-play communication is even more likely to happen. Only a government representing the consumers could implement entry taxes (the equivalent to entry fees) or price regulation (the equivalent to reservation prices).

5 Conclusions

In this paper, I analyze simultaneous weakly efficient auctions with entry costs, under the IPV assumption. I pay attention to the possibilities that pre-play communication among bidders brings in order to obtain collusive results, while I avoid other collusion mechanisms that are more clearly illegal and harder to manage. I focus on simple messages that could make the equilibrium realistic and easy to understand for the bidders. I find two kinds of interesting communication equilibria. One is the ranking revelation equilibrium, in which bidders publicly and truthfully order objects from the most preferred one to the least preferred. In this sense, I follow Campbell [6], who studies the two-bidder case. The other equilibrium is the most-preferred-object revelation equilibrium, which is self-defined. In either case, ex ante identical bidders improve their situation by distinguishing themselves from each other.

In both cases, any seller's profits asymptotically collapse to zero, if no reservation price is set. Therefore, I proceed to calculate optimal reservation prices. Despite setting an optimal reservation price, none of the sellers can obtain a higher expected profit than when no meaningful communication is taken. The most-preferred-object revelation equilibrium, though, is vulnerable to reservation price variability among objects. Moreover, a matter of equilibrium selection arises in this latter case when the number of bidders goes large, allowing sellers to "ignore" pre-play communication when setting reservation prices.

The problems produced by sellers' reaction to the anticipation of these equilibria are not that important when I reinterpret my model as a local markets one with entry cost, imperfect information and Bertrand competition among entrants. Here, firms play the role of "bidders", and local isolated markets play the role of "sellers". Firms could engage in pre-play communication, prior to taking any entrance decision, in such a way that similar equilibria can be found with respect to the auction model. In this setup, the local markets, that is, the consumers, can seldom react to the presence of a communication equilibrium. Only a local government could regulate a local market through entry taxes and price regulation.

There are several interesting issues open for future research. More analysis is needed in order to generalize these results to a context where the IPV assumption does not hold, like when there is a common value component. Another extension of my work could be addressed to the study of different auction mechanisms apart from the weakly efficient ones, such as the first-price auction with asymmetric bidders.

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6 Appendix

Proof. Lemma 1.

We assume, without loss of generality, that bidders are numbered according to the first order stochastic dominance ranking. Now, if $L = 2$, the proof is done in the following way: Define θ_1^+ as $F_2(\theta_1^+) \cdot \theta_1^+ \equiv c$, and θ_2^+ as $F_1(\theta_2^+) \cdot \theta_2^+ \equiv c$. Notice that, because of the first order domination, $\theta_1^+ \leq \theta_2^+$. It is easy to see that $\theta_1^+ = \theta_1^{\#}(\theta_1^+)$, $\theta_2^+ = \theta_2^{\#}(\theta_2^+)$, where # indicates "best cut-off response". Since $\theta_1^{\#}(\cdot)$ is strictly decreasing, it is defined under $\left[c, \theta_1^+\right] \times \left[\theta_1^+, v*\right]$. By the same reason, $\theta_2^{\#}(\cdot)$ is defined under $\left[c, \theta_2^+\right] \times \left[\theta_2^+, v*\right]$. Best responses are then closed under $\left[c, \theta_1^+\right] \times \left[\theta_2^+, v*\right]$, so we must find at least one equilibrium in this set. This equilibrium accomplishes with what is stated in this lemma.

We focus now on the cases where $L > 2$. First of all, define an ordered (k)-pseudo-equilibrium

$$\Theta^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_L) = \left(\theta_1^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_L), ..., \theta_k^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_L)\right)$$

as a k-first-players cut-off equilibrium that would appear if bidders $k+1, k+2, ..., L$ hold their threshold strategy constant at the values above indicated. Define continuous ordered (k)-pseudo-equilibrium line as a function that returns for each possible value $x \in [c, v*]$ and some fixed vector $(\theta_{k+2}, ..., \theta_L)$ an ordered (k)-pseudo-equilibrium

$$\Theta^k(x, \theta_{k+2}, ..., \theta_L) = \left(\theta_1^k(x, \theta_{k+2}, ..., \theta_L), ..., \theta_k^k(x, \theta_{k+2}, ..., \theta_L)\right)$$

in such a way that the resulting line is continuous and defined on $[c, v*]^L$. Pseudo-equilibria always exist in the same way as equilibrium does. A continuous pseudo-equilibrium path can always be found as optimal threshold strategies are continuous and hence a little change in one of the parameters change the function values smoothly. Then the following claims follow:

Claim 1 *If for any parameters* $(\theta_{k+2}, ..., \theta_L) \in [c, v*]^{L-k-1}$ *there is a continuous k-pseudo-equilibrium line such that for any* $x \in [c, v*]$

$$\theta_1^k(x,\theta_{k+2},...,\theta_L) \leq ... \leq \theta_k^k(x,\theta_{k+2},...,\theta_L) \tag{1}$$

then for the previously fixed parameters $(\theta_{k+3},...,\theta_L)$ *and whenever* $k+2 \leq L$ *there exists a continuous* $(k+1)$*-pseudo-equilibrium line such that for any* $y \in [c, v^*]$

$$\theta_1^{k+1}(y,\theta_{k+3},...,\theta_L) \leq ... \leq \theta_k^{k+1}(y,\theta_{k+3},...,\theta_L) \leq \theta_{k+1}^{k+1}(y,\theta_{k+3},...,\theta_L) \tag{2}$$

Proof of the Claim: Each (k)-pseudo-equilibrium meets

$$\int_0^{\theta_i^k(x,\theta_{k+2},...,\theta_L)} \prod_{\substack{j\neq i\\ j\leq k}} F_j\left(\max\{\theta_j^k(x,\theta_{k+2},...,\theta_L),\varepsilon\}\right)\cdot$$

$$\cdot F_{k+1}\left(\max\{x,\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c = (\leq)0, \quad \forall i\in\{1,...,k\}$$

We define the function $\pi_{k+1}(x)$ as equal to

$$\int_0^x \prod_{j\leq k} F_j\left(\max\{\theta_j^k(x,\theta_{k+2},...,\theta_L),\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c$$

If this function equals zero in x (or is non-positive for $x = v^*$), then we have a $(k+1)$-pseudo-equilibrium $\left(\theta_1^k(x,\theta_{k+2},...,\theta_L),...,\theta_k^k(x,\theta_{k+2},...,\theta_L),x\right)$ for the vector $(\theta_{k+2},...,\theta_L)$.

Define $\omega_k(\theta_{k+2},...,\theta_L)$ as the (unique) value satisfying

$$\int_0^x \prod_{\substack{j\leq k+1\\ j\neq k}} F_j\left(\max\{\omega_k(\theta_{k+2},...,\theta_L),\varepsilon\}\right)\cdot \prod_{j\geq k+2} F_j\left(\max\{\theta_j,\varepsilon\}\right)d\varepsilon - c = (\leq)0$$

Given (1), it is seen that $\theta_k^k(\omega_k(\theta_{k+2},...,\theta_L),\theta_{k+2},...,\theta_L) \geq \omega_k(\theta_{k+2},...,\theta_L)$. On the other hand, it is obvious that $\theta_k^k(v^*,\theta_{k+2},...,\theta_L) \leq v^*$. By continuity of this function, we deduce that

$$H_k \equiv \left\{x \in [\omega_k(\theta_{k+2},...,\theta_L), v^*] : \theta_k^k(x,\theta_{k+2},...,\theta_L) = x\right\} \neq \emptyset$$

Denote h_k as the supremum of H_k. Then, by first order dominance between bidders k and $k+1$, $\pi_{k+1}(h_k) \leq 0$. Hence, if $\pi_{k+1}(v*) < 0$, by continuity of $\pi_{k+1}(x)$ there exists some point $x^*\in [h_k, v^*]$ that yields the following $(k+1)$-pseudo-equilibrium

$$\Theta^{k+1}(\theta_{k+2}, ..., \theta_L) = \left(\theta_1^k(x^*, \theta_{k+2}, ..., \theta_L), ..., \theta_k^k(x^*, \theta_{k+2}, ..., \theta_L), x^*\right)$$

Notice that this pseudo-equilibrium meets (2). The reason is that for any $x^*\in [h_k, v^*]$, we have for sure $\theta_k^k(x, \theta_{k+2}, ..., \theta_L) \leq x$, due to the definition of h_k and the fact that $\theta_k^k(v^*, \theta_{k+2}, ..., \theta_L) \leq v^*$.

On the other hand, if $\pi_{k+1}(v*) \leq 0$ we already have a $(k+1)$-pseudo-equilibrium

$$\Theta^{k+1}(\theta_{k+2}, ..., \theta_L) = \left(\theta_1^k(v^*, \theta_{k+2}, ..., \theta_L), ..., \theta_k^k(v^*, \theta_{k+2}, ..., \theta_L), v^*\right)$$

clearly satisfying (2). Finally, it only remains to generalize (2) to all possible values of θ_{k+2}, and to state that the $(k+1)$-pseudo-equilibrium line obtained is also continuous, which is always possible as argued before. □

Claim 2 *For any* $(\theta_4, ..., \theta_L) \in [c, v^*]^{L-3}$*, whenever these parameters make sense, and in any case when* $L = 3$*, there exists a continuous* (2)*-pseudo-equilibrium line such that for any* $x \in [c, v^*]$ *(just ignore "*$\theta_4, ..., \theta_L$*" in the expression below if* $L = 3$*)*

$$\theta_1^2(x, \theta_4, ..., \theta_L) \leq \theta_2^2(x, \theta_4, ..., \theta_L)$$

Proof of the claim: The proof is almost identical to the case we studied where $L = 2$, and it is redundant to show it here. □

By Claims 1 and 2, we see by an induction argument that we can find a continuous $(N-1)$-pseudo-equilibrium line $\Theta^{L-1}(x)$ that meets the condition

$$\theta_1^{L-1}(x) \leq ... \leq \theta_{L-1}^{L-1}(x), \quad \forall x \in [c, v^*]$$

It remains to check that this line intersects with L^{th}-bidder's optimal threshold strategy in a point that meets $\theta_1^* \leq ... \leq \theta_L^*$. But mimicking the proof of Claim 1 (yet ignoring its last paragraph) readily does this. ■

Proof. Proposition 1.

We can assume that objects are ordered, so that we are analyzing a sequence of games, starting from a two-object one, then adding one object and playing the game again, and so on. I then proceed through six steps:

1) $\frac{r_{ih}^N}{N} \xrightarrow{prob} F(v_{ih})$, or equivalently, for any $\gamma, \eta > 0, \exists \bar{N}_0^h(\gamma, \eta) : \forall N \geq \bar{N}_0^h(\gamma, \eta)$,

$$\Pr\left(\left|F^{-1}\left(\frac{r_{ih}^N}{N}\right) - v_{ih}\right| > \gamma \text{ for some } i \in \{1, ..., L\}\right) < \eta$$

[5]

2) Define the set $W_h^N \equiv \underset{i\in\{1,...,L\}}{\arg\max} \frac{r_{ih}^N}{N}$, and WLOG choose $i_h^N \equiv \min W_h^N$.[6] I claim that for low enough values γ, η, if ranking values are revealed truthfully, for N big enough we have a cut-off equilibrium Θ_h^{*N} for object h (out of N) in which, with probability higher than $1-\eta$, only bidder i_h^N (if any, depending on whether or not $v_{i_h^N h} > c$) takes part in h^{th}-object auction.

I prove that. Recall that $\frac{r_{i_h^N h}^N}{N} \geq \frac{r_{jh}^N}{N}, \forall j \neq i_h^N$. Let θ_{jh}^N be bidder j's threshold strategy in the auction for object h (out of N objects). Assume $c \leq \theta_{i_h^N h}^N < F^{-1}\left(\frac{r_{i_h^N h}^N}{N}\right)$, $l \neq i_h^N$, and $\theta_{jh}^N = v^* \forall j \notin \{i_h^N, l\}$. Then, the expression

$$\int_0^{F^{-1}\left(\frac{r_{lh}^N}{N}\right)} \Pr\left(x > I\{v_{i_h^N h} > \theta_{i_h^N h}^N\} \cdot v_{i_h^N h} \;\middle|\; \left\{\frac{r_{ih}^N}{N}\right\}_{i\in\{1,...,L\}}\right) dx - c$$

is negative if γ, η are low enough, for N big enough, given the result in part 1) of the proof, even if $\frac{r_{lh}^N}{N} = \frac{r_{i_h^N h}^N}{N}$ (due to the participation cost). Therefore, $\forall l \neq i_h^N$, and

[5]WLOG we can define $F^{-1}(x) \equiv \begin{cases} \max\{v \in [0, v^*] : F(v) = x\}, \text{ if } x < 1 \\ \min\{v \in [0, v^*] : F(v) = 1\}, \text{ if } x = 1 \end{cases}$

[6]It shall be noticed that my selection simplifies notation, but any other selection criterion could work here, as long as it is identical among objects. For instance, the proof still works if a fair lottery is undertaken among any element of W_h^N, such that the winner is my selected i_h^N.

for N big enough, some $\theta_{lh}^{N} \geq F^{-1}\left(\frac{r_{lh}^{N}}{N}\right) + \alpha_{lh}^{N}$ is l's best response to some $\theta_{i_h^N h}^{N} \leq F^{-1}\left(\frac{r_{i_h^N h}^{N}}{N}\right) - \alpha_{i_h^N h}^{N} > c$, being $(\alpha_{1h}, ..., \alpha_{Lh}) \gg 0$. This holds regardless what cut-off any $j \notin \{l, i_h\}$ chooses. Consequently, for some $\beta_{i_h^N h}^{N} \geq 0$ small enough, for γ, η small enough and for N big enough, bidder i_h^N´s best response to $\left\{F^{-1}\left(\frac{r_{lh}^{N}}{N}\right) + \alpha_{lh}^{N}\right\}_{l \neq i_h^N}$ is equal to $c + \beta_{i_h^N h}^{N} \leq F^{-1}\left(\frac{r_{i_h^N h}^{N}}{N}\right) - \alpha_{i_h^N h}^{N}$ whenever $\theta_{lh}^{N} > F^{-1}\left(\frac{r_{lh}^{N}}{N}\right)$ $\forall l \neq i_h^N$. Then, the set

$$M_h^N \equiv \left[c,\, c + \beta_{i_h^N h}^{N}\right] \times \prod_{l \neq i_h} \left[\min\left\{F^{-1}\left(\frac{r_{lh}^{N}}{N}\right) + \alpha_{lh}, v^*\right\},\, v^*\right]$$

is closed under best responses. Hence, for N big enough, there exists at least one cut-off equilibrium Θ_h^{*N} in this set M_h^N. Particularly, notice that $\beta_{i_h^N h}^{N}$ becomes arbitrarily close to zero as N increases, and therefore $\theta_{i_h^N h}^{*N}$ asymptotically approaches c. Also, any other bidder l's equilibrium strategy is not below some value λ_{lh}^{N} , which in turn converges to $\min\left\{F^{-1}\left(\frac{r_{i_h^N h}^{N}}{N}\right) + c, v^*\right\}$. Such equilibrium results with probability higher than $1 - \eta$ in an outcome where nobody but i_h (if any) takes part in the auction.

The proof of the claim above is completed by considering cases where $F^{-1}\left(\frac{r_{i_h^N h}^{N}}{N}\right) \leq c$. We use Lemma 1 to select an equilibrium Θ_h^{*N} such that $c \leq \theta_{i_h^N h}^{*N} \leq \theta_{l1}^{*N}$, $\forall l \neq i_h^N$. In any case, it must happen $\theta_{l1}^{*N} > c$, $\forall l \neq i_h^N$, since no two bidders can both have the same cut-off equilibrium c. It is clear that i_h^N th-bidder's equilibrium strategy is arbitrarily close to c, as N grows large. Consequently, as N grows large, the probability of the desired result becomes higher than $1 - \eta$.

Denote $\Pr(A_h^{LN} \mid \left\{r_{ih}^{N}\right\}_{i \in \{1,...,L\}})$ to the probability that, in the cut-off equilibrium, no bidder $j \neq i_h^N$ takes part in the auction for object h out of N, with L potential bidders and the available information about ranking values. Notice that I have proved that there exists $\bar{N}_1^h$ such that, $\forall N \geq \bar{N}_1^h$, $\Pr(A_h^{LN} \mid \left\{r_{ih}^{N}\right\}_{i \in \{1,...,L\}}) > 1 - \eta$.

3) Set $\bar{N}_1(N) \geq \max_{h \in \{1,...,N\}} \bar{N}_1^h$ (we eliminate the arguments of $\bar{N}_1^h$ in order to avoid misunderstandings). For N big enough (say $N \geq \bar{N}_1$), we get $\bar{N}_1(N) \leq N$. Thus, for any $N \geq \bar{N}_1$, we have an equilibrium for each object h which satisfies the result of

part 3) of the proof.

4) Let, WLOG, $v_{i1} > v_{i2}$. Bidder i is to reveal either one of these two ranking profiles: $r_i^N \equiv (r_{i1}^N, ..., r_{iN}^N)$ such that $r_{i1}^N > r_{i2}^N$ or an alternative ranking profile $\tilde{r}_i^N \equiv (\tilde{r}_{i1}^N, ..., \tilde{r}_{iN}^N)$. This latter ranking profile is identical to the former one except for the fact that it switches positions between objects 1 and 2, so $\tilde{r}_{i1}^N = r_{i2}^N$ and $\tilde{r}_{i2}^N = r_{i1}^N$. I define the following ranking set: $r_{-ih}^N \equiv \{r_{jh}^N : j \neq i\}$. Assume for now that $r_{-i1}^N = r_{-i2}^N \equiv r_{-i}^N$. Then, in the cut-off equilibrium system described in part 3), respective expected profits regarding objects 1 and 2 are:

$$\pi_{i1}^N(r_{i1}^N, r_{-i}^N) = I\{i = \min W_1^N \mid r_{i1}^N, r_{-i}^N\}\cdot$$
$$\cdot \left[\Pr(A_1^{LN} \mid r_{i1}^N, r_{-i}^N) \cdot I\{v_{i1} > \theta_{i1}^{*N}\} \cdot (v_{i1} - c) + \left(1 - \Pr(A_1^{LN} \mid r_{i1}^N, r_{-i}^N)\right) \cdot g_{i1}^N(\cdot)\right] +$$
$$+ I\{i \neq \min W_1^N \mid r_{i1}^N, r_{-i}^N\} \cdot \left(1 - \Pr(A_1^{LN} \mid r_{i1}^N, r_{-i}^N)\right) \cdot d_{i1}^N(\cdot) \quad \xrightarrow{N \to \infty}$$
$$\xrightarrow{N \to \infty} \quad I\{i = \min W_1^N \mid r_{i1}^N, r_{-i}^N\} \cdot I\{v_{i1} > c\} \cdot (v_{i1} - c)$$
$$\pi_{i1}^N(\tilde{r}_{i1}^N, r_{-i}^N) = I\{i = \min W_1^N \mid r_{i2}^N, r_{-i}^N\}\cdot$$
$$\cdot \left[\Pr(A_1^{LN} \mid r_{i2}^N, r_{-i}^N) \cdot I\{v_{i1} > \tilde{\theta}_{i1}^{*N}\} \cdot (v_{i1} - c) + \left(1 - \Pr(A_1^{LN} \mid r_{i2}^N, r_{-i}^N)\right) \cdot \tilde{g}_{i1}^N(\cdot)\right] +$$
$$+ I\{i \neq \min W_1^N \mid r_{i2}^N, r_{-i}^N\} \cdot \left(1 - \Pr(A_1^{LN} \mid r_{i2}^N, r_{-i}^N)\right) \cdot \tilde{d}_{i1}^N(\cdot) \quad \xrightarrow{N \to \infty}$$
$$\xrightarrow{N \to \infty} \quad I\{i = \min W_1^N \mid r_{i2}^N, r_{-i}^N\} \cdot I\{v_{i1} > c\} \cdot (v_{i1} - c)$$
$$\pi_{i2}^N(r_{i2}^N, r_{-i}^N) = I\{i = \min W_2^N \mid r_{i2}^N, r_{-i}^N\}\cdot$$
$$\cdot \left[\Pr(A_2^{LN} \mid r_{i2}^N, r_{-i}^N) \cdot I\{v_{i2} > \theta_{i2}^{*N}\} \cdot (v_{i2} - c) + \left(1 - \Pr(A_2^{LN} \mid r_{i2}^N, r_{-i}^N)\right) \cdot g_{i2}^N(\cdot)\right] +$$
$$+ I\{i \neq \min W_2^N \mid r_{i2}^N, r_{-i}^N\} \cdot \left(1 - \Pr(A_2^{LN} \mid r_{i2}^N, r_{-i}^N)\right) \cdot d_{i2}^N(\cdot) \quad \xrightarrow{N \to \infty}$$
$$\xrightarrow{N \to \infty} \quad I\{i = \min W_2^N \mid r_{i2}^N, r_{-i}^N\} \cdot I\{v_{i2} > c\} \cdot (v_{i2} - c)$$
$$\pi_{i2}^N(\tilde{r}_{i2}^N, r_{-i}^N) = I\{i = \min W_2^N \mid r_{i1}^N, r_{-i}^N\}\cdot$$
$$\cdot \left[\Pr(A_2^{LN} \mid r_{i1}^N, r_{-i}^N) \cdot I\{v_{i2} > \tilde{\theta}_{i2}^{*N}\} \cdot (v_{i2} - c) + \left(1 - \Pr(A_2^{LN} \mid r_{i1}^N, r_{-i}^N)\right) \cdot \tilde{g}_{i2}^N(\cdot)\right] +$$
$$+ I\{i \neq \min W_2^N \mid r_{i1}^N, r_{-i}^N\} \cdot \left(1 - \Pr(A_2^{LN} \mid r_{i1}^N, r_{-i}^N)\right) \cdot \tilde{d}_{i2}^N(\cdot) \quad \xrightarrow{N \to \infty}$$
$$\xrightarrow{N \to \infty} \quad I\{i = \min W_2^N \mid r_{i1}^N, r_{-i}^N\} \cdot I\{v_{i2} > c\} \cdot (v_{i2} - c)$$

, where $g, d, \tilde{g}, \tilde{d}$ are nonnegative functions of several arguments. Arguments of the profit functions in parentheses constitute the object-specific information coming from the messages that are sent. $\theta_{ih}^{*N}, \tilde{\theta}_{ih}^{*N}$ are equilibrium cut-off strategies derived from the information that is revealed. Notice that $\Pr(A_1^{LN} \mid r_{i1}^N, r_{-i}^N) =$

$\Pr(A_2^{LN} \mid r_{i1}^N, r_{-i}^N)$, $\Pr(A_1^{LN} \mid r_{i2}^N, r_{-i}^N) = \Pr(A_2^{LN} \mid r_{i2}^N, r_{-i}^N)$, $\theta_{i1}^{*N} = \tilde{\theta}_{i2}^{*N}$ and $\tilde{\theta}_{i1}^{*N} = \theta_{i2}^{*N}$.

For any case where $i = \min W_h^N$ given what is revealed, θ_{ih}^{*N} (or $\tilde{\theta}_{ih}^{*N}$) tends to be arbitrarily close to c, since $\beta_{i_h^N h}^N$ tends to be arbitrarily close to zero, for N large enough. Together with the fact that $\left(1 - \Pr(A_h^{LN} \mid \cdot)\right)$ gets arbitrarily close to zero for η low enough and $N \geq \tilde{N}$ (being $\tilde{N}$ large enough), the former four equations prove that, asymptotically:

$$\pi_{i1}^N(r_{i1}^N, r_{-i}^N) - \pi_{i1}^N(r_{i2}^N, r_{-i}^N) - \left[\pi_{i2}^N(r_{i1}^N, r_{-i}^N) - \pi_{i2}^N(r_{i1}^N, r_{-i}^N)\right] \geq 0$$

5) Since bidder i does not know what messages he is going to receive from others when he decides what message to send, he has to calculate expected profits considering any (equally likely) combination of others' ranking profiles. I conclude that:

$$\pi_{i1}^N(r_i^N) - \pi_{i1}^N(\tilde{r}_i^N) =$$

$$= \frac{1}{\#\Omega_{-i}^N} \sum_{(r_{-i1}^N, \ldots, r_{-iN}^N) \in \Omega_{-i}^N} \left[\pi_{i1}^N(r_{i1}^N, r_{-i1}^N) - \pi_{i1}^N(r_{i2}^N, r_{-i1}^N) - \left[\pi_{i2}^N(r_{i1}^N, r_{-i2}^N) - \pi_{i2}^N(r_{i1}^N, r_{-i2}^N)\right]\right] \propto$$

$$\propto \sum_{r_{-i1}^N \in \Omega_{-i1}^N} \left[\pi_{i1}^N(r_{i1}^N, r_{-i1}^N) - \pi_{i1}^N(r_{i2}^N, r_{-i1}^N)\right] - \sum_{r_{-i2}^N \in \Omega_{-i2}^N} \left[\pi_{i2}^N(r_{i1}^N, r_{-i2}^N) - \pi_{i2}^N(r_{i1}^N, r_{-i2}^N)\right] \geq 0$$

Here, profit functions are taken in unconditional expectation form given that bidder i does not know what messages he is going to receive. Ω_{-i}^N is the set of all possible r_{-ih}^N-profiles. # is the number of elements in the set. Ω_{-ih}^N follows a similar definition, yet it just refers to the specific object h. $\propto$ means positive proportionality.

The last inequality follows from two facts: $\Omega_{-i1}^N = \Omega_{-i2}^N$ and the result from part 4). This proves the (asymptotic) existence of a truthful revelation equilibrium, since the potential bidder is always willing to give a higher ranking value to the object he prefers. Notice that this equilibrium only requires a limited amount of common knowledge: namely, that everybody knows that everybody is telling the truth, and that everybody knows that everybody knows that everybody is telling the truth.

6) I now prove efficiency. Concentrate on $N \geq \tilde{N}$. Under the equilibrium context seen so far, observe that

$$\forall \varphi_h > 0, \exists \bar{N}_2^h : \forall N \geq \bar{N}_2^h, \Pr\left(\frac{r_{ih}^N}{N} \neq \frac{r_{jh}^N}{N}, \forall i \neq j \in \{1, ..., L\}\right) > 1 - \varphi_h$$

, so we concentrate on the states where object ranking values never coincide among bidders. In these states, for $N \geq \bar{N}_2^h$, we get $\frac{r_{i_h h}^N}{N} > \frac{r_{jh}^N}{N}, \forall j \neq i_h$, for some $i_h \in \{1, ..., L\}$. Given that, it can be stated that $\forall \delta_h > 0, \exists \bar{N}_3^h \geq \bar{N}_2^h : \forall N \geq \bar{N}_3^h$,

$$\Pr\left(v_{i_h h} > v_{jh}, \forall j \neq i_h \;\middle|\; \left\{\frac{r_{ih}^N}{N}\right\}_{i \in \{1,...,L\}}, \frac{r_{ih}^N}{N} \neq \frac{r_{jh}^N}{N}, \forall i \neq j \in \{1, ..., L\}\right) > 1 - \delta_h$$

Therefore, for any object h, the probability that the object taker is the person that gives to it the highest value is higher than $1 - \delta_h$. Then, with a probability higher than

$$1 - \varepsilon \equiv \prod_{h \in O} [(1 - \eta)(1 - \varphi_h)(1 - \delta_h)]$$

, we obtain the desired result, for $N \geq \bar{N} \equiv \max\left\{\max_{h \in O} \bar{N}_3^h, \tilde{N}\right\}$, hence completing the proof. Observe that in the cut-off equilibria we are dealing with throughout the proof, the sellers of objects belonging to O finally face a unique buyer (if any) in each auction, and therefore they obtain zero profits, with a probability bigger than $(1 - \eta)^{\#O} > 1 - \varepsilon$. ■

Proof. Proposition 2.

We proceed again through several steps:

1) Let $\bar{W}_h^{LN}$ be the set of bidders that prefer object h among all N objects. $\Pr\left(\bar{W}_h^{LN} \neq \varnothing\right) = 1 - \left(\frac{N-1}{N}\right)^L$, so we concentrate on these states in parts 2) and 3) of the proof.

2) Choose $\bar{N}$ large enough to make the first order statistic distribution $F^{(\bar{N})}(\cdot)$ accomplish with the condition of Lemma 2. We can always find a value that makes $F^{(\bar{N})}(\cdot) = F(\cdot)^{\bar{N}}$ strong enough to first order dominate any distribution function already accomplishing the aforementioned condition. Thus, $F^{(\bar{N})}(\cdot)$ will also satisfy that condition, and so would $F^{(N)}(\cdot)$ for any $N \geq \bar{N}$.

3) I propose the following cut-off equilibrium allocation, in these states. Let S_h^{LN} be the set of bidders that claim to give the highest valuation to object h among all N objects. For this object, equilibrium cut-off strategies are:

$$\theta_{ih}^{*LN} = \begin{cases} c & if \quad i = \min S_h^{LN} \\ v^* & \quad otherwise \end{cases}$$

By Lemma 2, these strategies constitute a cut-off equilibrium.[7]

In cases where $S_h^{LN} = \emptyset$, any possible cut-off equilibrium Θ_G^{*L} can be played, under the condition of being object-invariant.[8]

4) We focus on the case where $v_{ih} \geq v_{ik} \, \forall k \in \{1, ..., N\}$, and bidder i is to reveal either a most-preferred object $n_i^N = h$ or an alternative $\tilde{n}_i^N = m \neq h$. Then, given that other bidders' most-preferred objects are not yet revealed, expected profits are calculated as:

$$\pi_{ih}^{LN}(n_i^N) = \Pr\left(\theta_{ih}^{*LN} = c \mid L, i, n_i^N, \right) \cdot I\{v_{ih} > c\} \cdot (v_{ih} - c)$$

$$\pi_{im}^{LN}(n_i^N) = \Pr(\bar{W}_m^{LN} \setminus \{i\} = \emptyset) \cdot I\{v_{im} > \theta_{iG}^{*L}\} \cdot \int_{\theta_{iG}^{*L}}^{v_{im}} \prod_{j \neq i} G\left(\max\{\theta_{jG}^{*L}, x\}\right) dx$$

$$\pi_{ih}^{LN}(\tilde{n}_i^N) = \Pr(\bar{W}_h^{LN} \setminus \{i\} = \emptyset) \cdot I(v_{ih} > \theta_{iG}^{*L}\} \cdot \int_{\theta_{iG}^{*L}}^{v_{ih}} \prod_{j \neq i} G\left(\max\{\theta_{jG}^{*L}, x\}\right) dx$$

$$\pi_{im}^{LN}(\tilde{n}_i^N) = \Pr\left(\theta_{im}^{*LN} = c \mid L, i, \tilde{n}_i^N\right) \cdot I\{v_{im} > c\} \cdot (v_{im} - c)$$

, where G is the distribution function for any not-most-preferred object. Since

[7]It shall be noticed again that the equilibrium I select simplifies notation, but any other selection criterion could work here, as long as it is identical among objects. For instance, the proof still works if a fair lottery is undertaken among any element of S_h^{LN}, such that the winner is the only participant (if any) in the equilibrium.

[8]Once again, a lottery could instead be undertaken among a set of cut-off equilibria in order to select the one that is to be played, and the proof would still hold, provided the lottery is the same among objects.

$$\begin{aligned}\Pr\left(\theta_{ih}^{*LN}=c \mid L,i,n_i^N\right) &= \Pr\left(\theta_{im}^{*LN}=c \mid L,i,\tilde{n}_i^N\right)=\left(\frac{N-1}{N}\right)^{i-1}\geq \\ &\geq \left(\frac{N-1}{N}\right)^{L-1}=\Pr(\bar{W}_h^{LN}\setminus\{i\}=\emptyset)=\Pr(\bar{W}_m^{LN}\setminus\{i\}=\emptyset)\end{aligned}$$

, and since it is always the case that

$$I\{v_{ih}>c\}\cdot(v_{ih}-c)-I\{v_{im}>c\}\cdot(v_{im}-c)\geq$$
$$\geq I(v_{ih}>\theta_{iG}^{*L}\}\cdot\int_{\theta_{iG}^{*L}}^{v_{ih}}\prod_{j\neq i}G\left(\max\{\theta_{jG}^{*L},x\}\right)dx-I\{v_{im}>\theta_{iG}^{*L}\}\cdot\int_{\theta_{iG}^{*L}}^{v_{im}}\prod_{j\neq i}G\left(\max\{\theta_{jG}^{*L},x\}\right)dx$$

, one concludes that $\pi_i^{LN}(n_i^N)\geq\pi_i^{LN}(\tilde{n}_i^N)$. A truthful revelation of the most preferred object constitutes a weakly dominant strategy given that bidder believes what the others reveal, yielding the desired equilibrium. Under this equilibrium any isolated seller will obtain zero profits with probability not lower than $1-\left(\frac{N-1}{N}\right)^L$. Notice again the few assumptions regarding common knowledge that this equilibrium imposes. ■

Proof. **Lemma 3.**

$$\begin{aligned}
\bar{F}_{NL}(x) &= \sum_{k=1}^{N}\left[\Pr\left(\max_{i\in\{1,\dots,L\}} r_{ih}^{N} \le k\right) - \Pr\left(\max_{i\in\{1,\dots,L\}} r_{ih}^{N} \le k-1\right)\right] F^{(k)}(x) = \\
&= \sum_{k=1}^{N}\left[\left(\frac{k}{N}\right)^{L} - \left(\frac{k-1}{N}\right)^{L}\right] \sum_{j=k}^{N} \binom{N}{j} F(x)^{j}\left[1-F(x)\right]^{N-j} = \\
&= \dots = \sum_{k=1}^{N}\left(\frac{k}{N}\right)^{L}\binom{N}{k} F(x)^{k}\left[1-F(x)\right]^{N-k} = \\
&= \sum_{k=1}^{N}\left(\frac{k}{N}\right)^{L}\binom{N}{k} F(x)^{k}\sum_{j=k}^{N}(-1)^{j-k}\binom{N-k}{j-k} F(x)^{j-k} = \\
&= \sum_{k=1}^{N}\sum_{j=k}^{N}\left(\frac{k}{N}\right)^{L}\binom{N}{k}\binom{N-k}{j-k}(-1)^{k}\left[-F(x)\right]^{j} = \\
&= \sum_{k=1}^{N}\sum_{j=k}^{N}\left(\frac{k}{N}\right)^{L}\binom{N}{j}\binom{j}{k}(-1)^{k}\left[-F(x)\right]^{j} = \dots = \\
&= \sum_{h=1}^{N}\left[-F(x)\right]^{h}\sum_{i=1}^{h}\binom{N}{h}\binom{h}{i}\left(\frac{i}{N}\right)^{L}(-1)^{i} = \\
&= \sum_{h=1}^{N} F(x)^{h}\frac{(-1)^{h}N!}{N^{L}(N-h)!}\sum_{i=1}^{h}\frac{(-1)^{i}i^{L}}{i!(h-i)!} \equiv \sum_{h=1}^{N} F(x)^{h}\cdot b_{h}(N,L)
\end{aligned}$$

First of all, it shall be noticed that $\sum_{h=1}^{N} b_h(N,L) = 1$. Otherwise, $\bar{F}_{NL}(v^*) \neq 1$. Next, I prove that $b_h(N,L) = 0$, $\forall h > L$. Define the sequence of functions:

$$g_0(x) \equiv (x-1)^h = (-1)^h h! \sum_{i=0}^{h} \frac{(-1)^i}{i!(h-i)!} x^i$$

$$g_n(x) \equiv x g'_{n-1}(x) = (-1)^h h! \sum_{i=1}^{h} \frac{(-1)^i}{i!(h-i)!} i^n x^i,\ n = 1, 2, \dots$$

Since for any $n < h$ we get $g_n(1) = 0$, it is clear that $\frac{g_L(1)}{(-1)^h h!} = \sum_{i=1}^{h} \frac{(-1)^i i^L}{i!(h-i)!} = 0$, proving what is claimed. Now, notice that $g_n(1) \geq 0$ for any $n = 1, 2, ...$ and for any fixed h. This implies that $b_h(N, L) \geq 0$ for any h. Therefore, $\bar{F}_{NL}(x)$ is a weighted average among $F(x), F(x)^2, ..., F(x)^L$. This proves the inequality of the Lemma. Finally, I prove that $b_h(N, L) \xrightarrow{N \to \infty} 0$, $\forall h < L$. This is straightforward, since $\lim_{N\to\infty} \frac{(-1)^h N!}{N^L (N-h)!} = 0$, $\forall h < L$. Hence, $\lim_{N\to\infty} b_L(N, L) = 1$, proving the desired limit result already stated in the Lemma. ■