Cardinal Bayesian Allocation Mechanisms without Transfers

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Abstract

Nontransfer allocation mechanisms are used in many contexts, from collusion to children-to-school assignment. This paper works on cardinal mechanisms, which elicit and use information on preference intensities, instead of the widely analyzed ordinal allocation mechanisms, where only ordinal preferences are used. I study the allocation of two objects under Bayesian incentive compatibility (BIC), symmetric agents and independent private valuations. I obtain two main results. One is the solution to the optimal (ex ante utilitarian welfare maximizing) cardinal mechanism problem, for a wide family of prior valuation distributions. The second one is a simple cardinal mechanism that improves over ordinal mechanisms and does not require an exact knowledge of priors, in two-agent cases. Additionally, if the problem is constrained to ensure exactly one object per agent, then BIC implies ordinality, and the optimal ordinal mechanism is optimal among cardinal mechanisms.

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1 Introduction

Allocating resources or objects to agents is one of the main issues in Economics. When money transfers are feasible, the mechanism designer or planner can readily create incentives so as to elicit

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truthful and sufficient information about agents' preferences, which would allow her to allocate resources efficiently. The way to obtain the desired results is commonly known as the Vickrey-Clarke-Groves mechanism after its pioneers (Vickrey, 1961; Clarke, 1971; Groves, 1973). Recent work has analyzed the validity of the VCG mechanism under more general assumptions such as interdependent valuations (McLean and Postlewaite, 2006).

Solving the problem of allocating several objects to several agents with no use of transfers is not as straightforward, but a solution is useful in many real scenarios:

1) In auctions and in other market institutions, *collusion* is frequently characterized by the absence of transfers among bidders/firms due to the risk of detection. If several objects (or markets) are at stake, a nontransfer allocation mechanism might be played by colluding bidders/firms (Campbell, 1998; Pesendorfer, 2000) that makes use of agents' preferences over objects (markets).

2) Also inside the firm, the *allocation of tasks* may not be always accompanied by monetary incentives. A team of workers may try to allocate tasks so as to minimize the work-driven disutility of employees, while ensuring that the tasks are done. Task allocation in households is also an interesting issue where monetary transfers are often constrained.

3) In *regulatory issues and public supply*, a price mechanism is sometimes not permitted, or is traditionally rejected. For instance, slot assignment in airport terminals does not involve monetary transfers in many cases[1]. The allocation of children to public schools does not use a market clearing price system. Other examples include the allocation of patients to medical doctors, and housing allocation in colleges.

The literature has extensively analyzed nontransfer allocation mechanisms that rely on agents' *ordinal preferences* (ordinal mechanisms for short). This has typically been the case in one-sided and two-sided matching problems, in the social choice literature and in allocation games where more than one object might be allocated to a single agent (Gibbard, 1973; Satterthwaite, 1975; Olson, 1991; Pápai, 2000; Pápai, 2001; Ehlers and Klaus, 2003).

The present paper analyzes instead so-called cardinal mechanisms, where the *intensity of preferences* is taken into account. Cardinal mechanisms increase welfare with respect to ordinal mechanisms, since more information on agents' preferences is utilized. In the paper, I study the allocation of two objects under Bayesian incentive compatibility (BIC), symmetric agents and independent private valuations. I present two main results. First, I derive a method to obtain the optimal (ex ante utilitarian welfare maximizing) cardinal mechanism. Second, I devise a sub-optimal cardinal

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[1] That is true for airports in the European Union and for noncongested airports in the United States.

mechanism that requires little amount of information about the priors in order to be implemented, in two-agent cases.

Requiring strong incentive compatibility conditions such as strategy-proofness leads to a natural focus on ordinal mechanisms. Linked to the strategy-proofness concept is the one of implementation in (weakly) dominant strategies. Olson (1991) proves that if a social allocation (or choice) function is truthfully nontransfer-implementable in dominant strategies, then this function must ignore the cardinality of preferences.[2] Some authors have studied strategy-proof social choice implementation with cardinal preferences, and have concluded that strategy-proofness and unanimity must lead to (weak) random dictatorships (Hylland, 1980; Nandeibam, 2004; Dutta, Peters and Sen, 2007). This is closely related to well-known results based on ordinal preferences, so the use of cardinal preferences do not add much as long as strategy-proofness is required.[3]

Therefore, if one wants to study cardinal mechanisms as a source of welfare gain with respect to ordinal mechanisms, the incentive compatibility concept must be less strict. A natural, less restrictive option is to require Bayesian (or interim) incentive compatibility (BIC). Following this concept, each agent optimally reveals her true preferences given her beliefs about others' preferences, and assuming the other agents truthfully reveal theirs.

The so-called *ranking revelation mechanism* (RRM) is an appealing ordinal nontransfer allocation mechanism benchmark. In the RRM, each agent ranks all objects from her most preferred to her least preferred, and the planner allocates each object to the agent that has given it the highest rank among all agents, with ties solved by even randomization. This mechanism is interim-efficient among BIC nontransfer ordinal mechanisms if agents' valuations are i.i.d. (Kwasnica, 2002). Asymptotically, as the number of objects grows large, the RRM approaches full efficiency (Campbell, 1998; Pesendorfer, 2000). In particular, under the i.i.d. assumption the RRM maximizes (among all BIC ordinal mechanisms) ex ante expected utilitarian welfare (the unweighted sum of agents' ex ante expected utilities), a welfare measurement that is used in the present paper.[4]

However, the RRM has nice properties among BIC nontransfer *ordinal mechanisms only.* It is

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[2] Olson's results are based on the assumption that each agent will be assigned a maximum of one object, an assumption that is not applied here. In addition, he assumes that an agent cannot be indifferent between two allocations that differ in the assignment the agent receives. If an agent has equal valuation for two objects, this assumption is violated.

[3] In a similar fashion, Hortalà-Vallvé (2007) shows that a decision mechanism over a set of public binary decisions must ignore the cardinality of preferences if both strategy-proofness and unanimity are to be preserved.

[4] It is easy to see that the RRM is ordinally efficient in the sense of Bogomolnaia and Moulin (2001). However, the RRM may not be envy-free.

worthwhile to investigate nontransfer mechanisms that account for the intensity of preferences, and not only about the ranking of objects, in order to make full use of the available tools in nontransfer mechanisms.

In this paper, I introduce a method to calculate the *optimal reduced-form nontransfer cardinal mechanism* (OCM). This method solves the problem of allocating two indivisible objects among several agents under the constraint that no transfers could be made. It is based on two ideas. First, allocation probabilities can WLOG depend on marginal rates of substitution between objects (or a one-to-one related variable). This reduces the dimensionality of the uncertainty problem. Second, the interim allocation probability for one object can be used as numeraire, so that the planner mimics a transfer mechanism. However, some complications have to be overcome, because this numeraire affects feasibility constraints. Fortunately, this problem is not binding for a wide family of underlying distributions of object valuations, including the uniform and the exponential. The implemented solution of the OCM problem is illustrated for two-agent cases. Numerical comparisons between the optimal mechanism and the RRM reveals substantial ex ante efficiency gains, more than 11¢ per dollar of expected object valuation.

Other sub-optimal yet simpler cardinal mechanisms could perform approximately as well as the OCM. I particularly discuss what I call *Extreme-Moderate Preference Mechanisms* (EMPM's). These mechanisms are "simple" in two respects. First, agents are only asked to declare both their ordinal preferences and whether their preference for one object over the other is "extreme" or "moderate" (where a previously fixed relative preference cutoff separates these intensities). Second, in two-agent scenarios, the mechanism designer may not need to know the underlying prior in order to implement the optimal EMPM. It just needs to be common knowledge that for any agent the valuation for one object and the valuation for the other are stochastically identical. Nothing else is has to be known about the prior distributions. Some numerical comparisons between the optimal EMPM and the RRM are also provided that yield as much welfare gain as 75 per cent of what would be gained with the OCM.

Finally, one extension of the cardinal BIC mechanism problem is analyzed, where a constraint is added by which each agent is to receive exactly one object. This constraint turns the problem into what I call the *two-school random assignment problem.* The additional constraint has dramatic consequences. I show that BIC is met if and only if an ordinal (or interim equivalent) mechanism is used and it gives each child more interim probability to be assigned to her parents' most-preferred school. The RRM turns out to be an ex ante optimal cardinal mechanism in this setup.

Several attempts to introduce cardinal mechanisms have been observed in the literature. A classic example is the *pseudomarket solution*. Hylland and Zeckhauser (1979) suggest the following idea. The mechanism designer, or planner, provides each agent with a hypothetical budget. The creation of budget sets generates a pseudomarket for objects (job positions in their study), in which agents buy units of probability to be assigned to each object. A *tatônement* process yields equilibrium prices and probabilities. The solution is naturally Pareto-efficient. The only drawback of this solution is that when the number of agents is finite, any agent might behave strategically in order to profitably alter equilibrium prices. In the corresponding direct mechanism, incentive compatibility may be violated.

Mechanisms accounting for the intensity of preferences are also studied in recent literature on *voting and political compromise* (Saari, 2000; Casella, 2005; Casella and Gelman, 2005; Börgers and Postl, 2006). Casella introduces the concept of *storable votes*. Each agent has a vote per proposal, plus a number of extra votes that can be placed as the agent wishes. In simultaneous voting, if agents are symmetric, each agent's extra votes are entirely used on the proposal for which agent's preferences are most intense. Casella and Gelman (2005) prove that this mechanism improves every agent's ex ante expected welfare relative to traditional voting, where votes are placed according to ordinal preferences only. Börgers and Postl (2006) prove that the Pareto-optimal allocation of public decisions that take into account the intensity of preferences is not BIC implementable. They numerically approach the optimal BIC mechanism.

Two recent papers are closely related to the one here presented. The first one is Abdulkadiroğlu, Che and Yasuda (2007). Applying Casella and Gelman's (2005) ideas to private good allocation, the authors suggest a student-to-school assignment algorithm, which is coined as *Choice-Augmented Deferred Acceptance* (CADA) algorithm. In it, each agent reports her ordinal preferences *plus a target* school. Slots are allocated following a Deferred Acceptance (DA) algorithm where the target serves as a tie-breaking. The target entails a cardinal preference component. If some agent prefers her second-best choice closely as much as her first choice, she may target that second-best school. It is shown that the CADA is welfare improving over the usual DA when ordinal preferences are equal among agents. Besides, the CADA is Pareto-optimal (i.e. not all possible agent types simultaneously prefer another alternative feasible mechanism) under mild conditions.

This paper differs from the one above in that is does not restrict attention to mechanisms that are constrained to ensure exactly one object (or slot) per agent. Also, it does not use the Nash equilibrium concept but the Bayesian Nash one. Contrarily to the aforementioned paper, it is not assumed here that each agent knows all other agents' preferences.

The second related paper is Abdulkadiroğlu and Loertscher (2007), based on Casella (2005). They suggest a *dynamic housing mechanism* in which housing (or some other scarce resource) has to be allocated to agents in each of two periods. Each agent knows her valuation for housing in period 1, and the valuation distribution in period 2. She either asks for allocation priority in period 1 or she opts out in favor of period 2. Housing is rationed according to the demanded priorities. This detail-free mechanism is shown to interim-dominate static mechanisms (where period-2 allocation is unrelated to period-1 allocation). Moreover, this mechanism is ex ante optimal among incentive-compatible ones, under mild assumptions.

The problem dealt with in the present paper is similar to the one above in that two objects are to be allocated, and in that the ex ante optimal mechanism is searched for. However, in the present paper the goods are simultaneously assigned, and more importantly, each agent already knows her valuations for both objects. This adds a two-dimensional uncertainty that is avoided in the Abdulkadiroğlu and Loertscher's dynamic problem.

The multi-object case is particularly complex because it introduces multidimensional information. For instance, the literature on revenue-maximizing auctions has not yet found a general solution even for the two-object auction![5] This paper precisely analyzes the two-object case. *The two-object nontransfer allocation problem is analytically tractable* and sufficiently illustrates on how to obtain information on cardinal utilities with no use of transfers.

By limiting the number of objects, I avoid the necessity of taking shortcuts. A typical example of shortcut is the use of unidimensional types to depict valuations for several objects (Krishna and Rosenthal, 1996; Levin, 1997; Menezes and Monteiro, 2003). This shortcut has its shortcomings, since valuations are forced to be perfectly correlated among objects. In the present paper, no shortcut is taken: only a vector of k elements can fully represent a vector of k valuations.

The next Section presents and solves the OCM problem. It also deals with implementation and examples. Section 3 develops and solves the EMPM problem. Section 4 analyzes the two-school random assignment problem. Section 5 concludes. Several Appendices present long proofs.

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[5] Armstrong (2000) constitutes the closest approach to that solution.

2 Optimal Cardinal Mechanism

This Sections deals with the OCM problem. The first subsection presents the problem and makes steps towards a new simplified version of it: the reduced-form allocation problem. The second subsection solves this simplified problem. The third subsection first analyzes how the reduced-form solution can be implemented, in two-agent cases. In a last subsection, I present examples showing the welfare gains given by the OCM, as compared to the RRM.

2.1 Notation and preliminary steps

Two objects are to be allocated among n agents. Each agent $i \in \{1, ..., n\}$ has a nonnegative valuation v_h^i for each object $h \in \{1, 2\}$. Let $v^i \equiv (v_1^i, v_2^i)$. Each valuation vector is independently drawn from the same atomless distribution function F, with associated density f which is assumed to be differentiable a.e.. For any values v and w, it is assumed that $f(v, w) = f(w, v)$, so the marginal distributions (denoted as $\tilde{F}$ throughout the paper) are identical.[6] It is also assumed that $(0, 0)$ is the infimum of the support of this distribution.

There are no synergies between the objects. Thus the utility an agent would enjoy from obtaining the two objects is equal to the sum of her valuations. For the moment, no restriction is placed in the number of objects an agent can obtain: zero, one or both. An alternative assumption is analyzed in Section 4.

Each agent i learns the realization of her valuation vector v^i only, following the Independent Private Valuations assumption. The planner just knows the prior distribution from which valuations are drawn.

The mechanism designer, or planner, designs a lottery mechanism to allocate the objects with the aim of maximizing the sum of agents' ex ante expected utilities. Following the Revelation Principle, the mechanism could be expressed as a probability matrix function $\tilde{q} : \mathbb{R}_+^{2n} \to [0, 1]^{2n}$ subject to the feasibility constraints $\sum_{i=1}^n \tilde{q}_h^i(v^1, ..., v^n) \leq 1$ and to incentive compatibility. The "reduced form" probabilities $\tilde{Q}_h^i(v^i) \equiv E_{-i}\tilde{q}_h^i(v^1, ..., v^n)$ [7] will be used to allow for an easier solution to the problem at hand. We define the vector $\tilde{Q}^i(v^i) \equiv (\tilde{Q}_1^i(v^i), \tilde{Q}_2^i(v^i))$. Agent i's interim expected utility from the allocation mechanism is denoted as $\tilde{U}^i(v^i) \equiv \tilde{Q}^i(v^i) \cdot v^i$. The objective function can be formulated as $\tilde{O} \equiv E \sum_{i=1}^n \tilde{U}^i(v^i) = E \sum_{i=1}^n \tilde{Q}^i(v^i) \cdot v^i$. The (Bayesian, or interim) incentive compatibility

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[6] Note that this includes i.i.d. object valuation distributions.

[7] Through this paper, $-x$ means "all elements that either do not belong to x or are not related to x".

constraints can be described as a set of inequalities $\tilde{Q}^i(v^i) \cdot v^i \geq \tilde{Q}^i(\tilde{v}^i) \cdot v^i$, $\forall i \in \{1, ..., n\}$, $\forall (v^i, \tilde{v}^i) \in \mathbb{R}^4_+$.

In order to obtain a more tractable problem, no participation constraints are considered, that is, agents' reservation utilities are set to 0. This assumption is not trivial in several real-world applications of nontransfer mechanisms. Collusion in auctions is a representative example. Obviously, the one-shot collusion mechanism has reservation utilities, namely the bidders' interim expected payoffs in the noncooperative-simultaneous-auction stage. However, under the IPV and risk-neutrality assumptions and with efficient auctions, McAfee and McMillan (1992) show that if each object valuation follows a marginal distribution with an increasing hazard rate, then the noncooperative auction mechanism is ex ante dominated by the Even Randomization Mechanism (ERM),[8] in which for each object one agent is chosen at random as the sole bidder. Clearly, the optimal nontransfer mechanism ex ante improves over the ERM. Therefore the optimal mechanism ex ante dominates the noncooperative mechanism under mild assumptions. In a (infinitely) repeated-game context in which there are two simultaneous auctions per period and each bidder only knows her valuations in the current-round auctions, repeated collusion can be supported as the equilibrium path of a subgame perfect equilibrium for high enough discount factors.

This subsection states a series of lemmas that allow for a restatement of the problem in a more useful way, the so-called reduced-form problem. The first lemma, from Pesendorfer (2000), states that any agent will be indifferent between reporting her true valuation vector and reporting another point on the same ray from the origin. The second one tells us that without loss of generality (WLOG) the planner can restrict herself to mechanisms that elicit information about rays from the origin only. The third result indicates that WLOG one can restrict attention to anonymous mechanisms. The fourth result states that, WLOG, the optimal mechanism is symmetric with respect to the 45^o line.[9] Finally, using a result from Border (1991), the allocation mechanism will be restated as a reduced-form mechanism.

Lemma 1 *(Pesendorfer, 2000). Any interim incentive compatible mechanism for this problem satisfies* $\tilde{Q}^i(v^i) \cdot v^i = \tilde{Q}^i(av^i) \cdot v^i$*, for any* i*,* v^i *and* $a > 0$*.*

Proof. By incentive compatibility, we have both $\tilde{Q}^i(v^i) \cdot v^i \geq \tilde{Q}^i(av^i) \cdot v^i$ and $\tilde{Q}^i(v^i) \cdot av^i \leq \tilde{Q}^i(av^i) \cdot av^i$. The scalar cancels out from the latter inequality, yielding the desired result. ■

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[8] This name is coined here and used throughout the paper. The authors use the term "weak cartel mechanism" instead.

[9] These four lemmas seemingly hold for the case of strategy-proof mechanisms, as Hortalà-Vallvé (2007) shows.

While the RRM only considers ordinal preferences, there is more information that could be profitably used. All points in the same ray from the origin share the same marginal rate of substitution v_2^i/v_1^i. Complete information on this variable could be then elicited and used. The RRM entails only partial revelation of this variable, namely whether the marginal rate of substitution is higher or lower than 1. This is the source of efficiency improvements for cardinal mechanisms.

In light of this lemma, it is useful to rescale valuation vectors into two new variables: a bundle valuation $B^i = v_1^i + v_2^i$, and a *relative preference parameter* $r^i = v_2^i/B^i \in [0,1]$.[10] B^i is the utility agent i would obtain if she were given both objects. r^i measures how intensely agent i prefers object 2 to object 1. Observe that $r^i \geq 1/2 \iff v_2^i \geq v_1^i$ and that $r^i = 1/2$ is equivalent to indifference between objects. There is a one-to-one relation between the relative preference parameter and the marginal rate of substitution.

The allocation mechanism can be regarded now as a function that depends on these new coordinates, namely $\hat{q} : \mathbb{R}_+^n \times [0,1]^n \rightarrow [0,1]^{2n}$. The reduced form probability $\hat{Q}_h^i$ is defined by $\hat{Q}_h^i(B^i, r^i) \equiv E_{-i}\hat{q}_h^i(B^1, ..., B^n, r^1, ..., r^n)$. Agent i's interim expected utility is denoted by $\hat{U}(B^i, r^i) \equiv \hat{Q}^i(B^i, r^i) \cdot (B^i[1-r^i], B^i r^i) = B^i[r^i \hat{Q}_2^i(B^i, r^i) + (1-r^i)\hat{Q}_1^i(B^i, r^i)]$, and the mechanism aims to maximize the sum of agents' ex ante expected utilities. The following lemma states that WLOG no useful and truthful information on bundle valuations is revealed in the optimal mechanism.

Lemma 2 *WLOG, the planner can restrict herself to allocation functions that depend only on relative preference parameters, namely $q(r^1, ..., r^n)$.*

Proof. See Appendix A. ■

Lemma 2 yields a simplification of the problem. The mechanism designer may ask for information about $(r^1, ..., r^n)$ only. Let Ψ denote the distribution function of any r^i. Its associated density function is then denoted by ψ. By the assumptions on F, the distribution of r^i is symmetric with respect to 1/2: $\psi(1-r^i) = \psi(r^i)$, $\Psi(1-r^i) = 1 - \Psi(r^i)$, and $\Psi(1/2) = 1/2$.

Since agents are ex ante identical, anonymity is a natural property to consider. Formally, a mechanism is anonymous if for any $(r^1, ..., r^n) \in [0,1]^n$ and any permutation π over an n-dimensional vector, we have $q_h(\pi(r^1, ..., r^n)) = \pi(q_h(r^1, ..., r^n))$ for any h.

Lemma 3 *Since agents are ex ante symmetric, the planner can WLOG restrict attention to anonymous mechanisms.*

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[10] Armstrong (2000) uses polar coordinates instead, when dealing with the two-object revenue-maximizing auction design problem. His approach and the one taken here are equivalent.

Proof. See Appendix A. ■

From this result, one can observe that $Q^i(r) = Q^j(r) \equiv Q(r)$, $\forall i, j \in \{1, ..., n\}$, $\forall r \in [0,1]$. Thus, superscripts in reduced form probabilities are dropped hereafter.

The following lemma states that the mechanism designer can restrict herself to allocation functions that are symmetric with respect to the 45^o line. This will be useful because it implies that optimal probabilities will only have to be calculated for r's either above or below 1/2.

Lemma 4 *Given the assumptions on* F*, the planner can WLOG restrict herself to allocation functions satisfying* $q_h^i(r^1, ..., r^n) = q_{-h}^i(1-r^1, ..., 1-r^n)$, $\forall i \in \{1, ..., n\}$, $\forall h \in \{1, 2\}$, *a.e., and therefore* $Q_h(r) = Q_{-h}(1-r)$ *a.e..*

Proof. See Appendix A. ■

The representative agent's interim expected utility is defined as $U(B, r) \equiv B[rQ_2(r) + (1-r)Q_1(r)]$. The planner's objective function can be expressed as $\int_0^1 \psi(r)E(B\,|r\,)u(r)dr$, where $u(r) \equiv rQ_2(r) + (1-r)Q_1(r)$ is the "normalized interim expected utility" (the attained utility if bundle valuation is normalized to 1). Due to the assumptions on F and to the symmetry of the solution, it can be seen that $E(B\,|r\,) = E(B\,|1-r\,)$ and $u(r) = u(1-r)$, for any $r \in [0,1]$. Additionally, for any such r it was already known that $\psi(r) = \psi(1-r)$. Under these circumstances, it is enough to maximize the following function:

$$O \equiv \int_{1/2}^{1} \psi(r)E(B\,|r\,)u(r)dr$$

and so the optimal reduced form probabilities need to be calculated only for $r \geq 1/2$.

Incentive compatibility is simplified to a set of constraints $rQ_2(r) + (1-r)Q_1(r) \geq rQ_2(\tilde{r}) + (1-\tilde{r})Q_1(r)$ for any pair $(r, \tilde{r}) \in [1/2, 1]^2$.[11] B disappears from these constraints, which is intuitive since $Q(\cdot)$ does not depend on the absolute valuation parameter.

Both the objective function and the incentive compatibility constraints depend on reduced form (or interim) probabilities, whereas any feasibility constraint is typically characterized as a relation among actual allocation probabilities. The following result, adapted from Border (1991), allows for a restatement of feasibility constraints as a set of constraints affecting reduced form probabilities:

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[11] One could wonder what happens if $r > 1/2$ and $\tilde{r} < 1/2$. Fortunately, it can be easily shown in that BIC implies $Q_2(r) \geq Q_1(r)$. With this and lemma 4, it is easy to show that BIC between r and $1-\tilde{r}$ implies BIC between r and $\tilde{r}$.

Lemma 5 *Under the assumption that $Q_2(\cdot)$ is nondecreasing (equivalently, $Q_1(\cdot)$ is nonincreasing), feasibility is equivalent to*

$$R_2(r) \equiv 1 - \Psi(r)^n - n \int_r^1 \psi(x) Q_2(x) dx \geq 0 \qquad (Q_2\text{-constraints})$$

$$\begin{aligned} R_1(r) \quad &\equiv \quad 1 - [1 - \Psi(r)]^n \\ &\quad -n \int_{1/2}^1 \psi(x) Q_2(x) dx - n \int_{1/2}^r \psi(x) Q_1(x) dx \\ &\geq \quad 0 \end{aligned} \qquad (Q_1\text{-}constraints)$$

for any $r \in [1/2, 1]$.

Proof. >From Border (1991), proposition 3.2, it is known that feasibility is equivalent to $1 - \Psi(r)^n - n \int_r^1 \psi(x) Q_2(x) dx \geq 0$ for any $r \in [0,1]$. The property is restated in such a way that it affects r's belonging to the $[1/2, 1]$ interval only. Particularly, notice that every Q_1-constraint for r is equivalent to a Q_2-constraint for $1-r$, by means of both the symmetry properties of the distribution of r and the fact that $Q_1(r) = Q_2(1-r)$ a.e.. ■

In light of the previous lemmas, the allocation problem can be solved first as a *reduced form allocation mechanism*, in which optimal probabilities are calculated at the interim level. Once solved this way, an actual mechanism may be obtained following Border (1991) again. He shows that any feasible reduced-form allocation mechanism can be arbitrarily approximated by a lottery over hierarchical mechanisms. In a *hierarchical mechanism*, for each object the planner partitions each agent's relevant type space (in the current case, the interval $[0, 1]$) into *hierarchy types.* These might be either singletons or intervals. Inside any hierarchy type, the planner makes no distinction (hierarchy type ties are solved by even randomization). Among hierarchy types, a strict priority level is set. The first hierarchy type has priority in being allocated the object. Should there not be any agent of this type, a second hierarchy type takes priority, and so on.[12]

2.2 Solution

Although the restatement of the problem as a reduced form symmetric mechanism has greatly reduced its complexity, the modified problem is still very difficult. A first obstacle to arise concerns

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[12] The hierarchical mechanism allows for the possibility that the last hierarchy type would not obtain the object even if no other hierarchy types are present. Such a variation could be regarded as a "wasting" hierarchical mechanism.

incentive compatibility. Some manipulations of incentive compatibility constraints allow for a restatement of them as a family of equations. The goal is to embed those into the model. The following result facilitates the simplification:

Lemma 6 *Bayesian (or interim) Incentive Compatibility (BIC) is equivalent to: 1)* $\frac{u(r)}{r} - \frac{u(\tilde{r})}{\tilde{r}} = \int_r^{\tilde{r}} \frac{Q_1(x)}{x^2}dx$, *for any pair* $(r, \tilde{r})$*; and 2)* $Q_1(\cdot)$ *is a nonincreasing function (or equivalently,* $Q_2(\cdot)$ *is nondecreasing).*

Proof. Standard manipulation of the $u(r)/r$ function, following theorems 1 and 2 in Milgrom and Segal (2002). ■

Some intuition is in order. Consider $u(r)/r = \frac{1-r}{r}Q_1(r) + Q_2(r)$, the "preference-weighted normalized interim expected utility". BIC constraints can be expressed in terms of $u(r)/r$ not being lower than the preference-weighted utility obtained by means of deception. This utility form resembles that of auction mechanisms, $u(v)Q(v) - T(v)$, where $u(v)$ is the utility derived from winning the object, $Q(v)$ is the interim probability of winning the object, and $T(v)$ is the interim expected payment. Therefore $Q_2(r)$ is being used as the numeraire that is transferred in exchange of more (or less) $Q_1(r)$. The nontransfer mechanism is dealt with as if it were indeed a transfer mechanism. The procedure resembles the one in Myerson (1981), yet some complications must be dealt with below.

The results of lemma 6 are incorporated into the reduced form problem. First, I use condition 1 to obtain a relation between any $Q_2(r)$, a reference value $Q_2(1)$ and all $Q_1(x)$'s for $x \in [r, 1]$:

$$Q_2(r) = Q_2(1) - Q_1(r)\frac{1-r}{r} + \int_r^1 \frac{Q_1(x)}{x^2}dx \tag{1}$$

This leads to $u(r) = rQ_2(1) + r\int_r^1 \frac{Q_1(x)}{x^2}dx$. Thus the objective function becomes

$$\begin{aligned} O &= \int_{1/2}^1 \psi(r)E(B\,|r)\left[rQ_2(1) + r\int_r^1 \frac{Q_1(x)}{x^2}dx\right]dr \\ &= Q_2(1)/2 \cdot E(v_2\,|\tilde{r} \geq 1/2) \\ &\quad + \int_{1/2}^1 Q_1(r)\frac{\Psi(r) - 1/2}{r^2}E(v_2\,|r \geq \tilde{r} \geq 1/2)dr \end{aligned} \tag{2}$$

The planner now faces the problem of maximizing this function subject to $Q_1(\cdot)$ being nondecreasing plus feasibility constraints where $Q_2(\cdot)$ has been properly substituted out. This problem is still more cumbersome than Myerson's (1981) optimal auction design problem for several reasons:

1) There is a reference value $Q_2(1)$ that has to be optimally chosen. This problem would be solved if the planner knew some r where either a Q_1-constraint or a Q_2-constraint is binding. The planner could use that to substitute $Q_2(1)$ out.

2) $Q_2(\cdot)$ is *not a proper numeraire* in that it affects the feasibility constraints. When $Q_2(\cdot)$ is substituted out in the feasibility constraints, those constraints become somewhat awkward. The derived Lagrangian cannot be maximized using the techniques developed by Myerson.

3) There are two families of feasibility constraints affecting the optimal choice of $Q(\cdot)$, instead of one.

Knowing where some constraints bind would be extremely convenient. In particular, if the Q_2-constraint at $r = 1/2$ bound, not only would problem 1 be solved, but also part of problem 2. Concretely, no Q_1-constraint would be affected by the function $Q_2(\cdot)$. Moreover, as it is shown in proposition 2, the Q_2-constraints may not bind at the optimum, hence completely solving problems 2 and 3. The optimal $Q_1(\cdot)$ could then be found by means of Myerson's method. $Q_2(1)$ could be calculated by means of the binding Q_2-constraint at $r = 1/2$. And the function $Q_2(\cdot)$ could be computed by the use of BIC.

The environment is said to satisfy the *regularity condition* (or is regular) if $E(v_2 \,|r\,)$ is nondecreasing in $r \in [0,1]$ (equivalently, $E(v_1 \,|r\,)$ is nonincreasing in $r \in [0,1]$). The density $\psi(r)$ is defined as *smooth* (or satisfies the smoothness condition) if for any triplet $(\underline{r}, \hat{r}, \bar{r})$, $1/2 \leq \underline{r} < \hat{r} < \bar{r} \leq 1$, we have a) $\frac{\Psi(\hat{r})-\Psi(\underline{r})}{\hat{r}-\underline{r}}(1-\underline{r}) \geq \frac{\Psi(\bar{r})-\Psi(\hat{r})}{\bar{r}-\hat{r}}(1-\bar{r})$ (i.e. $\psi(r)$ is not too increasing), and b) $\frac{\Psi(\hat{r})-\Psi(\underline{r})}{\hat{r}-\underline{r}}\underline{r} \leq \frac{\Psi(\bar{r})-\Psi(\hat{r})}{\bar{r}-\hat{r}}\bar{r}$ (i.e. $\psi(r)$ is not too decreasing). These conditions are explained further below. First, I present a very important technical result, namely that $R_2(1/2) = 0$.

Proposition 1 *Any solution to the allocation problem meets* $R_2(1/2) = 0$ *if both the regularity and smoothness conditions are satisfied.*

Proof. See Appendix B. ■

This technically convenient result has however a meaning itself. For some object h, if there is an agent that prefers it (i.e. her declared r is above 1/2), then no agent who prefers $-h$ (her declared r is below 1/2) may have a chance to obtain object h.

The regularity condition is quite intuitive. It simply postulates that if the relative preference for good h with respect to good $-h$ increases, the conditional expectation of v_h increases and the conditional expectation of v_{-h} decreases.

Also, a wide family of primitive valuation distributions satisify the smoothness condition. Consider the set of environments where agent's object valuations are i.i.d. (with distribution function $\tilde{F}$ and density $\tilde{f}$). Then the following marginal distributions (among others) allow $\psi(\cdot)$ to satisfy smoothness: the exponential; any gamma-type distribution with density proportional to $v^{a-1}e^{-bv}$, where $0 < a \leq 1$ and $b > 0$; the Weibull, $\tilde{f}(v) = abv^{b-1}e^{-av^b}$, with $a > 0$ and $0 < b \leq 1$; the uniform; the concave distributions $\tilde{F}(v) = v^k$ for $k \in (0, 1]$; and the distributions with linear densities $\tilde{f}(v) = a - 2(a-1)v$ supported on $[0, 1]$, where $1 \leq a \leq 2$.[13] These distributions also satisfy the regularity condition.

The following remark characterizes the smoothness condition.[14]

Remark 1 *Let $\psi'(\cdot)$ denote the derivative of $\psi(\cdot)$. Smoothness is equivalent to $\frac{-2}{r} \leq \frac{\psi'(r)}{\psi(r)} \leq \frac{2}{1-r}$ (in words, $\psi(r)r^2$ is nondecreasing and $\psi(r)(1-r)^2$ is nonincreasing) for any $r \in (1/2, 1)$.*

Basically, the density $\psi(\cdot)$ is required not to vary abruptly. In a similar fashion, much of the literature imposes conditions either on the hazard rate $\frac{\psi(\cdot)}{1-\Psi(\cdot)}$ (nondecreasing) or on the reverse hazard rate $\frac{\psi(\cdot)}{\Psi(\cdot)}$ (nonincreasing) or on both.[15] However, there is no close relation between the hazard rate condition and the condition in remark 1. The latter does not imply the former. The former does not imply the latter either, unless the additional condition $\psi(1) = 0$ (i.e. $f(0, v) = 0$) is assumed.

The rest of the paper focuses on environments satisfying the conditions of proposition 1. Using equation (1) and $R_2(1/2) = 0$,

$$Q_2(1)/2 = \frac{1-(1/2)^n}{n} + \int_{1/2}^{1} Q_1(x)\left[\psi(x)\frac{1-x}{x} - \frac{\Psi(x)-1/2}{x^2}\right]dx \tag{3}$$

and so the objective function becomes

$$\begin{aligned} O &= ctn + \int_{1/2}^{1} Q_1(r)\psi(r)\frac{1-r}{r}E(v_2\,|\tilde{r} \geq 1/2)dx \\ &\quad + \int_{1/2}^{1} Q_1(r)\frac{\Psi(r)-1/2}{r^2} \\ &\quad \cdot \left[E(v_2\,|r \geq \tilde{r} \geq 1/2) - E(v_2\,|\tilde{r} \geq 1/2)\right]dr \end{aligned} \tag{4}$$

where *ctn* denotes a constant. Due to $R_2(1/2) = 0$, the Q_1-constraints simplify to:

$$R_1(r) = (1/2)^n - [1-\Psi(r)]^n - n\int_{1/2}^{r} \psi(x)Q_1(x)dx \geq 0,\ r \in [1/2, 1] \tag{5}$$

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[13] While all these examples share the fact that marginal densities are nonincreasing, this property has not been shown to be either sufficient or necessary for smoothness. It is true though that increasing linear marginal densities or convex marginal distributions $v^k, k > 1$, induce a violation of the smoothness condition at some point.

[14] The result is obtained through three steps: 1) take logs of each inequality of that condition, 2) divide them by $\bar{r} - \underline{r}$, and 3) take limits as $\underline{r} \to \hat{r}$ and $\bar{r} \to \hat{r}$.

[15] In this context, the hazard rate is nondecreasing everywhere iff the reverse hazard rate is nonincreasing everywhere, due to the symmetry of the density with respect to 1/2.

The Q_2-constraints need not be rewritten here for the purposes of the solution. The problem of maximizing (4) subject to (5) and to nonincreasing $Q_1(\cdot)$ is analogous to the reduced-form version of an optimal auction problem. What the planner decides is how to allocate object 1 among types in $[1/2, 1]$, under the constraint that the presence of an agent whose type belongs to $[0, 1/2)$ makes it impossible for types in $[1/2, 1]$ to obtain that object.[16]

I define the following "*criterion function*":

$$\begin{aligned} C(r) &\equiv \frac{dO}{dQ_1(r)} \Big/ \psi(r) \\ &= \frac{1-r}{r} E(v_2 \,|\tilde{r} \geq 1/2\,) \\ &\quad + \frac{\Psi(r) - 1/2}{\psi(r) r^2} \left[E(v_2 \,|r \geq \tilde{r} \geq 1/2\,) - E(v_2 \,|\tilde{r} \geq 1/2\,) \right] \end{aligned}$$

Notice that $C(1/2) > C(1) = 0$, and that $C(\cdot)$ is continuous all over $[1/2, 1]$. This function is analogous to the "virtual valuation" in optimal auction design. $C(r)$ constitutes the density-weighted marginal contribution of $Q_1(r)$ to the objective function, once the corresponding incentive compatibility and binding feasibility constraints have been embedded. The idea is to allocate object 1 to the agent with the highest criterion function value, as long as this respects the constraint that $Q_1(\cdot)$ must be nonincreasing.

The final result of this subection is stated below. Subsequent corollaries point out interesting observations.

Proposition 2 *Assume that any optimal mechanism satisfies* $R_2(1/2) = 0$. *Let* $C(\cdot)$ *be such that* $E(C(\tilde{r}) \;|\tilde{r} \geq r\,) \geq 0$ *for any* $r \geq 1/2$. *Then, in order to solve for the function* Q_1, *the planner has to maximize* O *in equation (4) subject to (5) and to* $Q_1(\cdot)$ *being nonincreasing, ignoring the* Q_2*-constraints. In order to solve for the function* Q_2, *the planner has to use the already solved* Q_1^*, *jointly with equations (1) and (3).*

Proof. See Appendix C. ■

The basic idea is that by BIC, $Q_2(\cdot)$ has to be *smoother* than $Q_1(\cdot)$. For instance, if $Q_1(r)$ is differentiable, then $|Q_2'(r)| = \frac{1-r}{r} |Q_1'(r)| \leq |Q_1'(r)|$. This induces a single-crossing property: there exists a cutoff $\hat{r} \in [1/2, 1]$ such that $R_2(\cdot)$ is decreasing for $r > \hat{r}$ and increasing for $r < \hat{r}$. Hence, one just needs to find an initial value $Q_2(1) \leq 1$ such that $Q_2(1/2) \geq Q_1(1/2)$ and $R_2(1/2) = 0$. Thanks to the intermediate value theorem, the proof concludes by showing that if $Q_2(1/2) = Q_1(1/2)$ then

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[16] That is implied by $R_1(1/2) = R_2(1/2) = 0$.

$R_2(1/2) \geq 0$, and that if $Q_2(1) = 1$ then $R_2(1/2) < 0$. Q_1 can be solved independently from the Q_2-constraints, and Q_2 is calculated from $R_2(1/2) = 0$ and incentive compatibility.

The sign of $E(C(\tilde{r}) \, |\tilde{r} \geq r\,)$ determines whether is is optimal to give object 1 if all agents' types are inside $[r, 1]$. If that sign is negative, the planner may not give that object, and this possibility may make proposition 2 fail (see Appendix C).

Consequently, if the planner cannot commit to a mechanism that may not assign some of the objects to any agent, or if she is just constrained to give all the objects, then $R_1(1) = 0$ (the "final" feasibility constraint binds), and the proposition holds regardless the sign of conditionally expected $C(\cdot)$.

Corollary 1 *The Ranking Revelation Mechanism is not an optimal cardinal mechanism in any environment characterized by symmetrically distributed Private Values.*

The RRM is characterized $R_2(1/2) = 0$, since for each object it gives absolute priority to the agents who declare a preference for it over the other object. The RRM is characterized by a constant Q_1^* function over $[1/2, 1]$. A necessary condition for this to be optimal is that $C(\cdot)$ is weakly increasing over $[1/2, 1]$. But as mentioned, $C(1/2) > C(1)$, so this is impossible. This corollary is not surprising. Adding cardinality should optimally give strictly positive efficiency gains with respect to (even) the optimal ordinal mechanism.

Corollary 2 *Assume that any optimal mechanism satisifies $R_2(1/2) = 0$. Let $C(r)$ be nonincreasing in every $r \in [1/2, 1]$. Then there exists an optimal mechanism q^* such that $R_1(r) = 0$ for any $r \in [1/2, 1]$. Thus, $Q_1^*(r) = [1 - \Psi(r)]^{n-1}$, $\forall r \in [1/2, 1]$.*

The corollary follows from standard maximization given the Q_1-constraints. Both the uniform and the exponential (marginal) valuation distributions meet this condition. This will be useful when comparing the OCM with the optimal ordinal mechanism, which is the RRM.

2.3 Implementation

The implementation of the optimal reduced-form cardinal mechanism is characterized for two-agent environments, under the conditions of corollary 2. Calculating the implementation of a reduced-form mechanism with $n > 2$ is quite a cumbersome task.

Consider the set of environments where $n = 2$, $R_2(1/2) = 0$ is optimal, $C(r)$ is nonincreasing and $\psi(\cdot)$ is smooth. First of all, $Q_1^*(r) = 1 - \Psi(r)$, $\forall r \in [1/2, 1]$. From lemma 6, $Q_2^{*\prime}(r) = -\frac{1-r}{r} Q_1^{*\prime}(r) =$

$\frac{1-r}{r}\psi(r)$. So the reduced-form solution is completed by knowing the value of $Q_2^*(\cdot)$ at some point. In this case, $r = 1/2$ is chosen, although $r = 1$, as in the previous subection, would also suffice. We get

$$\begin{aligned} 0 &= R_2(1/2) \\ &= 3/4 - 2\int_{1/2}^{1} \psi(x) Q_2^*(x)dx \\ &= 3/4 - 2\int_{1/2}^{1} \psi(x) \left[Q_2^*(1/2) + \int_{1/2}^{x} \frac{1-y}{y}\psi(y)dy \right] dx \\ \Longrightarrow &\quad Q_2^*(1/2) = 3/4 - 2\int_{1/2}^{1} \frac{1-x}{x}\psi(x)[1-\Psi(x)]dx \qquad (6) \end{aligned}$$

Recalling Border (1991), the implementation can be approximated as a lottery over hierarchical mechanisms. For each $\hat{r} \in [1/2, 1)$, define the following (non-wasting)[17] hierarchical mechanism, denoted by $H(\hat{r})$. For object 2, make the following partition of the r-type space: $\{\{r\} \,|r \in [0, 1/2)\,\} \cup \{[1/2, \hat{r}]\} \cup \{(\hat{r}, 1]\}$. In this partition, higher sets of r's correspond to higher-ranked hierarchy types. For object 1, make the inverse partition: $\{[0, 1-\hat{r})\} \cup \{[1-\hat{r}, 1/2]\} \cup \{\{r\} \,|r \in (1/2, 1]\,\}$. In this partition, lower sets correspond to higher-ranked hierarchy types. Allocate objects following hierarchy type priorities, solving ties by even randomization.

I now show that the optimal mechanism can be implemented as a lottery over all these mechanisms, with a density function $\rho(\hat{r}) > 0$ for any $\hat{r} \in (1/2, 1)$ and a probability mass π in $\hat{r} = 1/2$.

Given this claim, for $r \geq 1/2$

$$\begin{aligned} Q_2^*(r) &= \int_{1/2}^{1} \rho(x) \left[I\{r \leq x\} \frac{\Psi(x) + 1/2}{2} + I\{r > x\} \frac{1 + \Psi(x)}{2} \right] dx \\ &\quad + \pi \cdot 3/4 \\ &= \pi/2 + 1/4 + \int_{1/2}^{1} \rho(x) \frac{\Psi(x)}{2} dx + \int_{1/2}^{r} (\rho(x)/4)dx \qquad (7) \end{aligned}$$

$I\{\cdot\}$ is an index function defined as usual. $\frac{\Psi(x)+1/2}{2} = 1/2 + \frac{\Psi(x)-1/2}{2}$ is the probability of winning object 2 if r belongs to the second-highest hierarchy type in $H(x)$, and $\frac{1+\Psi(x)}{2}$ is the probability of winning object 2 if r belongs to the highest one. If $H(1/2)$ is selected, then this probability reduces to 3/4. Differentiating this equation yields $\rho(r) = 4Q_2^{*\prime}(r) = 4\frac{1-r}{r}\psi(r)$.

Now if the claim is true, $1 - \pi = \int_{1/2}^{1} \rho(x)dx = \int_{1/2}^{1} 4\frac{1-x}{x}\psi(x)dx$. Supposing this value of π belongs to $[0, 1]$, it is easy to check that equation (6) and equation (7) evaluated at $r = 1/2$ are coincident, thus equation (7) is correct. Therefore, the claim holds if $\pi \in [0, 1]$. Clearly $\pi \leq 1$

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[17] See the note at the end of Section 2.

so only $\pi \geq 0$, or equivalently $\int_{1/2}^{1} \frac{1-x}{x}\psi(x)dx \leq 1/4$, has to be verified. Since $\frac{1-x}{x}$ is decreasing, the worst-case scenario to consider is the one that allocates the highest possible densities to the lowest r's. Under the smoothness condition, this is equivalent to $\psi(r)r^2$ constant for every $r \geq 1/2$, or $\psi(r) = \frac{\psi(1/2)}{4r^2}$. Since $\int_{1/2}^{1} \psi(x)dx = 1/2$, $\psi(1/2) = 2$, and $\psi(r) = \frac{1}{2r^2}$ for $r \geq 1/2$. Then the worst-case integral is $\int_{1/2}^{1} \frac{1-x}{2x^3}dx$, which evaluates to $1/4$. This proves the claim.

Proposition 3 *Let $n = 2$, $R_2(1/2) = 0$ be optimal, $C(\cdot)$ be nonincreasing and $\psi(\cdot)$ be smooth. Then an Optimal Cardinal Mechanism is implemented as follows. Let agent i report r^i and agent j report r^j. Now:*

1) If $r^i > 1/2 > r^j$, then allocate object 2 to i and object 1 to j. Do the opposite if $r^j > 1/2 > r^i$.

2) If $r^i > r^j > 1/2$, then give object 1 to j, and for object 2 allocate probabilities $q_2^i = \frac{1}{2}\left(1 + 4\int_{r^j}^{r^i} \frac{1-x}{x}\psi(x)dx\right)$ and $q_2^j = \frac{1}{2}\left(1 - 4\int_{r^j}^{r^i} \frac{1-x}{x}\psi(x)dx\right)$.

3) Proceed inversely when $r^i < r^j < 1/2$: allocate object 2 to j and give probabilities for object 1 as $q_1^i = \frac{1}{2}\left(1 + 4\int_{1-r^j}^{1-r^i} \frac{1-x}{x}\psi(x)dx\right)$ and $q_1^j = \frac{1}{2}\left(1 - 4\int_{1-r^j}^{1-r^i} \frac{1-x}{x}\psi(x)dx\right)$.

4) If $r^i = r^j$, split the objects by even randomization.

5) Arbitrarily decide whether to treat a report equal to $1/2$ as if it were lower or higher than $1/2$.

Proof. It is just a recalculation from the lottery over hierarchical mechanisms that has been previously derived. ■

2.4 Examples

Two-agent examples are discussed in this subsection. In the first example shown here, the marginal distribution of the valuation for each object is uniform. This example is also used in McAfee (1992), which is discussed below. In the second example, each object valuation follows the same exponential distribution. Both examples in this subsection satisfy the requirements of the preceding proposition.

Example 1 ***The uniform distribution:*** $\tilde{F}(v) = v$**,** $v \in [0,1]$**.** *For $r \geq 1/2$, the density of r is $\psi(r) = \frac{1}{2r^2}$. For this case, $\int_{r^j}^{r^i} \frac{1-x}{x}\psi(x)dx = \frac{2r^i-1}{4(r^i)^2} - \frac{2r^j-1}{4(r^j)^2}$. The per agent ex ante expected utility in the first best (perfect information) case is $2/3$. The OCM yields $\frac{11}{18}$, 91.67% of the perfect information benchmark. The RRM yields $\frac{7}{12}$, 87.5% of the first best. Finally, the naive Even Randomization Mechanism (ERM), the no-information case, yields $1/2$, 75 per cent of the first best.*

One third of the efficiency losses arising from the RRM could be saved if the OCM were implemented. Taking the first best as a benchmark, the OCM achieves 4.17 percentage points more in efficiency than the RRM. In monetary terms, if valuation distributions are rescaled to have expectation \$1, total ex ante expected (utilitarian) welfare gains are of the order of 11.11¢.

McAfee (1992) poses an example with two agents and twelve objects, with uniformly distributed valuations. Using the Alternating Selection Mechanism (ASM) he proposes, he obtains a 96.8% ex ante efficiency ratio. This is an outstanding efficiency result. However, it is already known that many ordinal mechanisms converge to full efficiency. It is interesting to evaluate its performance when the number of objects is lower.

The ASM and the RRM are interim payoff identical, in the two-object, two-agent case. Therefore, the ASM yields an 87.5% efficiency ratio with two objects. With few objects, the OCM would significantly improve over the ASM.

Example 2 ***The exponential distribution:*** $\tilde{F}(v) = 1 - e^{-\lambda v}$***,*** $v \in [0, \infty)$*,* $\lambda > 0$***.*** *In this case,* r *is distributed as a uniform* $U[0,1]$*, with density* $\psi(r) = 1$*. For this case,* $\int_{r^j}^{r^i} \frac{1-x}{x}\psi(x)dx = \ln \frac{r^i}{r^j} - (r^i - r^j)$*. The per agent ex ante expected utility in the first best (perfect information) case is* $\frac{3}{2\lambda}$*. The OCM gives* $\frac{2-\log 2}{\lambda}$*, 87.12% of the benchmark. The RRM yields* $\frac{5}{4\lambda}$*, 83.33% of the first best. Finally, the ERM yields* $1/\lambda$*, 66.67% of the first best.*

In this example, the reader can observe that the efficiency loss savings if the planner switches from the RRM to the OCM is approximately a 22.75% of ex ante expected losses, or 3.79 percentage points in efficiency differences. If once again valuation distributions are rescaled to have expectation \$1, total ex ante expected (utilitarian) welfare gains become 11.37¢.

3 Simpler cardinal mechanisms

This Section studies suboptimal cardinal mechanisms in which the efficiency loss is compensated by gains in other respects. Simplicity is the main motivation I consider.

The need for simplicity on the agents' side stems from the fact that agents may not have precise estimates of their valuations and hence of their relative preferences. Instead, they may have an approximate judgement on what object they prefer and whether their preference intensities are extreme or moderate (according to a cutoff criterion). In such a case, a mechanism asking only for this information would greatly simplify agents' decisions.

The value of simplicity on the planner's side comes from the information that is required in order to implement the OCM. It has been observed that even in simple cases ($n = 2$), the planner needs to know the exact distribution of the relative preference parameter. The question is whether there are Bayesian cardinal mechanisms that work with a planner's partial knowledge of priors.

The goal of this Section is to achieve gains in simplicity on both sides. The simplest alternative to RRM consists of splitting the (relative preference paramenter) type space into four subintervals, namely $\{[0, 1-\theta), [1-\theta, 1/2), [1/2, \theta], (\theta, 1]\}$, with $\theta \in (1/2, 1)$. Only four messages are relevant in equilibrium, and each message would indicate a specific subset of the partition. In that sense, each agents reveals which object she prefers and whether her preferences are "extreme" (either $r > \theta$ or $r < 1-\theta$) or "moderate" ($r \in [1-\theta, \theta]$). Mechanisms that elicit and use this kind of information only will be called *Extreme-Moderate Preference Mechanisms with cutoff* θ (EMPM$_\theta$). θ is taken to be a fixed parameter in most of the exposition. The reader could view it as a commonly known (or accepted) limit separating extreme preferences from moderate ones. However, see the end of next subsection for some discussion on how to optimally choose θ with a limited amount of information about priors.

The first subsection presents the interesting two-agent case, where the planner may implement a simple (and in many cases optimal) EMPM$_\theta$ without holding an exact knowledge on the priors. The second subsection shows the general solution to the Optimal EMPM$_\theta$ problem, for an arbitrary number of agents. Its derivation is left to the Appendix. A third subsection reanalyzes the examples provided in the previous Section.

3.1 Two agents

Assume that the regularity condition holds. Then, the next definition presents the mechanism the planner would ideally implement. This is true because the planner wishes to give an object to the agent with the highest expected valuation for it. And the regularity condition ensures that this expectation is monotone with respect to the partition of the agent's type space.

Definition 1 *The* general-rule hierarchical mechanism *is a hierarchical mechanism where:*

1) Each agent i reports a message $m^i \in \{2E, 2M, 1M, 1E\}$ *where the number represents agent i's preferred object and the letter expresses the intensity of her preference.*

2) $2E, 2M, 1M, 1E$ *is both the decreasing order of hierarchy types for object 2 and the increasing order of hierarchy types concerning object 1.*

$\{2E, 2M, 1M, 1E\}$ is interpreted in equilibrium as the partition $\{[0, 1-\theta), [1-\theta, 1/2), [1/2, \theta], (\theta, 1]\}$ of the relevant type space, where $\theta \in (1/2, 1)$. However, it is easy to check that this mechanism is not incentive compatible. One has to construct variations from it since BIC has to be preserved.

Consider this mechanism with two agents. In it, a $2E$ type wins object 2 if the other agent either has a lower type or has the same type but loses in the even randomization procedure. This accounts for an interim probability of winning object 2 equal to $Q_2^H(2E) = \Psi(\theta) + \frac{1-\Psi(\theta)}{2} = \frac{1+\Psi(\theta)}{2}$. Similarly, a $2M$ type wins object 2 with probability $Q_2^H(2M) = \frac{\Psi(\theta)+1/2}{2}$. Regarding object 1, the respective interim probabilities of winning are $Q_1^H(2E) = \frac{1-\Psi(\theta)}{2}$ and $Q_1^H(2M) = \frac{3/2-\Psi(\theta)}{2}$. An important consequence is that $Q_2^H(2E) - Q_2^H(2M) = Q_1^H(2M) - Q_1^H(2E) = 1/4$, regardless the cutoff value θ and the function $\Psi(\cdot)$. An analogous result holds for types $1I$, where $I \in \{M, E\}$.

Then, as shown in the next proposition, incentive compatibility is met in a variation of the general-rule hierarchical mechanism where with probability $1 - \frac{1-\theta}{\theta}$ types hM and hE are treated as if they belonged to the same hierarchy type when allocating object h, for each $h \in \{1, 2\}$. The following definition characterizes this mechanism.

Definition 2 *The* Two-Agent Simple EMPM$_\theta$ *is the mechanism which is implemented as follows. Let each agent reveal her type in* $\{2E, 2M, 1M, 1E\}$*. Proceed as in the general-rule hierarchical mechanism. However, apply the following exception: whenever a* hE *faces a* hM *type, allocate object* h *to the* hE *type with probability* $\frac{1}{2\theta}$ *and to the* hM *type with probability* $1 - \frac{1}{2\theta}$*.*

The following proposition summarizes an important property of the Two-Agent Simple EMPM$_\theta$.

Proposition 4 *Let* $n = 2$*. Then for any* θ *and for any symmetric (with respect to 1/2)* $\psi(\cdot)$*, the Two-Agent Simple EMPM*$_\theta$ *is Bayesian incentive compatible.*

Proof. A necessary condition for BIC is that a θ-type agent must hold indifferent between declaring $2M$ and $2E$. This condition can be expressed as $[Q_2(2E) - Q_2(2M)]\,\theta = [Q_1(2M) - Q_1(2E)]\,(1-\theta)$. By the previous discussion, $Q_2(2E) - Q_2(2M) = \frac{1-\theta}{\theta} \cdot 1/4 + \left[1 - \frac{1-\theta}{\theta}\right] \cdot 0$, and $Q_1(2M) - Q_1(2E) = 1/4$, so the condition holds. A similar reasoning follows for $1-\theta$, $1M$ and $1E$. Additionally, it is easy to check that each Q_h function is monotone in its respective ordering of hierarchy types, and that a 1/2-type agent is indifferent between declaring $1M$ and $2M$ (here the symmetry of $\psi(\cdot)$ is necessary). All these conditions together are sufficient for BIC. ■

There could be differences among agents' distributions of relative preferences, and the results would keep unchanged, as long as each distribution is symmetric with respect to 1/2. Also, there could be corretated valuations between agents, and the result would still be valid provided the aforementioned symmetry property holds in each conditional distribution of the relative preference.

A fair question is whether this simplicity also benefits the agents. Do they have to exactly and commonly know the priors of the game, in order to support incentive compatibility in the Two-Agent Simple EMPM$_\theta$? The answer is not entirely. What needs to be common knowledge is that each agent's pair of valuations follow a distribution with identical marginals. Common priors are not required.

The implementation of the Two-Agent Simple EMPM$_\theta$ just requires to commonly know that the object valuations are stochastically identical. That is the only aspect one has to be aware of when introducing this simple improvement over ordinal mechanisms.

To end this subsection, a digression on the optimal choice of θ is presented, under the original assumptions on F. Although θ has so far been set as fixed, the planner may have the chance to announce a different cutoff, as long as the new announcement does not alter the agents' capacity to discern between extreme and moderate preferences. In this case, the planner can, with limited information, move θ to a level that, while not necessarily optimal, at least ex ante dominates many other possible choices of the cutoff parameter.

In order to do so, the mechanism designer must only have precise estimates on the first moments of the order statistics, that is, the expected valuation of the most-preferred object $E(v_2\,|\tilde{r} \geq 1/2)$ and the expected valuation of the least-preferred one $E(v_1\,|\tilde{r} \geq 1/2)$. Adding up both expectations, one obtains the expected bundle valuation. With only this information, the following proposition states that $\theta^* = \frac{E(v_2|\tilde{r}\geq 1/2)}{EB}$ is an upper bound on the optimal cutoff. Additionally, low enough cutoffs are ex ante a worse election than θ^*.

Proposition 5 *Let $n = 2$, assume that the regularity condition holds, and define $\theta^* = \frac{E(v_2|\tilde{r}\geq 1/2)}{EB}$. Under the assumptions on F, the Two-Agent Simple EMPM$_{\theta^*}$ yields higher ex ante expected payoff than any other Two-Agent Simple EMPM$_\theta$ where $\theta > \theta^*$. Also, there exists $\underline{\theta} > 1/2$ such that for any $\theta \in [1/2, \underline{\theta}]$ the Two-Agent Simple EMPM$_{\theta^*}$ is ex ante superior to the Two-Agent Simple EMPM$_\theta$.*

Proof. See Appendix D. ■

3.2 General case: arbitrary n

With arbitrarily many agents, the implementation of an EMPM$_\theta$ typically requires the planner to know the distribution of types, as in the OCM case. The EMPM$_\theta$ still has an advantage with respect to the OCM. Under the regularity condition, the Optimal EMPM$_\theta$ and its implementation can be fully characterized for any symmetric $\psi(\cdot)$. An important corollary of this subsection is that when $n = 2$ the Two-Agent Simple EMPM$_\theta$ is an Optimal EMPM$_\theta$ under mild conditions.

The following function is crucial in the characterization of the solution:[18]

$$\tilde{C}(\theta) \equiv E(v_1 \,|\tilde{r} \geq \theta\,) - \frac{1-\theta}{\theta}\left[E(v_2 \,|\tilde{r} \geq \theta\,) - E(v_2 \,|\tilde{r} \geq 1/2\,)\right]$$

This function is the discrete version of the criterion function. It accounts for the marginal contribution of $Q_{-h}(hE)$ to the objective function after the incentive compatibility constraints and a technical result analogous to the one of proposition 1 ($R_2(1/2) = 0$) have been embedded. This function is important in determining whether the optimal mechanism includes the possibility of not giving some object, in which case I describe that mechanism as *possibly wasting.* If $\tilde{C}(\theta) < 0$, it is ex ante optimal to commit to this possibility.

The following proposition depicts the optimal EMPM$_\theta$ as an adequate set of exceptions from the ideal general-rule hierarchical mechanism, based on the value of $\tilde{C}(\theta)$.

Proposition 6 *Under the regularity condition, an Optimal EMPM$_\theta$ is implemented as follows, in environments with identical marginals. Each agent i is aware of θ and reports a message $m^i \in \{2E, 2M, 1M, 1E\}$. Let n_{hI}, $h \in \{1,2\}, I \in \{E, M\}$, be the number of agents reporting hI. The planner has to apply the general-rule hierarchical mechanism, with the following exceptions:*

1) If $\tilde{C}(\theta) \geq 0$, then if $n_{hE} > 0$ and $n_{hM} > 0$, with probability $\pi_{1,n} = \frac{1-\theta}{\theta} \cdot \frac{[1-\Psi(\theta)]\left[(1/2)^{n-1} - [1-\Psi(\theta)]^{n-1}\right]}{1-\Psi(\theta)^n - 2[1-\Psi(\theta)][1-(1/2)^n]}$ object h is randomly allocated among agents having reported hE, and with probability $1 - \pi_{1,n}$ the object is randomly allocated among all agents that have reported to prefer object h.

2) If $\tilde{C}(\theta) < 0$ and $\frac{(1/2)^n - [1-\Psi(\theta)]^n}{n[\Psi(\theta)-1/2]} \leq \frac{\theta}{1-\theta}\left[\frac{1-\Psi(\theta)^n}{n[1-\Psi(\theta)]} - \frac{\Psi(\theta)^n - (1/2)^n}{n[\Psi(\theta)-1/2]}\right]$, then: i) if $n_{hE} > 0$ and $n_{hM} > 0$, apply exception 1 but using $\pi_{2,n} = \frac{1-\theta}{\theta} \cdot \frac{2[1-\Psi(\theta)][(1/2)^n - [1-\Psi(\theta)]^n]}{1-\Psi(\theta)^n - 2[1-\Psi(\theta)][1-(1/2)^n]}$ instead of $\pi_{1,n}$; and ii) if $n_{hE} = n$, then object $-h$ is not allocated to any agent.

3) If $\tilde{C}(\theta) < 0$ and $\frac{(1/2)^n - [1-\Psi(\theta)]^n}{n[\Psi(\theta)-1/2]} > \frac{\theta}{1-\theta}\left[\frac{1-\Psi(\theta)^n}{n[1-\Psi(\theta)]} - \frac{\Psi(\theta)^n - (1/2)^n}{n[\Psi(\theta)-1/2]}\right]$, then if $n_{hE} = n$, with probability $\pi_{3,n} = \frac{(1/2)^n - [1-\Psi(\theta)]^n}{n[\Psi(\theta)-1/2]} - \frac{\theta}{1-\theta}\left[\frac{1-\Psi(\theta)^n}{n[1-\Psi(\theta)]} - \frac{\Psi(\theta)^n - (1/2)^n}{n[\Psi(\theta)-1/2]}\right]$ object $-h$ is randomly allocated among all agents, and with probability $1 - \pi_{3,n}$ it is not allocated to any agent.

Proof. Its derivation can be found in Appendix D. ■

If the planner cannot commit to a (possibly) wasting mechanism, or if she is just obliged to give all objects, then the the sign of $\tilde{C}(\theta)$ does not matter. The optimal mechanism in the EMPM$_\theta$ family would be the one that applies exception 1 regardless the value of $\tilde{C}(\theta)$.

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[18] More on this expression is explained in Appendix D, derivation of proposition 6.

However, if the planner can commit to a wasting mechanism, the following remarks state sufficient conditions under which she would optimally not do so, regardless the value of the cutoff θ.

Remark 2 *If $E(B\,|r)$ is nonincreasing for $r \in [1/2,1]$, then any of the following is a sufficient condition for $\tilde{C}(\theta) \geq 0$ for any $\theta \in [1/2,1]$:*
1) $\frac{\psi(r)(1-r)}{1-\Psi(r)} \geq \frac{E(v_1|\tilde{r}\geq 1/2)}{E(v_2|\tilde{r}\geq 1/2)}$ for any $r \in [1/2,1)$, and with strict inequality for $r=1$.
2) $\frac{\psi(r)(1-r)}{1-\Psi(r)}$ nondecreasing in $r \in [1/2,1]$.

Proof. See Appendix D. ■

Remark 3 *Condition 1 in Remark 2 includes the case where $\Psi(r)$ is concave on $[1/2,1]$.*

Remark 4 *Condition 2 in Remark 2 includes the case where $\psi(r)$ is increasing and log-concave on $[1/2,1]$.*

Under somewhat mild conditions, $\tilde{C}(\theta) \geq 0$ for any $\theta \in [1/2,1]$. Even if none of these are met, the planner can at least be sure that $\tilde{C}(\theta) \geq 0$ for θ close enough to $1/2$.

Notice that $\pi_{1,n}$ in proposition 6 reduces to $\frac{1-\theta}{\theta}$ when $n=2$. The reader would readily notice that the Optimal EMPM$_\theta$ when $\tilde{C}(\theta) \geq 0$ reduces to the Two-Agent Simple EMPM$_\theta$ if $n=2$.

Corollary 3 *If the environment is regular and characterized by identical marginals, and $n=2$, the Two-Agent Simple EMPM$_\theta$ is optimal among all EMPM$_\theta$'s if either $\tilde{C}(\theta) \geq 0$ or the planner cannot commit to (or cannot undertake) a possibly wasting mechanism.*

3.3 Examples

In this subsection, I compare the Two-Agent Simple EMPM$_{\theta^*}$ to the other mechanisms that have been discussed in subsection 2.4. For each of the cases discussed in that subsection, the suggested cutoff θ^* has been calculated as well as the corresponding Simple EMPM. As before, I consider ex ante expected (utilitarian) welfare.

In example 1, the uniform distribution case, $\theta^* = 2/3$, and for any $\theta \in (1/2,1)$ the Two-Agent Simple EMPM$_\theta$ is indeed optimal among all EMPM$_\theta$'s. The per agent ex ante expected payoff is $\frac{29}{48}$, 90.625% of what is achieved at the perfect information benchmark.

In example 2, the exponential distribution case, $\theta^* = 3/4$, and for any $\theta \in (1/2,1)$ the Two-Agent Simple EMPM$_\theta$ is again optimal among all EMPM$_\theta$'s. The per agent ex ante expected payoff is $\frac{31}{24\lambda}$, 86.11% of the benchmark.

Comparing to the preceding Section, the Two-Agent Simple EMPM$_{\theta^*}$ is only around one percentage point less efficient than the OCM. It roughly achieves the 75% of the efficiency gains that the OCM yields, with respect to the RRM. In both examples, if the marginal distributions are rescaled so that they generate a \$1 expected valuation, then the (ex ante) total welfare gains with respect to the RRM ascend to 8.33¢.

4 The two-school random assignment problem

One of the features of the problem so far analyzed is that it poses no restrictions on the number of objects that will be allocated to each agent. In particular, it may be the case that both objects are allocated to the same agent, or that some object is not allocated to any agent.

While this is fine in many settings, such as collusion mechanisms, in some cases this assumption is inappropriate. A paradigmatic case, from which this Section takes its name, consists of the assignment of children to public schools. Naturally, no child can be allocated to more than one school, and all children must be assigned.

The two-school random assignment problem is as follows. There is a number n of children, and each of them is to be assigned to exactly one of two schools. School 1 has $n_1 \in \{1, ..., n-1\}$ slots to be fulfilled, and school 2 has $n - n_1$ vacancies. It is assumed that schools' preferences will not play a role here, so this is a one-sided matching problem.[19] Child i's parents' preferences over schools are represented by valuations v_1^i and v_2^i, as in the original game, and these are private information. Valuations are i.i.d. among parents, and marginal distributions are identical between schools. Parents are indifferent among different slots in the same school.

The mechanism designer asks for information about parents' preferences, and allocates assignment probabilities $q_h^i(\cdot)$, $\forall i \in \{1, ..., n\}$, $\forall h \in \{1, 2\}$, as a function of revealed information. q_h^i is the probability that children i is allocated a slot in school h. The preliminary results of Section 2 still apply, so WLOG q_h^i will depend only on the vector $(r^1, ..., r^n)$, where r^i is defined as in Section 2. $Q_h^i(r^i)$, the reduced form probability, is also defined as in that Section, for any i and h.

The constraint that each child is assigned to one and only one school is translated to the property that for any $(r^1, ..., r^n) \in [0, 1]^n$ the matrix $q(r^1, ..., r^n) \equiv [q_h^i(r^1, ..., r^n)]_{i,h}$ must be bistochastic. That is, $\sum_i q_h^i(r^1, ..., r^n) = 1$ for any h, and $\sum_h q_h^i(r^1, ..., r^n) = 1$ for any i. By the Birkhoff - von Neumann theorem, any bistochastic matrix of assignment probabilities can be implemented as a

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[19] Some discussion of this assumption is given at the end of the section.

lottery among pure assignments that satisfy the required constraint.

An important consequence of the bistochastic nature of random assignment is that for any i and r^i, it must hold that $Q_2^i(r^i) = 1 - Q_1^i(r^i)$. Child i's parents' interim utility is therefore $U^i(r^i, B^i) \equiv B^i[Q_1^i(r^i)(1-r^i) + Q_2^i(r^i)r^i] = B^i[r^i + Q_1^i(r^i)(1-2r^i)]$. Since it only depends on one reduced form probability, the next result follows immediately.

Proposition 7 *In the two-school random assignment problem, BIC is equivalent to the following properties for any* $i \in \{1, ..., n\}$ *and any pair* $(r, \tilde{r}) \in [0,1]^2$, $r \geq \tilde{r}$*: 1) if* $sign(r-1/2) = sign(\tilde{r}-1/2)$*, then* $Q^i(r) = Q^i(\tilde{r})$*, and 2) if* $r \geq 1/2 \geq \tilde{r}$*, then* $Q_1^i(r) \leq Q_1^i(\tilde{r})$ *(and therefore* $Q_2^i(r) \geq Q_2^i(\tilde{r})$*).*

Proof. BIC is equivalent to the following: for any $i \in \{1, ..., n\}$ and any pair $(r, \tilde{r}) \in [0,1]^2$, where WLOG $r \geq \tilde{r}$, we have $Q_1^i(r)(1-2r) \geq Q_1^i(\tilde{r})(1-2r)$ and $Q_1^i(r)(1-2\tilde{r}) \leq Q_1^i(\tilde{r})(1-2\tilde{r})$. It is easy to check that this is equivalent to the stated properties. ■

The main consequence is apparent. In the problem under analysis, if a mechanism is Bayesian incentive compatible, then it must be interim equivalent to an ordinal mechanism.

Corollary 4 *The Ranking Revelation Mechanism (in a version that ensures exactly one slot per child) solves the two-school random assignment problem in environments characterized by Private Values and identical marginals.*

This result stems from the fact that the RRM is ex ante efficient among ordinal mechanisms under the assumptions on F. The way the RRM is implemented here is as follows. Let the number of parents[20] claiming to prefer school 1 be denoted by n_1^*. If $n_1^* = n_1$, then there is a perfect match between preferences and slots. If $n_1^* > n_1$, then all children whose parents claim to prefer school 2 are ensured a place in that school. The remaining places are allocated among the unassigned children by even randomization. If $n_1^* < n_1$, then the ones that prefer school 1 are the ones that are ensured a place in that school.

Not surprisingly, the CADA mechanism suggested by Abdulkadiroğlu, Che and Yasuda (2007) does not make any difference here, with respect to the RRM. If a student's parents prefer school 1 to 2, even if moderately, they do not gain anything by signalling school 2. A slot in their second-best choice is ensured anyway since there are only two schools. Therefore, signals coincide in equilibrium with declared ordinal preferences.

Although it is true that schools' preferences over children are not taken into account in many real world cases, it is also true that schools do have priorities over children. These priorities are

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[20] Each child's parents count both as one individual.

generally given by previous enrollment, by geographical proximity and by the presence of siblings. The optimal allocation mechanism would be a modification of the RRM that takes into account these priorities. For instance, the so-called "Boston mechanism" that has been used until recently is an example of such a modification of the RRM.

Recent literature has argued that the Boston mechanism violates strategy-proofness and has defended its substitution by the Deferred Acceptance (DA) algorithm (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth and Sönmez, 2005 and 2006; Chen and Sönmez, 2006; Ergin and Sönmez, 2006). Nevertheless, *the results of this Section are robust to strategy-proofness.* This is trivial with only two schools. The second-best allocation for the parents is indeed the worst allocation, so nothing is gained by misrepresenting ordinal preferences. Additionally, if the assumption of uncertainty about other agents' preferences is accepted, one can construct examples in which the Boston mechanism not only induces truthful revelation of preferences but also Pareto-dominates DA (Ergin and Sönmez, 2006). Finally, some recent theoretical and experimental studies cast doubt on the superiority of the proposed alternatives to the Boston mechanism, when for the sake of simplicity the list of ranked schools that parents have to declare is truncated below the total number of schools (Haeringer and Klijn, 2007; Calsamiglia, Haeringer and Klijn, 2007).

5 Conclusion

The literature has extensively analyzed the problem a planner faces when she is to allocate several indivisible objects to several agents, under the constraint that no monetary transfers can be made. The Ranking Revelation Mechanism (RRM) and other ordinal mechanisms, where object allocation is a function of ordinal preferences only, have been suggested as satisfactory solutions.

However, cardinal mechanisms are preferable, since information on the intensity of preferences is additionally used. When Bayesian (or interim) incentive compatible mechanisms are analyzed, information on agents' marginal rates of substitution between objects is valuable in determining the allocation decision. Bayesian cardinal mechanisms readily improve the results of ordinal mechanisms. This improvement is relatively more important the smaller is the number of objects to be allocated.

Solving for the optimal cardinal mechanism (OCM) is a cumbersome task due to the multidimensionality of types. This paper has analyzed the two-object, n-symmetric-agent, Independent-Private-Valuation case, and has provided a solution method. This method applies for a wide family of marginal valuation distributions, such as the uniform and the exponential. For the two-agent case, implementation has been characterized for a wide set of environments.

I also analyze examples comparing the OCM to the RRM (the ex ante optimal ordinal mechanism). In both, the number of agents is two. In the first case, the analyzed marginal valuation distribution is the uniform. In the second case, the marginal distribution is the exponential. Significant ex ante efficiency loss savings have been reported. An approximately 4 percentage point efficiency gain is found in both cases, where efficiency is measured as the percentage of ex ante welfare with respect to ex ante welfare in a first-best allocation. This accounts for a total ex ante (utilitarian) welfare gain of more than 11¢ per dollar of expected object valuation.

For the sake of simplicity, sub-optimal cardinal mechanisms are worthwhile studying. Sometimes agents do not have a precise idea about their respective marginal rates of substitution between objects. Nevertheless, each agent may probably have an idea of whether her preference in favor of one of the objects is "extreme" or "moderate", where some fixed cutoff defines the limit between the two preference intensities. A mechanism that just uses information on ordinal preferences and whether these are extreme or moderate is called Extreme-Moderate Preference Mechanism (EMPM). For each cutoff, the optimal EMPM is fully characterized.

In the two-agent case, a simple version of EMPM is particularly interesting in that the planner needs to know little about the prior distribution in order to implement it. So do agents in order to determine that truthful information revelation is already optimal. Common knowledge of identical marginals between objects suffices, no matter how these marginal valuation distributions look like. Some hints are also given to "optimally" fix the cutoff separating preference intensities, in a restricted information setup, where the planner has estimates on first moments of the order statistics of object valuations. It turns out that this mechanism (with "optimal" cutoff) satisfactorily approaches the efficiency level of the OCM. In the previously analyzed examples, the Two-Agent Simple EMPM roughly achieves the 75% of (ex ante) welfare gains the OCM would obtain with respect to the RRM.

Interestingly, when the problem is additionally constrained to ensure one (and only one) object per agent, incentive compatibility precludes any use of cardinal information, in the two-object case. The RRM turns out to be an optimal mechanism.

There is further interesting work to be done in this and related topics. A full solution to the optimal cardinal mechanism for the two-object case is still pending. This solution would also be extended to the problem of allocating more than two objects. Finally, the ideas of this model could be applied, after proper modifications, to public decisions and optimal referenda, which face different feasibility constraints and may generate negative utility.

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6 Appendices

APPENDIX A: Proofs in Section 2

Proof. Lemma 2

Take a mechanism $\hat{q}(\cdot)$ solving the problem, and hence maximizing $\sum_i E_i \hat{Q}^i(B^i, r^i) \cdot (B^i[1 - r^i], B^i r^i)$. Define $q(r^1, ..., r^n) \equiv E_{\tilde{B}^1,...,\tilde{B}^n} \hat{q}(\tilde{B}^1, ..., \tilde{B}^n, r^1, ..., r^n)$, that is, the expectation of $\hat{q}(\cdot)$ over all agents' B's. Define as usual $Q^i(r^i) \equiv E_{-i} q^i_h(r^1, ..., r^n)$. The new mechanism satisfies feasibility, since for any $h \in \{1, 2\}$ $\sum_{i=1}^n q^i_h(r^1, ..., r^n) = E_{\tilde{B}^1,...,\tilde{B}^n} \sum_{i=1}^n \hat{q}^i_h(\tilde{B}^1, ..., \tilde{B}^n, r^1, ..., r^n) \leq 1$. Moreover, it satisfies interim incentive compatibility, since

$$\begin{aligned} & Q^i(r^i) \cdot (B^i[1 - r^i], B^i r^i) \\ = \; & (E_{\tilde{B}^i} \hat{Q}^i(\tilde{B}^i, r^i)) \cdot (B^i[1 - r^i], B^i r^i) \\ = \; & \hat{Q}^i(B^i, r^i) \cdot (B^i[1 - r^i], B^i r^i) \end{aligned}$$

for any i and (B^i, r^i). The last equality follows from lemma 1. It is also apparent from these equalities that $q(\cdot)$ solves the original problem. ■

Proof. Lemma 3

Let $\bar{q}(\cdot)$ solve the maximization problem. Consider an anonymous mechanism $q(\cdot)$ defined as $q_h(r^1, ..., r^n) \equiv 1/n! \sum_{\pi(\cdot) \in \Pi^n} \pi^{-1}(\bar{q}_h(\pi(r^1, ..., r^n)))$, for any $(r^1, ..., r^n) \in [0, 1]^n$ and any $h \in \{1, 2\}$, where Π^n is the set of all possible permutations over an n-dimensional vector, and π^{-1} is the inverse permutation of π. This mechanism is feasible since, after proper rearrangements,

$$\begin{aligned} & \sum_{i=1}^n q^i_h(r^1, ..., r^n) \\ = \; & 1/n! \sum_{\pi(\cdot) \in \Pi^n} \sum_{i=1}^n \bar{q}^i_h(\pi(r^1, ..., r^n)) \\ \leq \; & 1/n! \sum_{\pi(\cdot) \in \Pi^n} 1 = 1 \end{aligned}$$

for any $h \in \{1, 2\}$. It is also true that $Q^i(r) = \sum_{j=1}^n 1/n \cdot \bar{Q}^j(r)$, for any i and r. Incentive compatibility holds then for $q(\cdot)$ because $\bar{Q}^j(r) \cdot (Br, B(1 - r)) \geq \bar{Q}^j(r) \cdot (Br, B(1 - r))$, for any r, r, B and j. Finally, it is seen that the proposed mechanism equals $\bar{q}(\cdot)$ in total ex ante expected payoff. Since $\sum_{i=1}^{n} Q^i(r) = \sum_{j=1}^{n} \bar{Q}^j(r)$ for any r, it follows that

$$\begin{aligned}
& \sum_{i=1}^{n} E_i Q^i(r^i) \cdot (B^i r^i, B^i(1-r^i)) \\
= & \sum_{i=1}^{n} \int_0^1 \psi(r) E(B\,|r) Q^i(r) \cdot (r, 1-r) dr \\
= & \int_0^1 \psi(r) E(B\,|r) \left[\sum_{i=1}^{n} Q^i(r) \right] \cdot (r, 1-r) dr \\
= & \ldots = \sum_{i=1}^{n} E_i \bar{Q}^i(r^i) \cdot (B^i r^i, B^i(1-r^i))
\end{aligned}$$

■

Proof. Lemma 4

Let $\bar{q}(\cdot)$ solve the maximization problem. Consider the alternative mechanism $q(\cdot)$ defined as $q_h(r^1, ..., r^n) \equiv \frac{1}{2}\bar{q}_h(r^1, ..., r^n) + \frac{1}{2}\bar{q}_{-h}(1-r^1, ..., 1-r^n)$, $\forall h \in \{1, 2\}$. This mechanism is obviously feasible and meets the properties stated in the lemma. It keeps the objective function constant, since

$$\begin{aligned}
& E \sum_{i=1}^{n} q^i(r^1, ..., r^n) \cdot (B^i[1-r^i], B^i r^i) \\
= & \frac{1}{2} E \sum_{i=1}^{n} \bar{q}^i(r^1, ..., r^n) \cdot (B^i[1-r^i], B^i r^i) \\
& + \frac{1}{2} E \sum_{i=1}^{n} \bar{q}^i(1-r^1, ..., 1-r^n) \cdot (B^i r^i, B^i[1-r^i]) \\
= & E \sum_{i=1}^{n} \bar{q}^i(r^1, ..., r^n) \cdot (B^i[1-r^i], B^i r^i)
\end{aligned}$$

The last equality comes from the fact that any r^i follows a symmetric distribution with respect to $r = 1/2$. Finally, the alternative mechanism meets incentive compatibility. This is so because, for any pair (r^i, r^i),

$$\begin{aligned}
& Q(r^i) \cdot (B^i[1-r^i], B^i r^i) \\
= & \frac{1}{2} \bar{Q}(r^i) \cdot (B^i[1-r^i], B^i r^i) \\
& + \frac{1}{2} \bar{Q}(1-r^i) \cdot (B^i r^i, B^i[1-r^i]) \\
\geq & \frac{1}{2} \bar{Q}(r^i) \cdot (B^i[1-r^i], B^i r^i) \\
& + \frac{1}{2} \bar{Q}(1-r^i) \cdot (B^i r^i, B^i[1-r^i]) \\
= & Q(r^i) \cdot (B^i[1-r^i], B^i r^i)
\end{aligned}$$

The inequality follows from the fact that $\bar{q}(\cdot)$ is incentive compatible. Therefore, the alternative mechanism $q(\cdot)$ is also optimal. ■

APPENDIX B: Proof of proposition 1

The following strategy is undertaken. First, the message space is discretized to a limited number of elements. For any such a number of possible messages, the desired result is obtained. Finally, this result is extended to the unrestricted model by a limit argument.

A discretization of the message space implies that no agents will exactly declare her parameter r to the planner but some interval (or region) r belongs to. Since the model is symmetric with respect to the 1/2 point, it only requires to analyze the regions in which the planner can split the $[1/2, 1]$ segment. Let us have a number T of pairwise disjoint subsets or *regions* in which split the aforementioned segment is split, in virtue of the limited number of possible messages. For each region $t \in \{1, ..., T\}$ the planner allocates reduced form probabilities $Q_1(t)$ and $Q_2(t)$, slightly abusing notation. Denote $Q \equiv (Q_1(1), Q_2(1), ..., Q_1(T), Q_2(T))$. There is a profile $\Theta \equiv (\theta_1, ..., \theta_{T-1})$ of region boundaries such that r is in region t in and only if $\theta_{t-1} \leq r < \theta_t$ (the only exception: $r = 1$ belongs to region T). WLOG set $\theta_0 = 1/2$ and $\theta_T = 1$, and $\theta_0 < \theta_1 < ... < \theta_{T-1} < \theta_T$.

The mechanism designer chooses a pair $(Q, \Theta)_T$ to maximize the function

$$\sum_{t=1}^{T} [\Psi(\theta_t) - \Psi(\theta_{t-1})] \left[Q_2(t) E(v_2 \left| t, \Theta \right.) + Q_1(t) E(v_1 \left| t, \Theta \right.) \right]$$

subject to

$$1/n[1 - \Psi(\theta_{t-1})^n] - \sum_{k=t}^{T} [\Psi(\theta_k) - \Psi(\theta_{k-1})] Q_2(k) \geq 0 \qquad ((\mu_t),\ t = 1, ..., T)$$

$$\begin{aligned} & 1/n\left[1 - [1 - \Psi(\theta_t)]^n\right] - && ((\gamma_t),\ t = 1, ..., T) \\ & \sum_{k=1}^{T} [\Psi(\theta_k) - \Psi(\theta_{k-1})] Q_2(k) - && (8) \\ & \sum_{k=1}^{t} [\Psi(\theta_k) - \Psi(\theta_{k-1})] Q_1(k) \geq 0 && \end{aligned}$$

$$\begin{aligned} & \theta_t [Q_2(t+1) - Q_2(t)] + && ((\lambda_t),\ t = 1, ..., T-1) \\ & (1 - \theta_t)[Q_1(t+1) - Q_1(t)] && \\ = \ & 0 && (9) \end{aligned}$$

, plus $Q_2(t+1) \geq Q_2(t)$ (multiplier m_t) and $Q_1(t+1) \leq Q_1(t)$ for any $t = 1, ..., T-1$, $Q_2(1) \geq Q_1(1)$ (multiplier M), and nonnegativity $Q_h(t) \geq 0$, $t = 1, ..., T$, $h = 1, 2$ (with associate multipliers Z_{ht}).

Greek letters stand for the corresponding multipliers. $\mu-$constraints are equivalent to the Q_2-group constraints, and $\gamma-$constraints correspond to the Q_1-group constraints. The rest of constraints are equivalent to incentive compatibility. Concretely, $\lambda-$constraints state that each borderline type is indifferent between the two regions it separates. WLOG we set $\lambda_0 = \lambda_T = 0$. The fact that $Q_1(\cdot)$ is nonincreasing is already implied by $Q_2(\cdot)$ being nondecreasing and the $\lambda-$constraints, therefore we can ignore those constraints. From now on, and in order to shorten notation, we will denote $P_t \equiv \Psi(\theta_t) - \Psi(\theta_{t-1})$.

The goal is to prove that the μ_1-constraint binds. If it does, then the feasibility constraint binds at $r = 1/2$, which is equivalent to Q_2-hierarchy at this point. The reader will see that in order to do so it is sufficient to show that $\lambda_1 \leq 0$. In the way to prove this, it will indeed be proved that none of the lambdas is strictly positive.

Let us assume that the optimal $(Q^*, \Theta^*)_T$ pair with T regions in the $[1/2, 1]$ interval has been found (associated multipliers will also be superscripted by *). WLOG it can be assumed that all $m-$constraints are nonbinding, that is, there are actually T regions. Otherwise, a case with $\tilde{T} > T$ relevant regions and optimally strictly increasing Q_2^* could be analized. If no such number exists, there must be a $\hat{T} < T$ and a $(Q^*, \Theta^*)_{\hat{T}}$ with no $m-$constraint binding such that a global maximum is reached, that is, allowing for a richer message space does not improve the objective function. In such a case, it suffices to analyze the case with $\hat{T}$ relevant regions.[21] For the same reason, nonnegativity constraints are WLOG nonbinding, except for the $Q_1(T)$ case. Therefore, only Z_{1T}^* is taken into account, and the other $Z-$multipliers are discarded.

Concerning $Q_2^*(t)$ and $Q_1^*(t)$, $t \in [2, T-1]$, first order conditions are, respectively,

$$\begin{aligned} & E(v_2\,|t, \Theta^*) - \frac{\lambda_t^* \theta_t^*}{P_t^*} + \frac{\lambda_{t-1}^* \theta_{t-1}^*}{P_t^*} \\ = & \sum_{k=1}^{T} \gamma_k^* + \sum_{k=1}^{t} \mu_k^* \end{aligned} \tag{FOC$_{2,t}$}$$

$$\begin{aligned} & E(v_1\,|t, \Theta^*) - \frac{\lambda_t^* (1-\theta_t^*)}{P_t^*} + \frac{\lambda_{t-1}^* (1-\theta_{t-1}^*)}{P_t^*} \\ = & \sum_{k=t}^{T} \gamma_k^* \end{aligned} \tag{FOC$_{1,t}$}$$

In the $t = T$ case, conditions are

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[21] In the special case $\hat{T} = 1$, it is easy to see that the optimal allocation mechanism is the Ranking Revelation Mechanism, where $Q_2^*(1) = \frac{1-(1/2)^n}{n1/2}$ and $Q_1^*(1) = \frac{(1/2)^n}{n1/2}$. It can be seen that in that case the μ_1-constraint binds.

$$E(v_2\,|T,\Theta^*\,) + \frac{\lambda^*_{T-1}\theta^*_{T-1}}{P^*_T} = \sum_{k=1}^{T}\gamma^*_k + \sum_{k=1}^{T}\mu^*_k \qquad (\mathrm{FOC}_{2,T})$$

$$E(v_1\,|T,\Theta^*\,) + \frac{\lambda^*_{T-1}(1-\theta^*_{T-1})}{P^*_T} + \frac{Z^*_{1T}}{P^*_T} = \gamma^*_T \qquad (\mathrm{FOC}_{1,T})$$

In the special case of $t = 1$, FOC's are

$$E(v_2\,|1,\Theta^*\,) - \frac{\lambda^*_1\theta^*_1}{P^*_1} + \frac{M^*}{P^*_1} = \sum_{k=1}^{T}\gamma^*_k + \mu^*_1 \qquad (\mathrm{FOC}_{2,1})$$

$$E(v_1\,|1,\Theta^*\,) - \frac{\lambda^*_1(1-\theta^*_1)}{P^*_1} - \frac{M^*}{P^*_1} = \sum_{k=1}^{T}\gamma^*_k \qquad (\mathrm{FOC}_{1,1})$$

The goal is to prove that $\mu^*_1 > 0$. By subtracting $\mathrm{FOC}_{1,1}$ from $\mathrm{FOC}_{2,1}$, one can see that it suffices to prove $\lambda^*_1 \leq 0$. This nonpositivity condition will indeed hold for any λ^*_t. Two cases are analyzed: one in which Z^*_{1T} is ignored and another in which it is assumed that $Z^*_{1T} > 0$. The latter is just an extension of the former.

The proof that lambdas are nonpositive: $Z^*_{1T} = 0$ case.

An inductive argument is used that starts from λ^*_{T-1} and finishes at λ^*_1. This argument is shown below, after having stated and proved the following lemma and claims, which are essential in the proof.

Lemma 7 *BIC implies that $Q_1(\cdot) + Q_2(\cdot)$ is decreasing on $(1/2, 1]$.*

Proof. By BIC, we have $\tilde{r}Q_2(r) + (1-\tilde{r})Q_1(r) \leq \tilde{r}Q_2(\tilde{r}) + (1-\tilde{r})Q_1(\tilde{r})$, for $r > \tilde{r} > 1/2$. That inequality is equivalent to $Q_1(\tilde{r}) - Q_1(r) \geq \frac{\tilde{r}}{1-\tilde{r}}[Q_2(r) - Q_2(\tilde{r})]$. Since $\frac{\tilde{r}}{1-\tilde{r}} > 1$, the desired result is reached. ∎

Claim 1 *For any $t \geq 2$, if $\mu_t > 0$ and $\gamma_t > 0$, then BIC is violated.*

Proof. In such a case, by constraints (μ_{t+1}) and (μ_t) one obtains $Q_2(t) \geq \frac{\Psi(\theta_t)^n - \Psi(\theta_{t-1})^n}{n[\Psi(\theta_t) - \Psi(\theta_{t-1})]}$. Analogously, by (γ_{t-1}) and (γ_t), it follows that $Q_1(t) \geq \frac{[1-\Psi(\theta_{t-1})]^n - [1-\Psi(\theta_t)]^n}{n[\Psi(\theta_t) - \Psi(\theta_{t-1})]}$.

By constraint (γ_{t-1}), and since (μ_t) binds,

$$\begin{aligned} & \sum_{k=1}^{t-1} [\Psi(\theta_k) - \Psi(\theta_{k-1})][Q_1(k) + Q_2(k)] \\ \leq \quad & 1/n[\Psi(\theta_{t-1})^n - [1 - \Psi(\theta_{t-1})]^n] \end{aligned}$$

This implies that there must exist $t' < t$ such that $Q_1(t') + Q_2(t') \leq \frac{\Psi(\theta_{t-1})^n - [1-\Psi(\theta_{t-1})]^n}{n[\Psi(\theta_{t-1}) - 1/2]}$. But it can be shown that[22]

$$\begin{aligned} & \frac{\Psi(\theta_t)^n - \Psi(\theta_{t-1})^n + [1 - \Psi(\theta_{t-1})]^n - [1 - \Psi(\theta_t)]^n}{n[\Psi(\theta_t) - \Psi(\theta_{t-1})]} \\ > \quad & \frac{\Psi(\theta_{t-1})^n - [1 - \Psi(\theta_{t-1})]^n}{n[\Psi(\theta_{t-1}) - 1/2]} \end{aligned}$$

Thus, it is implied that $Q_1(t) + Q_2(t) > Q_1(t') + Q_2(t')$, violating lemma 7. ■

Claim 2 *Let* $\lambda^*_{t-1} > 0$, $\lambda^*_t > 0$, $\lambda^*_{t+1} \in \mathbb{R}$, $\lambda^*_{t+1}\theta^*_{t+1} < \lambda^*_t\theta^*_t$ *and* $\mu^*_{t+1} = 0$. *Then, under the conditions stated in the proposition, it follows that* $\lambda^*_t\theta^*_t < \lambda^*_{t-1}\theta^*_{t-1}$ *and* $\gamma^*_t > 0$.

Proof. Since $\mu^*_{t+1} = 0$, it is satisfied that

$$\begin{aligned} & \frac{\lambda^*_{t+1}\theta^*_{t+1}}{P^*_{t+1}} + \frac{\lambda^*_{t-1}\theta^*_{t-1}}{P^*_t} - \frac{\lambda^*_t\theta^*_t}{P^*_{t+1}} - \frac{\lambda^*_t\theta^*_t}{P^*_t} \\ = \quad & E(v_2 \,|t+1, \Theta^*) - E(v_2 \,|t, \Theta^*) \end{aligned} \tag{C2.1}$$

, by comparison of $\text{FOC}_{2,t+1}$ and $\text{FOC}_{2,t}$. Since the RHS of (C2.1) is nonnegative and $\lambda^*_{t+1}\theta^*_{t+1} - \lambda^*_t\theta^*_t < 0$, it follows that $\lambda^*_t\theta^*_t < \lambda^*_{t-1}\theta^*_{t-1}$. It is left to show that

$$\begin{aligned} & \frac{\lambda^*_{t+1}(1-\theta^*_{t+1})}{P^*_{t+1}} + \frac{\lambda^*_{t-1}\theta^*_{t-1}}{P^*_t} - \frac{\lambda^*_t\theta^*_t}{P^*_{t+1}} - \frac{\lambda^*_t\theta^*_t}{P^*_t} \\ > \quad & E(v_1 \,|t+1, \Theta^*) - E(v_1 \,|t, \Theta^*) \end{aligned} \tag{C2.2}$$

(yielding $\gamma^*_t > 0$), or, by adding up (C2.1),

$$\begin{aligned} & \frac{\lambda^*_{t+1}}{P^*_{t+1}} + \frac{\lambda^*_{t-1}}{P^*_t} - \frac{\lambda^*_t}{P^*_{t+1}} - \frac{\lambda^*_t}{P^*_t} \\ > \quad & E(B \,|t+1, \Theta^*) - E(B \,|t, \Theta^*) \end{aligned} \tag{C2.3}$$

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[22] As a matter of fact, for any $1/2 \leq \underline{x} < x < \bar{x} \leq 1$, it can be shown that $\frac{\bar{x}^n - x^n + [1-x]^n - (1-\bar{x})^n}{n(\bar{x}-x)} > \frac{x^n - \underline{x}^n + (1-\underline{x})^n - (1-x)^n}{n(x-\underline{x})}$.

The procedure consists of showing that the RHS of (C2.3) is not higher than the corresponding side in (C2.1), and that the LHS in the inequality (C2.3) is higher than the LHS in the equation (C2.1). The RHS part follows directly from the regularity condition. Concerning the LHS, first notice that

$$\begin{aligned} & \frac{\lambda_{t+1}^{*}\theta_{t+1}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t-1}^{*}\theta_{t-1}^{*}}{P_{t}^{*}}-\frac{\lambda_{t}^{*}\theta_{t}^{*}}{P_{t+1}^{*}}-\frac{\lambda_{t}^{*}\theta_{t}^{*}}{P_{t}^{*}} \\ \leq\ & -\frac{\lambda_{t+1}^{*}\theta_{t+1}^{*}}{P_{t+1}^{*}\theta_{t}^{*}}-\frac{\lambda_{t-1}^{*}\theta_{t-1}^{*}}{P_{t}^{*}\theta_{t}^{*}}+\frac{\lambda_{t}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t}^{*}}{P_{t}^{*}} \end{aligned} \tag{C2.4}$$

, since the first term of this inequality is nonnegative. There is left to show then that $\frac{\lambda_{t+1}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t-1}^{*}}{P_{t}^{*}}>\frac{\lambda_{t+1}^{*}\theta_{t+1}^{*}}{P_{t+1}^{*}\theta_{t}^{*}}+\frac{\lambda_{t-1}^{*}\theta_{t-1}^{*}}{P_{t}^{*}\theta_{t}^{*}}$, or $\frac{\lambda_{t+1}^{*}(\theta_{t+1}^{*}-\theta_{t}^{*})}{P_{t+1}^{*}}<\frac{\lambda_{t-1}^{*}(\theta_{t}^{*}-\theta_{t-1}^{*})}{P_{t}^{*}}$. If $\lambda_{t+1}^{*}\leq 0$, the inequality holds trivially. If $\lambda_{t+1}^{*}>0$, since $\lambda_{t+1}^{*}\theta_{t+1}^{*}<\lambda_{t-1}^{*}\theta_{t-1}^{*}$, it is sufficient that $\frac{\theta_{t+1}^{*}-\theta_{t}^{*}}{P_{t+1}^{*}\theta_{t+1}^{*}}\leq\frac{\theta_{t}^{*}-\theta_{t-1}^{*}}{P_{t}^{*}\theta_{t-1}^{*}}$, which is equivalent to the smoothness condition. ■

Claim 3 *Let $\lambda_{t-1}^{*}\in\mathbb{R}$, $\lambda_{t}^{*}\leq 0$, $\lambda_{t+1}^{*}\leq 0$, and $\gamma_{t}^{*}=0$. Then, under the conditions stated in the proposition, if follows that $-\lambda_{t+1}^{*}(1-\theta_{t+1}^{*})+\lambda_{t}^{*}(1-\theta_{t}^{*})>0$ and $\mu_{t+1}^{*}>0$ if either one of these conditions is met: a) $\lambda_{t-1}^{*}>0$, or b) $\lambda_{t-1}^{*}\leq 0$ and $-\lambda_{t}^{*}(1-\theta_{t}^{*})>-\lambda_{t-1}^{*}(1-\theta_{t-1}^{*})$.*

Proof. Since $\gamma_{t}^{*}=0$,

$$\begin{aligned} & -\frac{\lambda_{t+1}^{*}(1-\theta_{t+1}^{*})}{P_{t+1}^{*}}-\frac{\lambda_{t-1}^{*}(1-\theta_{t-1}^{*})}{P_{t}^{*}}+\frac{\lambda_{t}^{*}(1-\theta_{t}^{*})}{P_{t+1}^{*}}+\frac{\lambda_{t}^{*}(1-\theta_{t}^{*})}{P_{t}^{*}} \\ =\ & E(v_{1}\,|t,\Theta^{*})-E(v_{1}\,|t+1,\Theta^{*}) \end{aligned} \tag{C3.1}$$

It is apparent from this equation that $-\lambda_{t+1}^{*}(1-\theta_{t+1}^{*})+\lambda_{t}^{*}(1-\theta_{t}^{*})>0$, since its RHS is nonnegative and $\lambda_{t}^{*}(1-\theta_{t}^{*})-\lambda_{t-1}^{*}(1-\theta_{t-1}^{*})<0$ in both parts a) and b) of the claim. It is left to show that

$$\begin{aligned} & -\frac{\lambda_{t+1}^{*}\theta_{t+1}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t}^{*}\theta_{t}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t}^{*}\theta_{t}^{*}}{P_{t}^{*}}-\frac{\lambda_{t-1}^{*}\theta_{t-1}^{*}}{P_{t}^{*}} \\ >\ & E(v_{2}\,|t,\Theta^{*})-E(v_{2}\,|t+1,\Theta^{*}) \end{aligned} \tag{C3.2}$$

(yielding $\mu_{t+1}^{*}>0$), or, by adding up the previous equation,

$$\begin{aligned} & -\frac{\lambda_{t+1}^{*}}{P_{t+1}^{*}}-\frac{\lambda_{t-1}^{*}}{P_{t}^{*}}+\frac{\lambda_{t}^{*}}{P_{t+1}^{*}}+\frac{\lambda_{t}^{*}}{P_{t}^{*}} \\ >\ & E(B\,|t,\Theta^{*})-E(B\,|t+1,\Theta^{*}) \end{aligned} \tag{C3.3}$$

We will proceed by showing that the RHS of (C3.3) is not higher than the corresponding side in (C3.1), and that the LHS in (C3.3) is higher than the LHS in the equation (C3.1). The RHSs part follows directly from regularity. Concerning the LHSs, notice that

$$\begin{aligned}
& -\frac{\lambda^*_{t+1}(1-\theta^*_{t+1})}{P^*_{t+1}} - \frac{\lambda^*_{t-1}(1-\theta^*_{t-1})}{P^*_t} + \frac{\lambda^*_t(1-\theta^*_t)}{P^*_{t+1}} + \frac{\lambda^*_t(1-\theta^*_t)}{P^*_t} \\
\leq \quad & -\frac{\lambda^*_{t+1}(1-\theta^*_{t+1})}{P^*_{t+1}(1-\theta^*_t)} - \frac{\lambda^*_{t-1}(1-\theta^*_{t-1})}{P^*_t(1-\theta^*_t)} + \frac{\lambda^*_t}{P^*_{t+1}} + \frac{\lambda^*_t}{P^*_t}
\end{aligned} \tag{C3.4}$$

, since the LHS of this latter inequality is nonnegative. It suffices then to show that $-\frac{\lambda^*_{t+1}}{P^*_{t+1}} - \frac{\lambda^*_{t-1}}{P^*_t} > -\frac{\lambda^*_{t+1}(1-\theta^*_{t+1})}{P^*_{t+1}(1-\theta^*_t)} - \frac{\lambda^*_{t-1}(1-\theta^*_{t-1})}{P^*_t(1-\theta^*_t)}$, or $-\frac{\lambda^*_{t+1}(\theta^*_{t+1}-\theta^*_t)}{P^*_{t+1}} > -\frac{\lambda^*_{t-1}(\theta^*_t-\theta^*_{t-1})}{P^*_t}$. This immediately follows in part a), since $\lambda^*_{t+1} \leq 0 < \lambda^*_{t-1}$. Concerning condition b), since $-\lambda^*_{t+1}(1-\theta^*_{t+1}) > -\lambda^*_{t-1}(1-\theta^*_{t-1})$, it is sufficient to show $\frac{\theta^*_{t+1}-\theta^*_t}{P^*_{t+1}(1-\theta^*_{t+1})} \geq \frac{\theta^*_t-\theta^*_{t-1}}{P^*_t(1-\theta^*_{t-1})}$. This inequality follows from the smoothness condition.

■

The inductive argument:

Step 1) Proof of $\lambda^*_{T-1} \leq 0$.

Suppose that $\lambda^*_{T-1} > 0$. Hence $\gamma^*_T > 0$, by $\text{FOC}_{1,T}$. By claim 1, it must happen that $\mu^*_T = 0$. By comparing $\text{FOC}_{2,T}$ to $\text{FOC}_{2,T-1}$, a necessary condition is $\lambda^*_{T-2}\theta^*_{T-2} > \lambda^*_{T-1}\theta^*_{T-1}$. This altogether implies $\gamma^*_{T-1} > 0$, by claim 2. Therefore, we must have $\mu^*_{T-1} = 0$, by claim 1 again. Because of this, a new use of claim 2 yields $\lambda^*_{T-3}\theta^*_{T-3} > \lambda^*_{T-2}\theta^*_{T-2} (> \lambda^*_{T-1}\theta^*_{T-1})$ and $\gamma^*_{T-2} > 0$. By applying claims 1 and 2 recursively, the following chain of implications of $\lambda^*_{T-1} > 0$ is found: $\gamma^*_T > 0$, $\mu^*_T = 0$, $\gamma^*_{T-1} > 0$, $\mu^*_{T-1} = 0$... It comes to a point where $\gamma^*_2 > 0$ and $\lambda^*_1\theta^*_1 > \lambda^*_2\theta^*_2 > ... > 0$.

By claim 1, the $\mu^*_2 = 0$ condition has to be met. Then the following contradiction arises. In order to guarantee $\mu^*_2 = 0$, it is needed $M^* > 0$ and $\frac{M^*}{P^*_1} - \frac{\lambda^*_1\theta^*_1}{P^*_1} > 0$. Therefore, by comparing $\text{FOC}_{2,1}$ to $\text{FOC}_{1,1}$, it follows that $\mu^*_1 > 0$. But both $M^* > 0$ ($\Longrightarrow Q^*_2(1) = Q^*_1(1)$) and $\mu^*_1 > 0$ cannot happen at the same time. $\mu^*_1 > 0$ and the feasibility constraints imply $Q^*_2(1) \geq \frac{\Psi(\theta^*_1)^n-(1/2)^n}{n(\Psi(\theta^*_1)-(1/2))} (>) \frac{(1/2)^n-[1-\Psi(\theta^*_1)]^n}{n(1/2-[1-\Psi(\theta^*_1)])} \geq Q^*_1(1)$. Therefore, we reach a contradiction and we conclude that $\lambda^*_{T-1} \leq 0$.

Step 2) Given that $\lambda^*_{T-1} \leq 0$, it is proved that $\lambda^*_{T-2} \leq 0$.

The procedure consists again of supposing the contrary, $\lambda^*_{T-2} > 0$, and proving contradiction. The $\gamma^*_{T-1} > 0, \mu^*_{T-1} = 0, \gamma^*_{T-2} > 0, ...$ argument, mimicking the case of λ^*_{T-1}, leads to the contradiction result.

Step 3) Proof that λ^*_{T-1} and λ^*_{T-2} are such that if $\lambda^*_{T-3} > 0$, then we must have $\gamma^*_{T-2} > 0$.

Suppose not, so $\gamma^*_{T-2} = 0$. Another induction argument, this time using claims 1 and 3, will lead to contradiction. The use of the latter claim for $t = T-2$ readily proves the goal of this step. On the one hand, it follows that $\mu^*_{T-1} > 0$. On the other, it is obtained that $-\lambda^*_{T-1}(1-\theta^*_{T-1}) +$

$\lambda^*_{T-2}(1-\theta^*_{T-2}) > 0$, which forces $\gamma^*_{T-1} > 0$, in view of $\lambda^*_{T-1} \leq 0$, $\text{FOC}_{1,T}$ and $\text{FOC}_{1,T-1}$. Thus, by invoking claim 1, a contradiction arises.

Step 4) Given that if $\lambda^*_{T-3} > 0$ then $\gamma^*_{T-2} > 0$, it is proved that $\lambda^*_{T-3} \leq 0$.

Suppose not. Then the use of the recursive argument used in Step 2 leads to contradiction.

Step 5) If $\lambda^*_{T-4} > 0$, then $\gamma^*_{T-3} > 0$.

Otherwise, a contradiction of the type $\mu^*_{T-1} > 0$ vs. $\gamma^*_{T-1} > 0$ is reached. The technique to do so is analogous to the one we used in Step 3. The repeated use of claims 1 and 3 generates a chain of results $\gamma^*_t = 0$, $\mu^*_{t+1} > 0$, $\gamma^*_{t+1} = 0$,... and $-\lambda^*_t(1-\theta^*_t) > -\lambda^*_{t+1}(1-\theta^*_{t+1}) > ...$ that ultimately leads to a situation where $\mu^*_{T-1} > 0$ and $\gamma^*_{T-1} > 0$, an impossible result.

...

Intermediate Steps)

The proof continues by showing that $\lambda^*_{T-4} \leq 0$ (using claims 1 and 2), that if $\lambda^*_{T-5} > 0$ then $\gamma^*_{T-4} > 0$ (using claims 1 and 3), that $\lambda^*_{T-5} \leq 0$, and so on. A point is reached where $\lambda^*_2 \leq 0$ and if $\lambda^*_1 > 0$ then $\gamma^*_2 > 0$.

...

Final Step) The proof finishes by showing that $\lambda^*_1 > 0$ is not possible.

If that were the case, then $\gamma^*_2 > 0$, and in order to avoid $\mu^*_2 > 0$ it is necessary that $M^* > 0$ and $\frac{M^*}{P^*_1} - \frac{\lambda^*_1\theta^*_1}{P^*_1} > 0 > -\frac{M^*}{P^*_1} - \frac{\lambda^*_1(1-\theta^*_1)}{P^*_1}$. This leads to $\mu^*_1 > 0$, which once again is incompatible with $M^* > 0$ (see second paragraph in Step 1). □

The proof that lambdas are nonpositive: $Z^*_{1T} > 0$ case.

The first one has to notice in this case is that it is mandatory to have $\gamma^*_T = 0$. Going to $\text{FOC}_{1,T}$, this directly implies $\lambda^*_{T-1} < 0$ (in fact, $\frac{\lambda^*_{T-1}(1-\theta^*_{T-1})}{P^*_T} + \frac{Z^*_{1T}}{P^*_T} < 0$). >From this point, the proof is equivalent to the one of the $Z^*_{1T} > 0$ case, only that the inductive argument here starts from λ^*_{T-2}. Particularly, claims 2 and 3 follow after minor changes that would include the Z^*_{1T} multiplier into the calculations. □

Since $\lambda^*_1 \leq 0$ (thus $-\lambda^*_1\theta^*_1 \geq -\lambda^*_1(1-\theta^*_1)$) and $E(v_2\,|1,\Theta^*) \equiv E(Br\,|1,\Theta^*) > E(B[1-r]\,|1,\Theta^*) \equiv E(v_1\,|1,\Theta^*)$, the desired result, $\mu^*_1 > 0$, is finally obtained. Thus, the feasibility constraint will bind at $r = 1/2$.

It remains to be argued that this result will hold in the continuous case, where the message space is unrestricted. But the continuous case is indeed approached as $T \to \infty$. Let $\mathbb{Q}_T$ be the set of all reduced-form mechanisms satisfying feasibility and incentive compatibility when T is the maximum number of regions in which the interval $[1/2, 1]$ is divided into. Let $\mathbb{Q}_T^*$ be the set of optimal mechanisms if T is the maximum number of regions. $\mathbb{Q}$ and $\mathbb{Q}^*$ are defined analogously, for the case where there is no restriction in the number of regions. Since $\mathbb{Q}_T \to \mathbb{Q}$ uniformly as $T \to \infty$, it follows that $\mathbb{Q}_T^* \to \mathbb{Q}^*$ uniformly as well. The reason is that the objective function $O = \int_{1/2}^{1} \psi(r)E(B\,|r)[Q_2(r)r + Q_1(r)(1-r)]dr$ is continuous with respect to the mechanism $Q = (Q_1, Q_2)$, being O a linear functional on $\mathbb{Q}$. (Since $\mathbb{Q}_1 \subset \mathbb{Q}_2 \subset ... \subset \mathbb{Q}$, O is also a linear functional on $\mathbb{Q}_T$ for any $T \in \mathbb{N}$). Formally, the continuity property implies that for any $Q^* \in \mathbb{Q}^*$ and any $\varepsilon > 0$ there exists a natural number $\tilde{T}$ and a sequence $\{Q_T^*\}_{T\in\mathbb{N}}$ such that $Q_T^* \in \mathbb{Q}_T^*$ for any $T \in \mathbb{N}$ and $\| Q_T^* - Q^* \|_\infty < \varepsilon$ for any $T > \tilde{T}$, where $\| A \|_\infty \equiv \sup\{\| A(r) \| : r \in [1/2, 1]\}$ is the uniform norm. If this were not the case there would be a violation of $Q^* \in \mathbb{Q}^*$. This result implies $\mathbb{Q}_T^* \to \mathbb{Q}^*$. **QED**

APPENDIX C: Proof of proposition 2

As in proposition 1 (see Appendix B), the proof is approached via a discretization of the message space, extending then the results to the unrestricted message space by a limit argument. The reader might refer to the Appendix B in order to understand notation matters that would be redundant here.

Let T be the number of elements of the message space. We thus partition the $[1/2, 1]$ interval into T pairwise disjoint subsets or *regions*. For each region $t \in \{1, ..., T\}$ we allocate reduced form probabilities $Q_1(t)$ and $Q_2(t)$, slightly abusing notation. There is a profile $\Theta \equiv (\theta_1, ..., \theta_{T-1})$ of region boundaries such that r is in region t in and only if $\theta_{t-1} \le r < \theta_t$ (the only exception: $r = 1$ belongs to region T). WLOG we set $\theta_0 = 1/2$ and $\theta_T = 1$. WLOG we set $\theta_0 < \theta_1 < ... < \theta_{T-1} < \theta_T$. We are then choosing a pair $(Q, \Theta)_T$ to solve the allocation problem. A complete statement of the problem will be found in the Appendix B.

WLOG, we assume that the optimal $(Q^*, \Theta^*)_T$ constitutes an actual partition in the sense that $t \neq t' \iff Q^*(t) \neq Q^*(t')$ (arguments for that are also found in the Appendix B). In such a case, since the Q_1 function is solved by ignoring other constraints but the own Q_1-constraints (denoted as $\gamma-$constraints in the Appendix B), its solution is

$$Q_1^*(t) = \frac{[1-\Psi(\theta_{t-1}^*)]^n - [1-\Psi(\theta_t^*)]^n}{n[\Psi(\theta_t^*) - \Psi(\theta_{t-1}^*)]} \qquad (t = 1, ..., T)$$

This happens because Q_1 is optimally strictly increasing, and therefore all $\gamma-$constraints (the only ones taken into account now) must bind. The property required to $C(\cdot)$ in the statement of the proposition ensures that $Q_1^*(T)$ follows the aforementioned formula. If the criterion function were negative in expectations on some interval including the point $r = 1$, then the optimal mechanism would give $Q_1^*(T) = 0$, and the proposition might not hold.

Now we prove that there exists a Q_2^* function that is constructed according to IC, meets feasibility and makes the Q_2-constraint at 1/2 bind. This would show that there was no loss in ignoring the Q_2-constraint when choosing the Q_1 function, thus we proceeded correctly. Define

$$Q_2^H(t) = \frac{\Psi(\theta_t^*)^n - \Psi(\theta_{t-1}^*)^n}{n[\Psi(\theta_t^*) - \Psi(\theta_{t-1}^*)]} \qquad (t = 1, ..., T)$$

, which is a function that would make all $\mu-$constraints (the discrete version of the Q_2-constraints, see the Appendix B) bind. It can be shown that $Q_1^*(t) + Q_2^H(t)$ is increasing in t (see footnote 5). Denoting $\Delta Q_2(t) \equiv Q_2(t) - Q_2(t-1)$ and $\nabla Q_1(t) \equiv Q_1(t-1) - Q_1(t)$, and using the BIC constraints (called $\lambda-$constraints in the Appendix B), we obtain

$$\Delta Q_2^*(t) = \frac{1 - \theta_{t-1}^*}{\theta_{t-1}^*} \nabla Q_1^*(t) < \Delta Q_2^H(t) \qquad (t = 2, ..., T)$$

This property ensures that if $Q_2^*(t) < Q_2^H(t)$ and $t' > t$, then $Q_2^*(t') < Q_2^H(t')$. Similarly, if $Q_2^*(t) > Q_2^H(t)$ and $t' < t$, then $Q_2^*(t') > Q_2^H(t')$. So once the two functions cross each other, they do not do it again.

Having a look at the constraint function $R_2(r)$, where $r \in t$, we can see that $signR_2'(r) = sign[Q_2^*(t) - Q_2^H(t)]$, and this derivative exists almost everywhere. Then, if a function Q_2 is chosen such that BIC is satisfied, $Q_2(1) = Q_2^H(1)$ and hence $Q_2(T) \equiv \underline{Q} < Q_2^H(T)$, we obtain, abusing notation, $R_2(1/2 \left| \underline{Q} \right.) > 0$. If, on the other hand, we choose Q_2 such that BIC is met and $Q_2(T) = Q_2^H(T) \equiv \bar{Q}$, we get $R_2(1/2 \left| \bar{Q} \right.) < 0$. Since $R_2(1/2 \left| Q \right.)$ is continuous in Q, by the intermediate value theorem there exists a function Q_2^* satisfying BIC such that $Q_2^*(T) \in (\underline{Q}, \bar{Q})$ and $R_2(1/2 \left| Q_2^*(T) \right.) = 0$. That is, the Q_2-constraint at 1/2 binds. Since the step functions Q_2^* and Q_2^H cross each other only once, no other feasibility constraint could be violated. Obviously, $Q_2^*(t)$ is a valid probability for any t. And the $Q_2^*(t)$ function meets IC. Therefore, a solution exists of the form proposed in this proposition.

The proof finishes by means of presenting the continuous, unrestricted-message-space model as the limit of this restricted model as $T \to \infty$. This has been shown at the end of Appendix B. **QED**

APPENDIX D: Proofs in Section 3

Proof. Proposition 5

The allocation probabilities in the general-rule $\{2E, 2M, 1M, 1E\}$ hierarchical mechanism account for interim probabilities $Q_2^H(E) = \frac{1+\Psi(\theta)}{2}$, $Q_2^H(M) = \frac{\Psi(\theta)+1/2}{2}$, $Q_1^H(E) = \frac{1-\Psi(\theta)}{2}$ and

$Q_1^H(M) = \frac{3/2-\Psi(\theta)}{2}$.

The RRM implies the following interim probabilities, superscripted by R: $Q_2^R(E) = Q_2^R(M) = 3/4$, and $Q_1^R(E) = Q_1^R(M) = 1/4$. The Two-Agent Simple EMPM$_\theta$ is characterized by the following interim probabilities: $Q_2^*(I) = \frac{1-\theta}{\theta} Q_2^H(I) + \left(1 - \frac{1-\theta}{\theta}\right) Q_2^R(I)$ and $Q_1^*(I) = Q_1^H(I)$, for any $I \in \{M, E\}$. Notice $Q_2^H(E) - Q_2^R(E) = \frac{\Psi(\theta)-1/2}{2} = Q_1^R(E) - Q_1^H(E)$, and $Q_2^R(M) - Q_2^H(M) = \frac{1-\Psi(\theta)}{2} = Q_1^H(M) - Q_1^R(M)$. Total ex ante expected gain per agent, when comparing the Two-Agent Simple EMPM$_\theta$ to the RRM, can be rearranged as

$$\begin{aligned} &[1-\Psi(\theta)]\,[\Psi(\theta)-1/2] \\ &\cdot \left[\frac{1-\theta}{\theta}\left(E(v_2\,|E,\theta\,) - E(v_2\,|M,\theta\,)\right) + E(v_1\,|M,\theta\,) - E(v_1\,|E,\theta\,)\right] \end{aligned}$$

If its derivative is taken *holding* $\frac{1-\theta}{\theta}$ *constant*, one obtains (after proper rearrangement) the following component:

$$\begin{aligned} & \frac{\psi(\theta)}{2}\{\frac{1-\theta}{\theta}\left[E(v_2\,|\tilde{r} \geq 1/2\,) - E(v_2\,|\tilde{r} = \theta\,)\right] \\ & -E(v_1\,|\tilde{r} \geq 1/2\,) + E(v_1\,|\tilde{r} = \theta\,)\} \\ = \quad & \frac{\psi(\theta)}{2}\{\frac{1-\theta}{\theta}E(v_2\,|\tilde{r} \geq 1/2\,) - E(v_1\,|\tilde{r} \geq 1/2\,)\} \end{aligned}$$

since $\frac{1-\theta}{\theta}E(v_2\,|\tilde{r} = \theta\,) = E(v_1\,|\tilde{r} = \theta\,)$. This component is positive when $\theta < \theta^* \equiv \frac{E(v_2|\tilde{r}\geq 1/2\,)}{E(v_2|\tilde{r}\geq 1/2\,)+E(v_1|\tilde{r}\geq 1/2\,)} = \frac{E(v_2|\tilde{r}\geq 1/2\,)}{EB}$, negative when $\theta > \theta^*$ and zero if $\theta = \theta^*$.

The second component of the derivative of (ex ante expected) gains comes when $\frac{1-\theta}{\theta}$ is differentiated and multiplied by $[1-\Psi(\theta)]\,[\Psi(\theta)-1/2]\,(E(v_2\,|E,\theta\,) - E(v_2\,|M,\theta\,))$. Clearly this component is nonpositive. Therefore, any $\theta > \theta^*$ performs worse than θ^*.

To prove the second statement of the proposition, one just needs to notice that ex ante expected gains are equal whether $\theta = 1$ or $\theta = 1/2$ is chosen (in both cases, gains collapse to zero). Since $\theta = 1$ performs worse than θ^*, so does $\theta = 1/2$, and by continuity of ex ante expected gains there must be a range of thetas higher than and close to $1/2$ that are also (ex ante) dominated by θ^*. ■

Derivation. Proposition 6

Denote $P_M \equiv \Psi(\theta) - 1/2$ and $P_E \equiv 1 - \Psi(\theta)$. M refers to moderate preferences in favor of object 2, and E denotes extreme ones. Given θ, the following discrete version of the OCM problem is solved:[23]

$$\max_{Q(\cdot)} \sum_{t=M,E} P_t \left[Q_2(t)E(v_2 \,|t,\theta\,) + Q_1(t)E(v_1 \,|t,\theta\,)\right]$$

subject to

$$\frac{1}{n}[1-\Psi(\theta)^n] - P_E Q_2(E) \geq 0 \tag{μ_E}$$

$$\frac{1}{n}[1-(1/2)^n] - P_E Q_2(E) - P_M Q_2(M) \geq 0 \tag{μ_M}$$

$$\frac{1}{n}\left[1-[1-\Psi(\theta)]^n\right] - P_E Q_2(E) - P_M[Q_2(M)+Q_1(M)] \geq 0 \tag{γ_M}$$

$$\frac{1}{n} - P_E[Q_2(E)+Q_1(E)] - P_M[Q_2(M)+Q_1(M)] \geq 0 \tag{γ_E}$$

$$\theta[Q_2(E)-Q_2(M)] + (1-\theta)[Q_1(E)-Q_1(M)] = 0 \tag{λ}$$

, plus $Q_2(E) \geq Q_2(M)$ (multiplier l) and $Q_1(E) \leq Q_1(M)$, $Q_2(M) \geq Q_1(M)$ (multiplier m), and nonnegativity $Q_1(E) \geq 0$, (with associate multiplier Z). Greek letters stand for the corresponding multipliers. $\mu-$constraints are analogous to the Q_2-constraints in the reduced-form OCM problem, and $\gamma-$constraints correspond to the Q_1-constraints, following Border's (1991) specification once again. The rest of constraints are equivalent to incentive compatibility. Concretely, the $\lambda-$constraint states that the θ type is indifferent between the two preference intensities it separates. The fact that $Q_1(\cdot)$ is nonincreasing is already implied by $Q_2(\cdot)$ being nondecreasing and the $\lambda-$constraint, therefore we can ignore it. It will also be assumed that $l = 0$, since otherwise the information on preference intensities is wasted.

First order conditions are

$$E(v_2 \,|E,\theta\,) + \frac{\lambda^*\theta}{P_E} = \sum_{t=E,M} (\gamma_t^* + \mu_t^*) \tag{FOC$_{2E}$}$$

$$E(v_2 \,|M,\theta\,) - \frac{\lambda^*\theta}{P_M} + \frac{m^*}{P_M} = \mu_M^* + \sum_{t=E,M} \gamma_t^* \tag{FOC$_{2M}$}$$

$$E(v_1 \,|M,\theta\,) - \frac{\lambda^*(1-\theta)}{P_M^*} - \frac{m^*}{P_M} = \sum_{t=E,M} \gamma_t^* \tag{FOC$_{1M}$}$$

$$E(v_1 \,|E,\theta\,) + \frac{\lambda^*(1-\theta)}{P_E} + \frac{Z^*}{P_E} = \gamma_E^* \tag{FOC$_{1E}$}$$

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[23] A generalized version of this discretized model is presented and solved in Appendix B. Notice that the maximization program just considers cases where object 2 is the preferred one, as lemma 4 dictates, emulating the reduced-form OCM problem.

Lemma 7 (see Appendix B) shows that if one out of γ_E^*, μ_E^* is strictly positive, then the other multiplier must be zero. With this and under the regularity condition, it can be shown that $\lambda^* \leq 0$. If this is not true, then $\gamma_E^* > 0$, and therefore $\mu_E^* = 0$. This implies $m^* > 0$ and $\frac{m^*}{P_M} - \frac{\lambda^* \theta}{P_M} > 0$. $m^* > 0$ implies $Q_2^*(M) = Q_1^*(M)$, which is incompatible with $\mu_M^* > 0$. But $\frac{m^*}{P_M} - \frac{\lambda^* \theta}{P_M} > 0$, $m^* > 0$ and $\lambda^* > 0$ imply precisely that $\mu_M^* > 0$ (by substracting FOC_{1M} to FOC_{2M}).[24] So by contradiction, $\lambda^* \leq 0$. The immediate consequence of this is $\mu_M^* > 0$ (once again, by substracting FOC_{1M} to FOC_{2M}). Therefore, the μ_M-constraint binds. A second consequence is $\gamma_M^* > 0$ (by substracting FOC_{1E} to FOC_{1M}), so the γ_M-constraint binds.

By the use of both the $\lambda-$constraint and the binding μ_M-constraint, $Q_2^*(\cdot)$ can be substituted out in terms of $Q_1^*(\cdot)$. By the binding μ_M- and γ_M-constraints, $Q_1^*(M) = \frac{(1/2)^n - [1-\Psi(\theta)]^n}{n[\Psi(\theta)-1/2]}$. So if $Q_1^*(E)$ is calculated, the solution is done. In doing so, one has to check whether the (probability-weighted) marginal contribution of $Q_1^*(E)$ to the objective function (after having substituted $Q_2^*(\cdot)$ out) is positive. After some algebra, this is denoted as

$$\tilde{C}(\theta) \equiv E(v_1 \,|\tilde{r} \geq \theta\,) - \frac{1-\theta}{\theta} \left[E(v_2 \,|\tilde{r} \geq \theta\,) - E(v_2 \,|\tilde{r} \geq 1/2\,)\right]$$

and it is the discrete version of the criterion function. If $\tilde{C}(\theta) > 0$, then the solution entails $\gamma_E^* > 0$ and (combining the binding γ_M- and γ_E-constraints) $Q_1^*(E) = [1 - \Psi(\theta)]^{n-1}/n$, a value that will also assumed if $\tilde{C}(\theta) = 0$. Following the proof of proposition 2, the resulting $Q_2^*(\cdot)$ is feasible, and the reduced-form problem is solved. If $\tilde{C}(\theta) < 0$, the solution is more problematic. In some cases, $Q_1^*(E) = 0$ and the corresponding $Q_2^*(\cdot)$ is feasible, and in some others $Q_2^*(\cdot)$ is not feasible and $Q_1^*(E)$ has to be slightly raised above 0 in order to make a feasible $Q_2^*(\cdot)$ incentive compatible.

I summarize. Exception 1 in the proposition covers the $\tilde{C}(\theta) \geq 0$ case. Since consequently the μ_E-constraint does not bind, when a hE type meets a hM type she may not always get object h. Exception 2 covers the $\tilde{C}(\theta) < 0$ + feasible $Q_2^*(\cdot)$ case. Then $Q_1^*(E) = 0$, leading to point *ii*, and the μ_E-constraint does not bind yet, leading to point *i*. Exception 3 arises when $\tilde{C}(\theta) < 0$ and ideally $Q_1^*(E) = 0$, but then $Q_2^*(\cdot)$ is not feasible (the ideal $Q_2^*(E)$ violates the μ_E-constraint). In that case, $Q_2^*(E)$ is pulled up as much as possible, until the μ_E-constraint binds[25] (so when a hE type meets a hM type she always gets object h), and $Q_1^*(E)$ has to be increased in order to satisfy incentive compatibility. That is why with some probability the optimal mechanism is not wasting. The implementation is calculated following Border's (1991) suggestion.

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[24] Notice that this is equivalent to $R_2(1/2) = 0$ in the continuous version of the problem. Remarkably, no smoothness condition is needed in this simple discrete case.

[25] In such a case, $Q_2^*(E) = \frac{1-\Psi(\theta)^n}{n[1-\Psi(\theta)]}$, and by the binding μ_M- constraint, $Q_2^*(M) = \frac{\Psi(\theta)^n-(1/2)^n}{n[\Psi(\theta)-1/2]}$.

Proof. Remark 2

Notation is shortened in the following way: $V_h(\theta) \equiv E(v_h \,|\tilde{r} \geq \theta\,)$, $v_h(\theta) \equiv E(v_h \,|\tilde{r} = \theta\,)$, $\Sigma(\theta) \equiv E(B\,|\tilde{r} \geq \theta\,)$, $B(\theta) \equiv E(B\,|\tilde{r} = \theta\,)$. Now, $\tilde{C}(\theta) \geq 0$ is equivalent to

$$\theta \leq \varphi(\theta) \equiv \frac{V_2(\theta) - V_2(1/2)}{\Sigma(\theta) - V_2(1/2)}$$

Observe that $\varphi(1/2) = 0$, $\varphi(1) = 1$ and $\varphi(\cdot)$ is differentiable on the domain $(1/2, 1)$. The derivative is

$$\begin{aligned}
\varphi'(\theta) &= \frac{\psi(\theta)}{1-\Psi(\theta)}\left[\frac{V_2(\theta) - v_2(\theta)}{\Sigma(\theta) - V_2(1/2)} - \varphi(\theta)\frac{\Sigma(\theta) - B(\theta)}{\Sigma(\theta) - V_2(1/2)}\right] \\
&= \frac{\psi(\theta)}{1-\Psi(\theta)}\left[\varphi(\theta)\frac{B(\theta) - V_2(1/2)}{\Sigma(\theta) - V_2(1/2)} - \frac{v_2(\theta) - V_2(1/2)}{\Sigma(\theta) - V_2(1/2)}\right]
\end{aligned}$$

The second equality follows after noticing that $\frac{V_2(\theta)-v_2(\theta)}{\Sigma(\theta)-V_2(1/2)} = \varphi(\theta) - \frac{v_2(\theta)-V_2(1/2)}{\Sigma(\theta)-V_2(1/2)}$. Now the strategy consists of considering critical values $\hat{\theta}$ for which $\varphi(\hat{\theta}) = \hat{\theta}$. In such cases, we have

$$\varphi'(\hat{\theta}) = \frac{\psi(\hat{\theta})(1-\hat{\theta})}{1-\Psi(\hat{\theta})}\frac{V_2(1/2)}{\Sigma(\hat{\theta}) - V_2(1/2)}$$

since $\theta B(\theta) = v_2(\theta)$.

If $B(\cdot)$ is (weakly) decreasing, then so is $\Sigma(\cdot)$, hence $\min\limits_{\hat{\theta}\in[1/2,1]} \frac{V_2(1/2)}{\Sigma(\hat{\theta})-V_2(1/2)} = \frac{V_2(1/2)}{V_1(1/2)}$ (recall that $B = v_1 + v_2$, thus $\Sigma(1/2) = V_1(1/2) + V_2(1/2)$). Condition 1 in the remark implies $\varphi'(\hat{\theta}) \geq 1$ for any $\hat{\theta}$ (strictly if $\hat{\theta} = 1$). This is incompatible with $\varphi(\theta)$ being above θ at some point.

Condition 2 ensures that for any two critical values $\hat{\theta}_1 > \hat{\theta}_2$, $\varphi'(\hat{\theta}_1) \geq \varphi'(\hat{\theta}_2)$. This is again incompatible with $\varphi(\theta)$ being above θ at some point. ■