Asset Market Equilibrium under Rational Inattention*

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Abstract

We propose a noisy rational expectations equilibrium model of asset markets with rationally inattentive investors. We incorporate any finite number of assets with arbitrary correlation. We also do not restrict the signal form and show that investors optimally choose a single signal, which is a noisy linear combination of all risky assets. This generates comovement of asset prices and contagion of shocks, even when asset payoffs are negatively correlated. The model also provides testable predictions of the impact of risk aversion, aggregate risk, and information capacity on the security market line, the portfolio dispersion, and the abnormal return.

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1 Introduction

Since the seminal contributions by Grossman and Stiglitz (1980), there has been a growing literature in economics and finance that incorporates asymmetric information in the noisy rational expectations equilibrium framework.[1] This literature typically assumes that information structure is exogenously given. However, agents may endogenously process information given their limited information capacity. For example, investors in financial markets typically pay attention to some individual assets or some portfolios (linear combinations) of assets. They can acquire signals about individual assets or portfolios of assets.

In this paper we propose a multiple asset, noisy rational expectations equilibrium model with rationally inattentive investors. The model features any finite number of risky assets with arbitrarily correlated payoffs and a continuum of ex ante identical investors who face information-processing constraints as in Sims (2003, 2011). Investors observe asset prices and acquire private signals about the assets to reduce uncertainty about their portfolio. We do not restrict the signal form except that it is a noisy linear transformation of asset payoffs. Investors optimally choose both the linear transformation and the precision of the signal subject to an entropy-based information constraint. After allocating their attention, investors incorporate the information from their private signals and asset prices through Bayesian updating to form their posterior beliefs about the asset payoffs and then choose their optimal asset holdings.

We show that each investor will optimally choose a one-dimensional signal that is a noisy linear combination of all risky assets. In a symmetric linear equilibrium, all investors choose the same signal form. We find that an unconditional CAPM holds in our model, but is rejected by econometricans (Type I error). As argued by Andrei, Cujean, and Wilson (2018), there is an information distance between econometricans and the average investor. Thus the security market line (SML) from the econometrican's point of view is different from the true SML in the model. We find that the econometrican's SML may not be linear and can be flatter or steeper than the true SML, unlike the result in Andrei, Cujean and Wilson (2018). Moreover, an increase in the information capacity reduces both the true SML and the econometrician's SML. The reason is that higher capacity allows investors to process more precise signals and hence reduce asset uncertainty more.

We also show that there is excess comovement in asset prices relative to asset fundamentals, as in Mondria (2010). We illustrate this result using a numerical example with three risky assets. One asset is independent of the other two and the other two are arbitrarily correlated. We find that the asset prices can be positively correlated even when asset payoffs are negatively correlated. The reason is that each investor receives a single signal, that is a noisy linear combination of all risky

[1] See Veldkamp (2011) and Angeletos and Chen (2016) for recent surveys of this literature.

assets with all positive coefficients. If there is good news about one asset, then investors observe a high realization of the private signal and they attribute part of the effect to one asset and the rest to the other two assets. This leads to an increase in the prices of all three assets and thus price comovement of these assets.

We also use the above numerical example to illustrate the contagion effect. We show that an increase in the variance of one independent asset payoff causes the prices of the other two assets to decline. This is true even when the payoffs of the other two assets are negatively correlated.

We finally study the implications for the portfolio dispersion. We find that the portfolio holdings dispersion declines in recessions when risk aversion (a proxy for the price of risk) or the aggregate volatility is high. The portfolio return dispersion declines with risk aversion, but increases with the aggregate volatility. Intuitively, higher risk aversion leads to more conservative portfolio choices, and hence a smaller dispersion of portfolio holdings. On the other hand, higher risk aversion or higher aggregate risk leads to higher market risk premium. The portfolio return dispersion reflects the combined effects of portfolio holdings and risk premium. The impact of the portfolio return dispersion depends on which effect dominates.

Extending our model to incorporate a fraction of uninformed investors, we study how informed investors can profit from their information advantage. We find that an informed investor earns an abnormal return. The abnormal return increases with the aggregate risk and the information capacity, and decreases with the fraction of informed investors. Moreover, it has a U-shaped relationship with the risk aversion.

Our paper is closely related to the literature on asset pricing models with rational inattention (Peng (2005), Peng and Xiong (2006), van Nieuwerburgh and Veldkamp (2009, 2010), Mondria (2010), and Kaperckyz, van Nieuwerburgh, and Veldkamp (henceforth KVV) (2016)).[2] Because of the difficulty in the case of multiple assets, this literature typically makes the signal independence assumption. That is, investors are assumed to process information about one asset at a time. The signal vector is equal to the unobservable asset payoff (or risk factor) vector plus a noise. An undesirable feature is that ex ante independent assets remain ex post independent, and hence such an assumption cannot explain asset comovement. KVV (2016) relax this assumption by imposing an invertibility assumption on the signal form. They derive some results different from ours as detailed in Sections 4 and 5.3. Mondria (2010) does not restrict the signal form, but his approach only applies to the two-asset case with ex ante independent assets.

An important contribution of our paper is to analyze the general case with any finite number of correlated assets. We solve for both the linear transformation and the precision of the signal vector. The difficulty is that the information choice problem is nonconvex, unlike the multivariate linear-

[2]See Matejka and Mackay (2005), Luo (2008), Woodford (2009), Maćkowiak and Wiederholt (2009), Caplin, Dean, and Leahy (2018), Miao (2019), Miao, Wu, and Young (2019) for other models with rational inattention. See Sims (2011) and Maćkowiak, Matějka and Wiederholt (2018) for surveys and additional references cited therein.

quadratic-Gaussian framework of Sims (2003), Miao (2019), and Miao, Wu, and Young (2019). We are able to derive a closed-form solution to this problem and find that the optimal signal is one dimensional. In particular, the optimal signal is a noisy linear combination of risky assets, as in Mondria (2010) for the two asset-case. Van Nieuwerburgh and Veldkamp (2010) and KVV (2016) also find that the information choice problem is nonconvex, but their problem is different from ours and their proofs are also different from ours. In particular, they find that investors specialize in learning about only one asset or only one risk factor.

Most studies in the literature on information acquisition assume that investors choose the signal precision only and do not consider the signal form as a linear transformation of states. Unlike the rational inattention framework of Sims (2003, 2011), some researchers impose information cost other than the entropy-based cost. This literature is too large for us to cite it all. Recent contributions include Huang and Liu (2007), Vives (2010), KVV (2016), Andrei and Hasler (2019), and Vives and Yang (2019), among others.

2 Model

Consider a three-date economy populated by a continuum of ex ante identical investors of measure one, indexed by $i \in [0, 1]$. Investors are endowed with initial wealth at date 1. They first choose their private signals given their limited capacity to process information at date 1. At date 2, they decide on the optimal portfolios given the observation of their private signals and the asset price. At the last date, investors consume the payoff of their portfolios.

There are n risky assets and one riskless asset. The riskless asset pays constant R_f units of the consumption good at date 3. The n risky assets pay F units of the consumption good at date 3, where F is a random column vector that is normally distributed with mean $\overline{F}$ and covariance matrix Σ_F (denoted by $F \sim N\left(\overline{F}, \Sigma_F\right)$). Assume that Σ_F is positive definite, denoted by $\Sigma_F \succ 0$.[3] Following KVV (2016), we may use the risk factor representation $F = \Gamma Y$, where $Y = [Y_1, ..., Y_n]'$ represents a column vector of n independent risk factors, and Γ is invertible and represents factor loadings. Let P denote the price vector of the n risky assets. To prevent the equilibrium price from being fully revealing, we introduce noisy asset supply. Suppose that the supply of the risky assets is given by a random vector $Z \sim N\left(\overline{Z}, \Sigma_Z\right)$, where $\Sigma_Z \succ 0$. Assume that Z and F are independent.

2.1 Information Cost

Investors want to acquire information about the risky assets to reduce the uncertainty of their portfolios. They have a limited capacity to process information about asset payoffs. They observe the asset prices and use asset prices and acquired signals to reduce payoff uncertainty. Assume that

[3] The notation $A \succeq B$ means that the matrix $A - B$ is positive semidefinite.

investor i can choose signals of the following form

$$S_i = C_i F + \epsilon_i, \tag{1}$$

where C_i is an n_s by n matrix, the noise $\epsilon_i \sim N(0, \Sigma_{\epsilon i})$ is independent of F and Z, and $\Sigma_{\epsilon i} \succ 0$. We call $(C_i, \Sigma_{\epsilon i})$ an information (signal) structure of investor i, which will be chosen endogenously. Notice that we do not impose any assumption on C_i. In particular, C_i may not be a square matrix in that $n_s \neq n$. Given F and ϵ_i are Gaussian, the signal vector S_i is also Gaussian. As is common in the literature, assume that investors do not process information about the asset supply.

Each investor i faces the following information-processing constraint

$$H(F) - H(F|S_i, P) \leq \kappa, \tag{2}$$

where $\kappa > 0$ is the parameter of the channel capacity, H denotes the Shannon entropy, and $H(\cdot|\cdot)$ denotes the conditional entropy.

For any n-dimensional multivariate normal random variable $X \sim N\left(\overline{X}, \Sigma\right)$, its Shannon entropy is given by

$$H(X) = \frac{1}{2} \log\left((2\pi e)^n \det \Sigma\right),$$

where $\det(\Sigma)$ denotes the determinant of Σ. We will show later that the equilibrium price P is Gaussian and hence F is Gaussian conditional on S_i and P. Thus we can simplify the constraint (2) as

$$\log\det(\mathrm{Var}(F)) - \log\det(\mathrm{Var}(F|S_i, P)) \leq 2\kappa, \tag{3}$$

where Var denotes the variance-covariance operator.

2.2 Decision Problem

Following van Nieuwerburgh and Veldkamp (2009, 2010) and Mondria (2010), assume that each investor i has the following utility function:

$$U_i = \mathbb{E}\left\{-\log\mathbb{E}\left[\exp\left(-\frac{W_i}{\rho}\right)|S_i, P\right]\right\}, \tag{4}$$

where $\rho > 0$ is the risk tolerance parameter and W_i denotes the final wealth level at date 3. The parameter $1/\rho$ represents risk aversion, which also measures the price of risk as in KVV (2016).

Investor i faces the following budget constraint

$$W_i = R_f W_{i0} + X_i' R^e, \tag{5}$$

where $R^e \equiv F - PR_f$ denotes the excess (dollar) return, W_{i0} denotes investor i's initial wealth level, and X_i denotes the vector of his/her risky asset holdings.

Each investor i first chooses a signal structure $(C_i, \Sigma_{\epsilon i})$ at date 1 and then chooses a portfolio demand X_i given the information conveyed by the signal S_i and the price vector P at date 2 to maximize his/her utility in (4) subject to the budget constraint in (5). When solving this problem, investor i takes the price vector P and other investors' information structures as given.

2.3 Equilibrium

An asset market equilibrium with rational inattention consists of a price vector P, a signal structure $(C_i, \Sigma_{\epsilon i})$, and a portfolio demand X_i, for each investor $i \in [0,1]$, such that (i) each investor i solves his/her decision problem in the previous subsection taking the price P and other investors' signal structures as given, and (ii) the market clears in that

$$\int_0^1 X_i di = Z. \tag{6}$$

3 Model Solution

When the information structure is exogenously given, the model is essentially the same as that of Admati (1985). Thus we solve our model in two steps. First, we derive equilibrium with a fixed information structure as in Admati (1985) and the associated utility level of each investor. Second, we solve for the optimal information structure for each investor to maximize his/her expected utility taken the other investors' information structures as given.

3.1 Equilibrium with Fixed Information Structure

Suppose that the information structure $(C_i, \Sigma_{\epsilon i})$ for each i is exogenously given. As in Admati (1985), we can show that the equilibrium asset price takes the following linear form

$$P = A_0 + A_1 F - A_2 Z, \tag{7}$$

where

$$A_0 = \frac{\rho}{R_f}\left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)^{-1}\left(\Sigma_F^{-1}\overline{F} + \Pi\Sigma_Z^{-1}\overline{Z}\right), \tag{8}$$

$$A_1 = \frac{1}{R_f}\left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)^{-1}\left(\Pi + \rho\Pi\Sigma_Z^{-1}\Pi\right), \tag{9}$$

$$A_2 = \frac{1}{R_f}\left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)^{-1}\left(I + \rho\Pi\Sigma_Z^{-1}\right). \tag{10}$$

Notice that A_2 is invertible and satisfies $A_2^{-1}A_1 = \Pi$, where

$$\Pi = \int_0^1 \rho C_i' \Sigma_{\epsilon i}^{-1} C_i di, \tag{11}$$

and $C_i'\Sigma_{\epsilon j}^{-1}C_i$ is called the signal-to-noise ratio (SNR) for investor i in the engineering literature of information theory. The SNR is a positive semidefinite matrix. The equilibrium price is determined by the aggregate SNR, but not a particular C_i or $\Sigma_{\epsilon i}$.

Solving for investor i's optimal portfolio choice yields the familiar mean-variance rule:

$$X_i = \rho \left[\mathrm{Var}\left(R^e | S_i, P\right)\right]^{-1} \mathbb{E}\left[R^e | S_i, P\right]. \tag{12}$$

Imposing the market-clearing condition in (6), computing the optimal wealth level in (5), and substituting the resulting expression in (4), we obtain the following result as in Mondria (2010):

Proposition 1 *Investor i's utility for a fixed information structure $(C_i, \Sigma_{\epsilon i})$ for all $i \in [0,1]$ is given by*[4]

$$U_i = \frac{W_{i0}R_f}{\rho} - \frac{n}{2} + \frac{1}{2}\mathrm{Tr}\left(V_i^{-1}\left(V_e + \overline{R}^e\overline{R}^{e\prime}\right)\right), \tag{13}$$

where

$$\overline{R}^e \equiv \mathbb{E}\left[R^e\right] = \left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)^{-1}\overline{Z}, \tag{14}$$

$$V_e \equiv \mathrm{Var}\left(R^e\right) = \Sigma_F + R_f^2 A_1 \Sigma_F A_1' + R_f^2 A_2 \Sigma_Z A_2' - R_f A_1 \Sigma_F - R_f \Sigma_F A_1', \tag{15}$$

$$V_i \equiv \mathrm{Var}\left(R^e | S_i, P\right) = \left(\Sigma_F^{-1} + \Pi\Sigma_Z^{-1}\Pi + C_i'\Sigma_{\epsilon i}^{-1}C_i\right)^{-1}. \tag{16}$$

Here $\overline{R}^e$ denotes the vector of unconditional expected excess returns, V_e denotes the unconditional covariance matrix of excess returns., and V_i denotes the conditional covariance matrix of excess returns given investor i's information. Proposition 1 gives investor i's ex ante expected utility in equilibrium given a fixed information structure.

3.2 Optimal Information Structure

In this subsection we solve for the optimal information structure $(C_i, \Sigma_{\epsilon i})$ for each infinitesimal investor i. The decision problem is given by

$$\max_{C_i, \Sigma_{\epsilon i} \succ 0} U_i$$

subject to (3), where U_i is given in (13) and equations (14), (15), and (16) hold. When solving this problem, investor i takes the other investors' information structures as given. In particular, he takes Π, $\overline{R}^e$, and V_e as given.

Define

$$\Omega = V_e + \overline{R}^e\overline{R}^{e\prime}. \tag{17}$$

[4]Notice that $\mathrm{Tr}(\cdot)$ denotes the trace operator.

Since $V_e \succ 0$, we have $\Omega \succ 0$. Define investor i's precision matrix of the excess return as $K_i = V_i^{-1}$. Then we use Proposition 1 to transform his/her information choice problem above into the following problem

$$\max_{K_i} \operatorname{Tr}(K_i \Omega) \tag{18}$$

subject to

$$\log\det(\Sigma_F) + \log\det(K_i) \leq 2\kappa, \tag{19}$$

$$K_i \succeq G, \tag{20}$$

where we define

$$G \equiv \Sigma_F^{-1} + \Pi \Sigma_Z^{-1} \Pi \succ 0. \tag{21}$$

Inequality (19) follows from the information-processing constraint (3) and inequality (20) is the no-forgetting constraint analogous to that in Sims (2003). The no-forgetting constraint says that the excess return after acquiring information S_i is less uncertain than that without acquiring information.

We have eliminated the choice of the matrix C_i in the above problem by using matrix inequality (20) to replace equality (16). After obtaining K_i or $V_i = K_i^{-1}$, we use (16) to recover the optimal information structure $(C_i, \Sigma_{\epsilon i})$:

$$C_i' \Sigma_{\epsilon i}^{-1} C_i = K_i - G. \tag{22}$$

Notice that the problem in (18) is not a concave optimization problem because the constraint set in (19) and (20) is not convex as noticed by van Niewerberg and Veldkemp (2010). This is different from the linear-quadratic-Gaussian framework studied by Sims (2003), Miao (2019), and Miao, Wu, and Young (2019). In particular, the solution to the problem in (18) may be at the corner.

Now we solve the problem in (18). Consider the eigen-decomposition:

$$G^{\frac{1}{2}} \Omega G^{\frac{1}{2}} = U \Omega_d U', \tag{23}$$

where U is an orthorgonal matrix and Ω_d is a diagonal matrix $\Omega_d = diag\,(d_j)_{j=1}^n$ with $d_1 > 0, ...,$ and $d_n > 0$ denoting eigenvalues of $G^{\frac{1}{2}} \Omega G^{\frac{1}{2}}$. Without loss of generality, let $d_1 = d_2 = ... = d_m$ be the identical largest elements of Ω_d.

Define the matrix

$$\widetilde{K}_i = U' G^{-\frac{1}{2}} K_i G^{-\frac{1}{2}} U.$$

Then $K_i = G^{\frac{1}{2}} U \widetilde{K}_i U' G^{\frac{1}{2}}$. Substituting this equation into (18), (19), and (20), we find that the problem in (18) becomes

$$\max_{\widetilde{K}} \operatorname{Tr}\left(\widetilde{K}_i \Omega_d\right) \tag{24}$$

subject to $\widetilde{K}_i \succeq I$ and

$$\log\det(\Sigma_F) + \log\det(G) + \log\det\left(\widetilde{K}_i\right) \leq 2\kappa.$$

Given the objective function in (24), only diagonal elements of $\widetilde{K}_i$ matters for the optimization. Thus we can focus only on the diagonal matrix for $\widetilde{K}_i$. We have the following result:

Proposition 2 *Suppose that $d_1, d_2, ..., d_m$ are the m identical largest eigenvalues of the matrix $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$, where G is given in (21) and $1 \leq m \leq n$. Suppose that*

$$\lambda^* \equiv \frac{\exp(2\kappa)}{\det\left(I + \Sigma_F \Pi \Sigma_Z^{-1} \Pi\right)} > 1. \tag{25}$$

If $m = 1$, then the solution to (24) is unique and given by

$$\widetilde{K}_i = I + diag(\lambda^* - 1, 0, ..., 0).$$

If $m \geq 2$, then there are multiple solutions to (24) given by

$$\widetilde{K}_i = I + (\lambda^* - 1) v^* v^{*\prime}$$

where $v^* = [a_1^*, a_2^*, ..., a_n^*]'$ is a column vector satisfying $\sum_{i=1}^m (a_i^*)^2 = 1$ and $a_i^* = 0$ for $i > m$. The optimal precision matrix is given by

$$V_i^{-1} = K_i = G^{\frac{1}{2}} U \widetilde{K}_i U' G^{\frac{1}{2}},$$

The optimal information structure $(C_i, \Sigma_{\epsilon i})$ satisfies

$$C_i' \Sigma_{\epsilon i}^{-1} C_i = (\lambda^* - 1) G^{\frac{1}{2}} U v^* v^{*\prime} U' G^{\frac{1}{2}}, \tag{26}$$

and the optimal signal is one-dimensional.

If the largest eigenvalue of $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$ is unique, then let d_1 be the unique largest eigenvalue without loss of generality. Every investor i will allocate all attention to d_1 to reduce uncertainty associated with d_1. Moreover, every investor i will not attend to other eigenvalues by setting $\widetilde{K}_{i\ell} = 1$ for $\ell \neq 1$. Thus the diagonal matrix $\widetilde{K}_i$ is the same for every investor i and hence the optimal precision matrix of the excess return is also the same for every investor i. It follows from (26) that the SNR $C_i' \Sigma_{\epsilon i}^{-1} C_i$ is also the same for every investor i as $v^* v^{*\prime} = [1, 0, ..., 0]' [1, 0, ..., 0]$.

Notice that the optimal information structure $(C_i, \Sigma_{\epsilon i})$ is not unique. A particular solution is given by

$$\Sigma_{\epsilon i} = \left(\tilde{K}_{i1} - 1\right)^{-1} > 0,$$

and

$$C_i = [1, 0, ..., 0]_{1 \times n} U' G^{\frac{1}{2}}.$$

Thus C_i is the first principal component of the matrix $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$. For any solution, the optimal signal is one dimensional. Investor i learns a linear combination of all risky assets in that C_i is a one-dimensional vector. We can write investor i's one-dimensional signal as

$$S_i = \sum_{j=1}^{n} C_{ij} F_j + \epsilon_i,$$

where C_{ij} and F_j are the jth components of the vectors C_i and F, respectively.

Following Mondria (2010), we can normalize the signal weight on the first risky asset to one by setting $\overline{C}_i = C_i / C_{i1}$ if $C_{i1} \neq 0$. Then the new signal structure is $\left(\overline{C}_i, \overline{\Sigma}_{\epsilon i}\right)$, where the noise variance is given by $\overline{\Sigma}_{\epsilon i} = \Sigma_{\epsilon i} C_{i1}^2$. Alternatively, we can normalize $\Sigma_{\epsilon i}$ to 1 and let $C_{i1} > 0$.

If $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$ has $m \geq 2$ identical largest eigenvalues, then the optimal information structure is not unique even after normalization. The optimal signal is still one dimensional because the dimension is determined by the rank of $v^* v^{*\prime}$ as shown in (26). Each investor acquires a signal that is a linear combination of the risky assets, but the normalized signal may not be identical for all investors. We will show next that this is the source of the existence of an asymmetric equilibrium.

3.3 Equilibrium under Rational Inattention

The existence of an equilibrium under rational inattention depends on the existence of an optimal information structure $(C_i, \Sigma_{\epsilon i})$ for all i that satisfies equation (26). This is a fixed point problem. We are unable to prove the existence for the general case.[5] Here we provide an algorithm to solve for an equilibrium. Specifically, let $\Phi_i \equiv C_i' \Sigma_{\epsilon i}^{-1} C_i$ denote the SNR for all $i \in [0, 1]$. The algorithm consists of the following steps:

Step 1. Given a guess for Φ_i for all i, we can determine Π in equation (11).

Step 2. Solve for A_0, A_1, and A_2 in equations (8), (9), and (10).

Step 3. Solve for $\overline{R}^e$, V_e, Ω, and G using equations (14), (15), (17), and (21).

Step 4. Compute the eigen-decomposition (23) and derive U and Ω_d.

Step 5. Derive $\widetilde{K}_i$ and K_i, and use (26) to determine an update of Φ_i for all i.

Step 6. Iterate the above steps until convergence.

As discussed in the previous subsection, if $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$ has a unique largest eigenvalue, then all investors choose the same information structure up to normalization. In this case, if an equilibrium exists, then it must be symmetric. If $G^{\frac{1}{2}}\Omega G^{\frac{1}{2}}$ has multiple largest eigenvalues, then different investors may choose different normalized information structures. Thus an asymmetric equilibrium may arise. We will focus on symmetric equilibrium in which $C_i = C$ and $\Sigma_{\epsilon i} = \Sigma_\epsilon$ for all i. As an accuracy check of our solution method, we find that our algorithm delivers almost the same numerical solutions as those in Mondria (2010), which provides a closed-form solution for the two-asset case.

[5] See Mondria (2010) for a proof for the case with two risky assets.

4 Properties of Equilibrium

In this section we analyze properties of the equilibrium under rational inattention. We will focus on the unique linear symmetric equilibrium only.

4.1 Asset Returns

Since the random supply vector Z and the market portfolio are not observable, the CAPM is difficult to test empirically. We thus derive an unconditional CAPM, similar to Andrei, Cujean, and Wilson (2018). As is well known, it is analytically more convenient to work with the dollar return instead of the rate of return in the CARA-normal framework because the rate of returns is not Gaussian. Recall that $R^e = F - R_f P$ is the vector of excess dollar returns on the n risky assets. Define $\overline{R}_m^e = \overline{Z}' R^e$ as the excess dollar return on the average market portfolio. We then obtain the following result.

Proposition 3 *In the linear symmetric equilibrium the excess (dollar) return and the average dollar market return satisfy the following unconditional CAPM:*

$$\mathbb{E}\left[R^e\right] = \beta_m \mathbb{E}\left[\bar{R}_m^e\right], \tag{27}$$

where

$$\beta_m = \frac{\mathrm{Cov}(R^e, \bar{R}_m^e | S_i, P)}{\mathrm{Var}(\bar{R}_m^e | S_i, P)} = \frac{\mathrm{Var}\left(R^e | S_i, P\right) \overline{Z}}{\overline{Z}' \mathrm{Var}\left(R^e | S_i, P\right) \overline{Z}}.$$

Notice that in the linear symmetric equilibrium the above expression for the vector β_m is independent of investor i. Computing β_m only requires to know the conditional variance of excess returns $\mathrm{Var}\left(R^e | S_i, P\right)$ for the average investor. If we plot $\mathbb{E}\left[R_j^e\right]$ against β_{mj} for different asset j, we obtain the security market line (SML). The slope of this line is the market risk premium $\mathbb{E}\left[\bar{R}_m^e\right]$. The market beta is equal to 1. From an econometrician's point of view, beta is computed as the linear regression coefficient of the realized excess return R^e on the average market return $\bar{R}_m^e$:

$$\widetilde{\beta}_m = \frac{\mathrm{Cov}\left(R^e, \bar{R}_m^e\right)}{\mathrm{Var}\left(\bar{R}_m^e\right)}.$$

This vector of betas is different from the true vector of betas β_m from the investors' point of view in the model. As Andrei, Cujean, and Wilson (2018) point out, there is an information distance between econometricans and investors because the unconditional covariance matrix of excess returns satisfies

$$\mathrm{Var}\left(R^e\right) = \mathrm{Var}\left[\mathbb{E}\left(R^e | S_i, P\right)\right] + \mathbb{E}\left[\mathrm{Var}\left(R^e | S_i, P\right)\right].$$

Investors' betas are computed based on the unexplained component $\mathbb{E}\left[\mathrm{Var}\left(R^e | S_i, P\right)\right] = \mathrm{Var}\left(R^e | S_i, P\right)$.

Andrei, Cujean, and Wilson (2018) show that their model delivers a linear relation between $\mathbb{E}\left[R_j^e\right]$ and $\widetilde{\beta}_{mj}$ but the slope is flatter than that of the true SML. The perceived SML rotates clockwise around the market portfolio, which flattens its slope and creates a positive intercept. We find that their result does not hold in our model. The main reason is that they assume that there is only one unobservable common risk factor in asset payoffs and investors receive signals about the common factor only. By contrast, we assume that investors receive signals about the unobservable asset payoffs which may contain several risk factors and asset specific idiosyncratic risks. Moreover, the information structure is endogenous in our model.

Now we use some numerical examples to study the impact of information choice on asset returns. Consider a factor specification of asset payoffs as in KVV (2016). There are five risky assets in the market, with asset payoffs given by $F = \Gamma Y$, where Y represents the risk factors and Γ represents the risk loadings. Let

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 0.2 \\ 0 & 0 & 1 & 0 & 0.3 \\ 0 & 0 & 0 & 1 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As in KVV (2016), we can interpret the last component Y_5 of Y as the aggregate risk factor because all of the risky assets are exposed to Y_5. The component Y_i represents the idiosyncratic risk for asset $i = 1, 2, 3, 4$. Since Γ is invertible, KVV (2016) construct from the original assets a set of synthetic assets whose payoffs are given by $Y = \Gamma^{-1} F$. The resulting payoff covariance matrix is diagonal. They also assume that the supply of each synthetic assets is independent and then work on the space of synthetic assets. Our solution method directly works with the original assets.

As baseline values, set the risk tolerance parameter as $\rho = 0.25$, and the information capacity parameter as $\kappa = 0.1$. The payoffs of synthetic assets are Gaussian with mean $[1, 1, 1, 1, 1]'$ and covariance matrix $diag(0.15^2, 0.15^2, 0.15^2, 0.15^2, 0.18^2)$. The supply of synthetic assets (denoted by Z_Y) is Gaussian with mean $[0.15, 0.15, 0.15, 0.15, 0.4]'$ and covariance matrix $diag(0.1^2, 0.1^2, 0.1^2, 0.1^2, 0.5^2)$. Following KVV (2016), we deduce that the supply of the original risky assets is given by $Z = \Gamma Z_Y$.

Panel A of Figure 1 shows that the observed SML is flatter than the true SML, a result that is discussed in Andrei, Cujean, and Wilson (2018). If we raise the degree of risk aversion from $1/\rho = 1/0.25$ to $1/\rho = 1/0.225$, both of the true SML and the observed SML shift up. This is because investors will demand higher excess returns if they are more risk averse. Panel B of Figure 1 shows the impact when the aggregate volatility is raised from 0.18 to 0.20. In this case, the market risk premium is higher and both the true SML and the observed SML shift up. This figure also shows that the observed SML may not be linear.

Figure 2 shows the impact of information capacity on asset returns. We find that an increase in information capacity κ causes both the true SML and the observed SML to shift down. Intuitively,

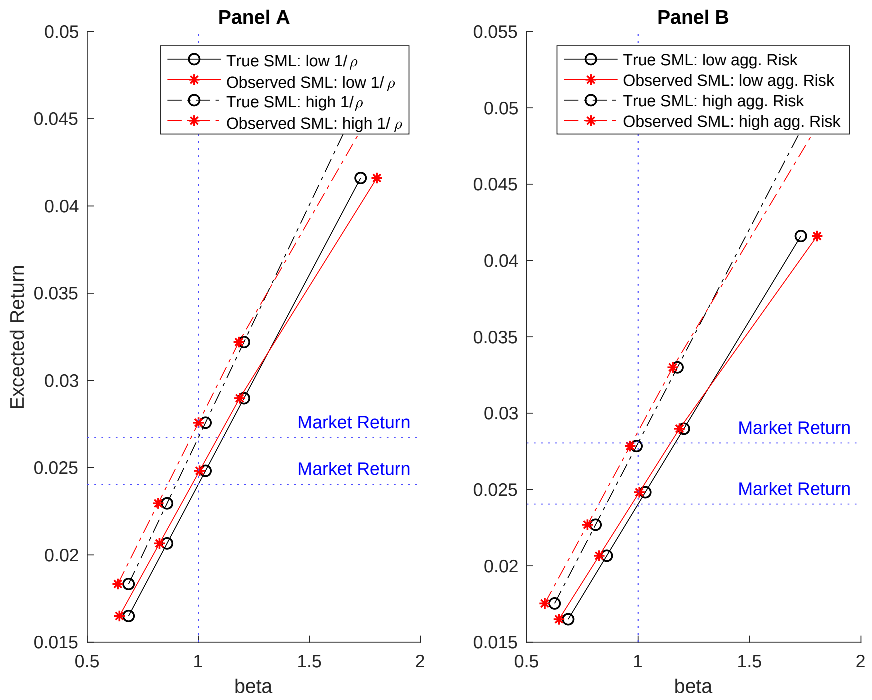

Figure 1: CAPM distortions

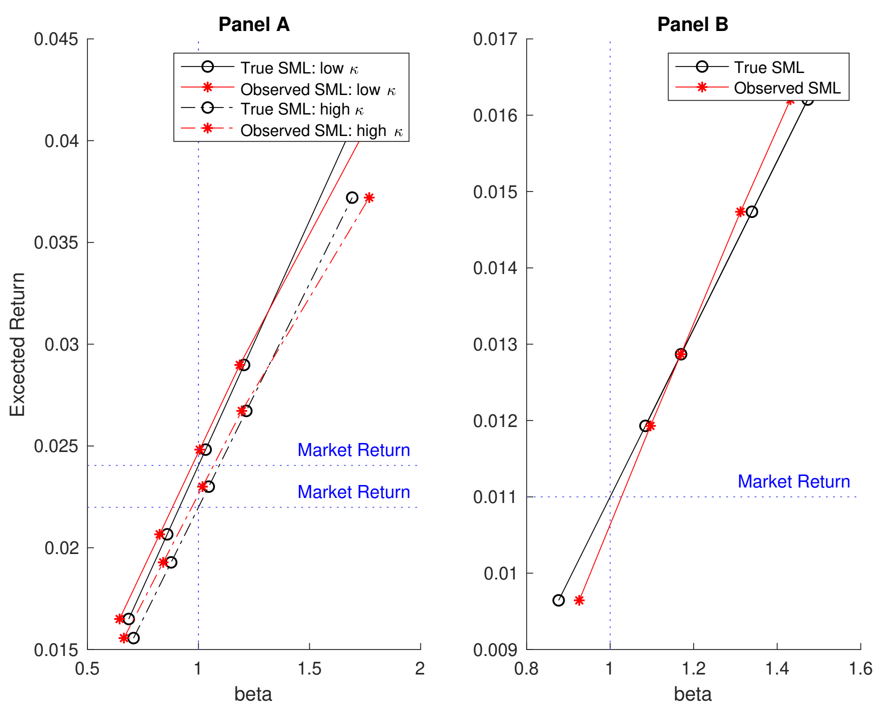

Figure 2: CAPM distortions

a higher information capacity helps investors to reduce uncertainty through learning. Therefore, investors demand lower risk premium on risky assets, leading to lower SMLs. Panel B of Figure 2 shows that the observed SML can be steeper than the true SML, unlike the result in Andrei, Cujean, and Wilson (2018). The parameter values of this panel are the same as the baseline values except that we set the aggregate volatility as 0.10 and the covariance matrix of Z_Y as $diag(0.4^2, 0.3^2, 0.2^2, 0.1^2, 0.5^2)$.

4.2 Comovement and Contagion

Since investors optimally acquire a noisy linear combination of asset payoffs as their private signals, investors are unable to distinguish among various sources of asset payoff shocks when processing private signals. This generates potential asset price comovement and provides a new channel of volatility transmission (Mondria, 2010). In this section we use some numerical examples to illustrate the role of endogenous information choice in generating asset price comovement and financial contagion.

By (7), we can derive the unconditional expectation of asset prices

$$\mathbb{E}[P] = A_0 + A_1\overline{F} - A_2\overline{Z}. \tag{28}$$

and the unconditional covariance matrix of asset prices

$$\mathrm{Var}\,(P) = A_1\Sigma_F A_1' + A_2\Sigma_Z A_2'. \tag{29}$$

Following Mondria (2010), we use (29) to derive the unconditional correlation of asset prices to characterize asset price comovement.

Consider the following three-asset example with the covariance matrix of asset payoffs given by

$$\Sigma_F = \begin{bmatrix} 0.15^2 & 0 & 0 \\ 0 & 0.15^2 & 0.15^2\phi \\ 0 & 0.15^2\phi & 0.15^2 \end{bmatrix},$$

where ϕ denotes the payoff correlation between assets 2 and 3, and asset 1's payoff is independent of these two assets. Other parameter values are set as $\kappa = 0.1$, $\rho = 0.25$, $\overline{F} = [1, 1, 1, 1, 1]'$, $\overline{Z} = [1/3, 1/3, 1/3]'$, and $\Sigma_Z = diag(0.10^2, 0.10^2, 0.10^2)$. We consider two cases with $\phi = 0.75$ and $\phi = -0.75$.

As shown in Section 3.2, investors choose the same information structure in the unique linear symmetric equilibrium. The optimal signal structure is a noisy linear combination of the three assets with the coefficient vector denoted by $C = [C_1, C_2, C_3]$. We normalize the noise variance Σ_ϵ to 1 and let $C_1 > 0$. Since assets 2 and 3 are symmetric ex-ante, they have the same equilibrium properties. More specifically, investors will allocate the same attention between these two assets, i.e. $C_2 = C_3$. Furthermore, assets 2 and 3 have the same unconditional expected prices, $\mathbb{E}[P_2] = \mathbb{E}[P_3]$, and the same correlation with asset 1, $\mathrm{Corr}(P_1, P_2) = \mathrm{Corr}(P_1, P_3)$.

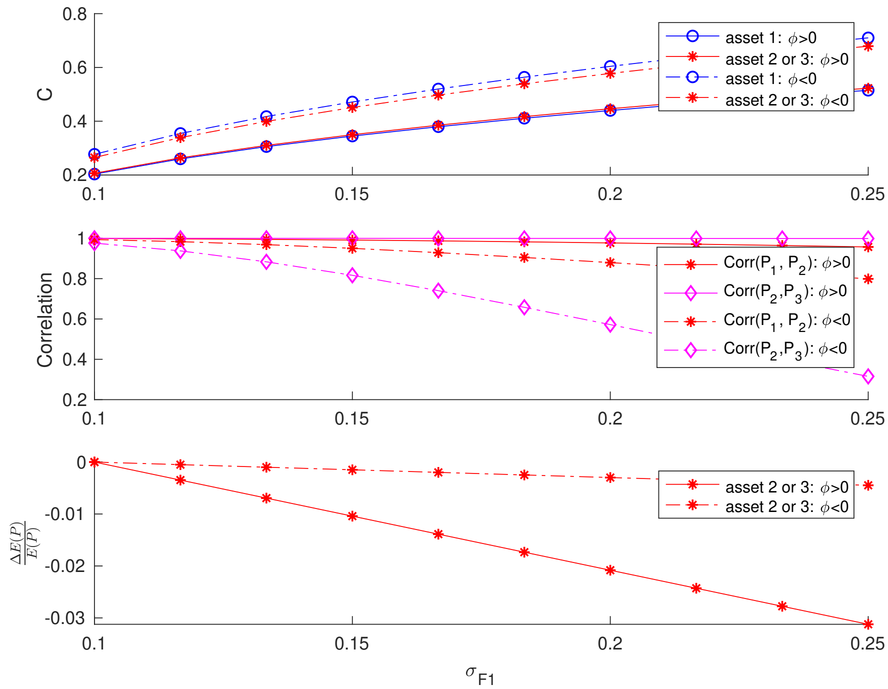

Figure 3: Correlation and contagion: transmission of payoff shocks

We study the impact of asset 1's payoff volatility σ_{F1} on the attention allocation C_1 and C_2, the prices of assets 2 and 3, and the correlations of asset prices. We also study the comparative statics with respect to ρ and κ.

Figure 3 shows that prices of assets 1 and 2 (or 3) are positively correlated when asset 1 is independent of assets 2 and 3. Mondria (2010) finds this result for the two-asset case. We also find that prices of assets 2 and 3 are positively correlated even when their payoffs are negatively correlated. To see the intuition, we first consider the attention allocation described by the endogenous signal structure. We find that investors acquire a one-dimensional signal that is a noisy linear combination of the three assets with positive coefficients $C_1 > 0$, $C_2 > 0$, and $C_3 = C_2 > 0$. Thus, conditional on the noisy asset supply shock Z, a high realization of the signal could be attributed to a high payoff realization for any of the three assets, leading to comovement of prices of these three assets. Since a positive supply shock Z decreases the asset price, it dampens the comovement effect. If the asset supply effect is weak enough, we will obtain the comovement result. Proposition 5 of Mondria (2010) gives an explicit condition for the two-asset case. Given the complexity of our three-asset case, we are unable to provide an analogous condition.

Figure 3 also shows that as asset 1's payoff volatility increases, the correlations become smaller. This is essentially because higher payoff uncertainty tightens the information-processing constraint (2) or (19). Due to a smaller information capacity, private signals become less precise. We also

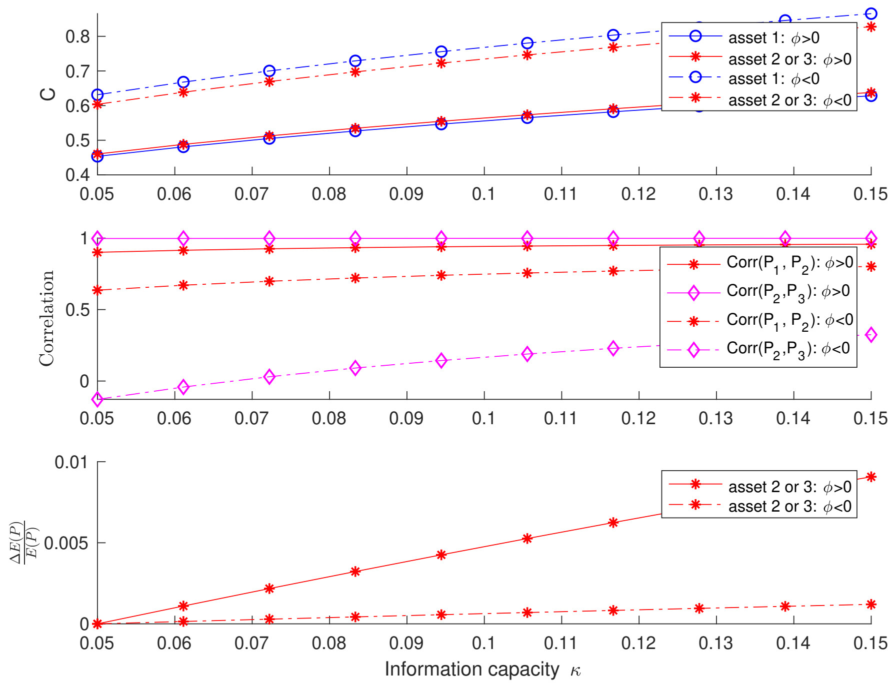

Figure 4: Correlation and contagion: information capacity κ

find that there exist transmissions of payoff risk. More specifically, when the payoff risk of asset 1 increases, not only the prices of asset 1 but also those of assets 2 and 3 decrease. This is due to the reallocation of investors' attention. Intuitively, investors pay more attention to asset 1 relative to the other two assets when asset 1's payoff volatility is relatively larger. As a result, the payoff risks of assets 2 and 3 are perceived to be higher. Higher posterior risks make assets 2 and 3 less desirable and reduce their prices. This result is analogous to Propositions 7 and 8 of Mondria (2010) for the two-asset case.

Figure 4 shows the impact of the information capacity κ. We find that asset price correlations increase with the information capacity κ. As κ increases, private signals become more precise and eventually become the dominating force, which leads to positive price correlations, even when the asset payoffs are negatively correlated. Moreover, as investors can process more information, the posterior variances of asset prices become smaller, leading to higher expected asset prices.

Figure 5 shows the impact of risk aversion. We find that as investors are more risk averse, asset price correlations are smaller. Intuitively, more risk averse investors will be less responsive to their private signals when making portfolio choice decisions. This makes asset prices less informative about future payoffs. Thus the price correlations become weaker. Moreover, since risk premium increases with risk aversion, expected asset prices decreases with risk aversion. We also find that asset price correlations are more sensitive to risk aversion if asset payoffs are negatively correlated.

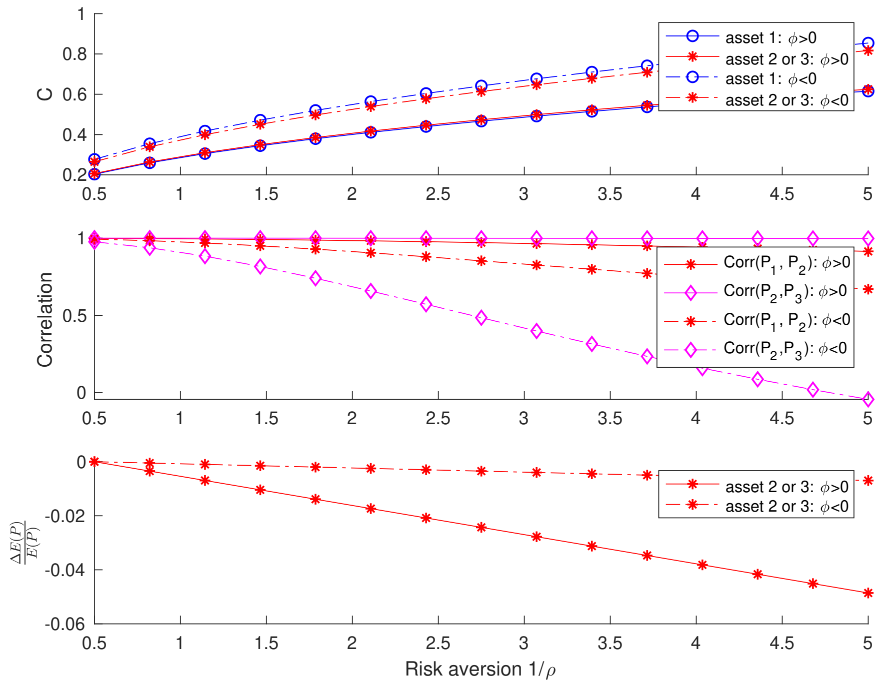

Figure 5: Correlation and contagion: risk aversion

This is because when payoffs are ex-ante negatively correlated, the asset payoff correlations and the private signals work towards different directions. With higher risk aversion, private signals are less important and so asset prices decrease faster due to the negative payoff correlation.

4.3 Portfolio Dispersion

In this section we study our model implications for the dispersion across investor portfolio holdings and portfolio excess returns. As KVV (2016) show, both portfolio dispersions would fall if investment strategies were passive during recessions. However, when investors endogenously process information and actively trade based on that information, the prediction may be different.

We first use equation (12) to derive the equilibrium portfolio strategies.

Proposition 4 *In the symmetric linear equilibrium investor i's holdings of risky assets are given by*

$$\begin{aligned} X_i &= \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1}\left[\mathbb{E}\left(F|S_i, P\right) - \mathbb{E}^m(F)\right] + Z \\ &= \rho C' \Sigma_\epsilon^{-1} \epsilon_i + Z, \end{aligned} \tag{30}$$

where $\mathbb{E}^m(F) = \int \mathbb{E}[F|S_i, P]di$ *denotes the market average expectation of the payoff vector.*

As in Biais, Bossaerts, and Spatt (2010), there is a winner's curse problem for the equilibrium portfolios: investor i invests more than the market portfolio Z in asset j when his/her expectation about the payoffs $\mathbb{E}(F|S_i, P)$ is greater than the average expectation $\mathbb{E}^m(F)$, while he invests less otherwise. Moreover, investor i's equilibrium portfolio responds to his/her idiosyncratic signal noise, conditional on the noisy supply Z.

Equation (30) holds for any fixed information structure (C, Σ_ϵ). Under rational inattention, information structure is endogenous in the sense that both C and Σ_ϵ are endogenously chosen. In particular, we have shown that the private signal is one dimensional so that ϵ_i is a scalar noise. In response to the signal noise, investor i will adjust holdings of all assets because he may believe the noise comes from the payoff shock to any asset in his/her portfolio. By contrast, if investors acquire a separate signal for each asset, i.e., C is an $n-$dimensional identity matrix, then they would adjust only one particular asset holdings in response to the signal noise on that asset.

To understand the aggregate implications, we follow KVV (2016) to define the dispersion of portfolio holdings as

$$\int \mathbb{E}\left[(X_i - Z)'(X_i - Z)\right] di, \tag{31}$$

and define the dispersion of the portfolio excess return as

$$\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di. \tag{32}$$

Then we have the following result:

Proposition 5 *In the symmetric linear equilibrium, the dispersion of portfolio holdings is given by*

$$\int \mathbb{E}\left[(X_i - Z)'(X_i - Z)\right] di = \rho^2 \mathrm{Tr}(C'\Sigma_\epsilon^{-1}C),$$

and the dispersion of portfolio returns is given by

$$\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di = \rho^2 \mathrm{Tr}\left(C'\Sigma_\epsilon^{-1}C\mathbb{E}[R^e R^{e\prime}]\right) = \rho \mathrm{Tr}\left(\Pi\Omega\right).$$

This proposition shows that the SNR $C'\Sigma_\epsilon^{-1}C$ shows up in both dispersion measures. The formulas above apply to any fixed information structure. When the information structure is chosen endogenously, changes in the SNR affect both dispersions. Since we are unable to derive analytical comparative statics results, we use numerical examples to illustrate the impact of information capacity κ, risk aversion $1/\rho$, and aggregate risk on portfolio dispersion. We still adopt the five-asset specification as in Section 4.1.

Panels A and B of Figure 6 show that a higher information capacity raises both the portfolio holdings dispersion and the portfolio return dispersion. Intuitively, a higher information capacity allows investors to process more precise private signals. Thus investors' portfolio holdings will be

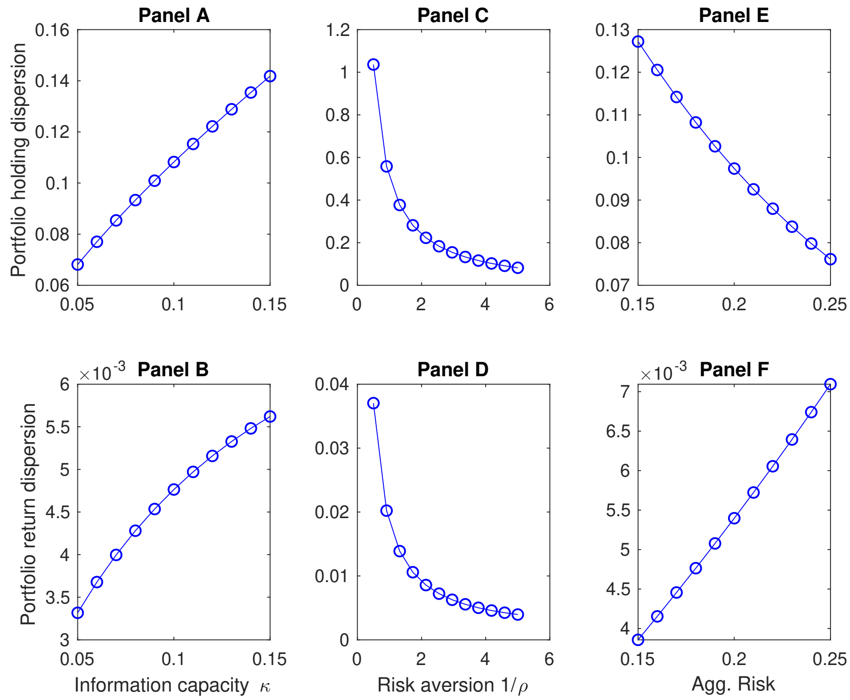

Figure 6: The impact of information capacity κ, risk aversion $1/\rho$, and aggregate risk on the portfolio dispersion.

more responsive to private signals, leading to a rise in the dispersion of portfolio holdings, and hence in the dispersion of portfolio excess returns.

Panels C and D of Figure 6 show that both the portfolio holdings dispersion and the portfolio return dispersion decrease with risk aversion. Intuitively, if investors are more risk averse, then their portfolio choices will be less responsive to their private signals. The common prior beliefs and the public price signal will drive investors to make similar portfolio choices. Hence there will be less heterogeneity in portfolio holdings. The portfolio return dispersion depends on both portfolio holdings and risk premium. Higher risk aversion raises risk premium. We find that the effect of a smaller portfolio holdings dispersion dominates the effect of higher risk premium, causing the dispersion of portfolio returns to decrease with risk aversion.

This result is qualitatively different from Proposition 4 in KVV (2016), which shows that under appropriate conditions, higher risk aversion leads to a higher dispersion of portfolio returns. In the KVV model, the impact of risk aversion on the portfolio return dispersion is mainly driven by the changes in risk premium. By contrast, our numerical examples show that this effect is dominated.

Finally we study the impact of aggregate risk on portfolio dispersions. All other things being equal, we increase the aggregate volatility from 0.15 to 0.25. Panels E and F of Figure 6 show that with higher aggregate risk, the dispersion of portfolio holdings is smaller but the dispersion of portfolio excess returns is larger.

To understand this result, we notice that with higher aggregate risk the information-processing constraint is effectively tighter. Investors have to reduce the precision of private signals to respect the information-processing constraint. Investors will then be less responsive to their private signals. Therefore there will be smaller dispersion of portfolio holdings. Moreover, with higher aggregate risk, investors will demand higher risk premium. The impact of higher risk premium dominates the impact of lower portfolio holdings dispersion, causing the dispersion of portfolio returns to increase.

This result is in contrast to Proposition 3 in KVV (2016), which shows that under suitable conditions, an increase in the payoff uncertainty for any risk factors will weakly increase both the portfolio holdings dispersion and the portfolio return dispersion. Their model differs from ours in several ways, which will be discussed further in Section 5.3.

5 Extensions and Discussions

In this section we discuss some of our model assumptions and an extension. We first introduce a fraction of uninformed investors into our model. Next we study a model in which asset prices are not in the information-processing constraint as in Mondria (2010). Finally we compare with the model of KVV (2016).

5.1 Uninformed Investors

In the model of Section 2 we have assumed that all investors can process information by acquiring signals. In this subsection we suppose that a fraction $1-\lambda \in (0,1)$ of investors are uninformed in the sense that they cannot acquire additional signals about asset payoffs. In particular, the signal precisions of these uninformed investors are given by $\Sigma_{\epsilon i}^{-1} = 0$. Both informed and uninformed investors observe the information in prices, which are public signals.

We focus on the symmetric linear equilibrium in which all informed investors choose the same information structure (C, Σ_ϵ). As discussed in Section 3.1, the equilibrium is still characterized by (7), (8), (9), (10) and (11), but the expression for Π becomes

$$\Pi = \lambda \rho C' \Sigma_\epsilon^{-1} C, \tag{33}$$

where $(C, \Sigma_\epsilon^{-1})$ is the information structure of the informed investors in a symmetric linear equilibrium. The optimal information structure of informed investors is still characterized by proposition 2.

It is interesting to understand whether informed investors outperform the market. Following KVV (2016), we use the abnormal return $\mathbb{E}\left[(X_i - Z)' R^e\right]$ to measure informed investor i's performance. This abnormal return is equal to informed investor i's expected portfolio excess return, minus the expected market excess return. The following proposition characterizes the abnormal return.

Proposition 6 *In the symmetric linear equilibrium with uninformed investors, we have*

$$\mathbb{E}\left[(X_i - Z)' R^e\right] = \frac{1-\lambda}{\lambda} \operatorname{Tr}(\Pi\Omega),$$

for any informed investor i.

Clearly, if there is no uninformed investors in the market (i.e., $\lambda = 1$), then all investors are symmetric and there is no abnormal return. Since we are unable to derive analytical comparative statics results, we use numerical solutions. We still choose the baseline parameter values given in Section 4.1. We also set $\lambda = 0.7$. Figure 7 presents the results.

We find that the abnormal return increases with the aggregate risk and information capacity κ, decreases with the fraction λ of informed investors. Moreover, it has a U-shaped relationship with the risk aversion $1/\rho$. The intuition is the following. The abnormal return depends on the combined effects of an informed investor's portfolio holdings and the excess returns. An increase in the aggregate risk raises the risk premium. Moreover, the value of information becomes more important as payoff uncertainty increases. As a result, informed investors gain more excess returns. Similarly, with a higher information capacity, informed investors have a larger information advantage over uninformed investors. Thus, informed investors earn higher abnormal returns. If there are more informed investors in the market, then there will be fewer uninformed investors from whom informed investors can make profits. Thus the abnormal return will be lower.

There are two effects of an increase in risk aversion. First, an informed investor's portfolio holdings will deviate less from the market portfolio Z in response to shocks. Second, higher risk aversion raises risk premium. For a low degree of risk aversion, the first effect dominates, but for a high degree of risk aversion, the second effect dominates. We then obtain the U-shaped relationship.

5.2 Prices Are Not in the Information-Processing Constraint

Mondria (2010) assumes that asset prices are not in an investor's information set when processing information. He specifies the following information-processing constraint

$$H(F) - H(F|S_i) \leq \kappa \text{ for all } i.$$

Mondria (2010) shows that using this constraint and (2) deliver similar results for the two-asset case. We will show that this is also true for the general multiple-asset case in the model of Section 2.

Using properties of the normal distribution, the above information-processing constraint can be rewritten as

$$\frac{1}{2}\log\det(\Sigma_F) + \frac{1}{2}\log\det\left(\Sigma_F^{-1} + C_i'\Sigma_{\epsilon i}^{-1}C_i\right) \leq \kappa.$$

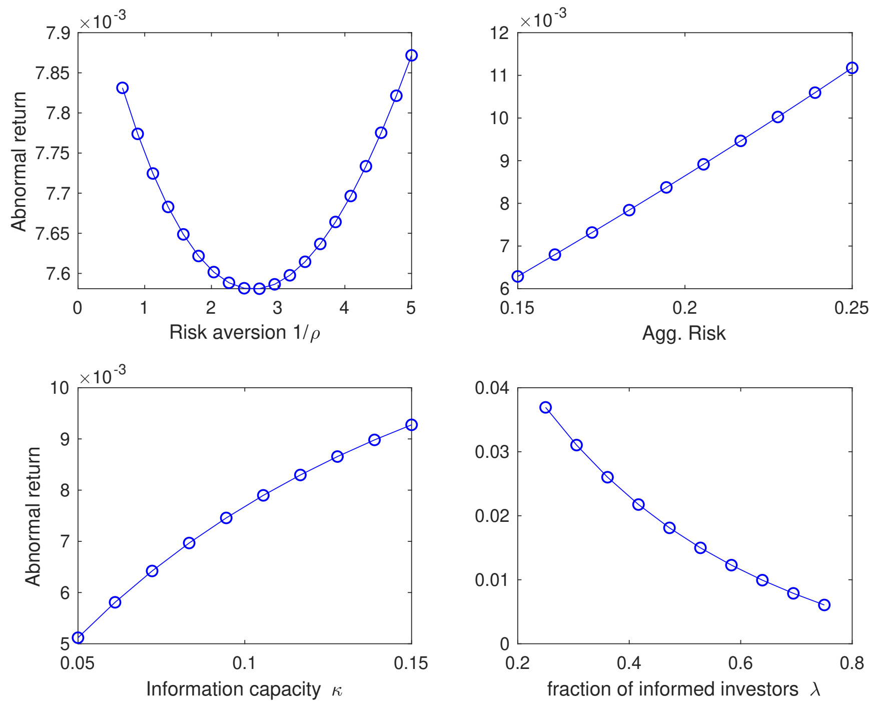

Figure 7: The abnormal return to the informed investors

For any given information structure, the optimal portfolio choice at date 2 is the same as in the model of Section 2. Then by Proposition 1, investor i chooses a signal structure $(\Sigma_{\epsilon i}, C_i)$ to solve the following problem

$$\max_{\Sigma_{\epsilon i}\succ 0, C_i} \operatorname{Tr}\left(K_i \Omega\right),$$

subject to

$$\frac{1}{2}\log\det\left(\Sigma_F\right) + \frac{1}{2}\log\det\left(K_i\right) \leq \kappa,$$

$$K_i \succeq G,$$

where the precision matrix is given by

$$K_i = \Sigma_F^{-1} + C_i' \Sigma_{\epsilon i}^{-1} C_i.$$

and the matrix G satisfies

$$G \equiv \Sigma_F^{-1} \succ 0. \tag{34}$$

This problem is the same as that described by (18), (19), (20) and (21), except that G has a different definition. Thus Proposition 2 still applies. We have verified numerically that the results in Section 4 also apply.

5.3 KVV Approach

Van Nieuwerburgh and Veldkamp (2010) and KVV (2016) study models with multiple risky assets under rational inattention by making the signal independence assumption.[6] Adapting their assumption in our model setup of Section 2, we assume that investors process information about one asset at a time. In particular, in the signal form (1), C_i is constrained to be an n-dimensional identity matrix, and $\Sigma_{\epsilon i}$ is diagonal with jth element $K_{ij}^{-1} \geq 0$.

KVV (2016) also assume that the information-processing constraint is given by

$$\sum_{j=1}^{n} K_{ij} \leq \kappa. \tag{35}$$

That is, the sum of signal precisions does not exceed an upper bound. Investor i's objective is to choose $K_{i1}, ...,$ and K_{in} to maximize $\mathrm{Tr}\left(\Sigma_{\epsilon i}^{-1}\Omega\right)$. Under the above assumptions, we can show that investors will specialize in learning only one asset (also see KVV (2016)). That is, the signal is one dimensional and is equal to the payoff of only one asset plus a noise. The specialized asset corresponds to any largest diagonal element of the matrix Ω in (17). Without loss of generality, let the first diagonal elemement is the largest. Then $K_{i1} = \kappa$ and $K_{ij} = 0$ for all $j \neq 1$.

An implication of the signal independence assumption is that ex ante independent asset remain ex post independent, as formalized by the following result.

Proposition 7 *Suppose that Σ_F and Σ_Z are diagonal matrices. Suppose that each investor i acquires a signal $S_i = F + \epsilon_i$, $\epsilon_i \sim N\left(0, \Sigma_{\epsilon i}\right)$, where $\Sigma_{\epsilon i}$ is diagonal with jth element $K_{ij}^{-1} \geq 0$. Then* $\mathrm{Var}\left(P\right)$ *is diagonal in equilibrium.*

By Proposition 5 and (35), we can show that in a symmetric linear equilibrium the dispersion of portfolio holdings is given by

$$\int \mathbb{E}\left[\left(X_i - Z\right)'\left(X_i - Z\right)\right] di = \rho^2 \mathrm{Tr}(C'\Sigma_{\epsilon}^{-1}C) = \rho^2\kappa,$$

which is a constant. Here C is an n-dimensional identity matrix. The dispersion of portfolio excess returns is given by

$$\int \mathbb{E}\left[\left(\left(X_i - Z\right)' R^e\right)^2\right] di = \rho \mathrm{Tr}\left(\Pi\Omega\right) = \rho^2\kappa\Omega_{kk},$$

where Ω_{kk} is the largest diagonal element of Ω. In this case the portfolio return dispersion increases with risk aversion and aggregate risk as in Propositions 3 and 4 of KVV (2016).

The model of KVV (2016) differs from ours in that they consider a risk factor specification of asset payoffs and assume that investors acquire information about one risk factor at a time. They also include both informed and uninformed investors. Thus their portfolio dispersion expressions in their Propositions 3 and 4 are different from our formulas.

[6] See Peng (2005), Peng and Xiong (2006), and Maćkowiak and Wiederholt (2009) for similar assumptions.

In Appendix B of KVV (2016), they consider a more general signal form in (1) that allows C_i and $\Sigma_{\epsilon i}$ to be invertible. After transformation, they argue that they can apply the method outlined above.[7] By contrast, our model does not restrict the form of matrix C_i and $\Sigma_{\epsilon i}$ and implies that investors will endogenously choose a signal, which is a noisy linear combination of risky assets, instead of specializing in a single risky asset or risk factor. In particular, C_i is not a square matrix.

6 Conclusion

We have analyzed a noisy rational expectations equilibrium model with rationally inattentive investors. We have solved the difficult problem with any finite number of assets with arbitrary correlation by relaxing the signal independence assumption. Our solution approach is useful to analyze other finance models with multiple assets. We have also derived some testable predictions that are different from the existing literature. It would be interesting to test these predictions for future research.

[7] In particular, the transformed signal takes the form $\tilde{S}_i = F + \tilde{\epsilon}_i$, where $\mathrm{Var}(\tilde{\epsilon}_i)$ is diagonal and invertible. A problem of this approach is that the optimal signal precision matrix is not invertible, inconsistent with the invertibility assumption of $\mathrm{Var}(\tilde{\epsilon}_i)$.

7 Proofs

Proof of Proposition 1: The proof essentially follows from Mondria (2010). For completeness, we sketch the key steps. First, by the projection theorem, we can compute

$$V_i = \text{Var}\left(R^e|S_i, P\right) = \text{Var}\left(F|S_i, P\right) = \left(\Sigma_F^{-1} + \Pi\Sigma_Z^{-1}\Pi + C_i'\Sigma_{\epsilon i}^{-1}C_i\right)^{-1}.$$

Taking expectations on the two sides of the market-clearing condition yields

$$\overline{Z} = \mathbb{E}\left[\int X_i di\right] = \left(\rho\int V_i^{-1}di\right)\mathbb{E}\left\{\mathbb{E}\left[R^e|S_i, P\right]\right\} = \left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)\overline{R}^e$$

where $\overline{R}^e \equiv \mathbb{E}\left[R^e\right]$. Thus we obtain

$$\overline{R}^e = \left(\rho\Sigma_F^{-1} + \rho\Pi\Sigma_Z^{-1}\Pi + \Pi\right)^{-1}\overline{Z}.$$

Using the budget constraint, we compute

$$\mathbb{E}\left[W_i|S_i, P\right] = W_{i0}R_f + X_i'\mathbb{E}\left[R^e|S_i, P\right] = W_{i0}R_f + \rho\mathbb{E}\left[R_e|S_i, P\right]'\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\mathbb{E}\left[R^e|S_i, P\right],$$

where we have plugged in the optimal portfolio rule given in (12). Similarly, we compute

$$\text{Var}\left[W_i|S_i, P\right] = X_i'\text{Var}\left[R^e|S_i, P\right]X_i = \rho^2\mathbb{E}\left[R^e|S_i, P\right]'\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\mathbb{E}\left[R^e|S_i, P\right].$$

Thus the initial utility at date zero is given by

$$\begin{aligned} U_i &= \frac{1}{\rho}\mathbb{E}\left(\mathbb{E}\left[W_i|S_i, P\right] - \frac{1}{2\rho}\text{Var}\left[W_i|S_i, P\right]\right) \\ &= \frac{W_{i0}R_f}{\rho} + \frac{1}{2}\mathbb{E}\left\{\mathbb{E}\left[R^e|S_i, P\right]'\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\mathbb{E}\left[R^e|S_i, P\right]\right\} \end{aligned}$$

Notice that $\mathbb{E}\left[R^e|S_i, P\right]$ is normal with mean $\overline{R}^e = \mathbb{E}\left[R^e\right]$ and variance $\text{Var}\left(\mathbb{E}\left[R^e|S_i, P\right]\right)$. Notice that

$$\text{Var}\left(\mathbb{E}\left[R^e|S_j, P\right]\right) = \text{Var}\left(R^e\right) - \text{Var}\left(R^e|S_i, P\right).$$

Moreover, if $x = (x_1, ..., x_n)' \sim N(\mu, V)$ and $q = x'Ax$, then $\mathbb{E}[q] = \text{Tr}(AV) + \mu'A\mu$. Using the preceding two formulas, we compute

$$\begin{aligned} &\mathbb{E}\left\{\mathbb{E}\left[R^e|S_i, P\right]'\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\mathbb{E}\left[R^e|S_i, P\right]\right\} \\ =\ &\text{Tr}\left(\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\text{Var}\left(\mathbb{E}\left[R^e|S_i, P\right]\right)\right) + \overline{R}^{e\prime}\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\overline{R}^e \\ =\ &\text{Tr}\left(\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\text{Var}\left(R^e\right) - I\right) + \overline{R}^{e\prime}\left[\text{Var}\left(R^e|S_i, P\right)\right]^{-1}\overline{R}^e \\ =\ &\text{Tr}\left(V_i^{-1}\left(V_e + \overline{R}^e\overline{R}^{e\prime}\right)\right) - n \end{aligned}$$

where we define $V_i \equiv \text{Var}\left(R^e|S_i, P\right)$ and

$$V_e \equiv \text{Var}\left(R^e\right) = \Sigma_F + R_f^2A_1\Sigma_FA_1' + R_f^2A_2\Sigma_ZA_2' - R_fA_1\Sigma_F - R_f\Sigma_FA_1'. \quad (36)$$

We then obtain the utility value in the proposition. Q.E.D.

Proof of Proposition 2: Given the discussion in Section 3.2, we only need to study the following problem:

$$\max_{\widetilde{K}_i} \operatorname{Tr}\left(\widetilde{K}_i \Omega_d\right) \tag{37}$$

subject to $\widetilde{K}_i \succeq I$ and

$$\log\det\left(\Sigma_F\right) + \log\det\left(G\right) + \log\det\left(\widetilde{K}_i\right) \leq 2\kappa. \tag{38}$$

Since $\widetilde{K}_i = U' G^{-\frac{1}{2}} K_i G^{-\frac{1}{2}} U$, $\widetilde{K}_i$ is a real symmetric matrix. If $\widetilde{K}_i$ is not a diagonal matrix, we consider the eigen-decomposition $\widetilde{K}_i = Q' \Lambda Q$, where Q is an orthogonal matrix and Λ is a diagonal matrix whose diagonal elements are the eigenvalues of $\widetilde{K}_i$, denoted by $\lambda_1, \lambda_2, ...,$ and λ_n. Since

$$\operatorname{Tr}\left(\widetilde{K}_i \Omega_d\right) = \operatorname{Tr}\left(Q' \Lambda Q \Omega_d\right) = \operatorname{Tr}\left(\Lambda Q \Omega_d Q'\right),$$

$$\widetilde{K}_i \succeq I \Longleftrightarrow Q' \Lambda Q \succeq I \Longleftrightarrow \Lambda \succeq QQ' = I,$$

$$\log\det\left(\widetilde{K}_i\right) = \log\det\left(Q' \Lambda Q\right) = \log\det\left(\Lambda\right),$$

we can rewrite the problem above as

$$\max_{\Lambda, Q} \operatorname{Tr}\left(\Lambda Q \Omega_d Q'\right) \tag{39}$$

subject to $\Lambda \succeq I$ and

$$\log\det\left(\Sigma_F\right) + \log\det\left(G\right) + \log\det\left(\Lambda\right) \leq 2\kappa. \tag{40}$$

Let the diagonal elements of the matrix $Q \Omega_d Q'$ be $\omega_1, \omega_2, ...,$ and ω_n. Then the problem in (39) is equivalent to the following problem:

$$\max_{\{\lambda_i\}_{i=1}^n, Q} \quad \sum_{i=1}^n \lambda_i \omega_i \tag{41}$$

subject to $\lambda_i \geq 1$, $i = 1, ..., n$,

$$\lambda_1 \lambda_2 \cdots \lambda_n \leq \frac{\exp\left(2\kappa\right)}{\det\left(\Sigma_F G\right)}.$$

Notice that the constraint set is not convex. Thus the solution must be at the corner. Without loss of generality, suppose that $\omega_1, ..., \omega_\ell$ are the identical maximum among $\omega_1, \omega_2, ...,$ and ω_n. Then the solution to problem (41) for any given $\omega_1, \omega_2, ...,$ and ω_n is not unique and given by

$$\lambda_k^* = \frac{\exp\left(2\kappa\right)}{\det\left(\Sigma_F G\right)} > 1 \text{ for } 1 \leq k \leq \ell \text{ and } \lambda_i^* = 1 \text{ for } i \neq k. \tag{42}$$

If $\ell = 1$, the solution is unique with $\lambda_1^* = \frac{\exp(2\kappa)}{\det(\Sigma_F G)}$ and $\lambda_i^* = 1$ for all $i > 1$.

Next we solve for Q. Let $\Lambda^* = diag\left(\lambda_i^*\right)_{i=1}^n$ and $Q' = \left(a_{ij}\right)_{n \times n}$. Since the jth diagonal element of the matrix $Q' \Lambda^* Q$ is $\sum_{i=1}^n \lambda_i^* a_{ji}^2$, we can rewrite (41) or (39) as

$$\max_Q \operatorname{Tr}\left(Q' \Lambda^* Q \Omega_d\right) = \max_Q \; d_1 \sum_{i=1}^n \lambda_i^* a_{1i}^2 + d_2 \sum_{i=1}^n \lambda_i^* a_{2i}^2 + ... + d_n \sum_{i=1}^n \lambda_i^* a_{ni}^2, \tag{43}$$

where $d_1, d_2, ...,$ and d_n are the diagonal elements of Ω_d. Since Q is an orthogonal matrix, we have $\sum_{i=1}^{n} a_{ji}^2 = 1$ for any $j = 1, 2..., n$. Thus it follows from (42) that

$$\sum_{i=1}^{n} \lambda_i^* a_{ji}^2 = \lambda_k^* a_{jk}^2 + \left(1 - a_{jk}^2\right).$$

The we can derive that

$$\begin{aligned} \max_Q \operatorname{Tr}\left(Q'\Lambda^* Q\Omega_d\right) &= \max_Q \; d_1 \; \left(\lambda_k^* a_{1k}^2 + 1 - a_{1k}^2\right) \\ &\quad + d_2 \left(\lambda_k^* a_{2k}^2 + 1 - a_{2k}^2\right) + ... + d_n \left(\lambda_k^* a_{nk}^2 + 1 - a_{nk}^2\right). \end{aligned} \tag{44}$$

Since Q is an orthogonal matrix, we have

$$\sum_{i=1}^{n} a_{ik}^2 = 1. \tag{45}$$

Without loss of generality, let $d_1, ..., d_m$ be the identical largest eigenvalues of Ω_d. Then the problem in (44) becomes

$$\begin{aligned} \max_Q \operatorname{Tr}\left(Q'\Lambda^* Q\Omega_d\right) &= \max_{a_{ik}} \; d_1 \left[\lambda_k^* \left(a_{1k}^2 + ... + a_{mk}^2\right) + m - \left(a_{1k}^2 + ... + a_{mk}^2\right)\right] \\ &\quad + d_{m+1} \left(\lambda_k^* a_{m+1,k}^2 + 1 - a_{m+1,k}^2\right) + ... + d_n \left(\lambda_k^* a_{nk}^2 + 1 - a_{nk}^2\right) \end{aligned} \tag{46}$$

subject to (45). Since $\lambda_k^* > 1$, the solution to the above problem is given by $a_{1k}^2 + ... + a_{mk}^2 = 1$ and thus $a_{m+1,k}^2 = ... = a_{nk}^2 = 0$. There is no restriction on the other elements of Q except that Q must be an orthogonal matrix. Thus

$$d_1 \lambda_k^* + (m-1)\, d_1 + d_{m+1} + ... + d_n = \max_Q \operatorname{Tr}\left(Q'\Lambda^* Q\Omega_d\right),$$

where Q is an orthogonal matrix.

Note that we can show that

$$\widetilde{K}_i = Q'\Lambda^* Q = Q'Q + Q'\left(\Lambda^* - I\right) Q = I + \left(\lambda_k^* - 1\right) v_k v_k',$$

where $Q = [v_1, ..., v_n]'$ with all $v_i = [a_{1i}, a_{2i}, ..., a_{ni}]'$ being column vectors. Let $v^* = v_k$. We then obtain the optimal signal structure stated in the proposition. Moreover, the dimension of the optimal signal is determined by the rank of $\widetilde{K}_i - I$ or $v^* v^{*\prime}$ by (22) or (26), which is equal to 1.

If $m = 1$, we have $a_{1k} = 1$ and $a_{jk} = 0$ for all $j \geq 1$. Then we have $\widetilde{K}_i = Q'\Lambda^* Q = diag\left(\lambda_k^*, 1, ..., 1\right)$, where λ_k^* is given by (42). The solution for $\widetilde{K}_i$ is unique.

If $m \geq 2$, the solution for $\widetilde{K}_i$ is not unique. For example, let $Q' = (a_{ij})$ be an elementary matrix where row 1 and row k are switched where $1 \leq k \leq \ell$. Then $Q\Omega_d Q'$ is the same as Ω_d except that the elements d_k and d_1 are switched. But the largest elements of $Q\Omega_d Q'$ and Ω_d are the same so that $\ell = m \geq 2$. We have $\widetilde{K}_i = Q'\Lambda^* Q$, which is the same as the diagonal matrix Λ^* except that the elements λ_k^* and λ_1^* are switched. Notice that a non-diagonal solution $\widetilde{K}_i = I + \left(\lambda_k^* - 1\right) v_k v_k'$ is also possible. Q.E.D.

Proof of Proposition 3: From the first-order condition for investor i's optimization problem, we have

$$\mathbb{E}\left[u'\left(W_i\right)R_j^e|S_i,P\right]=0,$$

where

$$u\left(W\right)=-\exp\left(-\frac{W}{\rho}\right).$$

Using the covariance decomposition yields

$$\mathbb{E}\left[u'\left(W_i\right)|S_i,P\right]\mathbb{E}\left[R_j^e|S_i,P\right]=-\mathrm{Cov}\left(u'\left(W_i\right),R_j^e|S_i,P\right).$$

By Stein's Lemma, we have

$$\mathbb{E}\left[u'\left(W_i\right)|S_i,P\right]\mathbb{E}\left[R_j^e|S_i,P\right]=-\mathbb{E}\left[u''\left(W_i\right)|S_i,P\right]\mathrm{Cov}\left(W_i,R_j^e|S_i,P\right).$$

By the specification of the CARA utility, we have

$$\frac{-\mathbb{E}\left[u''\left(W_i\right)|S_i,P\right]}{\mathbb{E}\left[u'\left(W_i\right)|S_i,P\right]}=\frac{1}{\rho}.$$

Thus we obtain

$$\mathbb{E}\left[R_j^e|S_i,P\right]=\frac{1}{\rho}\mathrm{Cov}\left(W_i,R_j^e|S_i,P\right).$$

By the budget constraint (5), we have

$$\mathbb{E}\left[R_j^e|S_i,P\right]=\frac{1}{\rho}X_i'\mathrm{Cov}\left(F,R_j^e|S_i,P\right),$$

where we have taken X_i' out of the conditional covariance operator as X_i is measurable with respect the investor i's information set $\{S_i,P\}$. Integrating over i yields

$$\int\mathbb{E}\left[R_j^e|S_i,P\right]di=\frac{1}{\rho}\int X_i'\mathrm{Cov}\left(F,R_j^e|S_i,P\right)di.$$

Notice that we have

$$\mathrm{Var}(F|S_i,P)=(\Sigma_F^{-1}+\Pi\Sigma_Z^{-1}\Pi+C_i'\Sigma_{\epsilon i}^{-1}C_i)^{-1}.$$

In a symmetric equilibrium $C_i'\Sigma_{\epsilon i}^{-1}C_i$ is identical for all investors and thus $\mathrm{Var}(F|S_i,P)$ and $\mathrm{Cov}\left(F,R_j|S_i,P\right)$ are independent of investor i. It follows from the market-clearing condition that

$$\int\mathbb{E}\left[R_j^e|S_i,P\right]di=\frac{1}{\rho}\mathrm{Cov}\left(F_j,F|S_i,P\right)Z. \tag{47}$$

Taking unconditional expectations on the two sides of equation (47) yields

$$\mathbb{E}\left[R_j^e\right]=\frac{1}{\rho}\mathrm{Cov}\left(F_j,F|S_i,P\right)\overline{Z}. \tag{48}$$

Using the Gaussian property we rewrite the above equation in the vector form as

$$\mathbb{E}\left[R^e\right]=\frac{1}{\rho}\mathrm{Cov}\left(F-R_fP,F-R_fP|S_i,P\right)\overline{Z}=\frac{1}{\rho}\mathrm{Var}\left(R^e|S_i,P\right)\overline{Z}. \tag{49}$$

Pre-multiplying both sides of the equation above by $\overline{Z}'$ yields

$$\mathbb{E}\left(\overline{R}_m^e\right)=\overline{Z}'\mathbb{E}\left(R^e\right)=\frac{1}{\rho}\overline{Z}'\mathrm{Var}(R^e|S_i,P)\overline{Z}.$$

Combining the above two equations yields (27). Q.E.D.

Proof of Proposition 4: From investor i's first-order condition, we have

$$\begin{aligned} X_i &= \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} \mathbb{E}\left[R^e|S_i, P\right] \\ &= \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} \left[\mathbb{E}\left(F|S_i, P\right) - PR_f\right]. \end{aligned} \tag{50}$$

From the market-clearing condition, we obtain

$$PR_f = \int \mathbb{E}(F|S_i, P) di - \frac{1}{\rho} \mathrm{Var}(F|S_i, P) Z. \tag{51}$$

Substituting this equation into (50) shows

$$\begin{aligned} X_i &= \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} \left[\mathbb{E}\left(F|S_i, P\right) - \int \mathbb{E}(F|S_i, P) di + \frac{1}{\rho} \mathrm{Var}(F|S_i, P) Z\right] \\ &= \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} \left[\mathbb{E}\left(F|S_i, P\right) - \int \mathbb{E}(F|S_i, P) di\right] + Z. \end{aligned} \tag{52}$$

Given that S_i, P and F are jointly normal, we know that $\mathbb{E}(F|S_i, P)$ has the following representation

$$\mathbb{E}(F|S_i, P) = B_0 + B_1 S_i + B_2 P, \tag{53}$$

where B_0, B_1 and B_2 are constant matrices. From Admati (1985), we have

$$B_1 = \mathrm{Var}(F|S_i, P) C' \Sigma_\epsilon^{-1}, \tag{54}$$

in the symmetric linear equilibrium.

Combining (53) and (54) yields

$$\mathbb{E}\left(F|S_i, P\right) - \int \mathbb{E}(F|S_i, P) di = B_1 \left[\epsilon_i - \int \epsilon_i di\right] = \mathrm{Var}(F|S_i, P) C' \Sigma_\epsilon^{-1} \epsilon_i, \tag{55}$$

where we have use the fact that the integration of noises is zero.

Substituting the above equation into (52) leads to the desired result. Q.E.D.

Proof of Proposition 5: We focus on the symmetric linear equilibrium in which $C_i = C$ and $\Sigma_{\epsilon i} = \Sigma_\epsilon$ for all i. To get a more explicit expression of the portfolio holdings dispersion, substituting (30) into (31) yields

$$\int \mathbb{E}\left[\left(X_i - Z\right)' \left(X_i - Z\right)\right] di = \rho^2 \int \mathbb{E}\left[\epsilon_i' \Sigma_\epsilon^{-1} C C' \Sigma_\epsilon^{-1} \epsilon_i\right] di = \rho^2 \mathrm{Tr}(\Sigma_\epsilon^{-1} C C'),$$

where we notice that $\epsilon_i' \Sigma_\epsilon^{-1} C C' \Sigma_\epsilon^{-1} \epsilon_i$ follows central χ^2 distribution. By the cyclic property of trace, the preceding equation can be rewritten as

$$\int \mathbb{E}\left[\left(X_i - Z\right)' \left(X_i - Z\right)\right] di = \rho^2 \mathrm{Tr}(C' \Sigma_\epsilon^{-1} C) = \rho \mathrm{Tr}(\Pi),$$

which says that the portfolio dispersion is simply the trace of signal-to-noise ratio adjusted by ρ^2.

For the dispersion of portfolio excess returns, we have

$$(X_i - Z)' R^e = \rho \epsilon_i' \Sigma_\epsilon^{-1} C R^e.$$

It follows that

$$\begin{aligned}\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di &= \rho^2 \int \mathbb{E}\left[\epsilon_i' \Sigma_\epsilon^{-1} C R^e R^{e\prime} C' \Sigma_\epsilon^{-1} \epsilon_i\right] di \\ &= \rho^2 \int \mathbb{E}\left[\mathbb{E}(\epsilon_i' \Sigma_\epsilon^{-1} C R^e R^{e\prime} C' \Sigma_\epsilon^{-1} \epsilon_i | Z, F)\right] di. \end{aligned} \tag{56}$$

Conditional on Z and F, $\epsilon_i' \Sigma_\epsilon^{-1} C R^e R^{e\prime} C' \Sigma_\epsilon^{-1} \epsilon_i$ follows a central χ^2 distribution. This implies that

$$\mathbb{E}(\epsilon_i' \Sigma_\epsilon^{-1} C R^e R^{e\prime} C' \Sigma_\epsilon^{-1} \epsilon_i | Z, F) = \mathrm{Tr}\left(\Sigma_\epsilon^{-1} C R^e R^{e\prime} C'\right).$$

Substitution of the preceding equation into (56) shows

$$\begin{aligned}\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di &= \rho^2 \int \mathbb{E}\left[\mathrm{Tr}\left(\Sigma_\epsilon^{-1} C R^e R^{e\prime} C'\right)\right] di \\ &= \rho^2 \int \mathbb{E}\left[\mathrm{Tr}\left(C' \Sigma_\epsilon^{-1} C R^e R^{e\prime}\right)\right] di.\end{aligned}$$

By the linearity of expectation and trace, we obtain

$$\begin{aligned}\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di &= \rho^2 \int \mathrm{Tr}\left(C' \Sigma_\epsilon^{-1} C \mathbb{E}\left[R^e R^{e\prime}\right]\right) di \\ &= \rho \mathrm{Tr}\left(\int \rho C' \Sigma_\epsilon^{-1} C di \mathbb{E}\left[R^e R^{e\prime}\right]\right) \\ &= \rho \mathrm{Tr}\left(\Pi \mathbb{E}[R^e R^{e\prime}]\right).\end{aligned} \tag{57}$$

Furthermore, we notice that

$$V_e = \mathrm{Var}(R^e) = \mathbb{E}[R^e R^{e\prime}] - \mathbb{E}[R^e]\mathbb{E}[R^{e\prime}] = \mathbb{E}[R^e R^{e\prime}] - \overline{R}^e \overline{R}^{e\prime}.$$

Replacing $\mathbb{E}[R^e R^{e\prime}]$ in (57) yields

$$\int \mathbb{E}\left[\left((X_i - Z)' R^e\right)^2\right] di = \rho \mathrm{Tr}\left(\Pi (V_e + \overline{R}^e \overline{R}^{e\prime})\right) = \rho \mathrm{Tr}\left(\Pi \Omega\right).$$

The proof is completed. Q.E.D.

Proof of Proposition 6 We focus on the symmetric linear equilibrium in which all informed investors choose the same information structure (C, Σ_ϵ). We also have $\Sigma_{\epsilon i}^{-1} = 0$ for any uninformed investor i.

Each investor i's first-order condition implies that

$$\begin{aligned} X_i &= \rho \left[\mathrm{Var}\left(R^e | S_i, P\right)\right]^{-1} \mathbb{E}\left[R^e | S_i, P\right] \\ &= \rho \left[\mathrm{Var}\left(R^e | S_i, P\right)\right]^{-1} \left[\mathbb{E}\left(F | S_i, P\right) - P R_f\right].\end{aligned}$$

From the market-clearing condition, we have

$$Z = \int_0^1 X_i di = \int_0^1 \rho \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} \left[\mathbb{E}\left(F|S_i, P\right) - PR_f\right] di. \tag{58}$$

From Admati (1985), we know

$$\mathbb{E}(F|S_i, P) = B_{0,i} + B_{1,i}S_i + B_{2,i}P,$$

where

$$B_{1,i} = \mathrm{Var}(R^e|S_i, P)C_i'\Sigma_{\epsilon i}^{-1},$$

$$B_{2,i} = R_f \mathrm{Var}(R^e|S_i, P)\Pi\Sigma_Z^{-1}A_2^{-1},$$

$$B_{0,i} = \mathrm{Var}_i(R^e|S_i, P)\tilde{B}_0.$$

Here Π is defined in (11) and $\tilde{B}_0$ is a constant that is independent of investor i. Note that $B_{1,i}$, $B_{2,i}$ and $B_{0,i}$ depend on whether investor i is informed or uninformed.

Substituting the above expressions into (58) yields

$$\begin{aligned}
Z &= \int_0^1 \rho \left[\tilde{B}_0 + C_i'\Sigma_{\epsilon i}^{-1}S_i + R_f\Pi\Sigma_Z^{-1}A_2^{-1}P\right] di - \rho\int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} diPR_f \\
&= \int_0^1 \rho \left[\tilde{B}_0 + C_i'\Sigma_{\epsilon i}^{-1}(C_iF + \epsilon_i) + R_f\Pi\Sigma_Z^{-1}A_2^{-1}P\right] di - \rho\int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} diPR_f \\
&= \rho\tilde{B}_0 + \Pi F + \rho R_f\Pi\Sigma_Z^{-1}A_2^{-1}P - \rho\int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} diPR_f,
\end{aligned}$$

where we notice that the integration of noises is equal to zero.

Similarly, for an informed investor i, we can derive

$$X_i = \rho(\tilde{B}_0 + C'\Sigma_\epsilon^{-1}S_i + R_f\Pi\Sigma_Z^{-1}A_2^{-1}P - \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} PR_f).$$

Combining the above two equations together yields

$$\begin{aligned}
X_i - Z =& \rho(\tilde{B}_0 + C'\Sigma_\epsilon^{-1}S_i + R_f\Pi\Sigma_Z^{-1}A_2^{-1}P - \left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} PR_f) \\
&- \left[\rho\tilde{B}_0 + \Pi F + \rho R_f\Pi\Sigma_Z^{-1}A_2^{-1}P - \rho\int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} diPR_f\right] \\
=& (\rho C'\Sigma_\epsilon^{-1}S_i - \Pi F) - \rho\left(\left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} - \int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} di\right) PR_f \\
=& (\rho C'\Sigma_\epsilon^{-1}C - \Pi)F + \rho C'\Sigma_\epsilon^{-1}\epsilon_i - \rho\left(\left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} - \int_0^1 \left[\mathrm{Var}(R^e|S_i, P)\right]^{-1} di\right) PR_f,
\end{aligned}$$

for an informed investor i.

Note that

$$\left[\mathrm{Var}\left(R^e|S_i, P\right)\right]^{-1} = \Sigma_F^{-1} + \Pi\Sigma_Z^{-1}\Pi + C_i'\Sigma_{\epsilon i}^{-1}C_i,$$

where $C_i'\Sigma_{\epsilon i}^{-1}C_i = C'\Sigma_\epsilon^{-1}C$ if investor i is informed, and $C_i'\Sigma_{\epsilon i}^{-1}C_i = 0$ if investor i is uninformed. We have

$$\int_0^1 [\mathrm{Var}(R^e|S_i,P)]^{-1}\,di = \Sigma_F^{-1} + \Pi\Sigma_Z^{-1}\Pi + \Pi/\rho,$$

where Π is given by (33). Thus we have

$$[\mathrm{Var}\,(R^e|S_i,P)]^{-1} - \int_0^1 [\mathrm{Var}(R^e|S_i,P)]^{-1}\,di = C'\Sigma_\epsilon^{-1}C - \Pi/\rho.$$

Substituting the condition above into the expression for $X_i - Z$ gives

$$\begin{aligned} X_i - Z &= \rho\left([\mathrm{Var}_i\,(R^e|S_i,P)]^{-1} - \int_0^1 [\mathrm{Var}_i(R^e|S_i,P)]^{-1}\,di\right)(F - PR_f) + \rho C'\Sigma_\epsilon^{-1}\epsilon_i \\ &= \rho\left(C'\Sigma_\epsilon^{-1}C - 1/\rho\Pi\right)(F - PR_f) + \rho C'\Sigma_\epsilon^{-1}\epsilon_i, \end{aligned}$$

for an informed investor i.

As a result, the expected portfolio excess return is given by

$$\mathbb{E}\left[(X_i - Z)'(F - PR_f)\right] = \mathbb{E}\left[(F - PR_f)'(\rho C'\Sigma_\epsilon^{-1}C - \Pi)(F - PR_f)\right].$$

By (33) and a standard result from statistics,[8] we have

$$\begin{aligned} \mathbb{E}\left[(X_i - Z)'(F - PR_f)\right] &= \mathrm{Tr}\left[(\rho C'\Sigma_\epsilon^{-1}C - \Pi)V_e\right] + \bar{R}^{e\prime}(\rho C'\Sigma_{\epsilon i}^{-1}C - \Pi)\bar{R}^e \\ &= \frac{1-\lambda}{\lambda}\mathrm{Tr}\,(\Pi V_e) + \frac{1-\lambda}{\lambda}\bar{R}^{e\prime}\Pi\bar{R}^e \\ &= \frac{1-\lambda}{\lambda}\mathrm{Tr}\,(\Pi V_e) + \frac{1-\lambda}{\lambda}\mathrm{Tr}(\bar{R}^{e\prime}\Pi\bar{R}^e) \\ &= \frac{1-\lambda}{\lambda}\mathrm{Tr}\,(\Pi V_e) + \frac{1-\lambda}{\lambda}\mathrm{Tr}(\Pi\bar{R}^e\bar{R}^{e\prime}) \\ &= \frac{1-\lambda}{\lambda}\mathrm{Tr}\left[\Pi(V_e + \bar{R}^e\bar{R}^{e\prime})\right] \\ &= \frac{1-\lambda}{\lambda}\mathrm{Tr}\,[\Pi\Omega], \end{aligned}$$

for an informed investor i, where $\bar{R}^e$ and V_e are the ex-ante mean and variance of the asset excess returns. Q.E.D.

Proof of Proposition 7: Recall that the variance-covariance matrix of asset prices is given by

$$\mathrm{Var}\,(P) = A_1\Sigma_F A_1' + A_2\Sigma_Z A_2',$$

where A_1 and A_2 satisfy (9) and (10). Given the signal independence assumption, Π is diagonal. Since Σ_F and Σ_Z are diagonal by assumption, A_1 and A_2 are also diagonal. Thus $\mathrm{Var}(P)$ is also diagonal. Q.E.D.

[8] If $x \sim N(\mu, V)$ and $q = x'Ax$, then $\mathbb{E}\,[q] = \mathrm{Tr}(AV) + \mu' A\mu$.

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