Monetary Policy and Economic Growth under Money Illusion*

Jianjun Miao[†] and Danyang Xie[‡]

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Abstract

Empirical and experimental evidence documents that money illusion is persistent and widespread. This paper incorporates money illusion into two stochastic continuous-time monetary models of endogenous growth. Motivated by psychology, we model an agent's money illusion behavior by assuming that he maximizes nonstandard utility derived from both nominal and real quantities. Money illusion affects an agent's perception of the growth and riskiness of real wealth and distorts his consumption/savings decisions. It influences long-run growth via this channel. We show that the welfare cost of money illusion is second order, whereas its impact on long-run growth is first order relative to the degree of money illusion. Monetary policy can eliminate this cost by correcting the distortions on a money-illusioned agent's consumption/savings decisions.

Key words: money illusion, inflation, growth, welfare cost, behavioral macroeconomics

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[†]Department of Economics, Boston University, 270 Bay State Road, Boston MA 02215, USA, and Department of Finance, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email: miaoj@bu.edu. Tel: (852) 2358 8298.

[‡]Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email: dxie@ust.hk. Tel. (852) 2358 7615.

1. Introduction

The term money illusion refers to the phenomenon where people confuse nominal with real magnitudes. It is widely believed that this term was coined by Irving Fisher who devoted an entire book to the subject (Fisher (1928)). The presence of money illusion has frequently been invoked to account for the short-run non-neutrality of money by Keynesian economists and by some quantity theorists such as Fisher.[1] However, to have long-run neutrality of money, money illusion must be absent. Money illusion is often regarded as irrational and costly to decision makers, and hence implausible. Since the rational expectations revolution, many economists have resisted to use money illusion to explain the effectiveness of monetary policy.

Recently, there has been renewed interest in both the empirical and theoretical implications of money illusion. First, there is enormous experimental and empirical evidence of money illusion. Shafir et al. (1997) conduct survey studies and provide psychological account of money illusion. Shiller (1997) conducts similar survey studies. Fehr and Tyran (2001, 2007) design laboratory experiments to show that money illusion is important for nominal price inertia and equilibrium selection. Furthermore, other evidence has provided empirical support for the effects of money illusion on housing markets (Brunnermeier and Julliard (2007), Genesove and Mayer (2001)) and stock markets (Campbell and Vuolteenaho (2004), Cohen, Polk and Vuolteenaho (2005)). Second, a growing number of studies incorporate money illusion into formal models to demonstrate that it can explain some empirical puzzles. For example, Akerlof and Yellen (1985a,b) study the money and output relation. Akerlof et al. (1996, 2000) analyze the Phillips curve. Basak and Yan (2007) examine asset pricing implications. Piazzesi and Schneider (2007) investigate housing markets. In summary, it seems that money illusion rides again as claimed by Blinder (2000).

While there exist extensive studies of the implications of money illusion for many aspects of macroeconomics and finance, its implications for the relationship between monetary policy and long-run growth have not been explored. Our paper intends to fill this gap by building two stochastic continuous-time monetary models of endogenous growth. Endogenous growth is introduced by adopting the one-sector AK framework of Jones and Manuelli (1990) and Rebelo (1991). Money is introduced via the money-in-the-utility (MIU) function framework of Sidrauski (1967a,b) or via a cash-in-advance (CIA) constraint on consumption purchases. Following Basak and Yan (2007), Dusansky and Kalman (1974), and Howitt and Patinkin (1980), we model a money illusioned agent's behavior by assuming that he maximizes a nonstandard utility function

[1] According to Keynesian economics (Keynes (1936), Leontief (1936)), workers suffer from money illusion. The labor supply depends on the nominal wage rate whereas the demand depends on the real wage. A rise in the price level will raise the equilibrium level of employment.

that violates a homogeneity of degree zero property. Namely, the agent derives utility from both nominal and real quantities. We show that money illusion affects an agent's perception of the growth and riskiness of real wealth and hence distorts his consumption/savings decisions. Our framework departs from the traditional modeling assumption that money illusioned agents and firms fail to maximize and follow rule-of-thumb decisions.

Both our MIU and CIA models are tractable and allow us to derive closed-form solutions. Yet, they are rich enough for calibration and quantitative assessments. We show that both our models yield the same implications of money illusion for inflation and long-run economic growth. Importantly, we establish that the welfare cost of money illusion is second order, whereas the impact on long-run economic growth is first order in terms of the degree of money illusion. This result is reminiscent of the Keynesian proposition that small deviations from rationality have small welfare losses, but can have a significant impact on economic outcomes (Akerlof (2002), Akerlof and Yellen (1985b), Mankiw (1985)). When calibrating our models to the U.S. annual data from 1960 to 2006 and setting the coefficient of relative risk aversion to 2, we find that a small degree of money illusion, in the sense that the representative agent puts 5 percent weight on nominal quantities in utility evaluation, results in a small welfare loss of 0.06 percent of real income, whereas it lowers the rate of economic growth by a non-negligible 0.11 percentage point. Is this growth effect significant? As an illustration, consider two identical economies starting in 1960 except that the representative agent in one economy has the preceding degree of money illusion and the agent in the other economy has no money illusion. Their income would differ by about 5 percent in 2007.

We show that the monetary authority can choose a growth rate of the money supply to eliminate the cost of money illusion by correcting the distortions on consumption/savings decisions. This monetary policy implements a specific nonzero expected inflation rate or a constant nominal interest rate such that the distortions arising from the agent's misperception of the growth and riskiness of real wealth offset each other.

One may argue that money illusion should not persist in the long run as agents can learn. However, as argued by Shafir et al. (1997), money illusion arises in large part because it is considerably easier and more natural for individuals to think in nominal rather than in real terms. This tendency is likely to persist despite economists' attempts to educate the public. Akerlof et al. (2000) use a variety of psychological evidence to argue that high inflation, not the passage of time, may dissipate money illusion. High inflation is salient so that people may take into account the difference between nominal and real values. Our quantitative results provide support for this psychological argument. We find that the welfare cost of money illusion is small for low inflation. It rises nonlinearly with the expected inflation rate and becomes large for high inflation.

Our models can shed light on the growth and inflation relationship.[2] While some researchers find evidence for a negative relationship (e.g., Barro (1996) and Chari et al. (1995)), other empirical studies show that this relationship is not robust (e.g., Bruno and Easterly (1998), Dotsey and Sarte (2000), Fischer et al. (2002), and Kahn and Senhadji (2001)). We prove that this relationship depends on the representative agent's risk attitudes or the elasticity of intertemporal substitution.[3] In particular, growth and inflation are negatively (positively) related if the degree of relative risk aversion is greater (less) than unity. They are independent if the degree of relative risk aversion is equal to unity.

The rest of the paper is organized as follows. In Section 2, we survey definition and evidence of money illusion, and discuss our modeling of money illusion. In Section 3, we study a MIU model of endogenous growth. In Section 4, we study a CIA model of endogenous growth. In Section 5, we analyze the quantitative effects of money illusion and discuss the monetary policy that eliminates the welfare cost of money illusion. We conclude in Section 6.

2. Definition and Evidence of Money Illusion

We first discuss the definition and our modeling of money illusion.[4] We then survey evidence of money illusion and discuss its underlying psychology.

2.1. Definition

Fisher (1928, p. 4) defines money illusion as "the failure to perceive that the dollar, or any other unit of money, expands or shrinks in value. " Leontief (1936) defines that there is no money illusion if demand and supply functions are homogeneous of degree zero in all nominal prices. This is what Leontief (1936) called the "homogeneity postulate." Beginning with Haberler (1941, p.460) other writers have used the term money illusion as synonymous with a violation of this homogeneity postulate. Patinkin (1949) objects this use on the grounds that it fails to take into account the real balance effect. Patinkin (1965, p.22) defines that "an individual will be said to be suffering from such an illusion if his excess-demand functions for commodities do not depend [...] solely on relative prices and real wealth."

In a static model, the absence of money illusion in Patinkin's sense is equivalent to the

[2] We note that both inflation and economic growth are endogenous. Their relationship essentially refers to the relationship between money growth and output growth since inflation is determined by the money growth.

[3] Under our power utility specification, the degree of relative risk aversion is equal to the inverse of the elasticity of intertemporal substitution.

[4] See Howitt (1987) for an excellent discussion of money illusion in an entry in the *New Palgrave Dictionary of Economics*.

assumption of rational behavior, in the following sense. Let an agent's demand functions $x_i^* (p_1, ..., p_n, W)$ for goods $i = 1, ..., n$, together with his money demand function $M^* (p_1, ..., p_n, W)$, be defined as the maximizers of the utility function $U (x_1, .., x_n; M, p_1, ..., p_n)$ subject to the budget constraint: $p_1 x_1 + ... + p_n x_n + M = W$, where W is initial nominal wealth. The utility function includes money M and the nominal prices p_i because money is assumed to yield unspecified services whose value depends on prices.[5] The function U is said to be illusion-free if it is homogeneous of degree zero in $(M, p_1, ..., p_n)$. It is easily verified that x_i^*'s are illusion-free in Patinkin's sense if and only if they can be derived from an illusion-free U (see Dusansky and Kalman (1974) and Howitt and Patinkin (1980)).

In our models studied below, there is only one good and agents make intertemporal choices. We consider the following time-additive expected utility function:

$$E \left\{ \int_0^\infty e^{-\rho t} U (c_t, M_t, P_t) \, dt \right\}, \tag{1}$$

where (c_t), (M_t), and (P_t) are consumption, money, and price processes, respectively. Analogous to the preceding definition, we define that U is illusion-free if it satisfies homogeneity of degree zero in M and P. Consequently, to have money illusion, this homogeneity of degree zero property must be violated.[6] Because we analyze long-run growth, we need the utility function to satisfy certain homogeneity property in consumption. This consideration leads us to take specific functional forms. First, when we study the MIU model in Section 3, we adopt the following CES function:

$$U (c, M, P) = \frac{1}{1-\gamma} \left(\alpha \left[c^{1-\theta} (Pc)^\theta \right]^{1-\varphi} + (1-\alpha) \left[(M/P)^{1-\theta} M^\theta \right]^{1-\varphi} \right)^{\frac{1-\gamma}{1-\varphi}}, \tag{2}$$

where $\gamma > 0$ is the risk aversion parameter, $1/\varphi > 0$ the elasticity of substitution between consumption and money, and $\alpha \in (0, 1)$ represents the relative weight of consumption and money. The most important parameter is $\theta \in [0, 1]$ which represents the degree of money illusion. The case with $\theta = 0$ is illusion-free and the case with $\theta = 1$ means complete money illusion. The interpretation is that the agent values both real and nominal terms, with weight θ being put on nominal terms. This interpretation is consistent with Shafir, Diamond and Tversky's (1997) psychological interpretation of money illusion. They argue that "people are generally aware that there is a difference between real and nominal values, but because at a single point in time, or over a short period, money is a salient and natural unit, people often think of

[5] There are other reasons why preferences depend on prices. For example, Pollak (1977) and Veblen (1899) argue that people may judge quality by price or a higher price enhances the snob appeal of a good.

[6] Using revealed preference theory in a static setting, Loffler (2001) provides an axiomatic foundation for a utility function to deliver a demand function that violates the homogeneity of degree zero property, and hence exhibits money illusion. He proves that this utility function depends on nominal wealth.

transactions in predominantly nominal terms. Consequently, the evaluation of transactions often represents a mixture of nominal and real assessments, which gives rise to money illusion." Note that one should not interpret our utility model as agents like expected inflation. Instead, our proposed utility function represents an illusioned agent's revealed preferences for nominal quantities because of his confusion between nominal and real quantities.

As is well known, the case with $\gamma = 1$ corresponds to the logarithmic utility function, and the case with $\varphi = 1$ corresponds to the following:

$$U(c, M, P) = \frac{1}{1-\gamma}\left(c^{\alpha}(M/P)^{1-\alpha}\right)^{1-\gamma} P^{\theta(1-\gamma)}. \tag{3}$$

When we study the CIA model in Section 4, money does not enter the utility function. We thus adopt the following specification:

$$U(c, M, P) = \frac{\left[c^{1-\theta}(Pc)^{\theta}\right]^{1-\gamma}}{1-\gamma}. \tag{4}$$

This specification is first adopted by Basak and Yan (2007) in a pure-exchange economy model.

Our modeling of money illusion follows the early literature that seeks micro-foundation of money illusion. There are alternative modeling approaches in the recent literature. Akerlof et al. (2000) assume that the productivity of a firm depends on the nominal wage paid by the firm relative to a reference nominal wage. An illusioned firm's reference nominal wage underestimates inflation. In an efficiency wage model, Shafir et al. (1997) model money illusion by assuming that an illusioned worker's effort depends not only on real wage, but also on the ratio of the current nominal wage to the previous nominal wage. In a two period model, Cohen et al. (2005) assume that (i) a money-illusioned agent maximizes expected utility over nominal wealth instead of real wealth, and (ii) this agent believes that the nominal growth of corporate assets do not depend on inflation. Piazzesi and Schneider (2007) assume that a money-illusioned agent has a standard utility function over real consumption, but this agent mistakenly believes that an asset's nominal payoffs do not depend on inflation.

2.2. Evidence and Psychology

Using survey questions, Shafir et al. (1997) present a series of studies that investigate the effects of nominal and real changes on people's stated choices and evaluation of economic conditions. Specifically, they study people's attitudes toward salary raises in times of inflation, people's evaluation of monetary transactions, effects of framing transactions in nominal or in real terms on a choice between indexed and unindexed contracts, investment in risky funds under inflation, effects of accounting practices, and judgments regarding fairness and moral. The consistency of their results observed across diverse subjects population (students, shoppers, airline passengers), and a variety of problem contexts (contracts, acquisitions, fairness perception, judgments about others, trading experiments, etc.) provide strong evidence for money illusion. They conclude that money illusion is a widespread phenomenon in reality. This phenomenon is not readily eliminated by learning. People may resort to an analysis in real terms when inflation is high, but may then go back to relying on nominal evaluations when the inflation subsides.

What causes money illusion? Shafir et al. (1997) provide a psychological account. They argue that economic agents often entertain both nominal and real representations of economic transactions, and money illusion is a bias in the assessment of the real value of transactions, induced by their nominal representation. People tend to think in terms of nominal value because it is salient, easy to gauge, and in many cases provides a reasonable estimate of real worth. The strength and persistence of this bias is likely to depend on several factors, notably the relative salience of the nominal and real representations, and the sophistication and experience of the decision maker.

Money illusion is closely related to other psychological judgement and decision biases. Without money illusion, money is a veil and only real prices and real wealth matter. However, psychological biases might prevent people from seeing through this veil. The framing effect is an important bias. It states that alternative representations (framing) of the same decision problem can lead to substantially different behavior (Tversky and Kahneman (1981)). Shafir et al. (1997) document that people's choices depend to a large degree on whether the problem is phrased in real terms or nominal terms. This framing effect has implications for both time preferences and risk attitudes. For example, in a contract choice experiment, Shafir et al. (1997) find the following result. If the problem is phrased in nominal terms, agents prefer the nominally less risky option to the alternative, which is less risky in real terms. That is, they avoid nominal risk rather than real risk. On the other hand, if the problem is stated in real terms, their preference ranking reverses.

Anchoring is a special form of the framing effect. It refers to the following phenomenon studied by Tversky and Kahneman (1974): In many situations, people make estimates by starting from an initial value that is adjusted to yield the final answer. The initial value, or starting point, may be suggested by the formulation of the problem, or it may be the result of a partial computation. In either case, adjustments are typically insufficient. That is, different starting points yield different estimates, which are biased toward the initial values. For example, the nominal purchasing price of a house can serve as an anchor for a reference price even when the real price can be easily derived. Genesove and Mayer (2001) document that people are reluctant to sell a house at a nominal loss. With changing relative prices, an effect of past nominal values on purchase or sale decisions is a form of money illusion that could be present even if the inflation rate is zero. Shafir et al. (1997) argue that people face mental accounting problems (Thaler (1980)) and also exhibit loss aversion behavior (Kahneman and Tversky (1979)). Loss aversion occurs relative to a reference point, which can often be nominal, yielding further manifestation of money illusion.

Shafir et al. (1997) also document that money illusion enters into people's perceptions of fairness because judgements of fairness are based largely on nominal rather than on real changes. The perception of fairness impinges on worker morale and hence may have implications for actual job decisions. This observation dates back to Keynes (1936, p.9).

Akerlof et al. (2000) argue that decision makers, far from making the best use of available information, readily ignore potentially relevant considerations and discard potentially relevant information in order to simplify their decision problems. This is the so called "editing" behavior (Kahneman and Tversky (1979)). When people "edit" decision problems, they rule out less important considerations in order to concentrate on the few factors that matter most. A related literature in the psychology of perception suggests that items must reach a threshold of salience before they are even perceived. Thus, when inflation is low, it may be at most a marginal factor in wage and price decisions, and decision makers may ignore it entirely.

Shiller (1997) documents evidence that the lay public fails to understand inflation as a general-equilibrium phenomenon. Workers and wage earners believe that inflation will make them poorer because it bids up the prices of the goods they consume, but they fail to appreciate fully, if at all, that inflation will also bid up the prices of other competing factors and other competing workers, thereby resulting in a rise in their own wages and salaries.

Does money illusion matter? Fehr and Tyran (2001) study the implications of money illusion for nominal inertia by conducting experiments from a price-setting game. They isolate money illusion from other potential determinants of nominal inertia. In these experiments, price setters are given payoffs derived from a monopolistic competition model. They find that negative changes in the money supply caused considerable output reductions when payoffs are denominated in nominal terms, rather than real terms. Subjects act as if other price setters suffered from money illusion, making them, in turn, reluctant to cut prices. That is, the reaction of prices to money supply changes involves the formation of expectations concerning the response of other price setters to the same shock. Fehr and Tyran's (2001) experimental results indicate that this indirect effect of money illusion is important in a strategic context. Fehr and Tyran (2007) conduct experiments to show that even if learning in the context of an individual optimization problem does eventually remove individuals' money illusion, there can be large permanent effects of money illusion in a strategic environment. These effects arise because money illusion induces individuals to coordinate on inferior equilibria.

Some researchers attribute real effects of inflation to money illusion in tax laws (e.g., Abel (1981), Auerbach (1979), and Feldstein (1983)). Specifically, in many countries interest income and expenses are taxed at the same rate independent of inflation, and historical money costs rather than current replacement costs are used for evaluating inventories and calculating depreciation allowances. Consequently, inflation can distort the after-tax cost of capital and hence influences investment. Based on this mechanism, Jones and Manuelli (1995) build an endogenous growth model to study the growth effect of inflation.

Some recent empirical studies have documented that money illusion has important implications in financial and real estate markets. This result is related to early empirical studies that document a negative correlation between nominal stock returns and inflation (e.g., Lintner (1975), Fama and Schwert (1977)). The early evidence appears puzzling since the Fisher-relation implies that nominal returns should increase with expected inflation. One interpretation is that higher inflation is associated with a grim economic condition (e.g., Fama (1981)). An alternative interpretation is that the negative correlation might be due to money illusion. Modigliani and Cohn (1979) hypothesize that prices significantly depart from fundamentals since investors make two inflation-induced judgement errors: (i) they tend to capitalize equity earnings at the nominal rate rather than the real rate, and (ii) they fail to realize that firms' corporate liabilities depreciate in real terms. Hence, stock prices are too low during high inflation periods.

Recently, some studies have found empirical support for the Modigliani-Cohn hypothesis. Ritter and Warr (2002) document that the value-price ratio is positively correlated with inflation and that this effect is more pronounced for leveraged firms. Campbell and Vuolteenaho (2004) decompose the dividend yield on stocks into three components: (1) a rational forecast of long-run real dividend growth, (2) the subjectively expected risk premium, and (3) residual mispricing attributed to the market's forecast of dividend growth deviating from the rational forecast. They show that the positive correlation of dividend yields with inflation is mostly due to the mispricing component. This indicates that stocks are undervalued by conventional measures when inflation is high. Cohen, Polk, and Vuolteenaho (2005) simultaneously examine the future returns of Treasury bills, safe stocks, and risky stocks in order to distinguish money illusion from any change in the attitudes toward risk. They find that when inflation is high (low), stock returns are higher (lower) than justified by an amount that is constant across stocks, irrespective of the riskiness of the particular stock. Their empirical evidence supports the hypothesis that investors suffer from money illusion. Brunnermeier and Julliard (2007) study the relationship between housing prices and inflation. They derive a decomposition of the price-rent ratio and identify an empirical proxy for the mispricing in the housing market.

Using data from the U.K. housing market, they show that it is largely explained by movements in inflation. In particular, a reduction in inflation can generate substantial increases in housing prices in a setting in which agents are prone to money illusion.

3. A MIU Model

We consider a monetary economy consisting of a representative agent and a monetary authority. Time is continuous and the horizon is infinite. To have a role of money, we start with the Sidrauski (1967a, b) formulation of the money-in-the-utility (MIU) function. Specifically, the agent derives utility from consumption and money. Unlike the standard formulation, we assume that the agent suffers from money illusion. To capture money illusion, his utility function is given by (1) and (2).

To generate long-run growth endogenously, we adopt a simple one-sector AK model (Jones and Manuelli (1990) and Rebelo (1991)). Unlike the standard deterministic AK model, we introduce uncertainty into the production technology as in Eaton (1981) to see how an uncertain environment may amplify the impact of money illusion. At each point in time, the agent decides how much to consume and how much of his wealth to invest in productive capital, nominal bonds, and money. The nominal bonds yield a constant nominal interest rate R and are in zero net supply. The nominal interest rate R is endogenous and will be verified to be constant in equilibrium. The agent operates a technology with constant returns to scale. In particular, with k_t units of real capital at time t, the technology yields stochastic output:

$$Ak_t dt + k_t \sigma_k dz_t, \tag{5}$$

during period dt, where $A, \sigma_k > 0$ and $(z_t)_{t \geq 0}$ is a standard Brownian motion. For simplicity, we assume that capital does not depreciate. The agent's budget constraint is given by

$$dk_t + \frac{dB_t}{P_t} + \frac{dM_t}{P_t} = (Ak_t - c_t)dt + k_t \sigma_k dz_t + \frac{RB_t}{P_t} dt + v_t dt, \tag{6}$$

where k_0 is given, P_t denotes the date t price level, B_t and M_t denote the date t bond and money holdings, and v_t denotes the date t real value of the government transfers. If $v_t < 0$, it is interpreted as lump-sum taxes.

The monetary authority sets a simple money growth rule such that the money supply, $\bar{M}_t$, satisfies:

$$d\bar{M}_t = \mu \bar{M}_t dt, \tag{7}$$

where $\mu > 0$ is the constant money growth rate. Any money supply policy must be implemented by a policy of fiscal transfers, open market operations, or both. As a starting point, we assume that the agent is given lump-sum transfers of money so that

$$v_t = \mu \frac{\bar{M}_t}{P_t}. \tag{8}$$

Alternative assumptions about the uses of growth of the money supply may lead to different conclusions about the relationship between inflation and growth. For example, using the growth of the money supply to subsidize capital formation or reduce other distortional taxes may stimulate growth.

A monetary competitive equilibrium can be defined in the usual fashion. That is, (i) the agent maximizes utility given by (1) and (2) subject to the budget constraint (6), (ii) all markets clear in that $B_t = 0$, $\bar{M}_t = M_t$ for any t, and

$$dk_t = (Ak_t - c_t)dt + k_t \sigma_k dz_t, \tag{9}$$

and (iii) monetary policy satisfies (7) and (8).

3.1. Consumption/savings decision

To solve for a monetary competitive equilibrium, we first solve the agent's decision problem by dynamic programming. To this end, we must derive the dynamics of the price level P_t and the transfers v_t. We conjecture that the equilibrium law of motion for P_t follows the following geometric Brownian motion process:

$$\frac{dP_t}{P_t} = \pi dt + \sigma_P dz_t, \ P_0 = 1, \tag{10}$$

where π is the constant expected inflation rate and σ_P is the constant volatility of the inflation rate. It follows from Ito's Lemma that

$$d\left(\frac{1}{P_t}\right) = -\frac{1}{P_t^2} dP_t + \frac{1}{P_t^3} (dP_t)^2 = \left(\frac{\sigma_P^2}{P_t} - \frac{\pi}{P_t}\right) dt - \frac{\sigma_P}{P_t} dz_t. \tag{11}$$

Define the agent's real wealth level $w_t = k_t + B_t/P_t + M_t/P_t$. Then, by definition,

$$dw_t = dk_t + d\left(\frac{B_t}{P_t}\right) + d\left(\frac{M_t}{P_t}\right). \tag{12}$$

Notice that it follows from Ito's Lemma that $d\left(\frac{M_t}{P_t}\right)$ and $d\left(\frac{B_t}{P_t}\right)$ can be written as:

$$d\left(\frac{M_t}{P_t}\right) = \frac{dM_t}{P_t} + M_t d\left(\frac{1}{P_t}\right), \tag{13}$$

and

$$d\left(\frac{B_t}{P_t}\right) = \frac{dB_t}{P_t} + B_t d\left(\frac{1}{P_t}\right). \tag{14}$$

We can now use (11), (13), and (14) to rewrite the agent's budget constraint (6) as

$$\begin{aligned} dw_t &= dk_t + \frac{dB_t}{P_t} + \frac{dM_t}{P_t} + (B_t + M_t)\, d\left(\frac{1}{P_t}\right) \\ &= (Ak_t - c_t)\, dt + k_t \sigma_k dz_t + \frac{RB_t}{P_t} dt + v_t dt \\ &\quad + (w_t - k_t)\left[\left(\sigma_P^2 - \pi\right) dt - \sigma_P dz_t\right]. \end{aligned} \tag{15}$$

Let $k_t = \phi_t w_t$ and $B_t/P_t = \psi_t w_t$, where ϕ_t and ψ_t are to be determined. Thus, $M_t/P_t = (1 - \phi_t - \psi_t)\, w_t$. We can then rewrite the budget constraint (15) as

$$dw_t = \left(w_t\left[A\phi_t + R\psi_t + (1-\phi_t)\left(\sigma_P^2 - \pi\right)\right] + v_t - c_t\right) dt + \left[\phi_t \sigma_k - (1-\phi_t)\,\sigma_P\right] w_t dz_t. \tag{16}$$

We conjecture that in equilibrium ϕ_t takes some constant value ϕ^* given in (24) below. Since in equilibrium $B_t = 0$ and $\psi_t = 0$, we thus have

$$\frac{M_t}{P_t} = w_t\,(1 - \phi^*)\,. \tag{17}$$

This equation is analogous to the money demand in the quantity theory of money since wealth is proportional to aggregate income in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion coefficients in (17), we obtain

$$\phi^* \sigma_k - (1 - \phi^*)\,\sigma_P = -\sigma_P.$$

Thus, $\sigma_P = -\sigma_k$.

We next turn to the dynamics of transfers v_t. The transfers depend on the equilibrium holdings of money, which in turn depend on the aggregate wealth level in the economy, denoted by $\bar{w}$. Therefore, it is important to derive the dynamics of aggregate wealth. We conjecture, as in Rebelo and Xie (1999), that aggregate real wealth follows the diffusion process:

$$d\bar{w}_t = f\left(\bar{w}_t\right) dt + h\left(\bar{w}_t\right) dz_t, \tag{18}$$

where f and h are functions to be determined such that it coincides with the representative agent's wealth level, $\bar{w}_t = w_t$. In this case, the lump-sum transfer satisfies:

$$v_t = \mu \bar{M}_t / P_t = \mu \bar{w}_t\,(1 - \phi^*)\,. \tag{19}$$

We are now ready to solve the agent's dynamic programming problem. It is natural that the agent's wealth level and the price level are state variables. When solving his decision problem, the agent's takes the lump-sum transfers as given. Thus, he must take into account the law of motion for the aggregate wealth level. This implies that the aggregate wealth level should be an additional state variable. Exploring the homogeneity property of the utility function, we conjecture that the value function takes the following form:

$$J(w,\bar{w},P)=b\frac{(w+\beta\bar{w})^{1-\gamma}}{1-\gamma}P^{\theta(1-\gamma)}, \tag{20}$$

where b and β are constants to be determined.

By the standard dynamic programming theory, the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned} 0 \;=\; & \max_{\phi,\psi} U\left(c,M,P\right)-\rho J(w,\bar{w},P) \qquad (21)\\ & +J_1\left(w,\bar{w},P\right)\left\{w\left[A\phi+R\psi+(1-\phi)\left(\sigma_k^2-\pi\right)\right]+v-c\right\}\\ & +J_2\left(w,\bar{w},P\right)f\left(\bar{w}\right)+J_3\left(w,\bar{w},P\right)P\pi\\ & +\frac{1}{2}J_{11}\left(w,\bar{w},P\right)\sigma_k^2w^2+\frac{1}{2}J_{22}\left(w,\bar{w},P\right)\left[h\left(\bar{w}\right)\right]^2+\frac{1}{2}J_{33}\left(w,\bar{w},P\right)\left[P\sigma_k\right]^2\\ & +J_{12}\left(w,\bar{w},P\right)h\left(\bar{w}\right)w\sigma_k-J_{13}\left(w,\bar{w},P\right)wP\sigma_k^2-J_{23}\left(w,\bar{w},P\right)h\left(\bar{w}\right)P\sigma_k, \end{aligned}$$

where U is given by (2). Solving this HJB equation, we obtain:

Proposition 1 *Suppose*

$$\eta\equiv\frac{\rho-(1-\gamma)A}{\gamma}+\frac{1}{2}(1-\gamma)\sigma_k^2+\frac{\theta(\gamma-1)}{\gamma}\left[\pi+\frac{\sigma_k^2}{2}\left((\gamma-1)(2-\theta)-1\right)\right]>0. \tag{22}$$

Let the price level, the aggregate wealth level, and the transfers satisfy equations (10), (18), and (19), respectively, with $\sigma_P=-\sigma_k$,

$$f\left(\bar{w}\right)=(A-\eta)\,\bar{w},\;\; h\left(\bar{w}\right)=\sigma_k\bar{w}, \tag{23}$$

and

$$\phi^*=1-\frac{\eta}{R+\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}-\mu}. \tag{24}$$

Then the value function is given by (20) where b *and* β *are given by*

$$b=\eta^{-\gamma}\left(\frac{R}{1-\alpha}\right)^{\gamma-1}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)}}, \tag{25}$$

$$\beta=\mu\left[R+\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}-\mu\right]^{-1}. \tag{26}$$

In addition, the nominal interest rate satisfies

$$R=A+\pi-\sigma_k^2, \tag{27}$$

the optimal consumption rule is given by

$$c = \frac{\left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} \eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}} \left(w + \beta \bar{w}\right), \tag{28}$$

and the optimal money demand rule is given by

$$\frac{M}{P} = \frac{\eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}} \left(w + \beta \bar{w}\right). \tag{29}$$

Equation (27) is a modified Fisher equation, which corrects for uncertainty. It shows that the nominal interest rate is equal to the real interest rate plus the expect inflation rate, minus the variance of the inflation rate. This equation follows from a simple no arbitrage argument.

Importantly, Proposition 1 verifies that the conjectured value function (20) is correct. In addition, it characterizes the agent's decision rules. To discuss these decision rules, we first analyze the special case with $\theta = 0$. This case characterizes the behavior of a fully rational agent without money illusion. In this case, as is well known from the portfolio choice theory (e.g., Merton (1969)), the fully rational agent consumes a constant fraction of his total wealth and allocates another constant fraction of his total wealth as cash. This total wealth, $w + \beta \bar{w}$, includes the discounted present value of transfers and wealth from holding capital, bonds, and cash. The two constant fractions in (28) and (29) depend on the nominal interest rate R, which is the opportunity cost of holding money. They also depend on the term η, which represents the marginal propensity to consume out of wealth in a consumption/savings model without money (e.g., Merton (1969)). As shown in (22), the return A to the investment and the variance of the return σ_k^2 are important determinants of η. Their effects on consumption through η depend on the degree of risk aversion γ, which determines the relative strength of substitution and wealth effects. In particular, the wealth effect dominates when $\gamma > 1$ so that an increase in A or a decrease in σ_k^2 raises current consumption. When $\gamma < 1$, the substitution effect dominates so that the opposite result holds. In the borderline case with $\gamma = 1$, the two effects cancel out so that changes in investment opportunities do not influence consumption.

We next turn to the behavior of a money illusioned agent with $\theta > 0$. Proposition 1 shows that the consumption and money demands exhibit similar functional forms to that for a fully rational agent. The key difference is that there is a new term in η as shown in equation (22):

$$\frac{\theta (\gamma - 1)}{\gamma} \left[\pi + \frac{\sigma_k^2}{2} \left((\gamma - 1)(2 - \theta) - 1\right)\right].$$

This term reflects the effect of money illusion. There are two components in the square bracket in this term. The first component is the expected inflation rate, which influences the growth of real wealth (see (16)). The second component captures precautionary motive to guard against inflation uncertainty. The latter component depends on the inflation volatility, $\sigma_P = -\sigma_k$, and the degrees of risk aversion and money illusion. It is positive if and only if $\gamma > 1/(2-\theta)+1$. In sum, money illusion distorts the agent's perception of the growth and riskiness of real wealth. The overall distortion is positively related to $\theta(\gamma-1)/\gamma$. We now focus on the distortion caused by the misperception of expected inflation, holding everything else constant. Equation (22) reveals that for $\gamma > 1$, higher expected inflation induces the money illusioned agent to consume more. The intuition is that a higher expected inflation rate π implies a higher nominal wealth growth rate, making a money-illusioned agent feel that his real wealth grows faster. This perception has two opposite wealth and substitution effects. When $\gamma > 1$, the wealth effect dominates so that the agent consumes more in response to an increased expected inflation rate. By contrast, when $\gamma < 1$, the substitution effect dominates so that the agent consumes less. When $\gamma = 1$, the wealth and substitution effects cancel out so that the agent does not respond to the change in the expected inflation rate. A similar analysis applies to the effect of expected inflation on the money demand.

3.2. Equilibrium

After analyzing the agent's decision problem, we are ready to solve for the equilibrium in which the price level, the aggregate wealth level, and the transfers satisfy (10), (18), and (19) with the restrictions given in Proposition 1. The key step is to solve for the expected inflation rate π. Using the market clearing conditions, we have:

Proposition 2 *Suppose condition (22) holds. Then in equilibrium the expected inflation rate π satisfies*

$$\pi = \frac{\gamma}{\theta + (1-\theta)\gamma}\left\{\mu - \frac{A-\rho}{\gamma} + \frac{1}{2}(3-\gamma)\sigma_k^2 + \frac{\theta(\gamma-1)\sigma_k^2}{2\gamma}((\gamma-1)(2-\theta)-1)\right\}. \tag{30}$$

In addition, the equilibrium consumption level and capital stock satisfy $c = \eta k$ and

$$\begin{aligned} c &= \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}(1-\phi^*)\,w, \\ M/P &= (1-\phi^*)\,w,\ \ k = \phi^* w, \end{aligned} \tag{31}$$

where ϕ^* is given by (24), and the expected growth rate g of capital is given by

$$g = A - \eta. \tag{32}$$

As in Proposition 1, we first discuss the equilibrium without money illusion ($\theta = 0$). In this case, η takes the value:

$$\eta^0 \equiv \frac{\rho - (1-\gamma) A}{\gamma} + \frac{1}{2}(1-\gamma)\sigma_k^2. \tag{33}$$

As a result, equation (32) implies that the expected growth rate of capital is given by

$$g^0 \equiv \frac{A-\rho}{\gamma} + \frac{1}{2}(\gamma-1)\sigma_k^2. \tag{34}$$

In addition, both real output and consumption also grow at this expected rate. Thus, monetary policy does not influence economic growth and real allocation. Its effect is to change inflation as revealed by equation (30). This result is consistent with other studies (e.g., Chang and Lai (2000), Marquis and Reffett (1991), Rebelo and Xie (1999), and Wang and Yip (1992)).

We now discuss the equilibrium with money illusion ($\theta > 0$). In this case, the expected inflation rate enters η, and thus influences real allocations and economic growth. This effect depends on the risk aversion parameter γ. In particular, when $\gamma > 1$, there is a negative relation between growth and inflation. Intuitively, an increase in the expected inflation rate makes the money illusioned agent to consume more and save less as shown in Proposition 1. Thus, it lowers economic growth. The opposite result holds for $\gamma < 1$. In the borderline case $\gamma = 1$, inflation does not affect economic growth. Our results are related to the empirical studies on the inflation and growth relationship. While some researchers find evidence for a negative relationship (e.g., Barro (1996) and Chari et al. (1995)), other empirical studies show that this relationship is not robust (e.g., Bruno and Easterly (1998), Dotsey and Sarte (2000), Fischer et al. (2002), and Kahn and Senhadji (2001)). Our results provide a partial rationale for this nonrobustness.

4. A CIA Model

In the previous section, we introduce a role of money by assuming the agent derives utility from holding money directly. This way of modeling suffers from the criticism of creating positive value for money by simply assuming the problem away. We value money because it is useful in facilitating transactions. To incorporate this role, we study a cash-in-advance (CIA) model in this section. We will show that both the MIU and CIA models yield similar results.

Instead of assuming money in the utility function as in (2), we assume that the agent has the utility function (4). This utility function also captures money illusion. We assume consumption purchases must be financed by cash so that the following cash-in-advance constraint must hold:

$$P_t c_t \leq M_t. \tag{35}$$

The rest of the model follows the same structure as in the previous section. We also use a similar procedure to solve for a monetary competitive equilibrium. In particular, we conjecture the price dynamics follow equation (10). Define the real wealth level $w_t = k_t + B_t/P_t + M_t/P_t$. Let $k_t = \phi_t w_t$ and $B_t/P_t = \psi_t w_t$, where ϕ_t and ψ_t are to be determined. Using (6), we derive the law of motion for the wealth level (16). We conjecture that in equilibrium ϕ_t takes a constant value ϕ^* given in (36) below. As a result, equation (17) holds so that $\sigma_P = -\sigma_k$. We conjecture that the equilibrium aggregate real wealth satisfies (18) and the lump-sum transfer is given by (19).

As in the previous section, given the above conjecture, we first study the agent's decision problem. We conjecture that the value function $J(w, \bar{w}, P)$ takes the form in (20), where b are β are constants to be determined. By the standard dynamic programming theory, this value function satisfies the HJB equation (21), where $U(c, M, P)$ is given by (4). Solving (21), we obtain:

Proposition 3 *Suppose assumption (22) holds. Let the price level, the aggregate wealth level, and the transfers satisfy equations (10), (18), and (19), respectively, where $\sigma_P = -\sigma_k$, $f(\bar{w})$ and $h(\bar{w})$ satisfy (23) and*

$$\phi^* = 1 - \frac{\eta}{R + 1 - \mu}. \tag{36}$$

Then the value function is given by (20) where b and β are given by

$$b = \eta^{-\gamma} (1 + R)^{\gamma - 1}, \tag{37}$$

$$\beta = \mu (R + 1 - \mu)^{-1}. \tag{38}$$

In addition, the nominal interest rate R satisfies (27), and the optimal consumption and money demand rules are given by

$$c = \frac{M}{P} = \frac{\eta}{R + 1} (w + \beta \bar{w}). \tag{39}$$

This proposition verifies that the conjectured value function in (20) is correct. In addition, this proposition characterizes a money illusioned agent's optimal consumption and money holdings policies. The equality of consumption and real money balances in (39) reflects the fact that the CIA constraint binds. As in Proposition 1, the agent consumes a constant fraction of his total wealth and holds an identical fraction of his total wealth as cash. This constant fraction depends on the interest rate R and the inflation rate π via η. The latter dependence reflects money illusion. The effect of inflation on optimal consumption and money holdings and its intuition are identical to those in the MIU model analyzed in the previous section. So we omit the discussion here.

We now solve for the equilibrium in which the price level, the aggregate wealth level, and the transfers satisfy (10), (18), and (19) with the restrictions given in Proposition 3. In addition, we must solve for the expected inflation rate π.

Proposition 4 *Suppose condition (22) holds. Then in equilibrium the expected inflation rate* π *satisfies (30). In addition, the equilibrium consumption level and capital stock satisfy* $c = \eta k$*,*

$$c = M/P = (1 - \phi^*) w, \text{ and } k = \phi^* w,$$

where ϕ^* *is given by (36), and the expected growth rate* g *of capital is given by (32).*

This proposition demonstrates that both the CIA and MIU models give an identical equilibrium consumption-capital ratio η and an identical growth rate g of capital. In particular, for a fully rational agent without money illusion, monetary policy has no real effects. It only affects the price level and inflation. With money illusion, monetary policy has real effects and generates nonzero correlation between inflation and economic growth as in the previous section.

5. Quantitative Effects of Money Illusion

As argued by Akerlof (2002), "a major contribution of behavioral macroeconomics is to demonstrate that, under sensible behavioral assumption, monetary policy does affect real outcomes just as Keynesian economics long asserted. Cognitive psychology pictures decision makers as 'intuitive scientists' who summarize information and make choices based on simplified mental frames." One critique often made by the neoclassical synthesis is that irrational behavior is costly to decision makers and hence implausible. We will demonstrate that money illusion is not only commonplace but also sensible: the welfare losses from decision rules under money illusion are extremely small, but the effects on economic growth are quite large.

5.1. Welfare cost

Following Lucas (2000), we define the welfare cost $\Delta(\theta, \pi)$ of money illusion to be the percentage income compensation needed to leave the agent indifferent between $\theta = 0$ and $\theta > 0$ when the expected inflation rate is π. Formally, consider an agent with the standard utility function $U^0(c, M, P)$ given by (2) in the MIU model or (4) in the CIA model, where $\theta = 0$ for both cases. Then the indirect utility function from the equilibrium consumption plan under money illusion is given by

$$V(k_0) = E\left\{\int_0^\infty e^{-\rho t} U^0(c_t, M_t, P_t)\, dt\right\}, \tag{40}$$

where k_0 is the initial capital holdings, $c_t = \eta k_t$ and $M_t/P_t = k_t(1-\phi^*)/\phi^*$ by Propositions 2 and 4, with $\theta > 0$ in both the MIU and CIA models. The indirect utility function from the equilibrium consumption plan without money illusion is given by

$$V^0(k_0) = E\left\{\int_0^\infty e^{-\rho t} U^0\left(c_t^0, M_t^0, P_t^0\right) dt\right\},$$

where $c_t^0 = \eta^0 k_t$, and $M_t^0/P_t^0 = k_t(1-\phi^*)/\phi^*$, with $\theta = 0$ in both the MIU and CIA models. Then the welfare cost $\Delta(\theta, \pi)$ is defined as the solution to the following equation:

$$V(k_0(1+\Delta(\theta,\pi))) = V^0(k_0). \tag{41}$$

Notice that this equation actually defines $\Delta(\theta,\pi)$ as percentage capital compensation. Since in equilibrium, capital, output and wealth are all proportional to each other, we may also interpret $\Delta(\theta,\pi)$ as percentage output or wealth compensation.

We shall emphasize that we define $\Delta(\theta,\pi)$ as a function of the expected inflation rate π, which is endogenous in equilibrium. It is determined by the money growth rate μ as shown in equation (30). Since the inflation rate is observable and commonly used in practice, we will focus on it directly, but have in mind that there is an underlying money growth rate determining this expected inflation rate in equilibrium.

The following proposition gives a closed-form solution for $\Delta(\theta,\pi)$ in both the MIU and CIA models.

Proposition 5 *Suppose*[7]

$$\rho - (1-\gamma)\left(A - \eta(\theta,\pi) - \gamma\sigma_k^2/2\right) > 0. \tag{42}$$

Then the welfare cost of money illusion in both the MIU and CIA models is given by

$$\Delta(\theta,\pi) = \left(\frac{\rho - (1-\gamma)\left(A - \eta(\theta,\pi) - \gamma\sigma_k^2/2\right)}{\rho - (1-\gamma)\left(A - \eta(0,\pi) - \gamma\sigma_k^2/2\right)}\right)^{\frac{1}{1-\gamma}} \frac{\eta(0,\pi)}{\eta(\theta,\pi)} - 1, \tag{43}$$

where we define $\eta(\cdot)$ in (22) as a function of θ and π. In addition, π is given by equation (30).

Using this proposition, we can apply the Taylor expansion theorem to show that the welfare cost of money illusion is second order.

Proposition 6 *Suppose conditions (22) and (42) hold. Then holding π fixed,*

$$\Delta(\theta,\pi) = \Delta_{11}(0,\pi)\theta^2 + o\left(\theta^2\right),$$

where $o\left(\theta^2\right)$ represents terms that have a higher order than θ^2.

[7]Condition (42) ensures that the indirect utility function $V(k_0)$ is finite. In standard models with $\theta = 0$, conditions (22) and (42) are equivalent.

The intuition behind this proposition follows from the implications of the Envelope Theorem as discussed in Akerlof and Yellen (1985b). In our model, without money illusion, its welfare cost is zero. Thus, $\theta = 0$ minimizes $\Delta(\theta, \pi)$. As a result, a small degree of money illusion or a small positive value of θ has a second-order effect on the welfare cost $\Delta(\theta, \pi)$. Does a small degree of money illusion have a significant effect on economic equilibria? We can see immediately from equation (22) and Proposition 2 that the growth rate g of output contains a first order term of θ. Thus, a small degree of money illusion has a first-order effect on economic growth.

We now provide some quantitative estimates by calibrating model parameters. We first calibrate a standard AK growth model without money. The parameters are γ, ρ, A, and σ_k. As is standard in the macroeconomics and finance literature, we set $\gamma = 2$, and $\rho = -\ln(0.98)$. The choice of ρ ensures that the subjective discount factor in a discrete time model is equal to 0.98. To calibrate A and σ_k, we use the US annual constant price GDP data and population data from 1960 to 2006 as provided by CEIC Data Company Ltd. Simple calculation shows that the average per capita real GDP growth rate is 0.0223 and the standard deviation of the per capita real GDP growth is 0.0204, which are consistent with the numbers in Ramey and Ramey (1995) for an earlier sample.[8] We thus set $g^0 = 0.0223$ and $\sigma_k = 0.0204$. We then use equation (34) to solve for $A = 0.0644$.

To gauge the welfare cost $\Delta(\theta, \pi)$ in (43) and the growth effect of money illusion, we need to assign a value of the expected inflation rate. We find the average annual inflation rate is 0.0425 from 1960 to 2006 in the US, as documented in the 2007 Economic Report of the President. Figure 1 plots the welfare cost $\Delta(\theta, \pi)$ and the percentage point change in the growth rate,

$$g - g^0 = \eta(0, \pi) - \eta(\theta, \pi),$$

as a function of $\theta \in [0, 1]$ for $\pi = 0.0425$. Notice that under our calibration money illusion always lowers the expected growth rate of output. Figure 1 reveals that the welfare cost of money illusion $\Delta(\theta, \pi)$ is generally small compared to the change in the rate of economic growth. When θ is large, both numbers are large. In the extreme case with complete money illusion $\theta = 1$, we find $\Delta(\theta, \pi) = 34.22\%$ and the rate of economic growth decrease by 2.13 percentage point. For small values of θ, the welfare cost is extremely small, but the change in growth rate can still be non-negligible. For example, when $\theta = 0.05$, the welfare cost $\Delta(\theta, \pi) = 0.06\%$, which is almost negligible, but the growth rate decreases by a non-negligible 0.11 percentage point relative to the growth rate without money illusion. As another example,

[8] Ramey and Ramey (1995) document that the average real per capita GDP growth rate in the USA between 1962 and 1985 was 0.0214 and the standard deviation of the per capita real GDP growth was 0.0259.

when $\theta = 0.1$, $\Delta(\theta, \pi) = 0.26\%$, but the growth rate decreases by 0.21 percentage point. Are these growth effects significant? As an illustration, consider the difference in real GDP between two economies that are otherwise the same. But the representative agent in one economy has money illusion with degree $\theta = 0.05$ and the agent in the other economy has no money illusion $\theta = 0$. Suppose both economies start in 1960 with the same level of income, their income would differ by about 5 percent in 2007. If the agent in the first economy has $\theta = 0.1$, then the income difference in 2007 would be 10 percent.

[Insert Figure 1 Here.]

The quantitative effects of money illusion depends crucially on the expected inflation rate. Figure 2 plots the effects of money illusion for $\pi = 0.02$, 0.05, 0.10, and 0.20 for small degrees of money illusion $\theta \in [0, 0.2]$. This figure reveals that the welfare cost and the growth effect of money illusion are very small for low values of inflation. However, they rise nonlinearly with the expected inflation rate. In particular, they are extremely large for high values of the expected inflation rate, even when the degree of money illusion is relatively small. For example, when $\pi = 0.2$ and $\theta = 0.1$, the welfare cost is about 6 percent of the per capita wealth and the rate of economic growth decreases by about 1 percentage point.

[Insert Figure 2 Here.]

An important parameter in our model is the coefficient of relative risk aversion γ. We now consider its effect. Propositions 1 and 4 show that the economic effects may be very different for $\gamma < 1$ and $\gamma > 1$. Since the case of $\gamma > 1$ is empirically more plausible, we conduct experiments with $\gamma > 1$. Figure 3 plots the welfare cost and the percentage point change in the growth rate for $\gamma = 1.5$, 2, 4, 6 and $\theta \in [0, 0.2]$. This figure reveals that the quantitative effects of money illusion is generally small for low risk aversion. These effects increase nonlinearly with risk aversion. When the degree of risk aversion is high, the economic effects of money illusion is significant. For example, when $\gamma = 6$ and $\theta = 0.05$, the welfare cost is 0.4 percent of income, which is small. But the growth rate decreases by a significant 0.18 percentage point. When $\theta = 0.1$, both numbers are large in that the welfare cost is 1.6 percent of income, and the growth rate decreases by 0.37 percentage point.

[Insert Figure 3 Here.]

5.2. Eliminating the cost of money illusion

Is there a monetary policy that minimizes the welfare cost of money illusion? In fact, for any θ, the monetary authority can choose an expected inflation rate π to make the welfare cost

$\Delta(\theta, \pi)$ equal to zero by selecting a money growth rate μ. Formally, by equation (22) we only need to set

$$\pi = \frac{\sigma_k^2}{2}(1 + (\gamma - 1)(\theta - 2)), \text{ for } \gamma \neq 1. \tag{44}$$

We can then use equation (30) to determine μ. One may intuitively argue that the monetary authority should set the expected inflation rate to zero in order to remove money illusion. This is true in our model when there is no uncertainty. That is, when $\sigma_k = 0$, (44) implies that $\pi = 0$. With uncertainty, even though the expected inflation rate is zero, the welfare cost of money illusion is positive because inflation is volatile and the variability of inflation distorts the money illusioned agent's perception of the riskiness of real wealth. To eliminate the welfare cost of money illusion, the monetary authority should set a policy such that the distortions on the growth and riskiness of real wealth offset each other, as shown in equation (44).

This equation also implies that the cost-minimizing inflation rate π increases with θ for $\gamma > 1$, and vice versa for $\gamma < 1$. The intuition follows from the utility functions in (2) and (4). For these utility functions, marginal utility of consumption decreases with θ if and only if $\gamma > 1$. Thus, when $\gamma > 1$, an agent with a higher degree θ of money illusion tends to consume less. To correct for this cost, the monetary authority should raise the expected inflation rate if π is positive (or lower the expected deflation rate if π is negative) to make the money-illusioned agent to feel wealthier and thus to consume more. Since $R = A + \pi - \sigma_k^2$, the nominal interest rate follows an identical monotonic relation with θ. Note that equation (44) implies that the cost-minimizing inflation rate may be negative depending on the values of γ and θ. Figure 4 plots the cost-minimizing inflation rate and nominal interest rate for $\theta \in [0, 1]$. This figure reveals that the cost-minimizing inflation rates are negative and small, and the cost-minimizing nominal interest rates are positive. In addition, both rates do not vary much with θ due to the small value of volatility σ_k under our benchmark calibration (see equation (44)).

[Insert Figure 4 Here.]

We now consider a sensitivity analysis for different values of the risk aversion parameter. Figure 5 plots the cost-minimizing interest rates and expected inflation rate. This figure reveals several features. First, the cost-minimizing expected inflation rates are generally small and close to zero. In addition, for most values of γ, they take negative values. For low values of γ, e.g., $\gamma = 1.5$, the cost minimizing expected inflation rates are positive. Second, both the cost-minimizing nominal interest rates and expected inflation rates are not sensitive to changes in risk aversion.

[Insert Figure 5 Here.]

We shall emphasize that the cost-minimizing nominal interest rate in our model is generally not equal to zero, as postulated by the Friedman rule. In the standard literature of optimal monetary policy, the objective of optimal monetary policy is to maximize individuals' utility. In our model, the money illusioned agent is irrational and we argue that the monetary authority should not maximize the utility function in (2) or (4). Instead, if the monetary authority maximizes $V(k_0)$ given in (40) with a standard $U^0(c, M, P)$, then this optimal monetary policy coincides with our cost-minimizing policy.

6. Conclusion

In this paper, we have presented two tractable stochastic continuous-time monetary models of endogenous growth. Our analysis has demonstrated that money illusion distorts an agent's perception of the growth and riskiness of real wealth and thus his consumption/savings choice. Its impact on long-run growth is via this channel. We have shown that the welfare cost of money illusion is second order, whereas its impact on long-run growth is first order in terms of the degree of money illusion. The monetary authority can choose a growth rate of the money supply to eliminate the cost of money illusion by correcting the distortions on consumption/savings decisions. This monetary policy implements a specific nonzero expected inflation rate or a constant nominal interest rate such that the distortions arising from the agent's misperception of the growth and riskiness of real wealth offset each other. Our models have implications for the empirical relationship between growth and inflation. Our results suggest this relationship crucially depends on the risk aversion parameter.

Our models can be varied in a number of dimensions. First, we may introduce labor-leisure choice and consider a type of the Lucas (1988) human-capital model. Second, we may separate risk aversion and intertemporal substitution in the utility specification as in Epstein and Zin (1989). We can then analyze how they interact with money illusion in an agent's consumption/savings decision problem. Finally, we may use our modeling of money illusion to study price and wage rigidities in a multi-agent or multi-firm model. We can then examine business cycle implications.

Appendix

A Proofs

Proof of Proposition 1: By definition,

$$M/P = (1-\phi-\psi)\,w. \tag{A.1}$$

Substituting this equation, the utility function (2), and the conjectured value function into the HJB equation (21), we cancel out $P^{\theta(1-\gamma)}$ and obtain:

$$\begin{aligned} 0 \;=\; & \max_{\phi,\psi}\ \frac{1}{1-\gamma}\left(\alpha c^{1-\varphi}+(1-\alpha)\left[(1-\phi-\psi)\,w\right]^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}}-\rho b\frac{(w+\beta\bar{w})^{1-\gamma}}{1-\gamma} && \text{(A.2)}\\ & +b\,(w+\beta\bar{w})^{-\gamma}\left(w\left[A\phi+R\psi+(1-\phi)\left(\sigma_k^2-\pi\right)\right]+v-c\right)\\ & +\beta b\,(w+\beta\bar{w})^{-\gamma} f(\bar{w})+b\theta\,(w+\beta\bar{w})^{1-\gamma}\pi\\ & -\frac{\gamma}{2}b\,(w+\beta\bar{w})^{-\gamma-1}\sigma_k^2 w^2\\ & -\frac{\gamma}{2}\beta^2 b\,(w+\beta\bar{w})^{-\gamma-1}\left[h(\bar{w})\right]^2+\frac{1}{2}\theta\,(\theta\,(1-\gamma)-1)\,b\,(w+\beta\bar{w})^{1-\gamma}\sigma_k^2\\ & -\gamma\beta b\,(w+\beta\bar{w})^{-\gamma-1}h(\bar{w})\,w\sigma_k\\ & -\theta\,(1-\gamma)\,b\,(w+\beta\bar{w})^{-\gamma}w\sigma_k^2\\ & -\theta\,(1-\gamma)\,b\beta\,(w+\beta\bar{w})^{-\gamma}h(\bar{w})\,\sigma_k. \end{aligned}$$

Taking the first-order conditions with respect to c, ϕ, and ψ yields:

$$\alpha c^{-\varphi}\left(\alpha c^{1-\varphi}+(1-\alpha)\left[(1-\phi-\psi)\,w\right]^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}-1}=b\,(w+\beta\bar{w})^{-\gamma}, \tag{A.3}$$

$$\begin{aligned} & (1-\alpha)\left[(1-\phi-\psi)\,w\right]^{-\varphi}\left(\alpha c^{1-\varphi}+(1-\alpha)\left[(1-\phi-\psi)\,w\right]^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}-1}\\ = \;& b\,(w+\beta\bar{w})^{-\gamma}\left(A+\pi-\sigma_k^2\right), && \text{(A.4)} \end{aligned}$$

$$\begin{aligned} & (1-\alpha)\left[(1-\phi-\psi)\,w\right]^{-\varphi}\left(\alpha c^{1-\varphi}+(1-\alpha)\left[(1-\phi-\psi)\,w\right]^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}-1}\\ = \;& b\,(w+\beta\bar{w})^{-\gamma}R. && \text{(A.5)} \end{aligned}$$

Equation (27) follows from (A.4) and (A.5). Using (A.3) and (A.5), we derive:

$$c=\left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}(1-\phi-\psi)\,w \tag{A.6}$$

We substitute (A.6) back into (A.5) to derive:

$$(1-\phi-\psi)\,w=(w+\beta\bar{w})\,b^{-\frac{1}{\gamma}}\left(\frac{R}{1-\alpha}\right)^{-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{\varphi-\gamma}{(1-\varphi)\gamma}}. \tag{A.7}$$

Substituting (A.6), (23) and (27) into (A.2) yields:

$$\begin{aligned}
0 \;=\;& \frac{1}{1-\gamma}\left[\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+1-\alpha\right]^{\frac{1-\gamma}{1-\varphi}}\left[(1-\varphi-\psi)\,w\right]^{1-\gamma}-\frac{\rho b\,(w+\bar{w})^{1-\gamma}}{1-\gamma} \qquad \text{(A.8)}\\
&+b\,(w+\beta\bar{w})^{-\gamma}\left[wA-w\,(1-\varphi-\psi)\left(R+\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}\right)+v\right]\\
&+\beta b\,(w+\beta\bar{w})^{-\gamma}\,\bar{w}\,(A-\eta)\\
&+\theta b\,(w+\beta\bar{w})^{1-\gamma}\,\pi\\
&-\frac{\gamma}{2}b\,(w+\beta\bar{w})^{-\gamma-1}\,\sigma_k^2w^2-\frac{\gamma}{2}\beta^2b\,(w+\beta\bar{w})^{-\gamma-1}\,\sigma_k^2\bar{w}^2-\gamma\beta b\,(w+\beta\bar{w})^{-\gamma-1}\,\sigma_k^2\bar{w}w\\
&+\frac{1}{2}\theta\,(\theta\,(1-\gamma)-1)\,b\,(w+\beta\bar{w})^{1-\gamma}\,\sigma_k^2\\
&-\theta\,(1-\gamma)\,b\,(w+\beta\bar{w})^{1-\gamma}\,\sigma_k^2.
\end{aligned}$$

Using (A.7), and substituting the expression for v in (19), we can simplify this equation to derive:

$$\begin{aligned}
0 \;=\;& \frac{(w+\beta\bar{w})\,b^{1-\frac{1}{\gamma}}}{1-\gamma}\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}}-\frac{\rho b\,(w+\bar{w})}{1-\gamma}\\
&+bwA-(w+\beta\bar{w})\,b^{1-\frac{1}{\gamma}}\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}}\\
&+\frac{b\bar{w}\mu\eta}{R+\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}-\mu}+\beta b\bar{w}\,(A-\eta)-\frac{\gamma}{2}b\,(w+\beta\bar{w})\,\sigma_k^2\\
&+\theta b\,(w+\beta\bar{w})\left[\pi+\frac{1}{2}\sigma_k^2\,((\theta-2)\,(1-\gamma)-1)\right]. \qquad \text{(A.9)}
\end{aligned}$$

Since (A.9) holds for all w and $\bar{w}$, we set the coefficients of w to zero to obtain:

$$\begin{aligned}
0 \;=\;& \frac{1}{1-\gamma}b^{1-\frac{1}{\gamma}}\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}}-\frac{\rho b}{1-\gamma}\\
&+b\left[A-b^{-\frac{1}{\gamma}}\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}}\right]-\frac{\gamma}{2}b\sigma_k^2\\
&+b\theta\left[\pi+\frac{1}{2}\sigma_k^2\,((\theta-2)\,(1-\gamma)-1)\right].
\end{aligned}$$

Using the definition of η in (22) and simplifying the preceding equation, we obtain equation (25). Similarly, we set the coefficients of $\bar{w}$ to zero to derive:

$$\begin{aligned}
0 \quad = \quad & \frac{b^{1-\frac{1}{\gamma}}\beta}{1-\gamma}\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}}-\frac{\rho b\beta}{1-\gamma} \\
& -b^{1-\frac{1}{\gamma}}\beta\left(\frac{R}{1-\alpha}\right)^{1-\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{(1-\gamma)\varphi}{(1-\varphi)\gamma}} \\
& +\frac{b\mu\eta}{R+\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}-\mu}+\beta b\left(A-\eta\right)-\frac{\gamma}{2}b\beta\sigma_k^2 \\
& +b\theta\beta\left[\pi+\frac{1}{2}\sigma_k^2\left(\left(\theta-2\right)\left(1-\gamma\right)-1\right)\right].
\end{aligned}$$

Using the preceding two equations, we can derive equation (26). Using equations (A.6), (A.7), and (25), we can derive the consumption rule (28). Using (A.1), (A.7), and (25), we can derive the money demand (29). Finally, we require $\eta > 0$ so that both consumption and money holdings are positive. Q.E.D.

Proof of Proposition 2: In equilibrium, bond holdings are zero so that $\psi = 0$. In addition, the representative agent's wealth is equal to aggregate wealth so that $w = \bar{w}$. Equation (A.7) then implies that ϕ takes a constant value ϕ^* in equilibrium, which satisfies:

$$1-\phi^* = (bR)^{-\frac{1}{\gamma}}\left(1+\beta\right)\left(1-\alpha\right)^{\frac{1}{\gamma}}\left(\alpha\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1-\varphi}{\varphi}}+(1-\alpha)\right)^{\frac{\varphi-\gamma}{(1-\varphi)\gamma}}.$$

Substituting the expressions for b and β in equations (25) and (26), we obtain equation (24), confirming our conjecture in Proposition 1.

We next verify the dynamics of transfers. Equation (A.1) implies that equation (17) holds in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion terms yield $\sigma_P = -\sigma_k$. In addition, the equilibrium lump-sum transfer satisfies:

$$v = \mu\bar{M}/P = \mu\bar{w}\left(1-\phi^*\right). \tag{A.10}$$

Since $w = \bar{w}$ in equilibrium, $\bar{w}$ and w must follow the same diffusion process. Substituting

(A.6), (A.10), $\phi = \phi^*$, and $\psi = 0$ into the drift of the agent's wealth dynamics (16), we obtain:

$$\begin{aligned} f(\bar{w}) &= \bar{w}\left[A\phi^* + (1-\phi^*)\left(\sigma_k^2 - \pi - \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}\right)\right] + v \\ &= \bar{w}\left[A - (1-\phi^*)\left(R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}\right)\right] + \mu\bar{w}(1-\phi^*) \\ &= \bar{w}\left[A - (1-\phi^*)\left(R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} - \mu\right)\right] \\ &= \bar{w}(A-\eta), \end{aligned}$$

where the last equality follows from the definition of ϕ^* in (24). Matching the diffusion terms of the wealth dynamics yields $h(\bar{w}) = \sigma_k \bar{w}$. Thus, we have verified (23).

We now determine the equilibrium inflation rate. Applying Ito's Lemma to the expressions on the two sides of the equation,

$$\frac{M}{P} = (1-\phi^*)\,\bar{w},$$

and matching the drift terms, we obtain:

$$\mu - \pi + \sigma_k^2 = f(\bar{w})/\bar{w} = A - \eta.$$

By the definition of η in (22), we can solve for the expected inflation rate given by (30). By (27), we obtain:

$$R - \mu = \eta. \tag{A.11}$$

In equilibrium, equation (A.6) implies that

$$\begin{aligned} \frac{c}{w} &= \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}(1-\phi^*) = \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} \frac{\eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}} - \mu} \\ &= \frac{\eta}{\eta + \left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}}\left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}, \end{aligned}$$

where we have substituted equation (24) and (A.11). Similarly, we can derive:

$$k = \phi^* w = \frac{\left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}}{\eta + \left(\frac{\alpha R}{1-\alpha}\right)^{\frac{1}{\varphi}}} w.$$

Thus, we obtain $c/k = \eta$. Using the resource constraint (9), we obtain the expected growth rate of capital in equation (32). Since output, consumption, capital and wealth are proportional to each other, they all have the same expected growth rate. Q.E.D.

Proof of Proposition 3: At optimum the CIA constraint (35) binds so that

$$c = M/P = (1-\phi-\psi)\, w, \tag{A.12}$$

where the last equality follows from (A.1). Substituting this equation and the conjectured value function into the HJB equation (21), we cancel out $P^{\theta(1-\gamma)}$ and obtain:

$$\begin{aligned} 0 \quad = \quad & \max_{\phi,\psi} \frac{[(1-\phi-\psi)\, w]^{1-\gamma}}{1-\gamma} - \rho b \frac{(w+\beta\bar{w})^{1-\gamma}}{1-\gamma} \\ & + b\,(w+\beta\bar{w})^{-\gamma} \left(w \left[A\phi + R\psi + (1-\phi)\left(\sigma_k^2 - \pi\right) - (1-\phi-\psi) \right] + v \right) \\ & + \beta b\,(w+\beta\bar{w})^{-\gamma} f(\bar{w}) + b\theta\,(w+\beta\bar{w})^{1-\gamma}\, \pi \\ & - \frac{\gamma}{2} b\,(w+\beta\bar{w})^{-\gamma-1}\, \sigma_k^2 w^2 \\ & - \frac{\gamma}{2}\beta^2 b\,(w+\beta\bar{w})^{-\gamma-1}\, [h(\bar{w})]^2 + \frac{1}{2}\theta\,(\theta\,(1-\gamma)-1)\, b\,(w+\beta\bar{w})^{1-\gamma}\, \sigma_k^2 \\ & - \gamma\beta b\,(w+\beta\bar{w})^{-\gamma-1}\, h(\bar{w})\, w\sigma_k \\ & - \theta\,(1-\gamma)\, b\,(w+\beta\bar{w})^{-\gamma}\, w\sigma_k^2 \\ & - \theta\,(1-\gamma)\, b\beta\,(w+\beta\bar{w})^{-\gamma}\, h(\bar{w})\, \sigma_k. \end{aligned} \tag{A.13}$$

Taking first-order conditions yields

$$(1-\phi-\psi)\, w \quad = \quad (w+\beta\bar{w}) \left[b\left(A+1+\pi-\sigma_k^2 \right) \right]^{-1/\gamma}, \tag{A.14}$$

$$(1-\phi-\psi)\, w \quad = \quad (w+\beta\bar{w})\, [b\,(R+1)]^{-1/\gamma}. \tag{A.15}$$

From these two equations, we obtain equation (27).

Substituting equation (A.15), (19), and (36) into (A.13), we cancel out $(w+\beta\bar{w})^{-\gamma}$ and obtain:

$$\begin{aligned} 0 \quad = \quad & \frac{(w+\beta\bar{w})\,[b\,(R+1)]^{1-1/\gamma}}{1-\gamma} - \frac{\rho b\,(w+\beta\bar{w})}{1-\gamma} \\ & + b\left[Aw - (w+\beta\bar{w})\,(R+1)\,[b\,(R+1)]^{-1/\gamma} + \frac{\mu\bar{w}}{R+1-\mu} \right] \\ & + \beta b\bar{w}\,(A-\eta) - \frac{\gamma}{2} b\,(w+\beta\bar{w})\, \sigma_k^2 \\ & + b\theta\,(w+\beta\bar{w}) \left[\pi + \frac{\sigma_k^2}{2}\left((1-\gamma)\,(\theta-2) - 1 \right) \right]. \end{aligned} \tag{A.16}$$

Matching the coefficients of w yields:

$$\begin{aligned} 0 \quad = \quad & \frac{[b\,(R+1)]^{1-1/\gamma}}{1-\gamma} - \frac{\rho b}{1-\gamma} \\ & + b\left(A - [b\,(R+1)]^{1-1/\gamma} \right) - \frac{1}{2}\gamma\sigma_k^2 b \\ & + b\theta \left[\pi + \frac{\sigma_k^2}{2}\left((1-\gamma)\,(\theta-2) - 1 \right) \right]. \end{aligned} \tag{A.17}$$

Using this equation and the definition of η in (22), we obtain (37). Matching the coefficients of $\bar{w}$ yields:

$$\begin{aligned} 0 &= \frac{[b(R+1)]^{1-1/\gamma}\beta}{1-\gamma} - \frac{\rho b\beta}{1-\gamma} \\ &\quad - (w+\beta\bar{w})\beta[b(R+1)]^{1-1/\gamma} + \frac{b\mu}{R+1-\mu} \\ &\quad - \frac{\gamma}{2}b\beta\sigma_k^2 + b\theta\beta\left[\pi + \frac{\sigma_k^2}{2}((1-\gamma)(\theta-2)-1)\right]. \end{aligned} \tag{A.18}$$

Using equations (A.17) and (A.18), we obtain equation (38). By (A.12), (A.15), and (37), we obtain (39). Finally, we require $\eta > 0$ to have positive consumption. Q.E.D.

Proof of Proposition 4: In equilibrium, $\psi = 0$ and $w = \bar{w}$. We then use equation (A.15), (37), and (38) to derive equation (36). We next verify the dynamics of transfers. Equation (A.1) implies that equation (17) holds in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion terms yield $\sigma_P = -\sigma_k$. In addition, the equilibrium lump-sum transfer satisfies:

$$v = \mu M/P = \mu\bar{w}(1-\phi^*). \tag{A.19}$$

Since $w = \bar{w}$ in equilibrium, $\bar{w}$ and w must follow the same diffusion process. Substituting (A.12), (A.10), $\phi = \phi^*$, and $\psi = 0$ into the drift of the agent's wealth dynamics (16), we obtain:

$$\begin{aligned} f(\bar{w}) &= \bar{w}\left[A\phi^* + (1-\phi^*)\left(\sigma_k^2 - \pi - 1\right)\right] + v \\ &= \bar{w}[A - (1-\phi^*)(R+1)] + \mu\bar{w}(1-\phi^*) \\ &= \bar{w}[A - (1-\phi^*)(R+1-\mu)] \\ &= \bar{w}(A-\eta), \end{aligned}$$

where the last equality follows from the definition of ϕ^* in (36). Matching the diffusion terms of the wealth dynamics yields $h(\bar{w}) = \sigma_k\bar{w}$. Thus, we have verified (23).

We now determine the equilibrium inflation rate. Applying Ito's Lemma to the expressions on the two sides of the equation,

$$\frac{M}{P} = (1-\phi^*)\bar{w},$$

and matching the drift terms, we obtain:

$$\mu - \pi + \sigma_k^2 = f(\bar{w})/\bar{w} = A - \eta.$$

By the definition of η in (22), we can solve for the expected inflation rate given by (30). By (27), we obtain (A.11).

We next solve for the real allocations. In equilibrium, equation (A.12) implies that

$$c = (1 - \phi^*) w = \frac{\eta w}{R + 1 - \mu} = \frac{\eta w}{\eta + 1},$$

where we have substituted equation (36) and (A.11). Similarly, we can derive:

$$k = \phi^* w = \frac{w}{\eta + 1}.$$

Thus, we obtain $c/k = \eta$. Using the resource constraint (9), we obtain the expected growth rate of capital in equation (32). Since output, consumption, capital and wealth are proportional to each other, they all have the same expected growth rate. Q.E.D.

Proof of Proposition 5: We start with the CIA model. In equilibrium the capital stock follows the dynamics:

$$dk_t/k_t = (A - c_t/k_t)\, dt + \sigma_k dz_t = (A - \eta(\theta, \pi))\, dt + \sigma_k dz_t.$$

Thus,

$$k_t = k_0 \exp\left\{ \left(A - \eta(\theta, \pi) - \frac{\sigma_k^2}{2} \right) t + \sigma_k z_t \right\}.$$

We can then solve:

$$\begin{aligned} V(k_0) &= E\left\{ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right\} = \frac{\eta(\theta, \pi)^{1-\gamma}}{1-\gamma} E\left\{ \int_0^\infty e^{-\rho t} k_t^{1-\gamma} dt \right\} \\ &= \frac{1}{1-\gamma} \frac{\eta(\theta, \pi)^{1-\gamma} k_0^{1-\gamma}}{\rho - (1-\gamma)\left(A - \eta - \gamma \sigma_k^2/2\right)}, \end{aligned}$$

where we have used condition (42) to ensure that the expectation is finite. We can similarly derive $V^0(k_0)$. By definition (41), we then obtain (43).

We next consider the MIU model. By Proposition 2,

$$\frac{M_t/P_t}{k_t} = \frac{1-\phi^*}{\phi^*} = \frac{c_t}{k_t} \left(\frac{\alpha R}{1-\alpha} \right)^{-\varphi} = \eta(\theta, \pi) \left(\frac{\alpha R}{1-\alpha} \right)^{-\varphi}.$$

Since

$$U^0(c, M, P) = \frac{1}{1-\gamma} \left(\alpha c^{1-\varphi} + (1-\alpha)(M/P)^{1-\varphi} \right)^{\frac{1-\gamma}{1-\varphi}},$$

we can compute

$$
\begin{aligned}
V(k_0) &= E\left\{\int_0^\infty e^{-\rho t}\frac{1}{1-\gamma}\left(\alpha c_t^{1-\varphi}+(1-\alpha)(M_t/P_t)^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}}dt\right\}\\
&= \frac{\eta(\theta,\pi)^{1-\gamma}}{1-\gamma}E\left\{\int_0^\infty e^{-\rho t}\left(\alpha k_t^{1-\varphi}+(1-\alpha)\left(\frac{\alpha R}{1-\alpha}\right)^{-\frac{(1-\varphi)}{\varphi}}k_t^{1-\varphi}\right)^{\frac{1-\gamma}{1-\varphi}}dt\right\}\\
&= \frac{\eta(\theta,\pi)^{1-\gamma}}{1-\gamma}E\left\{\int_0^\infty e^{-\rho t}k_t^{1-\gamma}dt\right\}\left(\alpha+(1-\alpha)\left(\frac{\alpha R}{1-\alpha}\right)^{-\frac{(1-\varphi)}{\varphi}}\right)^{\frac{1-\gamma}{1-\varphi}}.
\end{aligned}
$$

Thus,

$$
V(k_0)=\frac{\eta(\theta,\pi)^{1-\gamma}k_0^{1-\gamma}}{1-\gamma}\frac{\left(\alpha+(1-\alpha)\left(\frac{\alpha R}{1-\alpha}\right)^{-\frac{(1-\varphi)}{\varphi}}\right)^{\frac{1-\gamma}{1-\varphi}}}{\rho-(1-\gamma)\left(A-\eta(\theta,\pi)-\gamma\sigma_k^2/2\right)}.
$$

We finally use the definition (41) to derive (43). Q.E.D.

Proof of Proposition 6: We take ln on both sides of equation (43) to derive

$$
\ln[1+\Delta(\theta,\pi)]=\frac{1}{1-\gamma}\ln\left(\rho-(1-\gamma)\left(A-\eta(\theta,\pi)-\gamma\sigma_k^2/2\right)\right)-\ln(\eta(\theta,\pi))+I,
$$

where I is a term independent of θ. Holding π fixed and differentiating both sides with respect to θ, we obtain:

$$
\frac{1}{1+\Delta(\theta,\pi)}\Delta_1(\theta,\pi)=\frac{\eta_1(\theta,\pi)}{\left[\rho-(1-\gamma)\left(A-\eta(\theta,\pi)-\gamma\sigma_k^2/2\right)\right]}-\frac{\eta_1(\theta,\pi)}{\eta(\theta,\pi)}.
$$

Given the expression of $\eta(\theta,\pi)$ below:

$$
\eta(\theta,\pi)=\frac{\rho-(1-\gamma)A}{\gamma}+\frac{1}{2}(1-\gamma)\sigma_k^2+\frac{\theta(\gamma-1)}{\gamma}\left[\pi+\frac{\sigma_k^2}{2}((\gamma-1)(2-\theta)-1)\right],
$$

it is straightforward to verify that

$$
\rho-(1-\gamma)\left(A-\eta(0,\pi)-\gamma\sigma_k^2/2\right)=\eta(0,\pi).
$$

Thus,

$$
\begin{aligned}
\frac{1}{1+\Delta(\theta,\pi)}\Delta_1(0,\pi) &= \eta_1(0,\pi)\left[\frac{1}{\left[\rho-(1-\gamma)\left(A-\eta(0,\pi)-\gamma\sigma_k^2/2\right)\right]}-\frac{1}{\eta(0,\pi)}\right]\\
&= 0.
\end{aligned}
$$

This implies that $\Delta_1(0,\pi)=0$. In addition, it is straightforward to see that $\Delta(0,\pi)=0$. Thus, $\Delta(\theta,\pi)$ is second order of θ. Q.E.D.

References

Abel, Andrew B., 1981, "Taxes, Inflation and the Durability of Capital," *Journal of Political Economy* 89, 548-560.

Akerlof, George A., 2002, "Behavioral Macroeconomics and Macroeconomic Behavior," *American Economic Review* 92, 411-433.

Akerlof, George A., William T. Dickens, and George L. Perry, 1996, "The Macroeconomics of Low Inflation," *Brookings Papers on Economic Activity* 1, 1-59.

Akerlof, George A., William T. Dickens, and George L. Perry, 2000, "Near-Rational Wage and Price Setting and the Long-Run Phillips Curve," *Brookings Papers on Economic Activity* 1, 1-44.

Akerlof, George A. and Janet Yellen, 1985a, "A Near-Rational Model of the Business Cycle, with Wage and Price Inertia," *Quarterly Journal of Economics*, Supp., 100, 823-38.

Akerlof, George A. and Janet Yellen, 1985b, "Can Small Deviations from Rationality Make Significant Differences to Economic Equilibria," *American Economic Review* 75, 708-720.

Auerbach, Alan J., 1979, "Inflation and the Choice of Asset Life," *Journal of Political Economy* 87, 621-638.

Barro, Robert J., 1996, "Inflation and Growth," The *Federal Reserve Bank of St. Louis Review* 78, 153-169.

Basak, Suleyman and Hongjun Yan, 2007, "Equilibrium Asset Prices and Investor Behavior in the Presence of Money Illusion," *Working Paper*, London Business School.

Blinder, Alan S., 2000, "Comment on Near-Rational Wage and Price Setting and the Long-Run Phillips Curve?" *Brookings Papers on Economic Activity* 1, 2000, 50-55.

Brunnermeier, Markus K. and Christian Julliard, 2007, "Money Illusion and Housing Frenzies," *Working Paper*, Princeton University.

Bruno, Michael and William Easterly, 1998, "Inflation Crises and Long-Run Growth," *Journal of Monetary Economics* 41, 3-26.

Campbell, John Y. and Tuomo Vuolteenaho, 2004, "Inflation Illusion and Stock Prices," *American Economic Review Papers and Proceedings* 94, 19-23.

Chang, Wen-Ya and Ching-Chong Lai, 2000, "Anticipated Inflation in a Monetary Economy with Endogenous Growth," *Economica* 67, 399–417.

Chari, V.V., Larry E. Jones, and Rodolfo E. Manuelli, 1995, "The Growth Effects of Monetary Policy," *Fedreal Reserve Bank of Minneapolis Quarterly Review* 19, 18-31.

Cohen, Randolph B., Christopher Polk, and Tuomo Vuolteenaho, 2005, "Money Illusion in the Stock Market: The Modigliani-Cohn Hypothesis," *Quarterly Journal of Economics*, 120(2), 639-668.

Dotsey, Michael and Pierre Daniel Sarte, 2000, "Inflation Uncertainty and Growth in a Cash-in-Advance Economy," *Journal of Monetary Economics* 45, 631-655.

Dusansky, Richard and Peter J. Kalman, 1974, "The Foundations of Money Illusion in a Neoclassical Micro-Monetary Model," *American Economic Review* 64, 115-122.

Eaton, Jonathan, 1981, "Fiscal policy, Inflation and the Accumulation of Risky Capital," *Review of Economic Studies* 48, 435-445.

Epstein Larry G. and Stanley E. Zin, 1989, "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, pp. 937-969.

Fama, Eugene F., 1981, "Stock Returns, Real Activity, Inflation and Money," *American Economic Review*, 71(4), 545-565.

Fama, Eugene F. and G. William Schwert, 1977, "Asset Returns and Inflation," *Journal of Financial Economics*, 5(2), 115-146.

Fehr, Ernst and Jean Robert Tyran, 2001, "Does Money Illusion Matter?" *American Economic Review* 91, 1239-1262.

Fehr, Ernst and Jean Robert Tyran, 2007, "Money Illusion and Coordination Failure," forthcoming in *Games and Economic Behavior.*

Feldstein, Martin, 1983, *Inflation, Tax Rules, and Capital Formation*, Chicago: University of Chicago Press.

Fischer, Stanley, Ratna Sahay, and Carlos A. Végh, 2002, "Modern Hyper- and High Inflations," *Journal of Economic Literature* 40, 837-880.

Fisher, Irving, 1928, *The Money Illusion*, New York: Adelphi.

Genesove, David and Christopher Mayer, 2001, "Loss Aversion and Seller Behavior: Evidence from the Housing Market," *Quarterly Journal of Economics*, 116(4), 1233-1260.

Haberler, Gottfried, 1941, *Prosperity and Depression*, 3rd Edition Geneva, League of Nations.

Howitt, Peter, 1987, "Money Illusion," in *The New Palgrave: a Dictionary of Economics*, J. Eatwell, M. Milgate, and P. Newman, Eds, New York: W.W. Norton.

Howitt, Peter and Don Patinkin, 1980, "Utility Function Transformations and Money Illusion: Comment," *American Economic Review* 70, 819-822.

Jones, Larry E. and Rodolfo E. Manuelli, 1990, "A Convex Model of Equilibrium Grwoth: Theory and Policy Implications," *Journal of Political Economy* 98, 1008-1038.

Jones, Larry E. and Rodolfo E. Manuelli, 1995, "Growth and the Effects of Inflation," *Journal of Economic Dynamics and Control* 19, 1405-1428.

Khan, Mohsin S. and Abdelhak S. Senhadji, 2001, "Threshold Effects in the Relationship Between Inflation and Growth," *IMF Staff Papers.*

Kahneman, Daniel and Amos Tversky, 1979, "Prospect Theory: An Analysis of Decision under Risk," *Econometrica* 47(2): 263-92.

Keynes, John Maynard, 1936, *The General Theory of Employment, Interest, and Money*, Harcourt, Brace.

Leontief, Wassily, 1936, "The Fundamental Assumptions of Mr. Keynes' Monetary Theory of Unemployment," *Quarterly Journal of Economics*, 5(4), 192-97.

Lintner, John, 1975, "Inflation and Security Returns," *Journal of Finance*, 30(2), 259-280.

Loffler, Andreas, 2001, "A $\mu - \sigma -$Risk Aversion Paradox and Wealth Dependent Utility," *Journal of Risk and Uncertainty* 23, 57-73.

Lucas, Robert E. Jr., 1988, "On the Mechanics of Economic Development," *Journal of Monetary Economics* 22, 3–42.

Lucas, Robert E. Jr., 2000, "Inflation and Welfare," *Econometrica* 68, 247-274.

Mankiw, N. Gregory, 1985, "Small Menu Costs and Large Business Cycles: A Macroeconomic Model of Monopoly," *Quarterly Journal of Economics*, 100, 529–537.

Marquis, Milton H. and Kevin L. Reffett, 1991, "Real Interest Rates and Endogenous Growth in a Monetary Economy" *Economics Letters*, 37, 105-109.

Merton, Robert C., 1969, "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics* 51, 247-257.

Modigliani, Franco, and Richard Cohn, "Inflation, Rational Valuation and the Market," *Financial Analysts Journal*, XXXV 1979, 24-44.

Patinkin, Don, 1965, *Money, Interest, and Prices*, second edition, New York: Harper and Row.

Piazzesi, Monika and Martin Schneider, 2007, "Inflation Illusion, Credit and Asset Prices," In *Asset Prices and Monetary Policy*, ed. by J. Campbell. NBER, forthcoming.

Pollak, Robert A., 1977, "Price Dependent Preferences," *American Economic Review* 67, 64-75.

Ramey Garey and Valerie A. Ramey, 1995, "Cross-Country Evidence on the Link Between Volatility and Growth," *American Economic Review*, 85, pp. 1138-1151.

Rebelo, Sergio, 1991, "Long-Run Policy Analysis and Long-Run Growth," *Journal of Political Economy* 99, 500-521.

Rebelo, Sergio and Danyang Xie, 1999, "On the Optimality of Interest Rate Smoothing," *Journal of Monetary Economics* 43, 263-282.

Ritter, Jay R., and Richard S. Warr, 2002, "The Decline of Inflation and the Bull Market of 1982-1999," *Journal of Financial and Quantitative Analysis*, 37(1), 29-61.

Shafir Eldar, Peter Diamond, and Amos Tversky, 1997, “Money Illusion” *Quarterly Journal of Economics*, Vol. 112, No. 2, In Memory of Amos Tversky (1937-1996). pp. 341-374.

Shiller, Robert J., 1997, “Why Do People Dislike Inflation?” In *Reducing Inflation: Motivation and Strategy*, edited by Christina D. Romer and David H. Romer. University of Chicago Press.

Sidrauski, Miguel, 1967a, “Rational Choice and Patterns of Growth in a Monetary Economy,” *American Economic Reviw* 57, 534-544.

Sidrauski, Miguel, 1967b, “Inflation and Economic Growth,” *Journal of Political Economy* 75, 796-810.

Thaler, Richard H., 1980, “Toward a Positive Theory of Consumer Choice,” *Journal of Economic Behavior and Organization* 1, 39-60.

Tversky Amos and Daniel Kahneman , 1974, “Judgment under Uncertainty: Heuristics and Biases,” *Science*, 185, 1124–1131.

Tversky, Amos and Daniel Kahneman, 1981, “The Framing of Decisions and the Psychology of Choice,” *Science* 211, 453-458.

Veblen, Thorstein, 1899, *The Theory of the Leisure Class*, New York.

Wang, Ping, and Chong K. Yip, 1992, “Examining the Long-Run Effect of Money on Economic Growth,” *Journal of Macroeconomics* 14, 359–69.

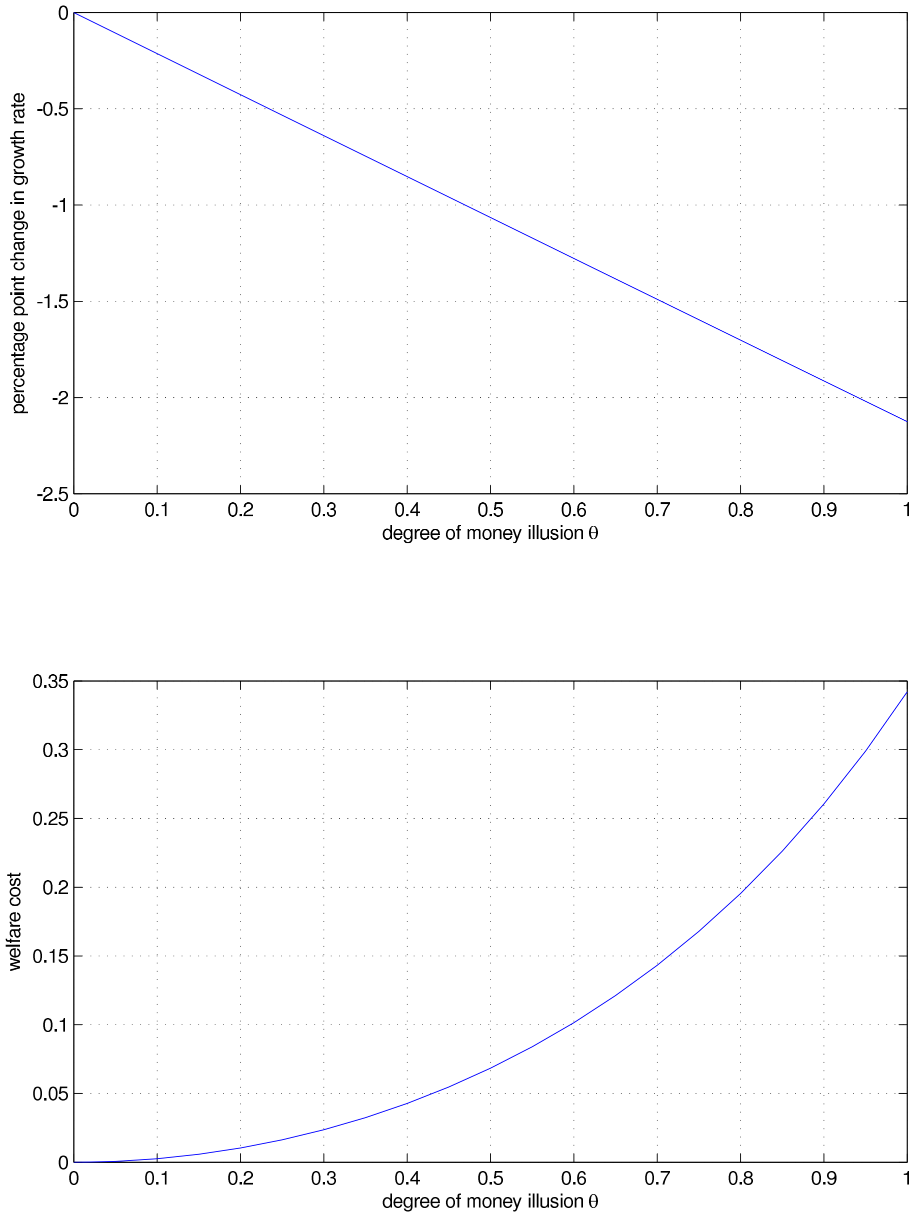

Figure 1: **The welfare cost and growth effects of money illusion.** We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, $\sigma_k = 0.0204$, and $\pi = 0.0425$.

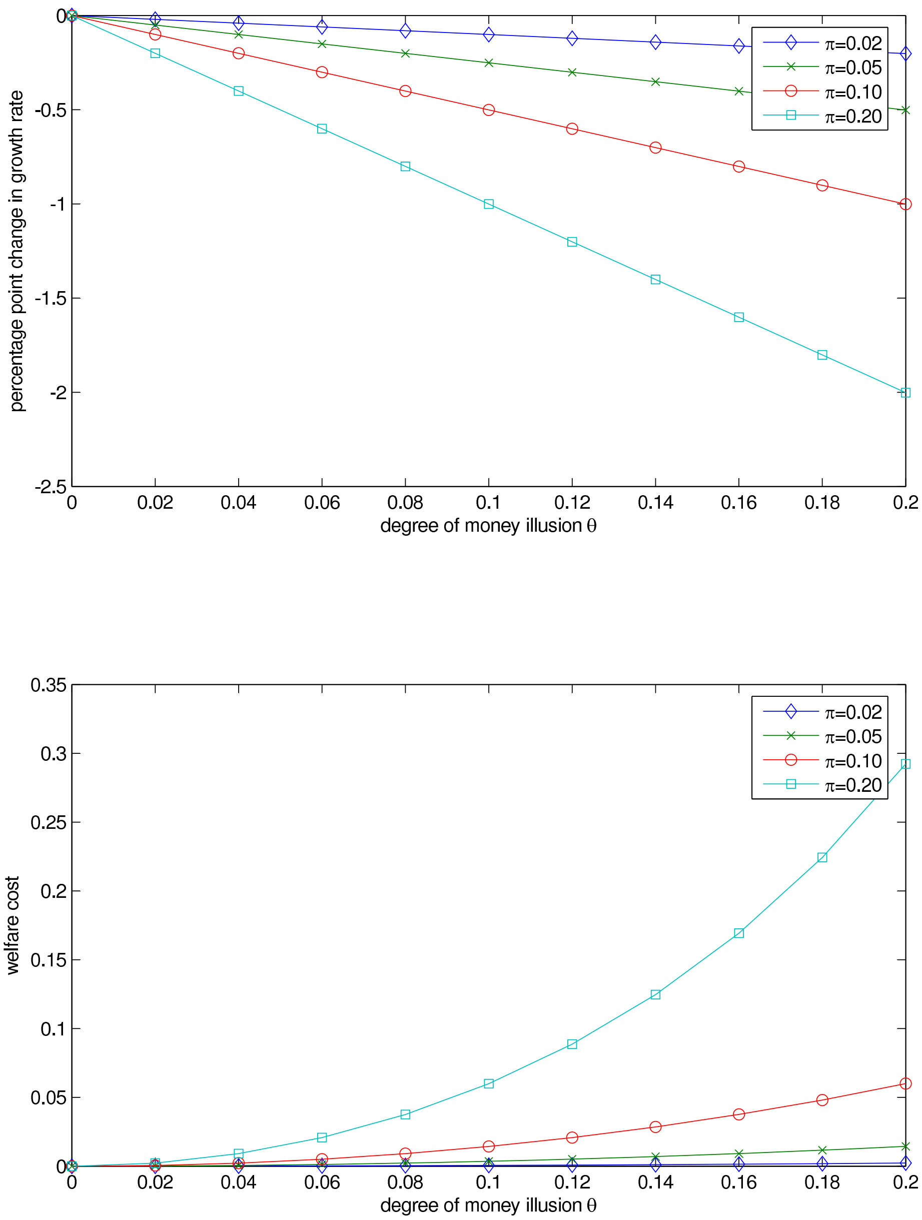

Figure 2: **Effects of money illusion for different expected inflation rates.** We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.

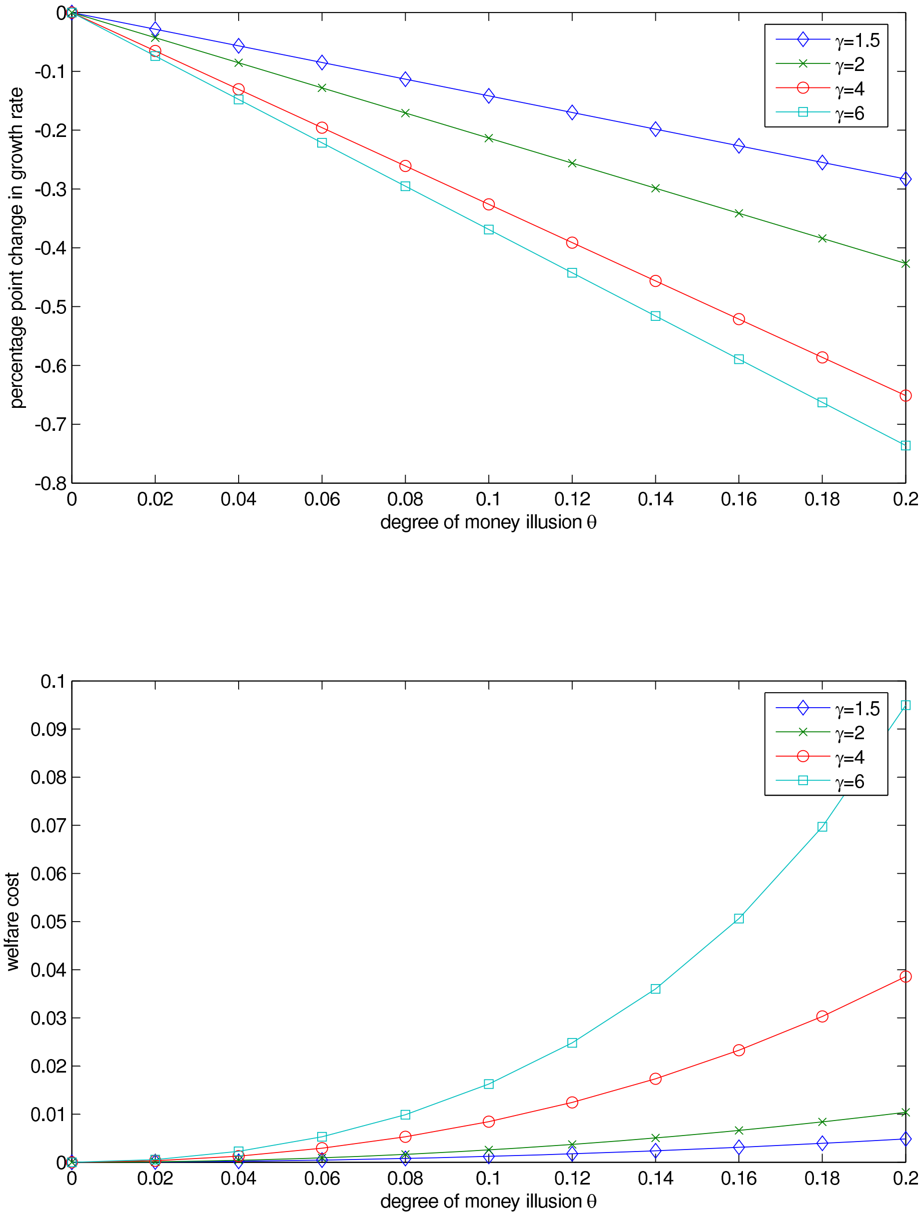

Figure 3: **Effects of money illusion for different degrees of risk aversion.** We set parameter values: $A = 0.0644$, $\rho = -\ln(0.98)$, $\sigma_k = 0.0204$, and $\pi = 0.0425$.

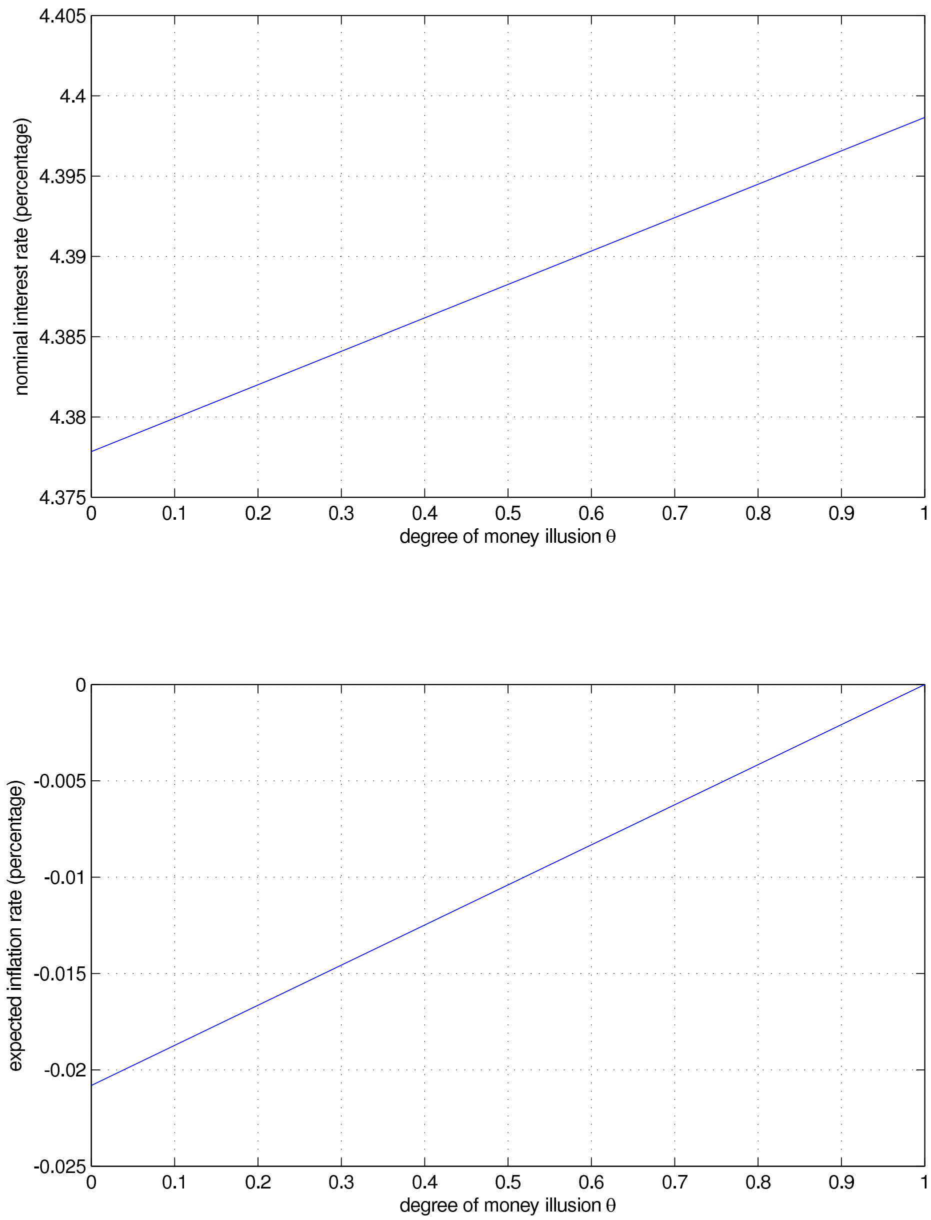

Figure 4: **Cost-minimizing nominal interest rates and expected inflation rates.** We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.

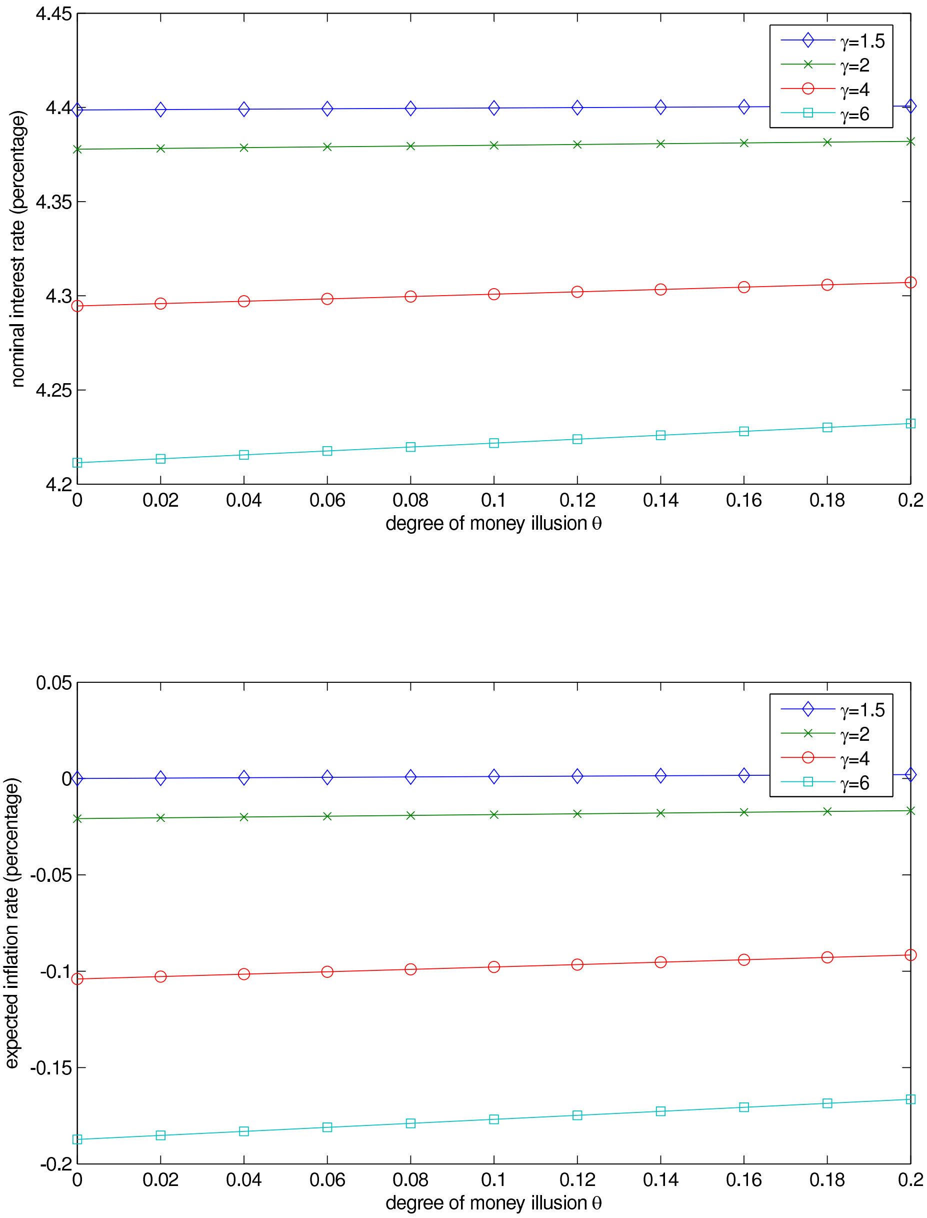

Figure 5: **Effects of risk aversion on the cost-minimizing nominal interest rates and expected inflation rates.** We set parameter values: $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.