Dynamic Contracts with Learning under Ambiguity

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Abstract

We study a continuous-time principal-agent problem with learning under ambiguity. The agent takes hidden actions to affect project output. The project quality is unknown to both the principal and the agent. The agent faces ambiguity about mean output, but the principal does not. We show that incentives are delayed due to ambiguity. While belief manipulation due to learning about unknown quality causes wages and pay-performance sensitivity to be front-loaded, ambiguity smoothes wages and causes the drift and volatility of wages to decline more slowly over time. When the level of ambiguity is sufficiently large, the principal fully insures the agent by allowing the agent to shirk forever.

Keywords: Dynamic Contract, Ambiguity, Learning, Incentive Compatibility, Recursive Multiple-Priors

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1 Introduction

The principal-agent model aims to study how to design a contract between a principal and an agent when there is information asymmetry. The agent may take hidden actions unknown to the principal. There may also exist a model parameter that is unknown to both parties. This parameter could represent the agent's ability or project quality. Both the principal and the agent have to learn about the unknown parameter. The standard approach adopts Bayesian learning. This approach rules out the role of confidence in probability assessments. A Bayesian decision marker does not distinguish between risky situations, where the odds are objectively known, and ambiguous situations, where they may have little information and hence also little confidence regarding the true odds. The Ellsberg (1961) paradox demonstrates that this distinction is behaviorally meaningful because people treat ambiguous bets differently from risky ones.

The goal of this paper is to study a principal-agent problem with learning under ambiguity in continuous time. Since there are two individuals in this problem, we have to consider who faces ambiguity and what he is ambiguous about. In this paper we suppose that the agent is ambiguity averse, but the principal is Bayesian. To capture ambiguity aversion, we adopt the recursive multiple-priors utility model of Chen and Epstein (2002), adapted to incorporate learning as in Miao (2009) and Leippold, Trojani, and Vanini (2008).[1] The contracting problem is based on Prat and Jovanovic (2014) in which the agent takes hidden actions to affect project output, which is governed by a diffusion process. Neither the principal nor the agent observes the project quality or productivity. Unlike Prat and Jovanovic (2014), we assume that the agent has ambiguous beliefs about the drift of the project output.

The key insight of Prat and Jovanovic (2014) is that the agent has an incentive to manipulate the principal's beliefs about the project quality. Observing the project output only, the principal may mistakenly attribute low output to low productivity instead of the agent's low effort. By inducing the principal to underestimate his productivity, the agent anticipates that he will benefit from overestimated inferences about his effort in the future and hence higher utility. Due to this belief manipulation effect, the optimal contract delays incentives. Specifically, the principal fully insures the agent by offering a constant wage and allowing the agent to shirk until a certain time. Starting from that time on, the principal recommends the agent to exert full effort and immediately raises his wage. The wage is stochastic and declines over time on average. To discourage the agent from manipulating his belief, the principal raises the agent's exposure to uncertainty by raising the volatility of the wage growth and making this volatility front-loaded.

[1] These models are based on the static model of Gilboa and Schmeidler (1989) and the dynamic extension of Epstein and Schneider (2007) in discrete time.

Our key insight is that there is an incentive and ambiguity sharing tradeoff when the agent is ambiguity averse. Ambiguity further delays incentive provision. The principal allows the agent to shirk for a longer period of time when the agent is more ambiguity averse. This is because ambiguity is costly to both the agent and the principal. To motivate the ambiguity-averse agent to work, the principal has to offer a higher wage, compared to the case without ambiguity. The principal would rather accept low output by avoiding paying a high wage to induce incentives. After this shirking period, the principal immediately raises the wage and the average wage gradually declines over time. Compared to the case without ambiguity, the wage jump is larger, the decline of the expected wage is slower, and the volatility of wage growth is lower. This is because the risk-neutral Bayesian principal wants to share ambiguity with the ambiguity-averse agent by allowing the agent to have less exposure to uncertainty. When the level of ambiguity is sufficiently large, the principal fully insures the agent by allowing the agent to shirk forever.

We show that risk aversion and ambiguity aversion have different impact on the optimal contract. In particular, given a certain level of ambiguity, the duration of shirking and the jump of expected wages may not monotonically increase with risk aversion. By contrast, they always increase with ambiguity aversion for any degree of risk aversion. This reflects the fact that risk aversion affects the curvature of the utility function, but ambiguity aversion affects only the probability assessments in the multiple-priors utility model.

Ambiguity aversion manifest itself as pessimistic behavior by distorting beliefs endogenously. The optimal contract under ambiguity is observationally equivalent to that when the agent has expected utility with some distorted belief. We show that ignoring the endogeneity of the worst-case beliefs by specifying a pessimistic belief exogenously can generate a different optimal contract.

While our model is too stylized to be confronted with data, it may help us understand the low pay-performance sensitivity and slow wage adjustment documented in some empirical studies (e.g., Jensen and Murphy (1990)). Jensen and Murphy (1990) find that the pay-performance sensitivity, defined as the dollar change in the CEO's compensation associated with a dollar change in shareholder wealth, is a very small positive number. Our model suggests that this may be consistent with an optimal contract when the CEO is averse to ambiguity. CEOs often face substantial amount of uncertainty not attributed to just risk in making daily business decisions and they are averse to such uncertainty. In this case making pay less sensitive to performance allows them to have less exposure to ambiguity.

Our paper is related to the recent literature on contracts in continuous time surveyed by Biais et al (2013), Sannikov (2013), and Cvitanic and Zhang (2013). DeMarzo and Sannikov (2006), Biais et al (2007), Sanikov (2008), Cvitanic, Wan, and Zhang (2009), and Williams (2009, 2011) have made important methodological contributions to solving continuous-time contracting problems. Our paper is more closely related to Prat and Jovanovic (2014), DeMarzo and Sannikov (2014), and He et al (2014) who introduce learning into the contracting problems. All of them emphasize the importance of the belief manipulation effect in different model setups. DeMarzo and Sannikov (2014) impose limited liability constraint on the risk-neutral agent and study the optimal payout and termination policies. He et al (2014) assume that the cost of effort is convex and show that the optimal effort is front-loaded and decreases stochastically over time. Like Prat and Jovanovic (2014), we obtain a different result because we assume that disutility is linear in effort.

Our main contribution is to introduce ambiguity into this literature. Our modeling of ambiguity follows Chen and Epstein (2002). Miao and Rivera (2015) adopt a different approach of Anderson et al (2003), Hansen et al (2006), and Hansen and Sargent (2008). This approach gives a smooth utility function, unlike the Chen-Epstein approach. Miao and Rivera (2015) assume that the principal faces ambiguity but the agent does not. They focus on capital structure implementation and asset pricing implications. They also emphasize the incentive and ambiguity sharing tradeoff, but with different implications. In their model the ambiguity-averse principal wants to transfer uncertainty to the ambiguity-neutral agent by raising the pay-performance sensitivity. By contrast, in this paper the risk-neutral Bayesian principal wants to transfer uncertainty from the ambiguity-averse agent by lowering the pay-performance sensitivity. Moreover, Miao and Rivera (2015) do not consider learning under unknown quality, which is the main focus of this paper.

In addition to the contributions above, we make a technical contribution by solving the contracting problem with recursive multiple-priors utility. We show that deriving the agent's incentive compatibility condition is equivalent to solving a control problem for forward-backward stochastic differential equations (FBSDEs). We adopt the calculus of variation approach presented in Cvitanic and Zhang (2013) to solve this problem. This approach typically requires that the drift and volatility terms in some stochastic differential equations be smooth so that differentiation can be used to perform small perturbations. But this condition is violated for the recursive multiple-priors utility model of Chen and Epstein (2002). We tackle this difficulty by suitably define derivatives at kink points as in the non-smooth analysis. We then derive the necessary and sufficient conditions for incentive compatibility. Our method will be useful for solving other contracting problems with recursive multiple-priors utility.

The remainder of the paper proceeds as follows. Section 2 presents the model. Section 3 presents the necessary and sufficient conditions for incentive compatibility. Section 4 provides explicit solutions under exponential utility. Section 5 analyzes properties of the optimal contracts. Section 6 concludes. All proofs are collected in the appendix.

2 The Model

Our model is based on Prat and Jovanovic (2014). Our novel assumption is that the agent has ambiguous beliefs about the output distribution and is averse to this ambiguity, but the principal does not face ambiguity. In this section we first introduce the information structure and then describe preferences. After this description we formulate the contracting problem.

2.1 Information Structure and Filtering

Time is continuous over the finite interval $[0, T]$. The principal and the agent commit to a long-term contract until time T. In the contract the agent exerts effort to run a project. The project generates cumulative output,

$$Y_t = \int_0^t (a_s + \eta) ds + \int_0^t \sigma dB_s^a, \tag{1}$$

where $a_t \in [0, 1]$ is the agent's effort level, η represents the project's time-invariant random profitability or productivity, $\sigma > 0$ is the constant output volatility, and $\{B_t^a\}$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P^a)$. Assume that $a_t = a\left(t, Y_0^t\right)$ for some functional a and is adapted to the filtration generated by Y, $\mathcal{F}_t^Y \triangleq \sigma(Y_s; 0 \leq s \leq t)$. By convention Y_0^t denotes the history $\{Y_s : 0 \leq s \leq t\}$. Note that (Y, B^a, P^a) is a weak solution to the stochastic differential equation in (1).

Neither the principal nor the agent observes η. But they share the common prior at time zero that η is normally distributed with mean m_0 and variance h_0^{-1}. We call h_0 the prior precision of η. The posterior belief about η depends on Y_t and on cumulative effort $A_t \triangleq \int_0^t a_s ds$. By standard filtering theory, conditional on (Y_t, A_t, t), η is normally distributed with mean

$$\widehat{\eta}(Y_t - A_t, t) \triangleq E^{P^a}[\eta \mid Y_t,\ A_t] = \frac{h_0 m_0 + \sigma^{-2}\left(Y_t - A_t\right)}{h_t} \tag{2}$$

and with precision

$$h_t \triangleq h_0 + \sigma^{-2} t. \tag{3}$$

Here E^{P^a} denotes the expectation operator with respect to P^a. In the long run as $t \to \infty$, h_t increases to infinity and hence $\widehat{\eta}(Y_t - A_t, t)$ converges to η almost surely.

The principal does not observe the agent's all possible effort choices, but only observes the equilibrium effort policy $a_t^* = a^*\left(t, Y_0^t\right)$ under the optimal contract. By contrast, effort is the agent private information and he may deviate from the equilibrium effort policy by choosing another effort policy $a_t = a\left(t, Y_0^t\right)$. Following Cvitanic, Wan, and Zhang (2009) and Cvitanic and Zhang (2013), we identify any effort functional $a(\cdot)$ as an $\left\{\mathcal{F}_t^Y\right\}$-predictable process. Then the effort policy under the agent's control is in the set $\mathcal{A} = \{a : [0,T] \times \Omega \to [0,1]\}$ of stochastic processes that are $\{\mathcal{F}_t^Y\}$-predictable.

Define the innovation process as

$$d\hat{B}_t^a \triangleq \frac{1}{\sigma}[dY_t - (\widehat{\eta}(Y_t - A_t, t) + a_t)dt] = \frac{1}{\sigma}[\eta - \widehat{\eta}(Y_t - A_t, t)]\, dt + dB_t^a. \tag{4}$$

Then $\{\hat{B}_t^a\}$ is a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^Y\}, P^a)$ and $\{\widehat{\eta}_t\}$ is a $(P^a, \mathcal{F}_t^Y)$-martingale with decreasing variance:

$$d\widehat{\eta}(Y_t - A_t, t) = \frac{\sigma^{-1}}{h_t} d\hat{B}_t^a. \tag{5}$$

A *contract* is a mapping $(a, w) : [0,T] \times \Omega \to [0,1] \times \mathbb{R}$ that is $\{\mathcal{F}_t^Y\}$-predictable as well as a terminal payment $W_T : \Omega \to \mathbb{R}$ that is $\mathcal{F}_T^Y$-measurable. The set of all contracts (a, w, W_T) is denoted by $\mathcal{C}$.

2.2 Preferences

Suppose that the principal is risk neutral and does not face ambiguity. He has discounted expected utility over profits derived from a contract $c = (a, w, W_T)$. His utility function is given by

$$U^p(a, w, W_T) = E^{P^a}\left[\int_0^T e^{-\rho t} dY_t - \int_0^T e^{-\rho t} w_t dt - e^{-\rho T} W_T\right],$$

where $\rho > 0$ is the subjective discount rate. Since the principal's information set is given by $\{\mathcal{F}_t^Y\}$ and his prior belief is P^a. His posterior beliefs about η are given by the normal distribution with mean in (2) and variance in (3). The principal's continuation value at date t is given by

$$U_t^p(a, w, W_T) = E_t^{P^a}\left[\int_t^T e^{-\rho(s-t)} dY_s - \int_t^T e^{-\rho(s-t)} w_s ds - e^{-\rho(T-t)} W_T\right],$$

where use $E_t^{P^a}$ to denote the conditional expectation operator for measure P^a given $\mathcal{F}_t^Y$.

Next consider the agent's preferences. To model the agent's learning under ambiguity, we adopt the model of Miao (2009), which extends the recursive multiple-priors utility model of Chen and Epstein (2002) to incorporate partial information. We suppose that the agent is not sure about the distribution of the innovation Brownian motion $\{\hat{B}_t^a\}$ for any effort process $a \in \mathcal{A}$. He has a set of priors $\mathcal{P}^{\Theta^a}$ induced by the set of density generators Θ^a. Each prior in the set is mutually absolutely continuous with respect P^a.[2] A *density generator* associated with an effort process $a \in \mathcal{A}$

[2]Here ambiguity is about the drift of the diffusion process. Epstein and Ji (2013) propose models of ambiguity about the volatility.

is an $\{\mathcal{F}_t^Y\}$-predictable process $\{b_t\}$. We will focus on the κ-ignorance specification of the set of density generators which satisfy $|b_t| \leq \kappa$ for each $b \in \Theta^a$. By the Girsanov theorem, the process $\{z_t^b\}$ defined by

$$z_t^b = \exp\left(\int_0^t b_s d\hat{B}_s^a - \frac{1}{2}\int_0^t b_s^2 ds\right)$$

is a $(P^a, \mathcal{F}_t^Y)-$martingale and the process $\{\hat{B}_t^{a,b}\}$ defined by

$$\hat{B}_t^{a,b} = \hat{B}_t^a - \int_0^t b_s ds$$

is a standard Brownian motion under the measure $Q^{a,b}$, where $dQ^{a,b}/dP^a|_{\mathcal{F}_t^Y} = z_t^b$. Under measure $Q^{a,b}$, output follows the process

$$dY_t = (a_t + \widehat{\eta}(Y_t - A_t, t))dt + \sigma(d\hat{B}_t^{a,b} + b_t dt),$$

and the posterior belief about η is

$$d\widehat{\eta}(Y_t - A_t, t) = \frac{\sigma^{-1}}{h_t} d\hat{B}_t^a = \frac{\sigma^{-1}}{h_t}(d\hat{B}_t^{a,b} + b_t dt).$$

For a contract $c = (a, w, W_T)$, we define the agent's continuation value as

$$v_t^c = \underset{Q^{a,b} \in \mathcal{P}^{\Theta^a}}{ess\inf} E_t^{Q^{a,b}}\left[\int_t^T e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} U(W_T)\right],$$

where u and U satisfy $u_w > 0$, $u_{ww} < 0$, $u_a < 0$, $u_{aa} < 0$, $U' > 0$ and $U'' < 0$.

By Chen and Epstein (2002), the pair (v^c, γ^c) is the unique $\{\mathcal{F}_t^Y\}$-adapted solution to the following backward stochastic differential equation (BSDE):

$$dv_t = \left[\rho v_t - u(w_t, a_t) - \sigma \min_{|b_t| \leq \kappa} b_t \gamma_t\right] dt + \sigma \gamma_t d\hat{B}_t^a, \tag{6}$$

with $v_T = U(W_T)$. Let (b_t^{c*}) denote a worse-case density generator that attains the minimum in (6). Then

$$b_t^{c*} = \begin{cases} -\kappa, & \text{if } \gamma_t^c > 0 \\ \kappa, & \text{if } \gamma_t^c < 0 \\ \text{any value in } [-\kappa, \kappa] & \text{if } \gamma_t^c = 0 \end{cases}.$$

We denote by $\mathcal{B}^c$ the set of all worst-case density generators for the contract c. We can then rewrite (6) as

$$dv_t = [\rho v_t - u(w_t, a_t) + \sigma\kappa |\gamma_t|] dt + \sigma \gamma_t d\hat{B}_t^a, \quad v_T = U(W_T), \tag{7}$$

or

$$dv_t = [\rho v_t - u(w_t, a_t)]\, dt + \sigma \gamma_t d\hat{B}_t^{a,b^{c*}}, \quad v_T = U(W_T), \tag{8}$$

where $\{\hat{B}_t^{a,b^{c*}}\}$ is the standard Brownian motion under the agent's worst-case belief $Q^{a,b^{c*}}$ induced by some density generator $b^{c*} \in \mathcal{B}^c$. One convenient choice is to set $b_t^{c*} = 0$ if $\gamma_t^c = 0$. We can then write $b_t^{c*} = -\kappa sgn(\gamma_t^c)$. Note that the drift term in (7) is not differentiable with respect to γ_t. This causes difficulty when applying calculus of variations.

The parameter κ can be interpreted as the degree of ambiguity aversion or the level of ambiguity. When $\kappa = 0$, (7) reduces to the standard expected utility model with the belief P^a. Since the agent has belief $Q^{a,b^{c*}}$ under ambiguity, which is different from the principal's belief P^a, ambiguity induces endogenous belief heterogeneity.

In the infinite-horizon limit as $T \to \infty$, we impose the transversality condition $\lim_{T\to\infty} e^{-\rho T} E[v_T] = 0$ to obtain the agent's utility process $\{v_t : t \geq 0\}$.

2.3 Contracting Problem

Suppose that the principal and the agent fully commit to the long-term contract. An optimal contract must be incentive compatible. A contract $(a, w, W_T) \in \mathcal{C}$ is incentive compatible if it solves the agent's problem:

$$v_0^{(a,w,W_T)} \geq v_0^{(\tilde{a},w,W_T)} \tag{9}$$

for any $\tilde{a} \in \mathcal{A}$ satisfying

$$dY_t = (\tilde{a}_t + \widehat{\eta}(Y_t - \tilde{A}_t, t))dt + \sigma d\hat{B}_t^{\tilde{a}}, \tag{10}$$

where $\tilde{A}_t = \int_0^t \tilde{a}_s ds$ and (10) is the filtered equation for Y associated with the agent's effort $\tilde{a}$. Denote the set of all incentive compatible contracts by $\mathcal{C}^{IC}$.

We now formulate the contracting problem as

$$\sup_{\{a,w,W_T\}\in\mathcal{C}^{IC}} E^{P^a}\left[\int_0^T e^{-\rho t}(dY_t - w_t dt) - e^{-\rho T} W_T\right] \tag{11}$$

subject to

$$v_0^{(a,w,W_T)} = v, \tag{12}$$

$$dY_t = (a_t + \widehat{\eta}(Y_t - A_t, t))dt + \sigma d\hat{B}_t^a, \tag{13}$$

where v is the principal's promised value to the agent, (12) is the initial promise-keeping constraint or the individual rationality constraint, and (13) is the filtered equation for Y associated with the recommended effort a.

We can simplify the principal's objective function. Since his continuation value $U_t^p(a, w, W_T)$ is equal to

$$
\begin{aligned}
& E_t^{P^a}\left[\int_t^T e^{-\rho(s-t)}[\widehat{\eta}(Y_s - A_s, s) + a_s - w_s]ds - e^{-\rho(T-t)}W_T\right] && (14)\\
= \;& \frac{1-e^{-\rho(T-t)}}{\rho}\widehat{\eta}(Y_t - A_t, t) + E_t^{P^a}\left[\int_t^T e^{-\rho(s-t)}(a_s - w_s)ds - e^{-\rho(T-t)}W_T\right],
\end{aligned}
$$

maximizing $U_t^p(a, w, W_T)$ is equivalent to maximizing the last expectation term. Thus we will use the last term as the principal's objective function.

3 Incentive Compatibility Conditions

In this section we present the necessary and sufficient conditions for incentive compatibility. In particular, we focus on the agent's problem in (9). We first present a result that an incentive compatible contract starting at time zero is also incentive compatible starting at any time $t > 0$. This result is analogous to that in discrete time established by Green (1987). In the appendix we use the theory of FBSDEs to prove it.

Lemma 1 *If the contract (a, w, W_T) is incentive-compatible, then given any history of the effort $\{a_s : s \in [0, t]\}$, from time t onward $\{a_s : s \geq t\}$ is optimal for (w, W_T), i.e.*

$$v_t^{(a,w,w_T)} = \underset{\tilde{a}\in\mathcal{A}_t}{ess\sup}\; v_t^{(\tilde{a},w,W_T)},$$

where $\mathcal{A}_t$ is a subset of $\mathcal{A}$ and all effort processes in $\mathcal{A}_t$ has the fixed history $\{a_s : s \in [0, t]\}$.

3.1 Necessary Conditions

The agent's problem in (9) cannot be analyzed using the standard dynamic programming method because the objective function depends on the process $\{w_t\}$, which is non-Markovian since it depends on the whole output history. We will use the stochastic maximum principle under the weak formulation of the agent's problem. The idea is to apply the method of the stochastic calculus of variations.

Theorem 1 *Under some technical assumptions in the appendix, if the contract $c = (a,\ w,\ W_T)$ is incentive compatible, then (a, γ^c) satisfies*

$$\begin{cases} \gamma_t^c + \frac{\sigma^{-2}}{h_t}\overline{p}_t + u_a(w_t,\ a_t) \geq 0, & \text{if } a_t > 0, \\ \gamma_t^c + \frac{\sigma^{-2}}{h_t}\underline{p}_t + u_a(w_t,\ a_t) \leq 0, & \text{if } a_t < 1, \end{cases} \qquad (15)$$

where (v^c, γ^c) *is the solution to BSDE (6) associated with the contract* c*,* $\overline{p}_t = \max_{p_t \in \mathcal{P}_t^c} p_t$*,* $\underline{p}_t = \min_{p_t \in \mathcal{P}_t^c} p_t$*,*

$$\mathcal{P}_t^c \triangleq \left\{ p_t : p_t \triangleq h_t E_t^{Q^{a,b^{c*}}} \left[-\int_t^T e^{-\rho(s-t)} \gamma_s^c \frac{1}{h_s} ds \right],\ b^{c*} \in \mathcal{B}^c \right\}, \tag{16}$$

and $Q^{a,b^{c*}}$ *is some worst-case measure for* v^c *with density* $b^{c*} \in \mathcal{B}^c$*, i.e.*

$$v_t^c = E_t^{Q^{a,b^{c*}}} \left[\int_t^T e^{-\rho(s-t)} u(w_s, a_s) ds + e^{-\rho(T-t)} U(W_T) \right].$$

As we point out in the introduction, the proof of this theorem is nontrivial because we cannot directly use the standard method of calculus of variations as in Cvitanic, Wan, and Zhang (2009), Prat and Jovanovic (2014), Williams (2009, 2011), Cvitanic and Zhang (2013), and He et al (2014). The standard method requires smoothness, but this condition is violated for recursive multiple-priors utility. Our solution is to suitably define derivative at the kink point so that we can get convergence when performing small perturbations.

Without ambiguity, this theorem reduces to Proposition 1 in Prat and Jovanovic (2014). In particular, the set $\mathcal{P}_t^c$ reduces to a singleton with element

$$p_t = h_t E_t^{P^a} \left[-\int_t^T e^{-\rho(s-t)} \gamma_s^c \frac{1}{h_s} ds \right].$$

The process (p_t) is attributed to learning about unknown quality. With ambiguity, we have to use the worst-case measure to compute the preceding expectation and there may exist multiple worst-case measures that attain the same minimum utility for the agent, as shown in Section 2.2. The set in (16) reflects this multiplicity.

As a special case, when η is known, $\mathcal{P}_t^c = \{0\}$ for all t. Condition (15) becomes

$$\left[\gamma_t^c + u_a(w_t,\ a_t)\right](\tilde{a}_t - a_t) \leq 0 \text{ for all } \tilde{a}_t \in [0,1].$$

This means that a_t maximizes $\gamma_t^c a_t + u(w_t, a_t)$. This is the necessary and sufficient condition in Sannikov (2008). It also holds under ambiguity without learning. This result is due to our special specification of the set of density generators. We can prove it using the comparison theorem in the BSDE theory. By this theorem, we can also prove a more general result for a general specification of the set of density generators Θ^a discussed in Chen and Epstein (2002). Formally, the necessary and sufficient condition for the contract (a, w, W_T) to be incentive compatible under ambiguity with full information is that a_t maximizes

$$\gamma_t^c a_t + u(w_t,\ a_t) + \sigma \min_{b_t \in \Theta_t^a} b_t \gamma_t^c.$$

When the last term is independent of a as in the case of κ-ignorance, the condition is that a_t maximizes $\gamma_t^c a_t + u(w_{t,}\ a_t)$.

Theorem 1 is too complex to be useful in applications. The following two lemmas simplify it significantly.

Lemma 2 *If the contract $c = (a, w, W_T)$ with $a_s > 0$ for $s \in [t, T]$ is incentive compatible, then $\gamma_s^c > 0$, the worst-case density generator satisfies $b_s^{c*} = -\kappa$ for $s \in [t, T]$, and $\mathcal{P}_s^c$ is a singleton with the element $p_s = h_s E_s^{Q^{a,b^{c*}}} \left[-\int_s^T e^{-\rho(\tau-t)} \gamma_\tau^c \frac{1}{h_\tau} d\tau \right]$ for $s \in [t, T]$.*

By this lemma, the necessary condition (15) implies that

$$\gamma_s^c \geq -\frac{\sigma^{-2}}{h_s} p_s - u_a(w_s, a_s), \text{ if } a_s > 0, \tag{17}$$

with equality when $a_s \in (0, 1)$, where p_s is the unique element in $\mathcal{P}_s^c$. The lemma below allows us to give an intuitive interpretation for p_t.

Lemma 3 *If $\gamma_s^c = -\frac{\sigma^{-2}}{h_s} p_s - u_a(w_s, a_s)$ for some $p_s \in \mathcal{P}_s^c$ for all $s \in [t, T]$, then $\gamma_s^c > 0$ and*

$$p_s = E_s^{Q^{a,b^{c*}}} \left[\int_s^T e^{-\rho(\tau-s)} u_a(w_\tau, a_\tau) d\tau \right] < 0, \tag{18}$$

where $Q^{a,b^{c*}}$ is the worst-case belief defined by the density generator (b_t^{c*}) with $b_s^{c*} = -\kappa$ for all $s \in [t, T]$.

This lemma says that whenever the incentive constraint binds in $[t, T]$, p_s for $s \in [t, T]$ is equal to the discounted marginal utility of effort, which is negative. We can then interpret condition (17) for $a_s > 0$ as follows: The expression γ_s^c on the left-hand side represents the marginal cost of deviating from the effort process a. If the agent exerts less effort, output will be lower and the agent will be punished by losing utility γ_t^c. But he will also enjoy a benefit of less disutility of working hard. This benefit is represented by $-u_a(w_{s,}\ a_s)$ on the right-hand side of (17). The remaining term $-\left(\sigma^{-2}/h_s\right) p_s > 0$ represents an additional benefit due to learning discussed in Prat and Jovanovic (2014) and He et al (2014), and is called the *information rent.* The inequality in (17) shows that the agent has no incentive to deviate.

The intuition behind the information rent is nicely explained by Prat and Jovanovic (2014). With unknown quality, past shirking behavior has a persistent effect on the incentive constraint. If the agent shirks in the past, he will overestimate the productivity of the project because he observes output but cannot separate project quality from the Brownian shock. The principal also observes output, but does not observe the agent's shirking behavior. He can only observe the recommended effort. Thus he will underestimate the productivity. This generates a persistent wedge between the principal's and the agent's posterior. This motivates the manipulation an agent might undertake. By shirking, the agent can benefit by lowering the discounted marginal utility of effort until the end of the contract. Unlike Prat and Jovanovic (2014), this discounted marginal utility is computed using the agent's worst-case belief $Q^{a,b^{c*}}$ under ambiguity. The worst-case belief is induced by the density generator $b_s^{c*} = -\kappa$ for $s \in [t, T]$. In particular, the agent pessimistically believes that the drift of output is shifted downward by κ.

By Lemma 3 we can use (6) and (18) to derive the BSDEs for v and p associated with the contract c as

$$\begin{aligned} dv_s &= [\rho v_s - u(w_s, a_s)]ds + \sigma\gamma_s d\hat{B}_s^{a,-\kappa} \\ &= [\rho v_s - u(w_s, a_s) + \kappa\sigma\gamma_s]ds + \sigma\gamma_s d\hat{B}_s^a, \end{aligned} \tag{19}$$

with $v_T^a = U(W_T)$ for $s \in [t, T]$ and

$$\begin{aligned} dp_s &= [\rho p_s - u_a(w_s, a_s)]ds + \sigma_s^p \sigma d\hat{B}_s^{a,-\kappa} \\ &= [\rho p_s - u_a(w_s, a_s) + \kappa\sigma\sigma_s^p]ds + \sigma\sigma_s^p d\hat{B}_s^a, \end{aligned} \tag{20}$$

with $p_T = 0$ for $s \in [t, T]$.

When $a_t = 0$, the condition in (15) becomes

$$\gamma_t^c \leq -\frac{\sigma^{-2}}{h_t}\underline{p}_t - u_a(w_t, 0). \tag{21}$$

The expression γ_t^c on the left-hand side represents the marginal benefit of an increase in effort. The two expressions on the right-hand side represent the total marginal cost of an increase in effort, consisting of disutility of working and the loss in the information rent. The condition above shows that the agent has no incentive to deviate.

3.2 Sufficient Conditions

To guarantee the necessary conditions to be sufficient, we need global concavity. Only then can we be sure that the agent finds it indeed optimal to provide recommended effort when assigned the wage function satisfying the local incentive constraint (15). Following Yong and Zhou (1999), Williams (2009), Cvitanic and Zhang (2013), and Prat and Jovanovic (2014), we impose conditions to ensure global concavity of the agent's Hamiltonian.

Theorem 2 *A contract $c = (a, w, W_T)$ is incentive compatible if there exists $p_t \in \mathcal{P}_t^c$ and the* *corresponding density generator* $b^{c*} \in \mathcal{B}^c$ *such that*

$$\left[\gamma_t^c + \frac{\sigma^{-2}}{h_t} p_t + u_a(w_t,\ a_t)\right](\tilde{a}_t - a_t) \le 0 \text{ for all } \tilde{a}_t \in [0,1] \text{ and}$$

$$-2u_{aa}(w_t,\ a_t) \ge e^{\rho t} \xi_t \sigma^2 h_t \ge 0, \tag{22}$$

for $t \in [0, T]$, *where* ξ *is the predictable process defined uniquely by*

$$E_t^{Q^{a,b^{c*}}}\left[-\int_0^T e^{-\rho s} \gamma_s^c \frac{\sigma^{-2}}{h_s} ds\right] - E^{Q^{a,b^{c*}}}\left[-\int_0^T e^{-\rho s} \gamma_s^c \frac{\sigma^{-2}}{h_s} ds\right] = \int_0^t \xi_s \sigma d\hat{B}_t^{a,b^{c*}}, \tag{23}$$

where $(\hat{B}_t^{a,b^{c*}})$ *is the standard Brownian motion under worst-case measure* $Q^{a,b^{c*}}$ *defined by* $b^{c*} \in \mathcal{B}^c$.

Since w_t, a_t, γ_t and ξ_t are all endogenous, we need to check whether the sufficient conditions are satisfied ex post for the contract derived from the necessary conditions. Without transparent solutions, these conditions are hard to verify. In the next section we will consider exponential utility to derive closed-form solutions.

4 Solutions under Exponential Utility

As in Prat and Jovanovic (2014), we restrict our attention to exponential utility functions of the form

$$u(w, a) = -\exp(-\alpha(w - \lambda a)) \text{ with } \lambda \in (0, 1), \tag{24}$$

where $\alpha > 0$ is the constant absolute risk aversion parameter. This utility has no wealth effect on leisure because $u_w/u_a = -\lambda^{-1}$. As will be shown below, assuming $0 < \lambda < 1$ ensures that $a = 1$ is the first-best action because the marginal utility of an additional unit of wage is larger than the marginal cost of effort.

For terminal utility, we set

$$U(W) = -\frac{\exp(-\alpha\rho W)}{\rho}. \tag{25}$$

This specification implies that an infinitely-lived agent retires at the termination date T under the contract and then consumes the perpetual annuity derived from W while providing zero effort. We will focus on the infinite-horizon limit as $T \to \infty$. Thus the particular specification of U will be immaterial.

Following Prat and Jovanovic (2014), we first focus on contracts in which recommended effort remains positive at all future dates before analyzing general contracts in which zero effort is possible. Such contracts with positive effort are called *incentive contracts*. We will solve for the optimal contract using the strong formulation by dynamic programming.

4.1 Known Quality under Ambiguity

When the quality η is known, there is no need to learn about η and there is no belief manipulation effect. This implies that the value of private information under ambiguity p is equal to 0 and the necessary condition (17) for an incentive contract (a, w) with $a_t > 0$ for all $t \geq 0$ to be incentive compatible becomes $\gamma_t \geq -u_a(w_t,\, a_t)$ with equality if $a_t \in (0, 1)$. This condition is also sufficient by the comparison theorem in the BSDE theory.

We first derive the first-best contract in which the agent's action is observable and hence the incentive constraint is absent.

Theorem 3 *Suppose that quality η is known. In the first-best infinite-horizon limit case where effort is observable and contractible, the recommended effort $a_{FB}(t) = 1$ for all t, the principal offers a constant wage to the agent*

$$w_{FB}(t) = -\frac{\ln(-\rho v_0)}{\alpha} + \lambda \tag{26}$$

and the agent's continuation value is $v_t = v_0$ for all t. The principal's value function $J_{FB}(v)$ is given by

$$\rho J_{FB}(v) = 1 - \lambda + \frac{\ln(-\rho v)}{\alpha}. \tag{27}$$

Under the first-best contract, the risk-neutral principal fully insures the risk- and ambiguity-averse agent by offering the agent a constant wage at all time. And the agent provides a full effort level at all time. The first-best contract under ambiguity is observationally equivalent to that without ambiguity. See Section 5.5 for a further discussion on this issue.

With hidden action, the principal's value function satisfies the following HJB equation in the infinite-horizon limit as $T \to \infty$:

$$\rho J_N(v) = \sup_{a>0,w,\gamma} a - w + J_N'(v)\left(\rho v - u(w,a) + \kappa\sigma\gamma\right) + \frac{1}{2} J_N''(v)\,\sigma^2\gamma^2 \tag{28}$$

subject to the incentive constraint $\gamma \geq -u_a(w, a)$. Since the incentive constraint implies that $\gamma > 0$, the impact of ambiguity is to raise the drift of the agent's continuation value by $\kappa\sigma\gamma > 0$. Since J_N is concave and decreasing in v, we can show that the incentive constraint always binds.

Lemma 4 *Assume that quality is known and any recommended effort level satisfies $a_t > 0$ for all t. Then the optimal contract with agency in the infinite-horizon limit recommends the first-best effort* *level* $a_t^* = 1$. *The principal's value function* $J_N(v)$ *satisfies*

$$\rho J_N(v) = F + \frac{\ln(-\rho v)}{\alpha}, \tag{29}$$

where

$$F = 1 - \lambda + \frac{\ln(K/\rho)}{\alpha} + \frac{1}{2\rho}\alpha(\lambda\sigma K)^2, \tag{30}$$

and K *is the positive root of the quadratic equation*

$$(\alpha\lambda\sigma)^2 K^2 + (1 + \kappa\alpha\sigma\lambda)K - \rho = 0. \tag{31}$$

The principal delivers the agent initial value v_0 *and the agent's continuation value satisfies*

$$dv_t = v_t(\alpha\lambda\sigma K)^2 dt - v_t\alpha\sigma\lambda K dB_t^1. \tag{32}$$

The optimal wage is given by

$$w_t^* = -\frac{\ln(-Kv_t)}{\alpha} + \lambda. \tag{33}$$

The optimal incentive contract offers the agent a wage w_t^* such that the agent's instantaneous utility $u(w_t^*, a_t^*)$ is proportional to his continuation value v_t. The factor of proportionality K is determined by equation (31). This equation comes from the first-order condition for the wage

$$1 = -J_N'(v)\, u_w(w, a) + J_N''(v)\, \sigma^2 \gamma \frac{\partial\gamma}{\partial w} + J_N'(v)\, \kappa\sigma \frac{\partial\gamma}{\partial w}.$$

The interpretation of this equation is as follows. A unit increase in the wage reduces the principal value by one unit. The marginal benefit consists of three components. First, an increase in the wage raises the agent's current utility and reduces his future continuation value, which raises the principal's value. This component is represented by $-J_N'(v)\, u_w(w, a)$. Second, an increase in the wage raises the marginal utility of effort $u_a(w, a)$ since $u_{aw} > 0$, and hence reduces the agent's utility or pay sensitivity to performance (γ) by the incentive constraint. Given that the value function is concave, this benefits the principal by $J_N''(v)\, \sigma^2\gamma\frac{\partial\gamma}{\partial w}$. Third, ambiguity raises the drift of the agent's continuation value by $\kappa\sigma\gamma$. An increase in the wage reduces the ambiguity adjustment $\kappa\sigma\gamma$. The reduced growth of the promised value to the agent benefits the principal by $J_N'(v)\, \kappa\sigma\frac{\partial\gamma}{\partial w}$. This component is specific to our model with ambiguity.

An important property of the agent's continuation value is that its drift $v_t(\alpha\lambda\sigma K)^2$ is negative, but its volatility $-v_t\alpha\sigma\lambda K$ is positive, since $v_t < 0$ for all t. The intuition is as follows. Raising the wage or promised value today reduces the volatility of the promised value since $\partial\gamma_t/w = -u_{aw}(w_t^*, a_t^*) < 0$, thereby reducing promised value in the future. This benefits the principal since his value decreases with the promised value to the agent and thus the agent's promise value converges to minus infinity in the long run. The agent's immiserization can be formally proved by noting that $J_N'(v_t)$ is a martingale since

$$dJ_N'(v_t) = \frac{\lambda\sigma K}{\rho v_t} dB_t^1.$$

By the martingale convergence theorem, $\lim_{t\to\infty} J_N'(v_t) = 0$ and hence $\lim_{t\to\infty} v_t = -\infty$.

It is straightforward to show that K decreases with the degree of ambiguity κ. Thus the principal's value function $J_N(v)$ also decreases with κ, but the optimal wage w_t^* increase with κ since $v_t < 0$. When the agent is averse to the ambiguity about the mean output, his utility decreases with the degree of ambiguity so that it is more costly to satisfy the promise-keeping constraint and the agent's initial participation constraint. The principal has to offer more wages to motivate the ambiguity-averse agent to commit to the contract. That is, the third component of the marginal benefit to the principal $J_N'(v)\kappa\sigma\partial\gamma/\partial w$ increases with κ.

Equation (32) shows that, given the same continuation value v_t, an agent who faces more ambiguity has a smaller volatility of utility, i.e., $-v_t\alpha\sigma\lambda K$ decreases with κ. This means that the wage is less sensitive to performance by (33). That is, the risk-neutral and ambiguity-neutral principal must provide more insurance to a more ambiguity-averse agent. This reflects the tradeoff between incentive and ambiguity sharing.

Is it possible to allow the agent to shirk for some time in an optimal contract with agency? The following lemma characterizes the optimal incentive compatible contract when the agent shirks at all time.

Lemma 5 *Suppose that quality η is known. When the agent shirks at all time $a_t = 0$ for all $t \geq 0$, the optimal incentive compatible contract in the infinite-horizon limit is to offer the constant wage $w_t = -\ln(-\rho v_0)/\alpha$ and promised value $v_t = v_0$ to the agent for all $t \geq 0$. The principal's value function is given by $\rho J_{NS}(v) = \ln(-\rho v)/\alpha$.*

For $a_t = 0$ to be incentive compatible, the constraint $\gamma_t \leq -u_a(w_t, a_t)$ must be satisfied. Since the agent is risk averse and ambiguity averse, it is optimal to set $\gamma_t = 0$ so that the risk-neutral and ambiguity-neutral principal provides full insurance to the agent. This contract is incentive compatible.

Combining Lemmas 4 and 5, we have the following result:

Theorem 4 *Suppose that quality η is known. When $F \geq 0$, the optimal contract with agency is described in Lemma 4. When $F \leq 0$, the optimal contract with agency is described in Lemma 5.*

To prove this theorem, we simply compare the principal's value functions $J_N(v)$ and $J_{NS}(v)$. The cost to the principal of allowing the agent to shirk is that output is lower, but the associated benefit is that the principal can offer a lower wage on average. The term F reflects the net benefit to the principal of not allowing the agent to shirk.

By Theorems 3 and 4, we deduce that the expression

$$1 - \lambda - F = -\frac{\ln(K/\rho)}{\alpha} - \frac{1}{2\rho}\alpha(\lambda\sigma K)^2$$

measures the per-period efficiency loss due to hidden action under ambiguity. Since K decreases with the degree of ambiguity κ, the efficiency loss increases with κ.

4.2 Unknown Quality under Ambiguity

We now study the case of unknown quality under ambiguity. We first describe the first-best contract. Next we analyze incentive contracts in which the agent never shirks. We then study the general optimal contract in which the agent can shirk for some time.

4.2.1 First-best Contract

It is straightforward to prove that Theorem 3 also applies to the first-best contract with unknown quality under ambiguity. This result is intuitive. When the agent is averse to risk and ambiguity, the risk-neutral Bayesian principal should fully insure the agent no matter whether there is unknown quality. The difference between the contracts under known and unknown quality lies in the principal's profits. Under symmetric information with known quality, the principal's discounted profits at any date t are given by

$$\frac{1}{\rho}\left[\eta + 1 - \lambda + \frac{\ln(-\rho v)}{\alpha}\right],$$

while they are equal to

$$\frac{1}{\rho}\left[\hat{\eta}\left(Y_t - t, t\right) + 1 - \lambda + \frac{\ln(-\rho v_t)}{\alpha}\right],$$

under unknown quality.

4.2.2 Incentive Contracts

In an incentive contract the recommended effort remains positive at all dates. By Lemma 2, the necessary condition for incentive compatibility is given by (17). Given that the agent is risk averse, it is natural to conjecture that the principal will minimize sensitivity γ_t so that the incentive constraint (17) binds. We can then apply Lemma 3 to deduce that $\gamma_t > 0$ and $\mathcal{P}_t^c$ contains a single element p_t, which satisfies (20). Moreover the agent's worst-case density generator is $b_t^* = -\kappa$. Now we can formulate the problem of solving for an optimal incentive contract by dynamic programming

$$J_t^T \triangleq \sup_{w,a>0,\gamma,\sigma^p} E_t^{P^a}\left[\int_t^T e^{-\rho(s-t)}(a_s - w_s)ds - e^{-\rho(T-t)}W_T\right] \tag{34}$$

subject to

$$dv_s = [\rho v_s - u(w_s, a_s) + \kappa\sigma\gamma_s]ds + \sigma\gamma_s d\hat{B}_s^a,\ v_T = U(W_T),$$

$$\begin{aligned} dp_s &= [\rho p_s - u_a(w_s, a_s) + \kappa\sigma\sigma_s^p]ds + \sigma\sigma_s^p d\hat{B}_s^a,\ p_T = 0, &(35)\\ \gamma_s &= -\frac{\sigma^{-2}}{h_s}p_s - u_a(w_s,\ a_s), &(36)\end{aligned}$$

for $s \in [t, T]$ and $v_t = v$. We can see that J_t^T depends on the state vector $(t,\ v,\ p)$. When $T \to \infty$, the time state t will not disappear because we consider the nonstationary learning case in which the learning precision h_t depends on time t. So time t about the agent age is a key role different from an infinite-horizon model without unknown quality and from an infinite-horizon model with stationary leaning.

Given (24), we have $u_a(w, a) = \alpha\lambda u(w, a)$. Using equations (19) and (20), we can show that

$$p_t = \alpha\lambda(v_t - e^{-\rho(T-t)}E_t^{Q^{a,-\kappa}}[v_T]). \tag{37}$$

Thus we can infer p_t from v_t given that $\{v_t\}$ is a one-dimensional diffusion process. In particular, when the contract goes to infinity and when the transversality condition $\lim_{T\to\infty} e^{-\rho T}E\left[v_T\right] = 0$ holds, we have $p_t = \alpha\lambda v_t$. This implies that tracking one of the state variables p_t and v_t is sufficient so that we can eliminate the state p_t.

Since the laws of motion for the variables v and p are Markovian, we can use a HJB equation to analyze the principal's optimal control problem. As discussed above, we can omit the state p and control σ^p so that the value function depends on (t, v) only. The HJB equation is given by

$$\rho J_t^T = \sup_{a>0,w,\gamma_t>0} a - w + \frac{\partial J_t^T}{\partial t} + \frac{\partial J_t^T}{\partial v}(\rho v - u(w, a) + \kappa\sigma\gamma_t) + \frac{1}{2}\frac{\partial^2 J_t^T}{\partial v^2}\sigma^2\gamma_t^2 \tag{38}$$

subject to (37),

$$\gamma_t = -\frac{\sigma^{-2}}{h_t}p_t - u_a(w, a), \tag{39}$$

and $J^T(T, v) = W_T = -U^{-1}(v)$.

Next we will show that the optimal incentive contract for the principal is to recommend the agent with full effort $a_t^* = 1$.

Lemma 6 *If the necessary conditions (35) and (36) hold for all $s \geq t$, then it is optimal to set effort equal to its first-best level, i.e., $a_s^* = 1$ for $s \geq t$.*

Now we solve the control problem (38) and give a condition to guarantee that the optimal solution satisfies the sufficient condition in (22) so that it is indeed an incentive-compatible optimal contract.

Lemma 7 *Assume that quality η is unknown and any recommended effort level satisfies $a_t > 0$ for all $t \geq 0$. Then the optimal incentive contract in the infinite-horizon limit as $T \to \infty$ recommends the first-best effort level $a_t^* = 1$. The principal's value function is given by*

$$\rho J_I(t, v) = f(t) + \frac{\ln(-\rho v)}{\alpha}, \tag{40}$$

where the function $f(t)$ is given by

$$f(t) = \int_t^\infty e^{-\rho(s-t)} \left[\rho \left(1 - \lambda + \frac{\ln(k_s/\rho)}{\alpha} \right) - \frac{1}{2} (\sigma\lambda)^2 \alpha \left(\left(\frac{\sigma^{-2}}{h_s} \right)^2 - k_s^2 \right) - \frac{\kappa \sigma^{-1} \lambda}{h_s} \right] ds, \tag{41}$$

and k_t is the positive solution of the quadratic equation,

$$(\alpha\lambda\sigma)^2 k_t^2 + \left(1 + \kappa\alpha\sigma\lambda + \frac{(\alpha\lambda)^2}{h_t} \right) k_t - \rho = 0. \tag{42}$$

The principal delivers the agent initial value v_0 and the agent's continuation value satisfies

$$dv_t = v_t \left[\rho - k_t - \kappa\alpha\sigma\lambda \left(k_t + \frac{\sigma^{-2}}{h_t} \right) \right] dt - v_t \alpha\sigma\lambda \left(k_t + \frac{\sigma^{-2}}{h_t} \right) d\hat{B}_t^1. \tag{43}$$

The agent's worst-case density generator is $b_t^* = -\kappa$. The wage is given by

$$w_t^* = \frac{-\ln(-k_t v_t)}{\alpha} + \lambda. \tag{44}$$

As in the case of known quality under ambiguity discussed in Lemma 4, the optimal incentive contract with unknown quality offers the agent a wage w_t^* such that his instantaneous utility is proportional to his continuation value. The difference is that the factor of proportionality k_t is time varying. Equation (42) shows that there is a new component $k_t(\alpha\lambda)^2/(\rho h_t)$ of marginal benefits from raising the wage. This component comes from learning. In particular, from the first-order condition for the wage as $T \to \infty$, we can compute that

$$\frac{\partial^2 J_I}{\partial v^2} \sigma^2 \gamma_t \frac{\partial \gamma_t}{\partial w} = -\frac{\partial^2 J_I}{\partial v^2} \sigma^2 \left[-\frac{\sigma^{-2}}{h_t} p_t - u_a(w, a) \right] u_{aw}(w, a),$$

where we have plugged in the incentive constraint (39). The term involving p_t reflects the effect of learning, which is positive since $u_{aw} > 0$, $p_t < 0$, and $\partial^2 J_I / \partial v^2 < 0$. The intuition is that the principal can minimize the agent's information rent by raising the wage when $u_{aw} > 0$. This effectively reduces the disutility of shirking and hence reduces the agent's incentive to manipulate information.

The following lemma characterizes properties of $f(t)$ and $k(t)$.

Lemma 8 *The function $f(t)$ is increasing over time and converges to F defined in (30). Moreover, $f(t)$, $k(t) > 0$, and $\dot{k}_t / k_t > 0$ decrease with κ.*

Applying Theorem 2 and the preceding lemma, we can verify incentive compatibility of the contract characterized in the preceding lemma.

Lemma 9 *The contract described in Lemma 7 is incentive compatible, i.e., meets condition (15) and (22) when*

$$\rho\sigma^2 > \frac{1 + \kappa\alpha\sigma\lambda}{h_t} + 2\left(\frac{\alpha\lambda}{h_t}\right)^2. \tag{45}$$

Because the precision h_t is increasing in t, the condition (45) holds at all subsequent dates $s \geq t$. This condition is more likely to hold when λ is low, volatility of output σ is high, the coefficient of absolute risk aversion α is low, the ambiguity parameter κ is small, and precision h_0 is high. In particular, when (45) holds in the case with ambiguity $\kappa > 0$, it also holds in the case without ambiguity ($\kappa = 0$). It always holds in the limit as $h_t \to \infty$, where the quality is known. This result confirms that the contract with known quality presented in Theorem 4 is indeed incentive compatible and hence optimal.

4.2.3 Optimal Contract

We have derived the optimal incentive contract by assuming that $a_t^* > 0$ for all t. But the principal can choose to perfectly insure the agent by giving zero effort recommendation. We now consider the case where full effort is not exerted for all t and the principal may choose to insure agent for a certain length of time.

Lemma 10 *Suppose that quality is unknown. When the agent shirks at all time $a_t = 0$ for all $t \geq 0$, the optimal incentive compatible contract is to offer the constant wage $w_t = -\ln(-\rho v_0)/\alpha$ and promised value $v_t = v_0$ to the agent for all $t \geq 0$. The principal's value function is given by $\rho J_S(t, v) = \ln(-\rho v)/\alpha$.*

Combining Lemmas 7 and 10, we deduce that $f(t)$ reflects the total benefit from not shirking. If $J_I(t,v) \leq J_S(t,v)$ or $f(t) \leq 0$ for all t, the optimal contract is to allow the agent to shirk at all time. This condition is satisfied if $F \leq 0$ because F is an upper bound of $f(t)$. What happens if $F > 0$? One may conjecture that the principal should allow the agent to shirk when $f(t) \leq 0$ and recommend full effort when $f(t) > 0$. As Prat and Jovanovic (2014) point out, this conjecture is incorrect because it ignores the option value to delay incentive provisions.

Since exponential utility does not have the wealth effect, the value of the full-insurance option does not depend on the current belief about η but is instead deterministic. The marginal benefits from delaying incentives are equal to $f'(t)$, while the costs due to discounting are given by $\rho f(t)$. Thus when

$$\psi(t) \triangleq \rho f(t) - f'(t) < 0$$

the principal perfectly insures the agent. But when $\psi(t) \geq 0$ he offers the incentive contract described in the previous subsection. Since $\psi(t)$ and $h(t)$ are increasing over time, there exists a function φ such that $\psi(t) = \varphi(h_t)$. Then there is at most one precision level $h_\tau = \bar{h}$ at a threshold time τ above which incentives provision is optimal. Such a precision level may not exist depending on parameter values.

Theorem 5 *Assume that quality η is unknown. Let F be defined in (30). (a) If $F > 0$, then there exists a unique $\bar{h} > 0$ such that $\varphi(\bar{h}) = 0$. Suppose further that*

$$\ln \varpi < \alpha(\lambda - 1) + \alpha\kappa\lambda\sigma\varpi, \text{ where } \varpi \triangleq \frac{2}{(1+\kappa\sigma\alpha\lambda) + \sqrt{(1+\kappa\sigma\alpha\lambda) + 8\alpha^2\lambda^2\rho\sigma^2}}. \tag{46}$$

Then for $h_0 < \bar{h}$, there exists a time $\tau > 0$ such that $h(\tau) = \bar{h}$ and the optimal contract with agency in the infinite-horizon limit recommends effort a^* such that $a_t^* = 0$ for $t \in [0,\ \tau)$ and $a_t^* = 1$ for $t \geq \tau$. The principal offers the agent initial value v_0 and the wage

$$w_t^* = \begin{cases} -\frac{\ln(-\rho v_0)}{\alpha} & \text{if } t \in [0,\ \tau) \\ -\frac{\ln(-k_t v_t)}{\alpha} + \lambda & \text{if } t \geq \tau \end{cases}.$$

The agent's continuation value satisfies $v_t = v_0$ for $t \in [0,\ \tau)$ and (43) for $t \geq \tau$ with $v_\tau = v_0$. His worst-case density generator is $b_t^* = 0$ for $t \in [0,\ \tau)$ and $b_t^* = -\kappa$ for $t \geq \tau$. The principal's value function J^* is given by

$$\rho J^*(t,v) = \begin{cases} e^{-\rho(\tau-t)} f(\tau) + \frac{\ln(-\rho v)}{\alpha} & \text{if } t \in [0,\ \tau) \\ f(t) + \frac{\ln(-\rho v)}{\alpha} & \text{if } t \geq \tau \end{cases}.$$

For $h_0 \geq \bar{h}$, $h_t \geq \bar{h}$ holds for all time $t \geq 0$ and the optimal contract is the incentive contract *described in Lemma 7.*

(b) If $F \leq 0$, the optimal contract with agency is the contract described in Lemma 10.

This theorem gives a complete characterization of the optimal contract. When $F > 0$, in the optimal contract the principal first perfectly insures the agent by allowing him to shirk until a certain time and then provides him incentives to exert full effort from that time on. The starting time τ of providing incentives depends on the parameter values. For $t < \tau$, $e^{-\rho(\tau-t)}f(\tau) > f(t)$ so that shirking is optimal. At time τ, $\lim_{t\uparrow\tau} e^{-\rho(\tau-t)}f(\tau) = f(t)$ and $\lim_{t\uparrow\tau} \frac{\partial e^{-\rho(\tau-t)}f(\tau)}{\partial t} = \rho f(\tau) = \lim_{t\downarrow\tau} f'(t)$. These two conditions are analogous to the value-matching and smooth-pasting conditions in the literature. The two segments of the principal's value function are smoothly pasted together. But the optimal wage has a jump at time τ. Before time τ, the agent shirks and receives a constant wage. To motivate the agent to exert full effort starting at time τ, the principal must reward the agent by raising the wage discretely. The jump size is given by

$$\lambda - \frac{\ln(k_\tau)}{\alpha} + \frac{\ln(\rho)}{\alpha} > 0.$$

Condition (46) ensures that the sufficient condition in (22) is satisfied for the optimal contract described in part (a). More specifically, let t_0 be the smallest time such that (22) holds. Condition (46) ensures $t_0 < \tau$ so that the recommended high effort starting from τ on is incentive compatible.

5 Properties of the Optimal Contract

In this section we analyze the properties of the optimal contract with unknown quality under ambiguity by conducting a comparative statics analysis with respect to the ambiguity aversion parameter κ. This parameter can also be interpreted as the level of ambiguity. We also compare with the case of risk aversion and with the case of exogenously given distorted beliefs.

5.1 Delayed Effort

How does ambiguity affect incentive provisions? We focus on the nontrivial case with $F > 0$. Then Theorem 5 shows that there is a threshold level of precision $\bar{h} = h_\tau$ at time τ such that full effort is provided if and only if $h_t > \bar{h}$ or $t > \tau$. The following result establishes the impact of κ on $\bar{h}$.

Proposition 1 *Suppose that the conditions in part (a) of Theorem 5 hold. Then $\bar{h}$ and τ increase with κ. Moreover, they approach infinity when κ is sufficiently large.*

This proposition shows that the incentive is delayed more if the agent faces more ambiguity. If the agent faces more ambiguity, the principal wants to provide more insurance to the agent by

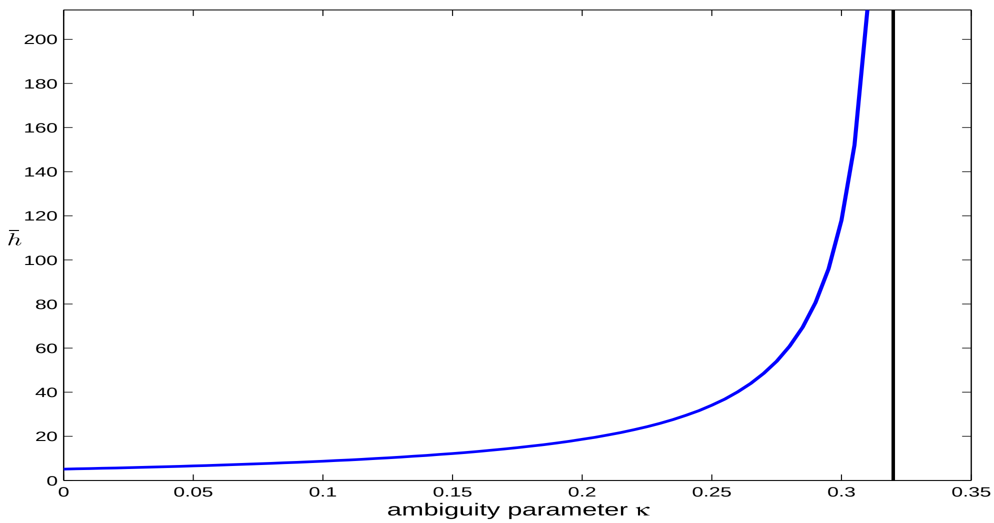

Figure 1: $\bar{h}$ as a function of the degree of ambiguity κ. We set $\alpha = 1, \rho = 0.5$, $\sigma = \sqrt{0.5}$, $\lambda = 0.785$, and $h_0 = 0.2$.

allowing him to provide zero effort for a longer period of time because allowing the agent to be exposed to uncertainty by eliciting full effort is costly to the principal.

Figure 1 plots $\bar{h}$ against κ. This figure shows that $\bar{h}$ increases with κ and approaches infinite when κ is sufficiently large. The main reason is that, when κ is sufficiently large, the net benefit F to the principal of allowing the agent to work approaches minus infinity in the limit when the project quality is known. Thus it is optimal to allow the agent to shirk forever. This is not the case for changes in risk aversion as will be shown below.

5.2 Sticky Wages

Theorem 5 shows that the wage is constant when effort is zero in the time interval $[0, \tau)$. Starting from time τ on, the agent exerts full effort and wages are stochastic. Applying Ito's Lemma to (44) and (43) yields

$$dw_t^* = \frac{1}{\alpha}\left[\underbrace{-\frac{1}{2}(\alpha\sigma\lambda k_t)^2}_{\text{immiserization}} \underbrace{-\frac{\dot{k}_t}{k_t}}_{\text{insurance}} + \underbrace{\frac{1}{2}(\alpha\lambda)^2\left(\frac{\sigma^{-1}}{h_t}\right)^2 + \kappa\alpha\lambda\frac{\sigma^{-1}}{h_t}}_{\text{information rent}}\right]dt + \underbrace{\lambda\left(k_t + \frac{\sigma^{-2}}{h_t}\right)}_{\text{PPS}}\sigma d\hat{B}_t^1 \tag{47}$$

for $t \geq \tau$. Following Prat and Jovanovic (2014), we can decompose the drift of the wage process into three components. First, wages decrease over time due to better insurance $(-\dot{k}_t/k_t < 0)$. Second, wages are driven downward by the agent's immiserization $\left(-0.5(\alpha\sigma\lambda k_t)^2\right)$. Third, when $h_t \in (0, \infty)$, the agent has information rents by shirking. An increase in the future wage lowers

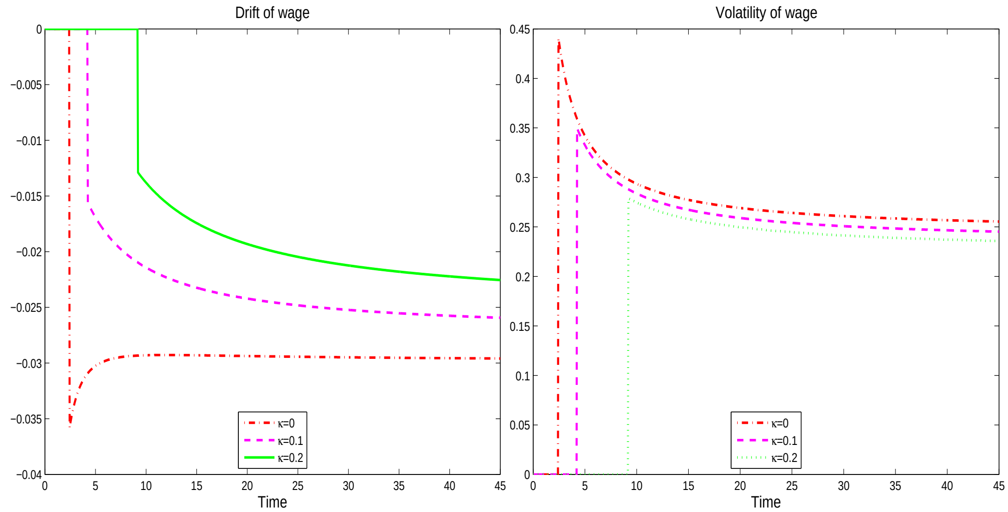

Figure 2: The drift and volatility of wages against time. We set $\alpha = 1$, $\rho = 0.5$, $\sigma = \sqrt{0.5}$, $\lambda = 0.785$, $h_0 = 0.2$, and $v_0 = -1.2$.

the information rents and hence strengthens the agent's incentives in the current period. This is because p is equal to the expected discounted marginal cost of future efforts. When $u_{wa} > 0$, raising future wages can reduce the agent's information rents. This is why the third component is positive and partially offsets the insurance and immiserization components.

We can interpret the diffusion term in the wage dynamics as the pay-performance sensitivity (PPS) as in Prat and Jovanovic (2014). It consists of two components. The first component reflects the marginal utility of effort (λk_t) and the second component reflects the impact of the information rents $\left(\lambda\sigma^{-2}/h_t\right)$. Both components come from the diffusion coefficient of the promised utility.

Proposition 2 *Suppose that the conditions in part (a) of Theorem 5 hold. Then the drift of the wage process increases with κ and is negative when t is sufficiently large. The PPS decreases with κ and decreases to the positive limit $\sigma\lambda K$ as $t \to \infty$.*

By Lemma 8 and equation (47), we find that the degree of ambiguity κ affects the drift and volatility of the wage process. It raises all three components of the drift, but lowers the volatility.

Figure 2 plots the drift and volatility of the wage process against time for three values of κ. Before an endogenously determined time τ, wages are constant and hence the drift and volatility are equal to zero. After time τ, the drift is negative and the volatility is positive. The drift increases with κ, but the volatility decreases with κ. Moreover, the drift increases over time without ambiguity ($\kappa = 0$), but decreases over time with ambiguity ($\kappa > 0$). For both cases the volatility decreases to some limit.

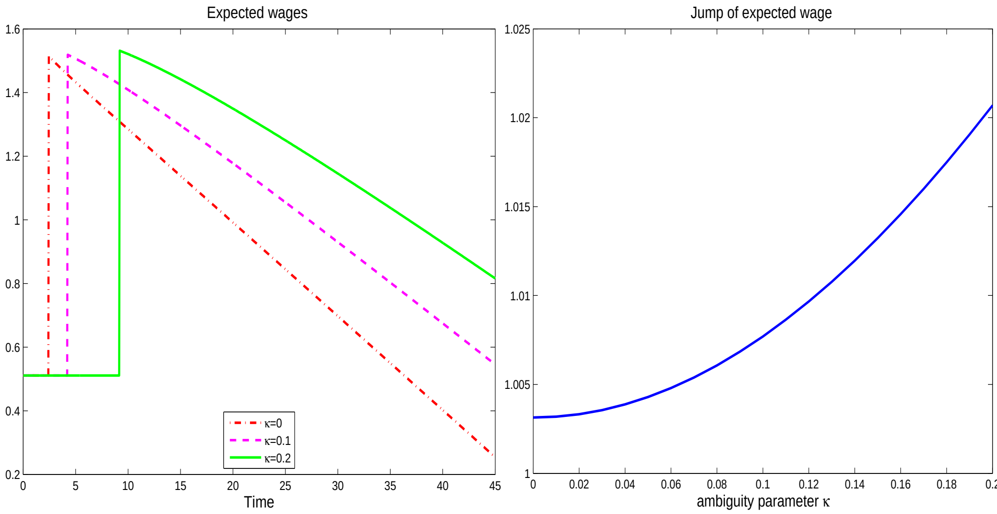

Figure 3: Expected wages against time (left panel) and the jump of expected wage against κ (right panel). We set $\alpha = 1, \rho = 0.5$, $\sigma = \sqrt{0.5}$, $\lambda = 0.785$, $h_0 = 0.2$, and $v_0 = -1.2$.

The left panel of Figure 3 plots the expected wage $E^{P^1}[w_t^*]$ against time for three values of κ. The expected wage is constant before time τ. It jumps up at time τ and then decreases over time. The jump size, $\lambda - \ln(k_\tau/\rho)/\alpha$, increases with κ as shown on the right panel of Figure 3, but the speed of declines in expected wages is slower when κ is larger. Thus the wage path is flatter.

5.3 Efficiency Loss

By comparing the principal's values in the first-best contract studied in Section 4.2.1 and in the contract with agency described in Theorem 5, we can compute the efficiency loss as $1 - \lambda - f(t)$ for $t \geq \tau$ and $1 - \lambda - e^{-\rho(\tau - t)} f(\tau)$ for $t < \tau$.

Proposition 3 *Suppose that the conditions in part (a) of Theorem 5 hold. Then the efficiency loss increases with κ and decreases to the limit $1 - \lambda - F > 0$ as $t \to \infty$. The principal's initial expected profits increase in belief precision h_0 and decreases with κ.*

Note that $1 - \lambda - F$ measures the efficiency loss with known quality. With unknown quality, the efficiency loss decreases over time as the principal and the agent gradually learn about the unknown quality. In the limit the principal knows the true quality because $\lim_{t\to\infty} \widehat{\eta}(Y_t - A_t, t) = E_t^{P^{a^*}}[\eta] = \eta$, but the agent believes the quality converges to $\lim_{t\to\infty} E_t^{Q^{a^*,b^*}}[\eta] = \eta - \sigma\kappa$ under his worst-case belief Q^{a^*,b^*}. Thus ambiguity does not disappears in the limit and the efficiency loss converges to $1 - \lambda - F$, which depends on κ. Since a larger degree of ambiguity is more costly to the principal, the efficiency loss increases with κ.

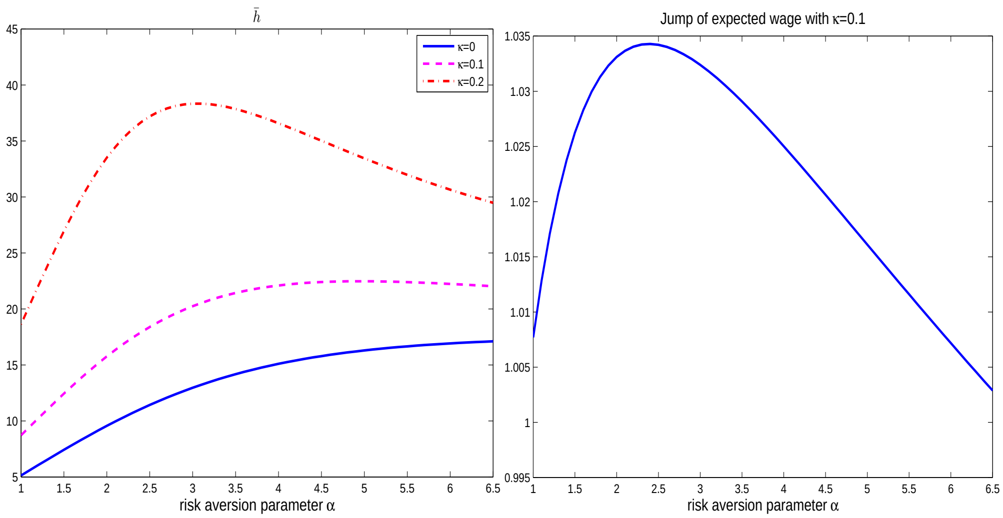

Figure 4: The impact of risk aversion. We set $\rho = 0.5$, $\sigma = \sqrt{0.5}$, $\lambda = 0.785$, $h_0 = 0.2$, and $v_0 = -1.2$.

5.4 Comparison with Risk Aversion

One may argue that ambiguity aversion is similar to risk aversion because both imply the agent dislikes uncertainty. There is an important difference between risk aversion and ambiguity aversion for the recursive multiple-priors utility model. The risk aversion parameter α affects the curvature of the utility function, but the ambiguity parameter κ does not affect it and only affects the agent's probability assessments. This allows for different attitudes toward risk (known odds) and ambiguity (unknown odds). To further examine this issue, we now study the impact of risk aversion numerically. The left panel of Figure 4 plots the threshold precision $\bar{h}$ against the risk aversion parameter α for three values of κ. We find that $\bar{h}$ may not be monotonic with α. In particular, it first increases and then decreases with α for $\kappa = 0.2$. An increase in α changes the curvature of the utility function, and hence raises the wage during the shirking period when $w_t = -\ln(\rho v_0)/\alpha > 0$. This is costly to the principal. Thus the principal may want to reduce the shirking duration when α is sufficiently large. By contrast, the ambiguity parameter κ does not affect the constant wage during the shirking phase and hence $\bar{h}$ always increases with κ for any value of α. An increase in κ is costly to the principal only if the wage is stochastic during the full-effort phase.

The right panel of Figure 4 plots the wage jump at $\bar{h}$ against the risk aversion parameter α for fixed $\kappa = 0.1$. This figure shows that the wage jump first increases with α and then decreases with α. This is different from the impact of κ for fixed α presented in Figure 3.

5.5 Exogenously Distorted Beliefs

Ambiguity aversion induces endogenous belief distortions in the sense that the agent chooses the worst-case belief Q^{a,b^*} associated with action a. One associated worst-case density generator is $b_t^* = -\kappa \text{sgn}(\gamma_t)$. The sign of the sensitivity γ_t of the agent's utility process determines the value of b_t^* endogenously. For the optimal contracts studied in Section 4, we find that $\gamma_t = 0$ until some time τ so that $b_t^* = 0$ for $0 \leq t \leq \tau$, and that $\gamma_t > 0$ from time τ on so that $b_t^* = -\kappa$ for $t > \tau$. The switching time τ is endogenously determined. Once we know this information, the optimal contract under ambiguity is observationally equivalent to that when the agent has expected utility with distorted belief defined by the density generator (b_t^*).

We emphasize that ambiguity generates pessimism endogenously through the choice of (b_t^*). This behavior is different from that with exogenous pessimism. To see this point in a simple way, we consider the first-best contract with known quality. Suppose that the agent pessimistically believe that the mean of the output process is lowered by $\kappa\sigma dt$ so that the utility process satisfies

$$dv_t = [\rho v_t - u(w_t, a_t)]\, dt + \kappa\sigma\gamma_t dt + \sigma\gamma_t dB_t^a. \tag{48}$$

Then

$$v_t = E^Q \left[\int_t^\infty e^{-\rho(s-t)} u\left(w_s, a_s\right) ds \right],$$

where Q is the distorted belief defined by

$$\frac{dQ}{dP^a}|_{\mathcal{F}_t^Y} = z_t = \exp\left(-\kappa B_t^a - \frac{1}{2}\kappa^2 t \right).$$

The HJB equation is

$$\rho J\left(v\right) = \sup_{a,w,\gamma} a - w + J'\left(v\right)\left(\rho v - u(w,a) + \kappa\sigma\gamma\right) + \frac{1}{2} J''\left(v\right) \sigma^2 \gamma^2. \tag{49}$$

Solving this equation shows that the first-best contract when the agent has an exogenously distorted belief recommends the agent to exert full effort $a_{FB}\left(t\right) = 1$ for all t and offers the agent the wage

$$w_t^{FB} = -\frac{\ln\left(-\rho v_t\right)}{\alpha} + \lambda,$$

where

$$dv_t = \kappa^2 v_t dt + \kappa v_t dB_t^a. \tag{50}$$

The principal's value function is given by

$$\rho J_{FB}\left(v\right) = 1 - \lambda + \frac{\ln\left(-\rho v\right)}{\alpha} + \frac{\kappa^2}{2\alpha\rho}.$$

Unlike the contract under ambiguity described in Theorem 3, the first-best contract does not offer full insurance to the agent if he has an exogenously distorted pessimistic belief. In particular, when there is a good shock to the output process, the agent should receive a lower wage in order to insure against a future bad shock. The principal benefits from the partial insurance in that his value is raised by $\kappa^2/\left(2\alpha\rho^2\right)$. The intuition is that the agent pessimistically believe that worse output is more likely and hence he should receive higher wages in bad times and lower wages in good times.

Why is the solution different under ambiguity? The distortion $\kappa\sigma\gamma$ is replaced by $\kappa\sigma\left|\gamma\right|$ in (49) for the case of ambiguity. If we assume $\gamma > 0$, the HJB equation under ambiguity is the same as (49). But the first-order condition gives $\gamma = \kappa v/\sigma < 0$. Similarly, if we assume $\gamma < 0$, the distortion $\kappa\sigma\gamma$ is replaced by $-\kappa\sigma\gamma$ in (49) and the first-order condition gives a positive γ. Both will lead to a contradiction. Thus the optimal γ must be zero under ambiguity. This implies that the optimal contract must fully insure the ambiguity-averse agent.

6 Conclusion

We have introduced ambiguity into the model of Prat and Jovanovic (2014). Our key insight is that there is a tradeoff between incentives and ambiguity sharing. This tradeoff is similar to, but distinct from the usual incentives and risk tradeoff. The risk-neutral Bayesian principal wants to transfer uncertainty from the ambiguity-averse agent by lowering pay-performance sensitivity. Ambiguity delays incentive provision and causes the expected wages to be smoother over time. When the level of ambiguity is sufficiently large, the principal fully insures the agent by allowing the agent to shirk forever.

As Hansen and Sargent (2012) point out, in a multi-agent framework, it is important to consider who faces ambiguity and what he is ambiguous about. Miao and Rivera (2015) study a problem in which the principal is ambiguity averse, but the agent is not. One might consider the case where both the principal and the agent are ambiguity averse. Of course different modeling assumptions apply to different economic problems. There are several different approaches to modeling ambiguity in the literature. Depending on the tractability and economic problems at hand, one may choose an approach that is convenient for delivering interesting economic implications.

Appendix

A Appendix: Proof

Proof of Theorem 1: Let

$$Y_t = \int_0^t \sigma d\bar{B}_s$$

be a Brownian motion under $\left(\Omega, \mathcal{F}, \bar{P}, \left\{\mathcal{F}_t^Y\right\}\right)$. Define

$$\Lambda_{t,\tau}^a \triangleq \exp\left(\int_t^\tau \frac{\widehat{\eta}(Y_s - A_s, s) + a_s}{\sigma} d\bar{B}_s - \frac{1}{2}\int_t^\tau \left|\frac{\widehat{\eta}(Y_s - A_s, s) + a_s}{\sigma}\right|^2 ds\right).$$

Since $Y_t = \sigma\bar{B}_t$, $|Y_t| \leq \sigma \left\|\bar{B}\right\|_t$, where $\left\|\bar{B}\right\|_t \triangleq \max\limits_{0\leq s\leq t}\left|\bar{B}_s\right|$. Since h_t, a_t and A_t are bounded for $t \in [0, T]$, we can find a constant C such that

$$\left|\frac{\widehat{\eta}(Y_t - A_t, t) + a_t}{\sigma}\right| = \left|\frac{h_0 m_0 + \sigma^{-2}(Y_t - A_t) + h_t a_t}{\sigma h_t}\right| \leq C(1 + \left\|\bar{B}\right\|_t)$$

for all $t \in [0, T]$. Then it follows from Karatzas and Shreve (1991, Corollary 5.16, p. 200) that $\Lambda_{t,\tau}^a$ is a martingale for $\tau \in [t, T]$ with $E_t\left[\Lambda_{t,\tau}^a\right] = 1$ and hence the Girsanov theorem and (4) ensure that

$$\hat{B}_t^a = \bar{B}_t - \int_0^t \frac{\widehat{\eta}(Y_s - A_s, s) + a_s}{\sigma} ds$$

is a Brownian motion under the new measure $dP^a/d\bar{P} = \Lambda_{0,T}^a$.

Consider the following forward and backward stochastic differential equations (FBSDE) for $(\widehat{\eta}_t^a, v_t^a, Z_t^a)$ under $(\bar{P}, \bar{B}_t)$:

$$d\widehat{\eta}_t^a = \frac{\sigma^{-1}}{h_t} d\hat{B}_t^a = -\frac{\sigma^{-2}}{h_t}(\widehat{\eta}_t^a + a_t)dt + \frac{\sigma^{-1}}{h_t} d\bar{B}_t, \tag{A.1}$$

$$dv_t^a = \left[\rho v_t^a - u(w_t, a_t) + \kappa\left|Z_t^a\right| - \sigma^{-1} Z_t^a(\widehat{\eta}_t + a_t)\right] dt + Z_t^a d\bar{B}_t, \tag{A.2}$$

with $\widehat{\eta}_0^a = m_0$ and $v_T^a = U(W_T)$. Then $\widehat{\eta}\left(Y_t - A_t, t\right)$ in (2) and v_t^a in (7) are the solutions to (A.1) and (A.2) with $Z_t^a = \sigma\gamma_t^a$. Thus the agent's problem (9) is equivalent to the following control problem:

$$\max_{a\in\mathcal{A}} v_0^a \tag{A.3}$$

subject to the FBSDE (A.1) and (A.2) for $(\widehat{\eta}_t^a, v_t^a)$. Note that (w_t, W_T) are not control variables so that we fix them in the proof and ignore their effects on the solution.

Since $\widehat{\eta}_t$ is unbounded, the Lipschitz condition for the existence of a solution to BSDE (A.2) is violated. We thus consider the following transformation. Define

$$\bar{B}_t^0 = \bar{B}_t - \int_0^t \frac{h_0 m_0 + \sigma^{-2} Y_s}{\sigma h_s} ds$$

and

$$\Lambda_{t,\tau}^0 \triangleq \exp\left(\int_t^\tau \frac{h_0 m_0 + \sigma^{-2} Y_s}{\sigma h_s} d\bar{B}_s - \frac{1}{2} \int_t^\tau \left| \frac{h_0 m_0 + \sigma^{-2} Y_s}{\sigma h_s} \right|^2 ds \right).$$

As in the proof above, $\Lambda_{t,\tau}^0$ is a martingale with $E_t\left[\Lambda_{t,\tau}^0\right] = 1$. The Girsanov theorem implies that $\bar{B}_t^0$ is a Brownian motion under a new measure $\bar{P}^0$ defined by $d\bar{P}^0/d\bar{P} = \Lambda_{0,T}^0$.

Under $(\bar{P}^0, \bar{B}_t^0)$, the FBSDEs (A.1) and (A.2) become

$$dA_s = a_s ds, \tag{A.4}$$

$$dv_t^a = \left[\rho v_t^a - u(w_t, a_t) + \kappa \left| Z_t^a \right| - \sigma^{-1} Z_t^a \left(\frac{-\sigma^{-2} A_t}{h_t} + a_t \right) \right] dt + Z_t^a d\bar{B}_t^0, \tag{A.5}$$

with $A_0 = 0$ and $v_T^a = U(W_T)$. The coefficient of Z_t^a is bounded on $t \in [0, T]$ and the Lipschitz condition is satisfied.

Let a be the optimal solution to (A.3) subject to (A.4) and (A.5). We can check that $\mathcal{A}$ is convex. For any $\tilde{a} \in \mathcal{A}$, define $\Delta a = \tilde{a} - a$ and $a^\epsilon \triangleq (1-\epsilon)a + \epsilon \tilde{a} \in \mathcal{A}$. For convenience we define functions

$$f_1(t, A_t, v_t, z_t, a_t) \triangleq -\rho v_t + u(w_t, a_t) + \frac{z_t}{\sigma} \left(\frac{-\sigma^{-2} A_t}{h_t} + a_t \right), \quad f_2(z_t) \triangleq -\kappa \left| z_t \right|.$$

Clearly, f_2 is not a differentiable function.

Let $(\nabla A, \nabla v, \nabla Z)$ solve the following linear FBSDE:

$$\begin{cases} \nabla A_t = \int_0^t \Delta a_s ds, \\ \nabla v_t = \int_t^T [(\partial_A f_1(s) \nabla A_s + \partial_v f_1(s) \nabla v_s + \partial_z f_1(s) \nabla Z_s + \partial_a f_1(s) \Delta a_s) + \partial_z f_2(Z_s^a, \nabla Z_s) \nabla Z_s] ds \\ \qquad - \int_t^T \nabla Z_s d\bar{B}_s^0, \end{cases} \tag{A.6}$$

where we can compute derivatives as

$$\begin{aligned}
\partial_A f_1(s) &\triangleq \partial_A f_1(s, A_s^a, v_s^a, Z_s^a, a_s) = \frac{-\sigma^{-2} Z_s^a}{h_s \sigma}, \\
\partial_v f_1(s) &\triangleq \partial_v f_1(s, A_s^a, v_s^a, Z_s^a, a_s) = -\rho, \\
\partial_z f_1(s) &\triangleq \partial_z f_1(s, A_s^a, v_s^a, Z_s^a, a_s) = \frac{1}{\sigma} \left(\frac{-\sigma^{-2} A_s}{h_s} + a_s \right), \\
\partial_a f_1(s) &\triangleq \partial_a f_1(s, A_s^a, v_s^a, Z_s^a, a_s) = u_a(w_s, a_s) + \frac{Z_s^a}{\sigma}.
\end{aligned}$$

We define

$$\partial_z f_2(Z_s^a, z) \triangleq \begin{cases} -\kappa & \text{if } Z_s^a > 0 \\ \kappa & \text{if } Z_s^a < 0 \\ -\kappa \cdot sgn(z) & \text{if } Z_s^a = 0 \end{cases}, \tag{A.7}$$

where

$$sgn(x) \triangleq \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then

$$\partial_z f_2(Z_s^a, \nabla Z_s)\nabla Z_s = \begin{cases} -\kappa \nabla Z_s & \text{if } Z_s^a > 0 \\ \kappa \nabla Z_s & \text{if } Z_s^a < 0 \\ -\kappa \left|\nabla Z_s\right| & \text{if } Z_s^a = 0 \end{cases}.$$

Note that the function $f_2(z_t) \triangleq -\kappa \left|z_t\right|$ has a link at zero. At this point we construct "derivative" in a special way as in (A.7). We will show later that this way allows us to get convergence when performing perturbations.

From the construction of FBSDEs (A.4), (A.5), and (A.6), we can see that the drift terms are uniformly Lipschitz continuous in (v, Z) and $(\nabla v, \nabla Z)$. We need the following integrability assumption to ensure that these two systems are well-posed.

Assumption 1 *Suppose that any admissible contract $(a, w, W_T) \in \mathcal{C}$ satisfies $E^{\bar{P}^0}[U(W_T)^2] < \infty$ and*

$$E^{\bar{P}^0}\left[\left(\int_0^T \left(\left|u(w_t, a_t)\right| + \left|u_a(w_t, a_t)\right|\right) dt\right)^2\right] < \infty.$$

By this assumption, we have $E^{\bar{P}^0}\left[\left(\int_0^T \left|u(w_t, a_t)\right| dt\right)^2\right] < \infty$. Since BSDE (A.5) satisfies the uniform Lipschitz continuity condition in (v, Z), we deduce that this BSDE is well-posed. Using the preceding expressions of derivatives,

$$E^{\bar{P}^0}\left(\int_0^T \left|Z_s^a\right|^2 ds\right) < \infty,$$

and Assumption 1, we can show that

$$E^{\bar{P}^0}\left[\left(\int_0^T \left(\left|\partial_A f_1(t)\nabla A_t\right| + \left|\partial_a f_1(t)\Delta a_t\right|\right) dt\right)^2\right] < \infty. \tag{A.8}$$

Since BSDE (A.6) satisfies the uniform Lipschitz continuity condition in $(\nabla v, \nabla Z)$, condition (A.8) ensures that (A.6) is well posed.

Define $(A^{a^\epsilon}, v^{a^\epsilon}, Z^{a^\epsilon})$ as the solution to FBSDEs (A.4) and (A.5) with respect to effort a^ϵ. Define

$$\begin{aligned}\Delta A^\epsilon &\triangleq A^{a^\varepsilon} - A^a,\ \Delta v^\epsilon \triangleq v^{a^\epsilon} - v^a,\ \Delta Z^\epsilon \triangleq Z^{a^\epsilon} - Z^a, \\ \nabla A^\epsilon &\triangleq \frac{\triangle A^\epsilon}{\epsilon},\ \nabla v^\epsilon \triangleq \frac{\triangle v^\epsilon}{\epsilon},\ \nabla Z^\epsilon \triangleq \frac{\triangle Z^\epsilon}{\epsilon}.\end{aligned}$$

We can check that $\nabla A = \nabla A^\epsilon$. Since $\triangle A_t^\epsilon = \epsilon \nabla A_t$ and ∇A_t are bounded, we have

$$\lim_{\epsilon\to 0} E^{\bar{P}^0}\left[\sup_{0\le t\le T} |\triangle A_t^\epsilon|^2\right] = 0.$$

We need $u(w_t, a_t^\epsilon)$ and $\partial_a u(w_t, a_t^\epsilon)$ to be square integrable uniformly in $\epsilon \in [0,1]$.

Assumption 2 *Suppose that*

$$\lim_{R\to\infty} \sup_{\epsilon\in[0,1]} E^{\bar{P}^0}\left[\left(\int_0^T [|u(w_t, a_t^\epsilon)|\, \mathbf{1}_{\{|u(w_t,a_t^\epsilon)|>R\}} + |\partial_a u(w_t, a_t^\epsilon)|\, \mathbf{1}_{\{|\partial_a u(w_t,a_t^\epsilon)|>R\}}]dt\right)^2\right] = 0,$$

where $\mathbf{1}$ *is an indicator function.*

By this assumption, we can check that

$$\lim_{R\to\infty} \sup_{\epsilon\in[0,1]} E^{\bar{P}^0}\{(\int_0^T [|f_1(t, A_t^\epsilon, 0, 0, a_t^\epsilon)|\, 1_{\{|f_1(t,A_t^\epsilon,0,0,a_t^\epsilon)|>R\}}$$

$$+ |\partial_a f_1(t, A_t^\epsilon, 0, 0, a_t^\epsilon)|\, \mathbf{1}_{\{|\partial_a f_1(t,A_t^\epsilon,0,0,a_t^\epsilon)|>R\}}]dt)^2\} = 0.$$

We then obtain the following important lemma:

Lemma 11 *Suppose that Assumption 2 holds. Then*

$$\lim_{\epsilon\to 0} E^{\bar{P}^0}\left\{\sup_{0\le t\le T} |\triangle v_t^\epsilon|^2 + \int_0^T |\triangle Z_t^\epsilon|^2\, dt\right\} = 0, \tag{A.9}$$

$$\lim_{\epsilon\to 0} E^{\bar{P}^0}\left\{\sup_{0\le t\le T} \left(|\nabla v_t^\epsilon - \nabla v_t|^2\right) + \int_0^T |\nabla Z_t^\epsilon - \nabla Z_t|^2\, dt\right\} = 0. \tag{A.10}$$

Proof. Since f_1 is continuous in a, we have $a_t^\epsilon \to a_t$ and $f_1(t, A_t^\epsilon, 0, 0, a_t^\epsilon) \to f_1(t, A_t, 0, 0, a_t)$ as $\epsilon \to 0$. Then the square uniform integrability assumption 2 implies that

$$\lim_{\epsilon\to 0} E^{\bar{P}^0}\left[\left(\int_0^T |f_1(t, A_t^\epsilon, 0, 0, a_t^\epsilon) - f_1(t, A_t, 0, 0, a_t)|\, dt\right)^2\right] = 0.$$

Also we have $f_1(t, A_t^\epsilon, y, z, a_t^\epsilon) \to f_1(t, A_t, y, z, a_t)$ as $\epsilon \to 0$ for all (y, z). Then applying Theorem 9.4.3 in Cvitanic and Zhang (2013) to (A.5), we obtain (A.9).

We can check that

$$\begin{aligned}
\nabla v_t^\epsilon &= \frac{v_t^{a^\epsilon} - v_t^a}{\epsilon} \\
&= \int_t^T [\partial_A f_1^\epsilon(s)\nabla A_s^\epsilon + \partial_v f_1^\epsilon(s)\nabla v_s^\epsilon + \partial_z f_1^\epsilon(s)\nabla Z_s^\epsilon + \partial_a f_1^\epsilon(s)\Delta a + \partial_z f_2^\epsilon(s, \nabla Z_s^\epsilon)\nabla Z_s^\epsilon]ds \\
&\quad - \int_t^T \nabla Z_s^\epsilon d\bar{B}_s^0
\end{aligned}$$

where we have used the following notation

$$\partial_A f_1^\epsilon(s) \triangleq \int_0^1 \partial_A f_1(s, A_s^a + \theta\Delta A_s^\epsilon, v_s^a + \theta\Delta v_s^\epsilon, Z_s^a + \theta\Delta Z_s^\epsilon, a_s + \theta\epsilon\Delta a_t)d\theta$$

and similar notations apply to $\partial_v f_1^\epsilon(s)$, $\partial_z f_1^\epsilon(s)$, and $\partial_a f_1^\epsilon(s)$. For $\partial_z f_2^\epsilon(s, \nabla Z_s^\epsilon)$, we define

$$\partial_z f_2^\epsilon(s,z) = \begin{cases} \int_0^1 \partial_z f_2(Z_s^a + \theta\Delta Z_s^\epsilon)d\theta & \text{if } Z_s^a \neq 0 \\ -\kappa \cdot sgn(z) & \text{if } Z_s^a = 0 \end{cases}.$$

Then we have

$$\partial_z f_2^\epsilon(s, \nabla Z_s^\epsilon)\nabla Z_s^\epsilon = \begin{cases} \int_0^1 \partial_z f_2(Z_s^a + \theta\Delta Z_s^\epsilon)d\theta \; \nabla Z_s^\epsilon & \text{if } Z_s^a \neq 0 \\ -\kappa\left|\nabla Z_s^\epsilon\right| = -\kappa\frac{\left|Z_s^{a^\epsilon}\right|}{\epsilon} & \text{if } Z_s^a = 0 \end{cases}.$$

Note that for $Z_s^a \neq 0$, the Lebesgue integral $\int_0^1 \partial_z f_2(Z_s^a + \theta\Delta Z_s^\epsilon)d\theta$ does not change value when there is a $\overline{\theta} \in [0,1]$ such that $Z_s^a + \overline{\theta} \bigtriangleup Z_s^\epsilon = 0$. By (A.9), we have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\partial_A f_1(s, A_s^a + \theta\Delta A_s^\epsilon, v_s^a + \theta\Delta v_s^\epsilon, Z_s^a + \theta\Delta Z_s^\epsilon, a_s + \theta\epsilon\Delta a_s) \\
= \;& \partial_A f_1(s, A_s^a, v_s^a, Z_s^a, a_s), \text{ a.s.}
\end{aligned}$$

Since $\theta, \epsilon \in [0,1]$, and by (A.9), we can apply the dominated convergence theorem to show that

$$\lim_{\epsilon\to 0}\partial_A f_1^\epsilon(s) = \partial_A f_1(s), \text{ a.s.}$$

We can derive similar limits for $\partial_v f_1^\epsilon(s)$ and $\partial_z f_1^\epsilon(s)$. We can also show that

$$\begin{aligned}
\lim_{\epsilon\to 0}\partial_z f_2^\epsilon(s,z) &= \partial_z f_2(Z_s^a, z), \text{ when } Z_s^a \neq 0, \\
\partial_z f_2^\epsilon(s,z)z = \partial_z f_2(Z_s^a, z)z &= -\kappa\left|z\right| \text{ for any } z,\ \epsilon > 0 \text{ when } Z_s^a = 0.
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \partial_A f_1^\epsilon(s) \nabla A_s^\epsilon + \partial_v f_1^\epsilon(s) v + \partial_z f_1^\epsilon(s) z + \partial_a f_1^\epsilon(s) \Delta a_s + \partial_z f_2^\epsilon(s, z) z \\
= \; & \partial_A f_1(s) \nabla A_s + \partial_v f_1(s) v + \partial_z f_1(s) z + \partial_a f_1(s) \Delta a_s + \partial_z f_2(Z_s^a, z) z
\end{aligned} \tag{A.11}
$$

for all constants v and z, almost surely under $dt \otimes d\bar{P}$. By the uniform square integrability assumption 2, $\lim_{\epsilon \to 0} \partial_a f_1^\epsilon(t) = \partial_a f_1(t)$ and (A.9), we have

$$
\lim_{\epsilon \to 0} E^{\bar{P}^0} \left[\left(\int_0^T |\partial_a f_1^\epsilon(t) - \partial_a f_1(t)| \cdot |\triangle a_t| \, dt \right)^2 \right] = 0. \tag{A.12}
$$

Also

$$
|\partial_A f_1^\epsilon(s) \nabla A_s^\epsilon - \partial_A f_1(s) \nabla A_s| = \frac{\sigma^{-2}}{\sigma h_t} \cdot \left| \frac{1}{2} \Delta Z_s^\epsilon \right| \cdot |\nabla A_s| \, .
$$

Since ∇A_s is bounded, we use (A.9) to derive

$$
\lim_{\epsilon \to 0} E^{\bar{P}^0} \left[\int_0^T \left(|\partial_A f_1^\epsilon(s) \nabla A_s^\epsilon - \partial_A f_1(s) \nabla A_s|^2 \right) dt \right] = 0. \tag{A.13}
$$

Finally combining (A.11), (A.12) and (A.13) and applying Theorem 9.4.3 in Cvitanic and Zhang (2013), we obtain

$$
\lim_{\epsilon \to 0} E^{\bar{P}^0} \left[\sup_{0 \le t \le T} (|\nabla v_t^\epsilon - \nabla v_t|^2) + \int_0^T |\nabla Z_t^\epsilon - \nabla Z_t|^2 \, dt \right] = 0.
$$

This completes the proof of the lemma. ■

Now we continue the proof of Theorem 1. In order to solve the linear FBSDE (A.6), we introduce the following adjoint process:

$$
\begin{aligned}
\Gamma_t^a &= 1 + \int_0^t \alpha_s ds + \int_0^t \beta_s d\bar{B}_s^0, \\
\overline{Y}_t^a &= \int_t^T \zeta_s ds - \int_t^T \overline{Z}_s^a d\bar{B}_s^0,
\end{aligned}
$$

where α, β and ζ will be determined later. Applying Ito's lemma, we have

$$\begin{aligned}
& d(\Gamma_t^a \nabla v_t - \overline{Y}_t^a \nabla A_t) \\
= \; & [\ldots] d\bar{B}_t^0 + \{-\Gamma_t^a[(\partial_A f_1(t)\nabla A_t + \partial_v f_1(t)\nabla v_t \\
& +(\partial_z f_1(t) + \partial_z f_2(Z_t^a, \nabla Z_t))\nabla Z_t + \partial_a f_1(t)\Delta a_t)] \\
& +\alpha_t \nabla v_t + \beta_t \nabla Z_t - \overline{Y}_t^a\,(\triangle a_t) + \zeta_t \nabla A_t\} dt \\
= \; & [\ldots] d\bar{B}_t^0 + \{[\nabla A_t[-\Gamma_t^a \partial_A f_1(t) + \zeta_t] + \nabla v_t[-\Gamma_t^a \partial_v f_1(t) + \alpha_t] \\
& +\nabla Z_t[-\Gamma_t^a(\partial_z f_1(t) + \partial_z f_2(Z_t^a, \nabla Z_t)) + \beta_t] - \triangle a_t[\Gamma_t^a \partial_a f_1(t) + \overline{Y}_t^a]\} dt.
\end{aligned}$$

We set

$$-\Gamma_t^a \partial_A f_1(t) + \zeta_t = -\Gamma_t^a \partial_v f_1(t) + \alpha_t = -\Gamma_t^a(\partial_z f_1(t) + \partial_z f_2(Z_t^a, \nabla Z_t)) + \beta_t = 0.$$

That is, we define

$$\zeta_t \triangleq \Gamma_t^a \partial_A f_1(t), \;\; \beta_t \triangleq \Gamma_t^a(\partial_z f_1(t) + \partial_z f_2(Z_t^a, \nabla Z_t)), \;\; \alpha_t \triangleq \Gamma_t^a \partial_v f_1(t).$$

Then we have

$$\begin{aligned}
\Gamma_t^a &= 1 + \int_0^t \Gamma_s^a \partial_v f_1(s, A_s^a, v_s^a, Z_s^a, a_s) ds \\
&\quad + \int_0^t \Gamma_s^a(\partial_z f_1(s, A_s^a, v_s^a, Z_s^a, a_s) + \partial_z f_2(Z_s^a, \nabla Z_s)) d\bar{B}_s^0, \\
\overline{Y}_t^a &= \int_t^T \Gamma_s^a \partial_A f_1(s, A_s^a, v_s^a, Z_s^a, a_s) ds - \int_t^T \overline{Z}_s^a d\bar{B}_s^0.
\end{aligned}$$

Note that Γ^a, $\overline{Y}^a$, and $\overline{Z}^a$ depend on a and $\triangle a$ because $\partial_z f_2(Z_s^a, \nabla Z_s)$ depend on a and $\triangle a$. Then we have

$$d(\Gamma_t^a \nabla v_t - \overline{Y}_t^a \nabla A_t) = [\ldots] d\bar{B}_t^0 - \triangle a_t[\Gamma_t^a \partial_a f_1(t) + \overline{Y}_t^a] dt.$$

By standard estimates, we know that the term [...] $d\bar{B}_t^0$ corresponds to a true martingale. Integrating over $[0, T]$ and using $\nabla v_T = 0$, $\nabla \hat{\eta}_0 = 0$, $\Gamma_0^a = 1$, and $\overline{Y}_T^a = 0$, we can show that

$$\nabla v_0 = E^{\bar{P}^0}\left\{\int_0^T \triangle a_t[\Gamma_t^a \partial_a f_1(t) + \overline{Y}_t^a] dt\right\}. \tag{A.14}$$

By (A.10),

$$\nabla v_0 = \lim_{\epsilon \to 0} \nabla v_0^\epsilon \leq 0.$$

Then

$$\nabla v_0 = E^{\bar{P}^0}\left\{\int_0^T \Gamma_t^a \triangle a_t[u_a(w_{t,}a_t) + Z_t^a\sigma^{-1} + \frac{\overline{Y}_t^a}{\Gamma_t^a}]dt\right\} \leq 0.$$

Next we express the term $u_a(w_{t,}a_t) + Z_t^a\sigma^{-1} + \frac{\overline{Y}_t^a}{\Gamma_t^a}$.

We define $Z_t^a\sigma^{-1} = \gamma_t^a$ and compute the last term in the square bracket. Using the derivatives above, we can show that

$$\begin{aligned}\Gamma_t^a &= e^{-\rho t}\exp\left\{\int_0^t \left(\frac{1}{\sigma}\left(\frac{-\sigma^{-2}A_s}{h_s} + a_s\right) + \partial_z f_2(Z_s^a, \nabla Z_s)\right) d\bar{B}_s^0\right.\\ &\qquad \left. -\frac{1}{2}\int_0^t \left|\frac{1}{\sigma}\left(\frac{-\sigma^{-2}A_s}{h_s} + a_s\right) + \partial_z f_2(Z_s^a, \nabla Z_s)\right|^2 ds\right\}\end{aligned}$$

and

$$\overline{Y}_t^a = E_t^{\bar{P}^0}\left[\int_t^T \frac{-\sigma^{-2}Z_s^a}{h_s\sigma}\Gamma_s^a ds\right].$$

Define

$$\Gamma_{t,\tau}^a \triangleq \exp\left(\int_t^\tau \left(\frac{-\sigma^{-2}A_s}{\sigma h_s} + \frac{a_s}{\sigma} + b_s^{c*}\right) dB_s^0 - \frac{1}{2}\int_t^\tau \left|\frac{-\sigma^{-2}A_s}{\sigma h_s} + \frac{a_s}{\sigma} + b_s^{c*}\right|^2 ds\right),$$

where $b_s^{c*} = \partial_z f_2(Z_s^a, \nabla Z_s)$. Then we can verify that $\Gamma_{t,\tau}^a$ is a martingale for $\tau \in [t, T]$ with $E_t\left[\Gamma_{t,\tau}^a\right] = 1$. By the Girsanov theorem,

$$\begin{aligned}\hat{B}_t^{a,b^{c*}} &= \hat{B}_t^a - \int_0^t b_s^{c*} ds = \bar{B}_t - \int_0^t \left[\frac{\widehat{\eta}(Y_s - A_s, s) + a_s}{\sigma} + b_s^{c*}\right] ds\\ &= \bar{B}_t^0 - \int_0^t \left[\frac{-\sigma^{-2}A_s}{\sigma h_s} + \frac{a_s}{\sigma} + b_s^{c*}\right] ds\end{aligned}$$

is a Brownian motion under $dQ^{a,b^{c*}}/d\bar{P}^0 = \Gamma_{0,T}^a$. Note that the measure $Q^{a,b^{c*}}$ is a worst-case belief for the agent since the density generator $b_s^{c*} \in \mathcal{B}^c$. Formally, $\kappa\sigma\left|\gamma_s^a\right| = \kappa\left|Z_s^a\right| = \min_{|b_s|\leq\kappa} b_s Z_s^a = b_s^{c*}Z_s^a$. In particular, when $Z_s^a = 0$, b_s^{c*} can take any value in $[-\kappa, \kappa]$. Our choice of $b_s^{c*} = \partial_z f_2(Z_s^a, \nabla Z_s)$ enables us to perform calculus of variations. But it implies that b_s^{c*} depends on $\widetilde{a}$ because $\partial_z f_2(Z_s^a, \nabla Z_s)$ depends on $\triangle a$.

Now we can compute that

$$\begin{aligned}\frac{\overline{Y}_t^a}{\Gamma_t^a} &= E_t^{\bar{P}^0}\left[\int_t^T \frac{-\sigma^{-2}Z_s^a}{h_s\sigma}\frac{\Gamma_s^a}{\Gamma_t^a}ds\right] = E_t^{\bar{P}^0}\left[\int_t^T \frac{-\sigma^{-2}Z_s^a}{h_s\sigma}e^{-\rho(s-t)}\Gamma_{t,s}^a ds\right]\\ &= -\frac{\sigma^{-2}}{h_t}E_t^{Q^{a,b^{c*}}}\left[\int_t^T e^{-\rho(s-t)}\gamma_s^a\frac{h_t}{h_s}ds\right].\end{aligned}$$

So $\frac{\overline{Y}_t^a}{\Gamma_t^a} = \frac{\sigma^{-2}}{h_t} p_t^{\widetilde{a}}$ where $p_t^{\widetilde{a}} \triangleq h_t E_t^{Q^{a,b^{c*}}} \left[\int_t^T -e^{-\rho(s-t)} \gamma_s^a \frac{1}{h_s} ds\right] \in \mathcal{P}_t^c$. Then we have

$$\nabla v_0 = E^{\bar{P}^0} \left\{ \int_0^T \Gamma_t^a \triangle a_t [u_a(w_t, a_t) + \gamma_t^a + \frac{\sigma^{-2}}{h_t} p_t^{\widetilde{a}}] dt \right\} \leq 0 \text{ for all } \widetilde{a} \in \mathcal{A}.$$

Note that $p_t^{\widetilde{a}}$ depends on the perturbation $\widetilde{a}$. We need to select elements in $\mathcal{P}_t^c$ that are independent of $\widetilde{a}$. Because $\Gamma_t^a > 0$ for all $\widetilde{a} \in \mathcal{A}$, we have

$$E^{\bar{P}^0} \left\{ \int_0^T \Gamma_t^a \triangle a_t [u_a(w_t, a_t) + \gamma_t^a + \frac{\sigma^{-2}}{h_t} \widehat{p}_t^{\widetilde{a}}] dt \right\} \leq \nabla v_0 \leq 0 \text{ for all } \widetilde{a} \in \mathcal{A}, \tag{A.15}$$

where

$$\widehat{p}_t^{\widetilde{a}} = \begin{cases} \underline{p}_t, & \text{if } \widetilde{a}_t \geq a_t; \\ \overline{p}_t, & \text{if } \widetilde{a}_t < a_t, \end{cases}$$

$\overline{p}_t$ and $\underline{p}_t$ are defined in Theorem 1. Then we can see that the condition in (A.15) is equivalent to (15). Q.E.D.

Proof of Lemma 1: We use the notation in the proof of Theorem 1. For given time t, we consider the following FBSDEs for $s \in [t, T]$ when the history of the effort is $\{a_\tau : \tau \in [0, t]\}$:

$$\begin{cases} d\widetilde{A}_s = \widetilde{a}_s ds \\ dv_s^{\widetilde{a}} = \left[\rho v_s^{\widetilde{a}} - u(w_s, \widetilde{a}_s) + \kappa \left|Z_s^{\widetilde{a}}\right| - \sigma^{-1} Z_s^{\widetilde{a}} \left(\frac{-\sigma^{-2}\widetilde{A}_s}{h_s} + \widetilde{a}_s\right)\right] dt + Z_s^{\widetilde{a}} d\bar{B}_s^0 \\ v_T^{\widetilde{a}} = U(W_T) \end{cases}, \tag{A.16}$$

where $\widetilde{A}_t = A_t = \int_0^t a_s ds$, and $\{\widetilde{a}_s\}_{s\in[0,T]} \in \mathcal{A}_t \triangleq \{\widetilde{a} \in \mathcal{A} : \widetilde{a}_\tau = a_\tau,\ \tau \in [0, t]\}$. The preceding system for $s \in [t, T]$ has a unique solution for $\{\widetilde{A}_s, v_s^{\widetilde{a}}, Z_s^{\widetilde{a}}\}_{s\in[t,T]}$. Let us define the output history $Y_0^t \triangleq \{Y_\tau : 0 \leq \tau \leq t\}$. The wage w_t is a function of Y_0^t and is written as $w(Y_0^t)$. Then we consider the control problem

$$\underset{\widetilde{a}\in\mathcal{A}_t}{ess\sup}\ v_t^{(\widetilde{a},w,W_T)}(Y_0^t, A_t)$$

subject to (A.16), for any history (Y_0^t, A_t).

If $\underset{\widetilde{a}\in\mathcal{A}_t}{ess\sup} v_t^{(\widetilde{a},w,W_T)}(Y_0^t, A_t) > v_t^{(a,w,W_T)}(Y_0^t, A_t)$, then we can find some $\widehat{a} \in \mathcal{A}_t$ such that

$$v_t^{(\widehat{a},w,W_T)}(Y_0^t, A_t) \geq v_t^{(a,w,W_T)}(Y_0^t, A_t)$$

and

$$\bar{P}^0 \left(v_t^{(\widehat{a},w,W_T)}(Y_0^t, A_t) > v_t^{(a,w,W_T)}(Y_0^t, A_t) \right) > 0.$$

So we have $v_t^{(\widehat{a},w,W_T)} \geq v_t^{(a,w,W_T)}$ and $\bar{P}^0 \left(v_t^{(\widehat{a},w,W_T)} > v_t^{(a,w,W_T)} \right) > 0$. Then from the comparison theorem of BSDEs we have $v_0^{\widehat{a}} > v_0^a$, contradicting the fact that a is incentive compatible. Thus $\underset{\widetilde{a}\in\mathcal{A}_t}{ess\sup} v_t^{(\widetilde{a},w,W_T)}(Y_0^t, A_t) = v_t^{(a,w,W_T)}(Y_0^t, A_t)$. Q.E.D.

Proof of Lemma 2: Fix the contract $c = (a, w, W_T)$. When $a_s > 0$ for $s \in [t, T]$, from (15) we have $\gamma_s^c \geq -\frac{\sigma^{-2}}{h_s}\overline{p}_s - u_a(w_s,\, a_s)$ for $s \in [t, T]$. We know that if $\gamma_s \equiv 0$ for $s \in [t, T]$, we have $\overline{p}_s \equiv 0$. Let $\gamma_s^c = -\frac{\sigma^{-2}}{h_s}\overline{p}_s - u_a(w_s, a_s) + \Delta_s$, where $\Delta_s \geq 0$ for $s \in [t, T]$. By Theorem 1, $\overline{p}$ satisfies BSDE:

$$\begin{aligned} d\overline{p}_s &= \left[\overline{p}_s\left(\rho + \frac{\sigma^{-2}}{h_s}\right) + \gamma_s^c - \overline{\sigma}_s^p \sigma \overline{b}_s^{c*}\right] dt + \overline{\sigma}_s^p \sigma d\hat{B}_s^a \\ &= \left[\overline{p}_s \rho - u_a(w_s,\, a_s) + \Delta_s - \overline{\sigma}_s^p \sigma \overline{b}_s^{c*}\right] dt + \overline{\sigma}_s^p \sigma d\hat{B}_s^a, \qquad \text{(A.17)} \\ \overline{p}_T &= 0, \end{aligned}$$

where $\overline{b}^{c*}$ is the worst-case density generator associated with $\overline{p}$. We define the negative drift of this BSDE as

$$g(s, \overline{p}, \overline{\sigma}^p) \triangleq -\left[\overline{p}_s\rho - u_a(w_s,\, a_s) + \Delta_s - \overline{\sigma}_s^p \sigma \overline{b}_s^{c*}\right].$$

We then have $g(s,\, 0,\, 0) < 0$ for $s \in [t, T]$ since $u_a(w_s,\, a_s) < 0$ and $-\Delta_s \leq 0$. From the comparison theorem in the BSDE theory, we have $\overline{p}_s \leq 0$ for $s \in [t, T]$. Thus $\gamma_s^c > 0$ and any worst-case density generator b^{c*} must satisfy $b_s^{c*} = -\kappa$ for $s \in [t, T]$. Moreover, p_s defined by this density generator is the unique element in $\mathcal{P}_s^c$ for $s \in [t, T]$. Q.E.D.

Proof of Lemma 3: Define $p_s' = \frac{\sigma^{-2}}{h_s} p_s$. Then we have $p_s' = -E_s^{Q^{a,b^{c*}}}\left[\int_s^T e^{-\rho(v-s)} \gamma_v^c \frac{\sigma^{-2}}{h_v} dv\right]$ for some worst-case density generator b^{c*} by Theorem 1. By Ito's Lemma,

$$dp_s' = (\rho p_s' + \frac{\sigma^{-2}}{h_s}\gamma_s^c) ds + [\ldots] d\hat{B}_s^{a,b^{c*}} = \left[\rho p_s' - \frac{\sigma^{-2}}{h_s}(u_a(w_s, a_s^*) + p_s')\right] ds + [\ldots] d\hat{B}_s^{a,b^{c*}},$$

where we use the binding condition $\gamma_s^c = -u_a(w_s, a_s) - p_s$. From the equation above, we can derive

$$p_s' = E_s^{Q^{a,b^{c*}}}\left[\int_s^T e^{[-\rho(v-s) + \int_s^v \frac{\sigma^{-2}}{h_\tau} d\tau]} u_a(w_v, a_v) \frac{\sigma^{-2}}{h_v} dv\right].$$

Noting that

$$\exp\left(\int_s^v \frac{\sigma^{-2}}{h_\tau} d\tau\right) = \frac{h_v}{h_s},$$

we can show that

$$\begin{aligned} p_s' &= E_s^{Q^{a,b^{c*}}}\left[\int_s^T e^{-\rho(v-s)} \frac{h_v}{h_s} u_a(w_v, a_v)(\frac{\sigma^{-2}}{h_v}) dv\right] \\ &= \frac{\sigma^{-2}}{h_s} E_s^{Q^{a,b^{c*}}}\left[\int_s^T e^{-\rho(v-s)} u_a(w_v, a_v) dv\right]. \end{aligned}$$

Since $u_a(w_s,\, a_s) < 0$, we have $p_s' < 0$ and hence $p_s < 0$. Furthermore, since $\gamma_s^c = -\frac{\sigma^{-2}}{h_s}p_s - u_a(w_s,$ $a_s)$ by assumption, we have $\gamma_s^c > 0$ for $s \in [t, T]$. Thus the worst-case density satisfies $b_s^{c*} = -\kappa$ for $s \in [t, T]$ so that there is only one element p_s in $\mathcal{P}_s^c$ for $s \in [t, T]$. We then obtain

$$p_s = E_s^{Q^{a,b^{c*}}} \left[\int_s^T e^{-\rho(\tau - s)} u_a(w_\tau, a_\tau) d\tau \right],$$

which is (18). Q.E.D.

Proof of Theorem 2: For notation convenience, we use $\{a_t^*\}_{t=0}^T$ to denote an incentive compatible effort process. We also attach an asterisk to any stochastic process associated with the incentive compatible contract. Given any effort path $\{a_t\}_{t=0}^T \in \mathcal{A}$, define $\delta_t \equiv a_t - a_t^*$ and $\Delta_t \equiv \int_0^t \delta_s ds = A_t - A_t^*$. We can show that

$$\begin{aligned} dY_t &= \left(\widehat{\eta}(Y_t - A_t^*, t) + a_t^*\right) dt + \sigma d\hat{B}_t^{a^*} \\ &= \left(\widehat{\eta}(Y_t - A_t^*, t) + a_t^*\right) dt + \sigma \left(d\hat{B}_t^{a^*, b^*} + b_t^* dt \right), \end{aligned} \tag{A.18}$$

where $\{b_t^* = -\kappa \cdot sgn\left(\gamma_t^{a^*}\right) : t \geq 0\} \in \mathcal{B}^c$ is some worst-case density generator associated with the optimal effort a^*. Fix this generator and use the Girsanov theorem to deduce that $\hat{B}^{a,b^*}$ defined by

$$d\hat{B}_t^{a,b^*} = d\hat{B}_t^a - b_t^* dt$$

is a standard Brownian motion under the measure Q^{a,b^*} defined by

$$\frac{dQ^{a,b^*}}{dP^a} = \exp\left(\int_0^T b_s^* d\hat{B}_t^a - \frac{1}{2} \int_0^T |b_s^*|^2 \, ds \right).$$

Then we obtain

$$\begin{aligned} dY_t &= \left(\widehat{\eta}(Y_t - A_t, t) + a_t\right) dt + \sigma d\hat{B}_t^a \\ &= \left(\widehat{\eta}(Y_t - A_t, t) + a_t\right) dt + \sigma \left(d\hat{B}_t^{a,b^*} + b_t^* dt \right). \end{aligned} \tag{A.19}$$

Using (A.18) and (A.19), we can show that

$$\begin{aligned} \sigma d\hat{B}_t^{a^*, b^*} &= \sigma d\hat{B}_t^{a,b^*} + [\widehat{\eta}(Y_t - A_t, t) + a_t - \widehat{\eta}(Y_t - A_t^*) - a_t^*] dt \\ &= \sigma d\hat{B}_t^{a,b^*} + \left(\delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t \right) dt. \end{aligned} \tag{A.20}$$

Define

$$v_0^{a,b^*} = E^{Q^{a,b*}} \left[\int_0^T e^{-\rho t} u(w_t, a_t) dt + e^{-\rho T} U(W_T) \right]$$

for any effort level a. Then $v_0^{a^*} = v_0^{a^*,b^*}$ since b^* is the worst-case density generator associated with the optimal effort process a^*.

We consider the total reward from the optimal strategy a^*

$$I^{a^*}(T) = \int_0^T e^{-\rho t} u(w_t, a_t^*) dt + e^{-\rho T} U(W_T).$$

From the martingale representation theorem, there exists a process ϕ^* such that

$$\begin{aligned} I^{a^*}(T) &= v_0^{a^*} + \int_0^T \phi_t^* \sigma d\hat{B}_t^{a^*,b^*} \\ &= v_0^{a^*} + \int_0^T \phi_t^* \left[\delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t \right] dt + \int_0^T \phi_t^* \sigma d\hat{B}_t^{a,b^*}, \end{aligned}$$

where we have substituted (A.20) into the equation above. By the BSDE for v^{a^*} in (7) with $a = a^*$, we have $\phi_t^* = e^{-\rho t} \gamma_t^{a^*}$. Hence, the total reward from the arbitrary policy is given by

$$\begin{aligned} I^a(T) &= \int_0^T [e^{-\rho t} u(w_t, a_t) - e^{-\rho t} u(w_t, a_t^*)] dt + I^{a^*}(T) \\ &= \int_0^T [e^{-\rho t} u(w_t, a_t) - e^{-\rho t} u(w_t, a_t^*)] dt + v_0^{a^*} \\ &\quad + \int_0^T \phi_t^* \left[\delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t \right] dt + \int_0^T \phi_t^* \sigma d\hat{B}_t^{a,b^*}. \end{aligned}$$

We can show that

$$\begin{aligned} -\int_0^T \phi_t^* \frac{\sigma^{-2}}{h_t} \Delta_t dt &= -\int_0^T \phi_t^* \frac{\sigma^{-2}}{h_t} \left(\int_0^t \delta_s ds \right) dt \\ &= \int_0^T \delta_t \left(-\int_t^T \phi_s^* \frac{\sigma^{-2}}{h_s} ds \right) dt \\ &= \int_0^T \delta_t \left(-\int_t^T e^{-\rho s} \gamma_s^{a^*} \frac{\sigma^{-2}}{h_s} ds \right) dt. \end{aligned}$$

Since $E_t^{Q^{a^*,b^*}} \left[-\int_0^T e^{-\rho s} \gamma_s^{a^*} \frac{\sigma^{-2}}{h_s} ds \right]$ is a martingale and from the martingale representation theorem and definition (23), there exists a predictable process ξ^* such that

$$\begin{aligned} \int_t^T -e^{-\rho s} \gamma_s^{a^*} \frac{\sigma^{-2}}{h_s} ds &= E_t^{Q^{a^*,b^*}} \left[-\int_t^T e^{-\rho s} \gamma_s^{a^*} \frac{\sigma^{-2}}{h_s} ds \right] + \int_t^T \xi_s^* \sigma d\hat{B}_t^{a^*,b^*} \\ &= e^{-\rho t} \frac{\sigma^{-2}}{h_t} p_t^* + \int_t^T \xi_s^* \sigma d\hat{B}_t^{a^*,b^*} \\ &= e^{-\rho t} \frac{\sigma^{-2}}{h_t} p_t^* + \int_t^T \xi_s^* \left(\sigma d\hat{B}_t^{a,b^*} + \left(\delta_t - \frac{\sigma^{-2}}{h_t} \Delta_t \right) dt \right), \end{aligned}$$

where we have substituted (A.20) into the last equation. Using the equations above, we can derive that

$$\begin{aligned}
v_0^{a,b^*} - v_0^{a^*} &= E^{Q^{a,b^*}}[I^a(T)] - v_0^{a^*} \\
&= E^{Q^{a,b^*}}\left[\int_0^T \left(e^{-\rho t}u(w_t,a_t) - e^{-\rho t}u(w_t,a_t^*) + \delta_t\left(\phi_t^* + e^{-\rho t}\frac{\sigma^{-2}}{h_t}p_t^*\right)\right)dt\right] \\
&\quad + E^{Q^{a,b^*}}\left[\int_0^T \delta_t\left(\int_t^T \xi_s[\delta_s - \frac{\sigma^{-2}}{h_s}\Delta_s]ds\right)dt\right] \\
&= E^{Q^{a,b^*}}\left[\int_0^T (e^{-\rho t}u(w_t,a_t) - e^{-\rho t}u(w_t,a_t^*) + \delta_t(e^{-\rho t}\gamma_t^{a^*} + e^{-\rho t}\frac{\sigma^{-2}}{h_t}p_t^*))dt\right] \\
&\quad + E^{Q^{a,b^*}}\left[\int_0^T \xi_t^*\Delta_t\left(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t\right)dt\right].
\end{aligned}$$

Since $u(w_t,a_t) - u(w_t,a_t^*) \leq u_a(w_t,a_t^*)\,\delta_t$ by the concavity of u and since a^* satisfies the necessary condition (15), the first expectation is at most equal to zero.

Next we define the predictable process $\chi_t^* \triangleq \gamma_t^{a^*} - e^{\rho t}\xi_t^* A_t^*$ and define the Hamiltonian function:

$$\mathcal{H}(t,a,A;\chi^*,\xi^*,p^*) \triangleq u(w,a) + (\chi^* + e^{\rho t}\xi^* A)a - e^{\rho t}\xi^*\frac{\sigma^{-2}}{h_t}A^2 + \frac{\sigma^{-2}}{h_t}p^* a.$$

We may simply write it as $\mathcal{H}_t(a,A)$. Then we have

$$\begin{aligned}
&\mathcal{H}_t(a_t,A_t) - \mathcal{H}_t(a_t^*,A_t^*) - \partial_A\mathcal{H}_t(a_t^*,A_t^*)\Delta_t \\
= \; & e^{\rho t}\left[e^{-\rho t}u(w_t,a_t) - e^{-\rho t}u(w_t,a_t^*) + \delta_t(\phi_t^* + e^{-\rho t}\frac{\sigma^{-2}}{h_t}p_t^*) + \xi_t^*\Delta_t(\delta_t - \frac{\sigma^{-2}}{h_t}\Delta_t)\right].
\end{aligned}$$

Thus

$$v_0^{a,b^*} - v_0^{a^*} = E^{Q^{a,b^*}}\left[\int_0^T e^{-\rho t}(\mathcal{H}_t(a_t,A_t) - \mathcal{H}_t(a_t^*,A_t^*) - \partial_A\mathcal{H}_t(a_t^*,A_t^*)\Delta_t)dt\right].$$

If $\mathcal{H}_t(a,A)$ is jointly concave in (a,A) for any (t,χ^*,ξ^*,p^*), we can use condition (15) to show that

$$\begin{aligned}
&\mathcal{H}_t(a_t,A_t) - \mathcal{H}_t(a_t^*,A_t^*) - \partial_A\mathcal{H}_t(a_t^*,A_t^*)\Delta_t \\
\leq \; & \mathcal{H}_t(a_t,A_t) - \mathcal{H}_t(a_t^*,A_t^*) - \partial_A\mathcal{H}_t(a_t^*,A_t^*)\Delta_t - \partial_a\mathcal{H}_t(a_t^*,A_t^*)\delta_t \\
\leq \; & 0
\end{aligned}$$

for (t,χ^*,ξ^*,p^*). Then $v_0^{a,b^*} - v_0^{a^*} \leq 0$ and we have

$$v_0^{a^*} \geq v_0^{a,b^*} \geq \inf_{Q^{a,b}\in\mathcal{P}^{\Theta_a}} v_0^{a,b} = v_0^a.$$

We then conclude that the strategy a^* is the optimal control.

The Hessian matrix of $\mathcal{H}_t(a, A)$ is

$$\begin{pmatrix} u_{aa}(w_t, a_t^*) & e^{\rho t}\xi_t^* \\ e^{\rho t}\xi_t^* & -2e^{\rho t}\xi_t^* \frac{\sigma^{-2}}{h_t} \end{pmatrix}.$$

The joint concavity of $\mathcal{H}_t(a, A)$ at (a^*, A^*) is equivalent to the negative semidefiniteness of the Hessian matrix of $\mathcal{H}_t(\cdot)$ at (a_t^*, A_t^*), which is equivalent to $u_{aa}(w_t, a_t^*) \leq 0$ and

$$\begin{vmatrix} u_{aa}(w_t, a_t^*) & e^{\rho t}\xi_t^* \\ e^{\rho t}\xi_t^* & -2e^{\rho t}\xi_t^* \frac{\sigma^{-2}}{h_t} \end{vmatrix} \geq 0,$$

or

$$-2\frac{\sigma^{-2}}{h_t} u_{aa}(w_t, a_t^*) \geq e^{\rho t}\xi_t^* \geq 0.$$

as stated in (22). Note that in the statement of the theorem we use a to denote the optimal strategy instead of a^*. Q.E.D.

Proof of Theorem 3: In the first-best case there is no incentive constraint and the HJB equation is

$$\rho J_{FB}(v) = \sup_{a,w,\gamma} a - w + J'_{FB}(v)\left(\rho v - u(w, a) + \kappa\sigma|\gamma|\right) + \frac{1}{2}J''_{FB}(v)\,\sigma^2\gamma^2.$$

By the first-order condition for w,

$$1 + J'_{FB}(v)\,\alpha\exp\left(-\alpha\left(w - \lambda a\right)\right) = 0.$$

Since $\lambda \in (0, 1)$, the first-order condition for a gives

$$1 + J'_{FB}(v)\,\lambda\alpha\exp\exp\left(-\alpha\left(w - \lambda a\right)\right) > 0.$$

Thus the optimal effort $a_{FB} = 1$.

Conjecture that $\rho J_{FB}(v) = 1 - \lambda + \ln(-\rho v)/\alpha$. Using the preceding first-order condition, we obtain the optimal wage policy

$$w_{FB} = -\frac{\ln(-\rho v)}{\alpha} + \lambda \Longrightarrow u\left(w_{FB}, 1\right) = \rho v.$$

If the optimal sensitivity satisfies $\gamma > 0$, then the first-order condition gives $\gamma = \frac{-\kappa J'_{FB}(v)}{\sigma J''_{FB}(v)} < 0$, a contradiction. If the optimal sensitivity satisfies $\gamma < 0$, then the first-order condition gives $\gamma = \frac{\kappa J'_{FB}(v)}{\sigma J''_{FB}(v)} > 0$, also a contradiction. Thus the optimal sensitivity must be $\gamma = 0$. This implies that $dv_t = 0$ so that $v_t = v_0$ for all t. Thus the first-best wage is also constant over time. We can verify the preceding value function J_{FB} and policies for a, w, and γ satisfy the HJB equation above so that they are indeed optimal. Q.E.D.

Proof of Lemma 4: The value function $J_N(v)$ satisfies the HJB in (28). Conjecture that it takes the form in (29). Since J_N is concave and decreasing in v, we can show that the incentive constraint $\gamma \geq -u_a(w, a)$ always binds so that $\gamma > 0$ and the worst-case density satisfies $b_t^* = -\kappa$. We next show that the optimal effort satisfies $a^* \equiv 1$. By the first-order condition for w, we obtain

$$-1 - J_N'(v)\left[\kappa\sigma u_{aw}(w,a) + u_w(w,a)\right] - J_N''(v)\,\sigma^2\gamma u_{aw}(w,a) = 0.$$

Given the exponential form of u, we have

$$1 + J_N'(v)\left(\kappa\sigma\lambda\alpha^2 + \alpha\right)\exp\left(-\alpha\left(w - \lambda a\right)\right) + J_N''(v)\,\sigma^2\gamma\lambda\alpha^2\exp\left(-\alpha\left(w - \lambda a\right)\right) = 0. \tag{A.21}$$

Differentiating with a in (29) yields

$$\begin{aligned}
& 1 - J_N'(v)\left[\kappa\sigma u_{aa}(w,a) + u_a(w,a)\right] - J_N''(v)\,\sigma^2\gamma u_{aa}(w,a) \\
= \ & 1 + J_N'(v)\left[\kappa\sigma\left(\lambda\alpha\right)^2 + \lambda\alpha\right]\exp\left(-\alpha\left(w - \lambda a\right)\right) + J_N''(v)\,\sigma^2\gamma\left(\lambda\alpha\right)^2\exp\left(-\alpha\left(w - \lambda a\right)\right) \\
= \ & 1 + \lambda\cdot\left[J_N'(v)\left(\kappa\sigma\lambda\alpha^2 + \alpha\right)\exp\left(-\alpha\left(w - \lambda a\right)\right) + J_N''(v)\,\sigma^2\gamma\lambda\alpha^2\exp\left(-\alpha\left(w - \lambda a\right)\right)\right] \\
= \ & 1 - \lambda > 0,
\end{aligned}$$

where the inequality follows from the fact that $\lambda \in (0,1)$ and the third equality follows from (A.21). Thus we must have $a^* = 1$.

We now conjecture that the optimal wage policy is given by

$$w^*(v) = -\frac{\ln(-Kv)}{\alpha} + \lambda \Longrightarrow u(w^*(v), 1) = Kv, \tag{A.22}$$

where K is a positive constant to be determine soon. The binding incentive constraint becomes

$$\gamma = -u_a(w,1) = -\alpha\lambda Kv. \tag{A.23}$$

For $a^* = 1$, the first-order condition for wage (A.21) gives equation (31). The constant K must be the positive root of this equation.

Plugging the expressions $\gamma_t = -\alpha\lambda K v_t$, $u(w_t^*, 1) = Kv_t$ in the dynamics for v_t

$$dv_t = \left(\rho v_t - u\left(w_t^*, 1\right) + \kappa\sigma\gamma_t\right) + \gamma_t\sigma dB_t^1,$$

and using equation (31), we obtain (32).

Plugging (29), (A.22), (A.23) into the HJB equation (28) and matching coefficients, we obtain (30), where we have also used equation (31). We can verify that the preceding policies for a, w and γ attain the maximum in the HJB equation (28) and hence they are indeed optimal. Q.E.D.

Proof of Lemma 5: Suppose that shirking $a = 0$ is incentive compatible. Then the HJB equation is

$$\rho J_{NS}(v) = \sup_{w,\gamma} - w + J'_{NS}(v)(\rho v - u(w,0) + \kappa\sigma|\gamma|) + \frac{1}{2}J''_{NS}(v)\sigma^2\gamma^2$$

subject to $\gamma \leq -u_a(w_t, 0)$. We ignore the incentive constraint for now. We conjecture that the value function takes the form

$$\rho J_{NS}(v) = \ln(-\rho v)/\alpha.$$

From the first-order condition for w, we can derive that

$$w = -\frac{\ln(-\rho v)}{\alpha} \Longrightarrow u(w,0) = \rho v.$$

Suppose that $\gamma > 0$. Then the first-order condition for γ gives

$$J'_{NS}(v)\kappa\sigma + J''_{NS}(v)\sigma^2\gamma = 0.$$

Since $J'_{NS} < 0$ and $J''_{NS} < 0$, it follows that $\gamma < 0$, a contradiction. Similarly, we can show that $\gamma < 0$ is impossible. Thus we must have $\gamma = 0$. Since $u_a < 0$, the solution $\gamma = 0$ satisfies the incentive constraint $\gamma < -u_a(w,a)$. Then v_t satisfies $dv_t = 0$ so that $v_t = v_0$ for all $t \geq 0$. Finally, we can verify that the preceding conjectured value function $J_{NS}(v)$ and the wage policy solve the HJB equation above and thus they are indeed optimal. Q.E.D.

Proof of Lemma 6: Without loss of generality we consider the case of $s \in [0, T]$. Given (35), (36), and exponential utility u, we deduce that p_t is a function of v_t and hence we can write γ_t as $\gamma(t, v, w, a)$. Then the first-order condition for effort $a \in (0, 1]$ in HJB (38) is

$$1 - \frac{\partial J_t^T}{\partial v}u_a(w,a) + \kappa\sigma\frac{\partial J_t^T}{\partial v}\frac{\partial\gamma(t,v,w,a)}{\partial a} + \sigma^2\frac{\partial^2 J_t^T}{\partial v^2}\gamma(t,v,w,a)\frac{\partial\gamma(t,v,w,a)}{\partial a} \geq 0.$$

The first-order condition for the wage is

$$-1 - \frac{\partial J_t^T}{\partial v}u_w(w,a) + \kappa\sigma\frac{\partial J_t^T}{\partial v}\frac{\partial\gamma(t,v,w,a)}{\partial w} + \sigma^2\frac{\partial^2 J_t^T}{\partial v^2}\gamma(t,v,w,a)\frac{\partial\gamma(t,v,w,a)}{\partial w} = 0. \tag{A.24}$$

When the incentive constraint binds, we have

$$u_w(w,a) = -\frac{1}{\lambda}u_a(w,a) \text{ and } \frac{\partial\gamma(t,v,w,a)}{\partial w} = -\frac{1}{\lambda}\frac{\partial\gamma(t,v,w,a)}{\partial a}.$$

It follows from (A.24) that

$$\begin{aligned}
& -\frac{\partial J_t^T}{\partial v}u_a(w,a) + \kappa\sigma\frac{\partial J_t^T}{\partial v}\frac{\partial\gamma(t,v,w,a)}{\partial a} + \sigma^2\frac{\partial^2 J_t^T}{\partial v^2}\gamma(t,v,w,a)\frac{\partial\gamma(t,v,w,a)}{\partial a} \\
= & -\lambda\left[-\frac{\partial J_t^T}{\partial v}u_w(w,a) + \kappa\sigma\frac{\partial J_t^T}{\partial v}\frac{\partial\gamma(t,v,w,a)}{\partial w} + \sigma^2\frac{\partial^2 J_t^T}{\partial v^2}\gamma(t,v,w,a)\frac{\partial\gamma(t,v,w,a)}{\partial w}\right] \\
= & -\lambda.
\end{aligned}$$

Since $\lambda \in (0,1)$, we can deduce that

$$1 - \frac{\partial J_t^T}{\partial v}u_a(w,a) + \kappa\sigma\frac{\partial J_t^T}{\partial v}\frac{\partial\gamma(t,v,w,a)}{\partial a} + \sigma^2\frac{\partial^2 J_t^T}{\partial v^2}\gamma(t,v,w,a)\frac{\partial\gamma(t,v,w,a)}{\partial a} > 0,$$

Thus optimal effort is always at the full level $a^* = 1$. Q.E.D.

Proof of Lemma 7: We adapt the proof in Prat and Jovanovic (2014) and divide the proof into six steps.

1. Initial guess. By Lemma 6, the optimal incentive effort $a^* = 1$. We conjecture that the value function takes the form:
$$\rho J_I^T(t,v) = f_t^T + \frac{\ln(-\rho v)}{\alpha},$$
where f_t^T is a term to be determined. We also guess the form of the optimal wage and the information rent:
$$\begin{aligned}
w^*(t,v) &= -\frac{\ln(-k_t^T v)}{\alpha} + \lambda \Longrightarrow u(w_t^*,1) = k_t^T v, \\
p^*(t,v) &= \alpha\lambda\phi_t^T v,
\end{aligned}$$
where k_t^T and ϕ_t^T are continuously differentiable functions to be determined. Since $U(W_T) = -\exp(-\alpha\rho W_T)/\rho$, the terminal boundary condition for the value function is given by
$$\rho J_I^T(T,v) = -\rho W_T = \frac{\ln(-\rho v)}{\alpha}.$$

2. Deriving k_t^T. By Lemma 6, $a^* = 1$. The incentive constraint becomes
$$\gamma_t^1 = -u_a(w_t^*,1) - \frac{\sigma^{-2}}{h_t}p_t^* = -\alpha\lambda\left(k_t^T + \frac{\sigma^{-2}}{h_t}\phi_t^T\right)v_t,$$

where we have substituted the conjectured solutions for w_t^* and p_t^*. Then we can derive the derivative of γ_t^1 with respect to the wage:

$$\frac{\partial \gamma(t, v, w^*, 1)}{\partial w} = -u_{aw}(w_t^*, 1) = \alpha^2 \lambda v k_t^T.$$

The first-order condition for the wage (A.24) is then given by

$$\begin{aligned} & -1 + \frac{\partial J_I^T(t,v)}{\partial v} \alpha k_t^T v + \frac{\partial J_I^T(t,v)}{\partial v} \kappa \sigma \frac{\partial \gamma^1}{\partial w} \\ & -\sigma^2 \frac{\partial^2 J_I^T(t,v)}{\partial v^2} \alpha^3 (\lambda v)^2 (k_t^T (k_t^T + \frac{\sigma^{-2}}{h_t} \phi_t^T)) \\ = \quad & \frac{1}{\rho} \left(-\rho + (1 + \kappa\sigma\alpha\lambda) k_t^T + (\alpha\lambda\sigma)^2 [k_t^T (k_t^T + \frac{\sigma^{-2}}{h_t} \phi_t^T)] \right) = 0. \end{aligned}$$

Simplifying yields an equation for k_t^T :

$$(\alpha\lambda\sigma)^2 (k_t^T)^2 + \left(1 + \kappa\sigma\alpha\lambda + (\alpha\lambda)^2 \frac{\phi_t^T}{h_t}\right) k_t^T - \rho = 0. \tag{A.25}$$

Since utility is negative, we need k_t^T to be the positive root of this equation. Then the law of motion for agent's continuation value (7) becomes:

$$dv_t = v_t \left[\rho - (1 + \kappa\alpha\sigma\lambda) k_t^T - \kappa\sigma\alpha\lambda \frac{\sigma^{-2}}{h_t} \phi_t^T \right] dt - v_t \alpha\sigma\lambda \left(k_t^T + \frac{\sigma^{-2}}{h_t} \phi_t^T \right) d\hat{B}_t^1.$$

3. Deriving f_t^T. Now we need that the value function satisfies the dynamic programming equation:

$$\begin{aligned} \rho J_I^T(t,v) \quad = \quad & 1 - w + \frac{\partial J_I^T(t,v)}{\partial t} + \frac{\partial J_I^T(t,v)}{\partial v} (\rho v - u(w_t^*, 1) + \kappa\sigma\gamma_t^1) \\ & + \frac{\partial^2 J_I^T(t,v)}{\partial v^2} (\frac{1}{2}\sigma^2) \left(\gamma_t^1\right)^2 \\ = \quad & 1 + \frac{\ln(-v)}{\alpha} + \frac{\ln(k_t^T)}{\alpha} - \lambda + \frac{1}{\rho} \frac{df_t^T}{dt} + \frac{\rho - k_t^T}{\rho\alpha} \\ & - \frac{\kappa\sigma\lambda}{\rho} \left(k_t^T + \frac{\sigma^{-2}}{h_t} \phi_t^T \right) - \frac{\alpha(\sigma\lambda)^2}{2\rho} \left(k_t^T + \frac{\sigma^{-2}}{h_t} \phi_t^T \right)^2. \end{aligned}$$

From equation (A.25), we can see the HJB equation holds for all promised value v as long as

$$\begin{aligned} \frac{df_t^T}{dt} - \rho f_t^T \quad = \quad & -\rho \left(1 - \lambda + \frac{\ln(k_t^T/\rho)}{\alpha} \right) \\ & + \frac{1}{2} (\sigma\lambda)^2 \alpha \left((\frac{\sigma^{-2}}{h_t} \phi_t^T)^2 - (k_t^T)^2 \right) + \kappa\sigma\lambda \frac{\sigma^{-2}}{h_t} \phi_t^T. \end{aligned}$$

Define

$$\psi_s^T \triangleq \rho(1-\lambda+\frac{\ln(k_s^T/\rho)}{\alpha}) - \frac{1}{2}(\sigma\lambda)^2\alpha((\frac{\sigma^{-2}}{h_s}\phi_s^T)^2 - (k_s^T)^2) - \kappa\sigma\lambda\frac{\sigma^{-2}}{h_s}\phi_s^T. \tag{A.26}$$

We then integrate f_t^T with respect to time t to obtain

$$f_t^T = \int_t^T e^{-\rho(s-t)}\psi_s^T ds.$$

4. Deriving ϕ_t^T. We need to verify the guess of the form p_t^*. First we need to characterize the law of motion for p_t^*. Because

$$\gamma_t^1 = -\alpha\lambda\left(k_t^T + \frac{\sigma^{-2}}{h_t}\phi_t^T\right)v_t,$$

we reinsert this expression of γ_t^1 into the BSDE for v_t (8) to obtain

$$\begin{aligned} dv_s &= [\rho v_s - u(w_s^*,1)]ds + \sigma\gamma_s^1 d\hat{B}_s^{1,-\kappa} \\ &= v_s\left[(\rho - k_s^T)ds - \alpha\lambda(k_s^T + \frac{\sigma^{-2}}{h_s}\phi_s^T)\sigma d\hat{B}_s^{1,-\kappa}\right]. \end{aligned}$$

If $\alpha\lambda(k_t^T + \frac{\sigma^{-2}}{h_t}\phi_t^T)$ is bounded (this can be verified below), then we have

$$E_t^{Q^{1,-\kappa}}[v_T] = v_t \exp\left(\int_t^T (\rho - k_s^T)ds\right)$$

and so

$$p_t^* = \alpha\lambda(v_t - e^{-\rho(T-s)}E_t^{Q^{1,-\kappa}}[v_T]) = \alpha\lambda\left(1-\exp\left(-\int_t^T k_s^T ds\right)\right)v_t.$$

So we should define $\phi_t^T \triangleq 1-\exp\left(-\int_t^T k_s^T ds\right)$. Since $k_t^T > 0$, $\phi_t^T \in (0,1)$ for all $t<T$. Let K and k_t be the positive solution to equations (31) and (42), respectively. Then we can show that

$$\begin{aligned} & (\alpha\lambda\sigma)^2K^2 + \left(1+\kappa\sigma\alpha\lambda + (\alpha\lambda)^2\frac{\phi_t^T}{h_t}\right)K - \rho \\ > \;& (\alpha\lambda\sigma)^2K^2 + (1+\kappa\sigma\alpha\lambda)K - \rho = 0 \end{aligned}$$

and

$$\begin{aligned} & (\alpha\lambda\sigma)^2(k_t)^2 + \left(1+\kappa\sigma\alpha\lambda + (\alpha\lambda)^2\frac{\phi_t^T}{h_t}\right)k_t - \rho \\ < \;& (\alpha\lambda\sigma)^2(k_t)^2 + \left(1+\kappa\sigma\alpha\lambda + \frac{(\alpha\lambda)^2}{h_t}\right)k_t - \rho = 0. \end{aligned}$$

This implies that $k_t^T \in (k_t, K)$ for all $t < T$. Thus we have verified that $\alpha\lambda(k_t^T + \frac{\sigma^{-2}}{h_t}\phi_t^T)$ is bounded and also verified the guess of the expression for p_t^*. Also we have $p_T^* = \phi_T^T = 0$, which means that there is no remaining information rent.

5. Verification of the HJB equation. It is straightforward to show that the solution above for the value function J_I^T and policy functions for a^*, w^*, and γ^* solve the HJB equation in (38) and the maximum is attained at the solution. Thus the usual verification theorem holds and the value function for the control problem is $J_I^T(t, v)$ with the optimal Markovian controls $w_t^* = -\frac{\ln(-k_t^T v)}{\alpha} + \lambda$ and $a_t^* = 1$.

6. Convergence of T to infinity. Remember that k_t^T and ϕ_t^T are deterministic where k_t^T is defined in (A.25) and $\phi_t^T \triangleq 1 - \exp\left(-\int_t^T k_s^T ds\right)$ with $\phi_T^T = 0$. We have shown that $k_t^T \in (k_t, K)$. Moreover, we can show that $k'(h) > 0$, where

$$k(h) \triangleq \frac{\sqrt{(1 + (\alpha\lambda)^2/h + \kappa\alpha\lambda\sigma)^2 + 4\rho(\alpha\lambda\sigma)^2} - 1 - (\alpha\lambda)^2/h - \kappa\alpha\lambda\sigma}{2(\alpha\lambda\sigma)^2}.$$

Thus we have

$$\frac{dk_t}{dt} = \frac{dk\,(h_t)}{dh_t}\frac{dh_t}{dt} > 0.$$

It follows that $\int_t^T k_s^T ds > \int_t^T k_s ds > k_t(T - t)$, which means that $\lim_{T\to\infty} \int_t^T k_s^T ds = \infty$. Then

$$\lim_{T\to\infty} \phi_t^T = \lim_{T\to\infty} (1 - e^{-\int_t^T k_s^T ds}) = 1.$$

Plugging this limit into (A.25), we obtain $\lim_{T\to\infty} k_t^T = k_t$. Then the agent's continuation value satisfies

$$dv_t = v_t \left[\rho - (1 + \kappa\alpha\sigma\lambda)k_t - \kappa\sigma\alpha\lambda\frac{\sigma^{-2}}{h_t}\right] dt - v_t\alpha\sigma\lambda \left(k_t + \frac{\sigma^{-2}}{h_t}\right) d\hat{B}_t^1,$$

when $T \to \infty$. Then the pointwise convergence of k_t^T to k_t and ϕ_t^T to 1 implies that ψ_t^T converges pointwise to

$$\psi_t \triangleq \lim_{T\to\infty} \psi_t^T = \rho(1 - \lambda + \frac{\ln(k_t/\rho)}{\alpha}) - \frac{1}{2}(\sigma\lambda)^2\alpha \left((\frac{\sigma^{-2}}{h_t})^2 - (k_t)^2 \right) - \kappa\sigma\lambda\frac{\sigma^{-2}}{h_t}. \qquad \text{(A.27)}$$

Note that we can define $f_t^T = \int_t^T e^{-\rho(s-t)}\psi_s^T ds = \int_t^\infty e^{-\rho(s-t)}\psi_s^T ds$ because $\psi_t^T = 0$ for all $t > T$. Given that $\left|\psi_t^T\right|$ is bounded, $e^{-\rho(s-t)}\psi_s^T$ is dominated by some integrable function and we can apply dominated convergence theorem to get that $\lim_{T\to\infty} f_t^T = f_t = \int_t^\infty e^{-\rho(s-t)}\psi_s ds$. This completes the proof. Q.E.D.

Proof of Lemma 8: As in the roof of Lemma 7, we define

$$\psi_t \triangleq \rho\left[1-\lambda+\frac{\ln(k_t/\rho)}{\alpha}\right]-\frac{1}{2}(\sigma\lambda)^2\alpha\left[\left(\frac{\sigma^{-2}}{h_t}\right)^2-(k_t)^2\right]-\kappa\sigma\lambda\frac{\sigma^{-2}}{h_t} \tag{A.28}$$

and

$$f_t=\int_t^\infty e^{-\rho(s-t)}\psi_s ds.$$

Differentiating the expression of ψ_t with respect to time t leads to

$$\psi_t'=\frac{\rho}{\alpha}\frac{k_t'}{k_t}-(\sigma\lambda)^2\alpha\left(-\frac{\sigma^{-6}}{h_t^3}-k_t'k_t\right)+\frac{\kappa\sigma\lambda\sigma^{-4}}{h_t^2}.$$

From the definition of k_t, which is a positive root of (42), we have $k_t'>0$ and $k_t>0$; so we have $\psi_t'>0$. Also since $\lim_{s\to\infty} h_s=\infty$, we get that $\lim_{t\to\infty} k_t=K$ and $\lim_{s\to\infty}\psi_s=\rho F$, where F is defined in (30) and K is the solution to equation (31). Note that $f_t=\int_t^\infty e^{-\rho(s-t)}\psi_s ds$, so we have the differential equation $f'(t)=\rho f(t)-\psi(t)$. Because ψ_t converges when t goes to infinity, we have the boundary condition $\lim_{s\to\infty}\rho f(s)=\lim_{s\to\infty}\psi(s)=\rho F$, which means $f(t)$ converges to F. Since

$$f_t=\int_t^\infty e^{-\rho(s-t)}\psi_s ds>\psi_t\int_t^\infty e^{-\rho(s-t)}ds=\frac{\psi_t}{\rho},$$

we have $f'(t)=\rho f(t)-\psi(t)>0$.

It is easily known that k_t decreases with κ, and because ψ_t increases with k_t, from the definition of ψ_t we can show that ψ_t decreases with κ for all t. So $f(t)$ also decreases with κ.

We differentiate k_t with respect to t to obtain

$$\frac{\dot{k}_t}{k_t}=-\frac{1}{\sqrt{(1+\kappa\alpha\sigma\lambda+\frac{(\alpha\lambda)^2}{h_t})^2+4\rho(\alpha\lambda\sigma)^2}}\frac{d\left(\frac{(\alpha\lambda)^2}{h_t}\right)}{dt}>0.$$

So $\dot{k}_t/k_t$ decreases with κ. Q.E.D.

Lemma 9: Because k_t^T and ϕ_t^T are deterministic, we directly let the horizon T go to infinity. To derive the law of motion p_t, we define the auxiliary process

$$\mu_t \triangleq E_t^{Q^{1,-\kappa}}\left[-\int_0^T e^{-\rho s}\gamma_s^1\frac{\sigma^{-2}}{h_s}ds\right]=\mu_0+\int_0^t \xi_s\sigma d\hat{B}_t^{1,-\kappa} \text{ for all } t\in[0,T],$$

where the process ξ is derived from the martingale representation theorem. Then the definition of p_t in (18) can be expressed as

$$p_t = e^{\rho t}\sigma^2 h_t \left[\mu_t + \int_0^t e^{-\rho s}\gamma_s^1 \frac{\sigma^{-2}}{h_s} ds\right].$$

It follows from Ito's lemma that

$$\begin{aligned} dp_t &= \left[\rho p_t + \frac{d(\sigma^2 h_t)}{dt}\frac{\sigma^{-2}}{h_t}p_t + \gamma_t^1\right]dt + e^{\rho t}\sigma^2 h_t d\mu_t \\ &= \left[p_t\left(\rho + \frac{\sigma^{-2}}{h_t}\right) + \gamma_t^1\right]dt + \sigma_t^p \sigma d\hat{B}_t^{1,-\kappa}, \end{aligned}$$

where $\sigma_t^p \triangleq e^{\rho t}\sigma^2 h_t \xi_t$.

When T goes into infinity, we have $p_t = \alpha\lambda v_t$, $a_t^* = 1$, $u_{aa}(w_t^*, 1) = (\alpha\lambda)^2 v_t k_t$ and then

$$e^{\rho t}\sigma^2 h_t \xi_t = \sigma_t^p = \alpha\lambda\gamma_t^1 = -(\alpha\lambda)^2\left(k_t + \frac{\sigma^{-2}}{h_t}\right)v_t > 0.$$

In order to satisfy the sufficient condition (22), we should have

$$-2(\alpha\lambda)^2 v_t k_t \geq -(\alpha\lambda)^2\left(k_t + \frac{\sigma^{-2}}{h_t}\right)v_t \geq 0.$$

This is equivalent to

$$k_t > \frac{\sigma^{-2}}{h_t}.$$

Since k_t is the positive solution to (42), we can show that this condition is equivalent to the condition

$$(\alpha\lambda\sigma)^2\left(\frac{\sigma^{-2}}{h_t}\right)^2 + \left(1 + \kappa\alpha\sigma\lambda + \frac{(\alpha\lambda)^2}{h_t}\right)\frac{\sigma^{-2}}{h_t} - \rho < 0, \tag{A.29}$$

which is also equivalent to (45). Since $dk_t/dt > 0$ and $dh_t/dt > 0$, when this condition is satisfied for a fixed t, it will be satisfied for all $s \geq t$. Q.E.D.

Proof of Lemma 10: The proof is similar to that for Lemma 5. The key is to first consider the relaxed problem by ignoring the incentive constraint (21) and then show that the solution to the relaxed problem satisfies (21). It turns out $\gamma = \underline{p} = 0$ so that this condition is easy to verify. We omit the detail here. Q.E.D.

Proof of Theorem 5: We consider the infinite-horizon limit directly. Allowing for shirking, we can write the HJB equation as

$$\rho J_t = \max\{H_1(t), H_2(t)\}, \tag{A.30}$$

where

$$H_1(t) = \max_{w,\gamma_t} -w + \frac{\partial J_t}{\partial t} + \frac{\partial J_t}{\partial v}(\rho v - u(w,0) + \kappa\sigma|\gamma_t|) + \frac{1}{2}\frac{\partial^2 J_t}{\partial v^2}\sigma^2\gamma_t^2$$

subject to (16) and $\gamma_t \leq -\frac{\sigma^{-2}}{h_t}\underline{p}_t - u_a(w,0)$, and

$$H_2(t) = \max_{a>0,w,\gamma_t} a - w + \frac{\partial J_t}{\partial t} + \frac{\partial J_t}{\partial v}(\rho v - u(w,a) + \kappa\sigma\gamma_t) + \frac{1}{2}\frac{\partial^2 J_t}{\partial v^2}\sigma^2\gamma_t^2$$

subject to

$$\gamma_t = -\frac{\sigma^{-2}}{h_t}p_t - u_a(w,a) > 0,\ p_t = \alpha\lambda v_t.$$

(a) Suppose that $F > 0$. Consider the case where $\rho J_t = H_1(t) \geq H_2(t)$. We conjecture that the function

$$\rho J_t = g_t + \frac{\ln(-\rho v)}{\alpha}$$

solves the HJB equation $\rho J_t = H_1(t)$, where g_t is a function to be determined. The optimal wage satisfies

$$\frac{1}{\alpha\rho v}u_w(w,0) = -1 \Longrightarrow w = \frac{-\ln(-\rho v)}{\alpha}.$$

Thus $u(w,0) = \rho v$. Ignoring the incentive constraint, we can verify that the optimal sensitivity is $\gamma_t = 0$ so that $dv_t = 0$. Thus $\underline{p}_t = 0$ and $\gamma_t = 0$ satisfies the incentive constraint. Substituting the conjectured J_t back into the HJB equation, we obtain

$$\rho g(t) = g'(t).$$

Consider the case where $\rho J_t = H_2(t) \geq H_1(t)$. By Lemma 7,

$$\rho J_t = f_t + \frac{\ln(-\rho v)}{\alpha}.$$

We then obtain two segments of the value function. These two segments must be smoothly connected. We thus have the value-matching condition

$$g(\tau) = f(\tau)$$

and the smooth-pasting condition

$$g'(\tau) = f'(\tau) \Longrightarrow \rho f(\tau) = f'(\tau),$$

at some time τ. We can then show that

$$g(t) = e^{-\rho(\tau - t)}f(\tau)$$

for any t.

We next show that if $F > 0$, there exists a unique $\tau > 0$ such that $\rho f(\tau) = f'(\tau)$ or $\psi(\tau) = \rho f(\tau) - f'(\tau) = 0$. By (42) and (A.28), we deduce that ψ_t is a function of h_t. Since $d\psi_t/dt > 0$ and $dh_t/dt > 0$, ψ_t is an increasing function of h_t. Define this function as

$$\varphi(h) \triangleq \rho\left[1 - \lambda + \frac{\ln(k(h)/\rho)}{\alpha}\right] - \frac{1}{2}(\sigma\lambda)^2\alpha\left((\frac{\sigma^{-2}}{h})^2 - k(h)^2\right) - \frac{k\sigma^{-1}\lambda}{h},$$

where

$$k(h) = \frac{-(1 + \kappa\alpha\sigma\lambda + \frac{(\alpha\lambda)^2}{h}) + \sqrt{(1 + \kappa\alpha\sigma\lambda + \frac{(\alpha\lambda)^2}{h})^2 + 4\rho(\alpha\lambda\sigma)^2}}{2(\alpha\lambda\sigma)^2}.$$

Then we have $\varphi(h_t) = \psi(t)$ and $k(h_t) = k_t$. We can check that $\lim_{h_t\to\infty}\varphi(h_t) = \rho F$ and $\lim_{h_t\to 0}\varphi(h_t) = -\infty$. Thus if $F > 0$ there exists a unique $\bar{h}$ such that $\varphi(\bar{h}) = 0$. Then $\tau > 0$ is the unique solution to $h_\tau = \bar{h}$ when $h_0 < \bar{h}$.

We now derive the sufficient condition (45), which is equivalent to $k_\tau > \frac{\sigma^{-2}}{h_\tau}$. Suppose that $k_{t_0} = \frac{\sigma^{-2}}{h_{t_0}}$ at some time t_0. Using this equation to eliminate h_{t_0} in (42), we can derive the positive solution for k_{t_0} :

$$\frac{\sigma^{-2}}{h_{t_0}} = k_{t_0} = \frac{2\rho}{(1 + \kappa\sigma\alpha\lambda) + \sqrt{(1 + \kappa\sigma\alpha\lambda) + 8\alpha^2\lambda^2\rho\sigma^2}}.$$

Since $k_t - \sigma^{-2}/h_t$ and $\varphi(h_t)$ are increasing functions of h_t or time t and since $\varphi(h_\tau) = 0$, we know $k_\tau > \frac{\sigma^{-2}}{h_\tau}$ if we can show that $\varphi(h_{t_0}) < 0$. Substituting $k_{t_0} = \frac{\sigma^{-2}}{h_{t_0}}$ into (A.27) for $t = t_0$, we obtain

$$\rho(1 - \lambda + \frac{\ln(k_{t_0}/\rho)}{\alpha}) - \kappa\sigma\lambda\frac{\sigma^{-2}}{h_{t_0}} < 0.$$

Substituting the preceding expressions for k_{t_0} and σ^{-2}/h_{t_0} into the inequality above, we obtain the parameter condition (46), which guarantees that when $\varphi(h_\tau) = 0$, (45) holds for all $s \geq \tau$. By Proposition 9, we deduce that the optimal full effort contract is incentive-compatible for $s \geq \tau$.

We now show that

$$g(t) = \frac{g'(t)}{\rho} > f(t) \text{ for } t < \tau,$$

and

$$g(t) = \frac{g'(t)}{\rho} < f(t) \text{ for } t > \tau,$$

so that $H_1(t) > H_2(t)$ on $[0, \tau)$ and $H_1(t) < H_2(t)$ on (τ, ∞). We only need to show that

$$e^{-\rho(\tau-t)} f(\tau) > f(t) \text{ on } t < \tau,$$

or

$$G(t) = e^{-\rho t} f(t) - e^{-\rho \tau} f(\tau) < 0 \text{ on } t < \tau.$$

Since $G'(t) = e^{-\rho t}(f'(t) - \rho f(t)) > 0$ if and only if

$$\psi(t) = \rho f(t) - f'(t) < 0,$$

and since $\psi_t' > 0$ and $\psi(\tau) = 0$, we obtain the desired result.

(b) When $F \leq 0$, we know that $\psi(t) \leq 0$ for all t. This implies that $H_1(t) \geq H_2(t)$ or $f(t) \leq 0$ for all $t \geq 0$. Thus shirking forever is optimal. The result then follows from Lemma 10. Q.E.D.

Proof of Proposition 1: Let $k(h)$ and $\varphi(h)$ be defined as in the proof of Theorem 5. Fixed h, we can see that $k(h)$ is decreasing with respect to κ. Since $\varphi(h)$ is increasing with respect to $k(h)$, we deduce that $\varphi(h)$ is decreasing with respect to κ. Since $\varphi(h)$ is an increasing function of h and since there is a unique $\bar{h}$ such that $\varphi(\bar{h}) = 0$, we deduce that $\bar{h}$ increases with κ and hence τ increases with κ.

By (31), we can show that

$$K = \frac{2\rho}{\sqrt{b^2 + 4a\rho} + b},$$

where we define $a \triangleq (\alpha\lambda\sigma)^2$ and $b \triangleq (1 + \kappa\alpha\sigma\lambda) \in [1, \infty)$ for $\kappa \in [0, \infty)$. When $\kappa \to \infty$, we have $b \to \infty$ and hence $K \to 0$. Thus $F \to -\infty$ by (30) so that $\bar{h} \to \infty$. That is, there is no solution $\bar{h}$ such that $\varphi(\bar{h}) = 0$. When κ is sufficiently large, the principal will fully insure the agent and allow him shirk forever. Q.E.D.

Proof of Proposition 2: From Lemma 8 we know that $k(t)$ and $\dot{k}_t/k_t$ decrease with κ. It follows from equation (47) that the drift of the process for w_t increases with κ and the volatility decreases with κ. Also, the limit of the drift is the constant drift of wages in case of known quality under ambiguity: $\frac{1}{\alpha}[-\frac{1}{2}(\alpha\sigma\lambda K)^2] < 0$. So the drift of wage process will be negative when t is large. Since

$$\frac{dk_t}{dt} = \frac{1}{2}\left[\frac{(1+\kappa\alpha\sigma\lambda)/(\alpha\sigma\lambda)^2 + \sigma^{-2}/h_t}{\sqrt{((1+\kappa\alpha\sigma\lambda)/(\alpha\sigma\lambda)^2 + \sigma^{-2}/h_t)^2 + 4\rho/(\alpha\sigma\lambda)^2}} - 1\right]\frac{d(\sigma^{-2}/h_t)}{dt},$$

we have

$$\begin{aligned}\frac{d(k_t + \sigma^{-2}/h_t)}{dt} &= \frac{1}{2}\left[\frac{(1+\kappa\alpha\sigma\lambda)/(\alpha\sigma\lambda)^2 + \sigma^{-2}/h_t}{\sqrt{((1+\kappa\alpha\sigma\lambda)/(\alpha\sigma\lambda)^2 + \sigma^{-2}/h_t)^2 + 4\rho/(\alpha\sigma\lambda)^2}} + 1\right]\frac{d(\sigma^{-2}/h_t)}{dt} \\ &< 0.\end{aligned}$$

Thus the PPS term decreases with time t and converges to $\lambda\sigma K$. Q.E.D.

Proof of Proposition 3: We differentiate $\varphi(h)$ with respect to h to obtain

$$\varphi'(h) = (\frac{\rho}{\alpha})\frac{k'(h)}{k(h)} - (\sigma\lambda)^2\alpha\left[-\frac{\sigma^{-4}}{h^3} - k'(h)k(h)\right] + \frac{k\sigma\lambda\sigma^{-2}}{h^2} > 0.$$

Since $\varphi(h_0 + \sigma^{-2}t) = \psi(t)$, we deduce that $\psi(t)$ increases with h_0. So we get $f(t)$ increases with h_0. Other results are directly derived from Lemma 8. Q.E.D.

References

Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent, 2003, A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection, Journal of the European Economic Association, 1, 68-123.

Biais, Bruno, Thomas Mariotti, Guillaume Plantin, and Jean-Charles Rochet, 2007, Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications, Review of Economic Studies 74, 345-390.

Biais, Bruno, Thomas, Mariotti, Jean Charles Rochet, and Stephane Villeneuve, 2013, Dynamic Financial Contracting, in *Advances in Economics and Econometrics: Theory and Applications*, Tenth World Congress of the Econometric Society, edited by Daron Acemoglu, Manuel Arellano, and Eddie Dekel, Cambridge university Press.

Chen, Zengjing and Larry G. Epstein, 2002, Ambiguity, Risk, and Asset Returns in Continuous Time, Econometrica 70, 1403–1443.

Cvitanic, Jaksa, Xuhu Wan, and Jianfeng Zhang, 2009, Optimal Compensation with Hidden Action and Lump-sum Payment in a Continuous-Time Model, Applied Mathematics and Optimization 59, 99-146.

Cvitanic, Jaksa and Jianfeng Zhang, 2013, Contract Theory in Continuous-Time Models, Springer-Verlag, Berlin.

DeMarzo, Peter M. and Yuliy Sannikov, 2006, Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model, Journal of Finance 61, 2681–2724.

DeMarzo, Peter M. and Yuliy Sannikov, 2014, Learning in Dynamic Incentive Contracts, working paper, Princeton University.

Ellsberg, Daniel, 1961, Risk, Ambiguity and the Savage Axiom, Quarterly Journal of Economics 75, 643-669.

Epstein, Larry G. and Shaolin Ji, 2013, Ambiguous Volatility and Asset Pricing in Continuous Time, Review of Financial Studies 26, 1740-1786.

Epstein, Larry G. and Martin Schneider, 2007, Learning under Ambiguity, Review of Economic Studies 74, 1275-1303.

Gilboa, Itzhak and David Schmeidler, 1989, Maxmin Expected Utility with Non-unique Prior, Journal of Mathematical Economics 18, 141–153.

Green, Edward, 1987, Lending and the Smoothing of Uninsurable Income, in Contractual Arrangements for Intertemporal Trade, ed. by E.C. Prescott and N. Wallace. Minneapolis: University of Minnesota Press, 3-25.

Hansen, Lars Peter and Thomas J. Sargent, 2008, Robustness, Princeton University Press, Princeton and Oxford.

Hansen, Lars Peter and Thomas J. Sargent, 2012, Three Types of Ambiguity, Journal of Monetary Economics 59, 422-445.

Hansen, Lars Peter, Thomas J. Sargent, Gauhar A. Turmuhambetova, and Noah Williams, 2006, Robust Control and Model Misspecification, Journal of Economic Theory 128, 45-90.

He, Zhiguo, Bin Wei, Jianfeng Y, and Feng Gao, 2014, Optimal Long-term Contracting with Learning, working paper, University of Chicago.

Jensen, Michael C. and Kevin J. Murphy, 1990, Performance Pay and Top Management Incentives, Journal of Political Economy 98, 225 264.

Karatzas, Ioannis and Steven E. Shreve, 1991, Brownian Motion and Stochastic Calculus, 2nd Edition, Springer-Verlag, New York.

Leippold, Markus, Fabio Trojani, Paolo Vanini, 2008, Learning and Asset Prices Under Ambiguous Information, Review of Financial Studies 21, 2565-2597.

Miao, Jianjun, 2009, Ambiguity, Risk and Portfolio Choice under Incomplete Information, Annals of Economics and Finance 10, 257-279.

Miao, Jianjun, and Alejandra Rivera, 2015, Robust Contracts in Continuous Time, forthcoming in Econometrica.

Prat, Julien, and Boyan Jovanovic, 2014, Dynamic Contracts when the Agent's Quality is Unknown, Theoretical Economics 9, 865-914.

Sannikov, Yuliy, 2008, A Continuous-Time Version of the Principal-Agent Problem, Review of Economic Studies 75, 957–984.

Sannikov, Yuliy, 2013, Dynamic Security Design and Corporate Financing, in George M. Constantinides, Milton Harris, and Rene M. Stulz eds, Handbook of the Economics of Finance, Volume 2A, North Holland.

Williams, Noah, 2009, On Dynamic Principal-Agent Problems in Continuous Time, working paper, University of Wisconsin at Madison.

Williams, Noah, 2011, Persistent Private Information, Econometrica 79, 1233-1274.

Yong, Jiongmin and Xunyu Zhou, 1999, *Stochastic Controls*, Springer-Verlag, New York.