Monetary Policy and Rational Asset Price Bubbles Redux*

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This version: January 2018
First Version: November 2017

Abstract

We show that Galí's (2014) counterintuitive results are driven by his choice of a backward-looking sunspot solution around a stable bubbly steady state. Under adaptive learning, such a steady state and the associated sunspot solutions under optimal monetary policy are not E-stable. When deriving the unique forward-looking minimum stable variable (MSV) solution around a bubbly steady state, we obtain results that are consistent with the conventional views. Such a steady state and the associated MSV solution are E-stable.

JEL Classification: E13, E32, E44, E52, G12

Keywords: asset bubble, monetary policy, learning, multiple equilibria, E-stability

*We thank Jess Benhabib for helpful comments.

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1 Introduction

Due to the recent financial crisis during 2018-2019, there is a renewed interest in understanding the role of asset bubbles for business cycles and the associated policy implications. Galí (2014) presents an elegant overlapping generations model with nominal rigidities to study the impact of monetary policy on rational asset bubbles. He finds some provocative results that are inconsistent with conventional views. These results are summarized below:

- A stronger interest rate response to bubble fluctuations (i.e., a "leaning against the wind policy") may raise the volatility of asset prices and of their bubble component.
- The optimal monetary policy strikes a balance between stabilization of current aggregate demand and the bubble. If the average size of the bubble is sufficiently large, the latter motive will be dominant, making it optimal for the central bank to lower interest rates in the face of a growing bubble.

In this paper we revisit Galí's analysis. We argue that his results are driven by his particular choice of the equilibrium solution. In his model there are multiple steady states and multiple equilibria. In particular, there is a continuum of stable bubbly steady states and a continuum of unstable bubbly steady states. He focuses on a backward-looking sunspot solution around a stable bubbly steady state. For this solution the asset bubble is predetermined and does not respond to shocks on impact. Thus an increase in interest rates raises the future bubble size. By contrast we analyze the forward-looking minimal state variable (MSV) solution around an unstable bubbly steady state. For this solution the asset bubble responds to shocks on impact just like any asset prices. An increase in interest rates dampens the asset bubble on impact. We find results that are consistent with conventional views and are different from Galí's results mentioned above. In particular, the optimal policy calls for a leaning-against-the-wind rule.

Given that there are multiple steady states and multiple equilibria in Galí's model, which one should we pick? Following the methodology surveyed by Evans and Honkapohja (1999, 2001), we use learning as a selection device to select a particular steady state and a particular equilibrium.[1] The idea is that agents of the model do not initially have rational expectations and they instead form forecasts by using some adaptive learning rules such as recursive least squares based on the data. The question is whether the agents can learn a particular equilibrium or a particular steady state. Marcet and Sargent (1989) and Evans and Honkapohja (1999, 2001) show that the notion of expectational stability (E-stability) determines local convergence of real time recursive learning algorithms in a wide variety of economic models.

[1] See Bullard and Mitra (2002), Adam (2003), Woodford (2003), Duffy and Xiao (2007), Benhabib, Evans, and Honkapohja (2012), among others, for the application of learning to select equilibrium in macroeconomic models.

We find that the sunspot equilibrium solution adopted by Galí (2014) is not E-stable under his optimal monetary policy rule, but the forward-looking MSV solution is E-stable. We also find that the unstable bubbly steady state Pareto dominates the stable bubbly steady state. Moreover the former steady state is E-stable, but the latter is not. Our analysis suggests that the MSV equilibrium around the unstable bubbly steady state studied in this paper is more appealing from the welfare and learning perspective and should be the focus for policy analysis.

2 Solving Galí's Model

We first present the equilibrium system of Galí's model without spelling out its detailed setup. A key feature of his model is that once an old bubble bursts, a new bubble can be created as in Martin and Ventura (2012) and Miao, Wang, and Xu (2015). The equilibrium system consists of eight equations

$$\begin{aligned}
C_{1t} + C_{2t} &= 1, \\
D_t + W_t &= 1, \\
C_{1t} + Q_t &= W_t + U_t, \\
B_t + U_t &= \beta E_t \left[\frac{C_{1t}}{C_{2t+1}} B_{t+1} \right], \\
Q_t &= B_t + U_t, \\
0 &= E_{t-1} \left[\frac{\beta C_{1t-1}}{C_{2t}} \left(1 - \mathcal{M} W_t\right) \right], \\
\ln\left(1 + i_t\right) &= \ln R + \phi_\pi \ln\left(\frac{\Pi_t}{\Pi}\right) + \phi_b \ln\left(\frac{Q_t}{Q}\right) + \ln E_t \Pi_{t+1}, \\
1 &= \beta\left(1 + i_t\right) E_t \left[\frac{C_{1t}}{C_{2t+1}} \frac{1}{\Pi_{t+1}} \right],
\end{aligned}$$

for nine stochastic processes $\{C_{1t}\}, \{C_{2t}\}, \{D_t\}, \{W_t\}, \{\Pi_t\}, \{i_t\}, \{Q_t\}, \{B_t\}$, and $\{U_t\}$, where these variables denote respectively the young agent's consumption, the old agent's consumption, dividends, the real wage, the inflation rate, the nominal interest rate, the aggregate bubble, the old bubble, and the new bubble. Moreover, $\mathcal{M} = \epsilon / (\epsilon - 1)$ denotes the markup. A variable without a time subscript denotes its deterministic steady state value. Define the gross real interest rate as

$$R_t = \left(1 + i_t\right) E_t \frac{1}{\Pi_{t+1}}.$$

Since there are eight equilibrium conditions for nine variables, the equilibrium system cannot determine the size of the old bubble and the new bubble independently. Following Galí (2014), we assume that the value of the new bubble U_t follows an exogenous stochastic process. In the deterministic case where $U_t = U > 0$ for all t, Galí (2014) shows that the old bubble $\{B_t\}$ satisfies the difference equation

$$B_{t+1} = \frac{\left(1 - 1/\mathcal{M}\right)\left(B_t + U\right)}{\beta/\mathcal{M} - \left(1 + \beta\right) B_t - U} \equiv H\left(B_t, U\right). \tag{1}$$

He also shows that a necessary and sufficient condition for the existence of a deterministic bubbly steady state is given by

$$\mathcal{M} < 1 + \beta. \tag{2}$$

Furthermore, when this condition is satisfied there exists a continuum of stable bubbly steady states indexed by U,

$$\left\{ (B_s(U), U) : B_s(U) = H(B_s(U), U) \text{ for } U \in \left(0, \bar{U}\right) \right\},$$

and a continuum of unstable bubbly steady states also indexed by U,

$$\left\{ (B_u(U), U) : B_u(U) = H(B_u(U), U) \text{ for } U \in [0, \bar{U}),\ B_u(U) > B_s(U) \right\},$$

where

$$\bar{U} = \beta + (1+\beta)(1-W) - 2\sqrt{\beta(1+\beta)(1-W)} > 0 \text{ and } W = \frac{1}{\mathcal{M}}.$$

The economy also has a bubbleless steady state in which $B = U = 0$. In this steady state we can show

$$C_1 = \frac{1}{\mathcal{M}},\ C_2 = 1 - \frac{1}{\mathcal{M}},\ \text{and } R = \frac{\mathcal{M} - 1}{\beta}.$$

Thus condition (2) is the same as $R < 1$, which is the standard condition in the literature (Tirole (1985)), i.e., the bubbleless equilibrium is dynamically inefficient.

Galí (2014) shows that the log-linearized system around a bubbly steady state for any fixed $U \in \left(0, \bar{U}\right)$ is given by

$$0 = c_{1,t} + \beta R c_{2,t}, \tag{3}$$

$$c_{1,t} = E_t c_{2,t+1} - r_t, \tag{4}$$

$$c_{2,t} = (1-\Gamma) d_t + \Gamma b_t, \tag{5}$$

$$q_t = R b_t + (1-R) u_t, \tag{6}$$

$$q_t = E_t b_{t+1} - r_t, \tag{7}$$

$$E_{t-1} w_t = E_{t-1} d_t = 0, \tag{8}$$

$$r_t = \phi_\pi \pi_t + \phi_b q_t, \tag{9}$$

where we use a lower case variable x_t to denote the log deviation from its steady state value, $x_t = \ln(X_t/X)$. Moreover, we define $\Gamma = \epsilon B/(\epsilon B + 1)$ and show that the bubbly steady-state real interest rate is given by

$$R = \frac{1}{\beta} \frac{1 - 1/\mathcal{M} + B}{1/\mathcal{M} - B} = \frac{B}{B + U}.$$

Note that there are two bubbly steady states for a fixed $U \in \left(0, \bar{U}\right)$. Without risk of confusion, we use the same notation B to represent either one of the steady-state size of the old bubble in the analysis below.

In Appendix A we show that the above log-linearized equilibrium system can be reduced to a unidimensional system

$$
\begin{aligned}
b_t = {} & \frac{1}{R(\phi_b+1)} E_t b_{t+1} + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1} E_{t-1} b_t \\
& + \frac{R-1}{R} u_t + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R} E_{t-1} u_t.
\end{aligned} \tag{10}
$$

Our objective is to solve for a rational expectations equilibrium (REE) using (10). Galí (2014) assumes that u_t is IID. We consider a more general AR(1) process

$$
u_t = \rho u_{t-1} + e_t, \quad \rho \in (0,1), \tag{11}
$$

where e_t is an IID random variable with mean zero and variance σ_e^2.

Galí (2014) focuses his analysis on a sunspot solution around a stable bubbly steady state. Given (11), we can derive the following more general solution. Its proof and the proofs of the remaining propositions in the paper are given in Appendix B.

Proposition 1 *Fix $U \in \left(0, \bar{U}\right)$. For any b_0, there is a linear sunspot solution in a neighborhood of the bubbly stable steady state given by*

$$
b_t = \chi b_{t-1} + (1-R)(1+\epsilon B)\rho u_{t-2} + \varphi_2^* e_t + \varphi_3^* e_{t-1} + \varphi_4^* \xi_t + \varphi_5^* \xi_{t-1},
$$

where ξ_t denotes a sunspot shock satisfying $E_{t-1}\xi_t = 0$, φ_3^* and φ_5^* are arbitrary real numbers, and

$$
\varphi_2^* = \frac{\varphi_3^* + (R-1)(1+\phi_b)}{R(\phi_b+1) - \chi},
$$

$$
\varphi_4^* = \frac{\varphi_5^*}{R(\phi_b+1) - \chi},
$$

$$
\chi = R(1 + \epsilon B(1+\beta)) \in (0,1).
$$

Galí (2014) shows that $\chi = \partial H(B,U)/\partial B$. For a stable bubbly steady state, we must have $\chi \in (0,1)$, which also implies that the backward-looking solution in Proposition 1 is stationary. Galí (2014) defines a sunspot variable $\xi_t = b_t - E_{t-1} b_t$. Substituting this variable into (10) yields a particular solution

$$
\begin{aligned}
b_t \quad = \quad & \chi b_{t-1} + (\phi_b+1)(1-R) u_{t-1} - (\phi_b - \epsilon B)(1-R)\rho u_{t-2} \\
& + \xi_t + (\phi_b - \epsilon B(1+\beta)) R \xi_{t-1},
\end{aligned} \tag{12}
$$

which can also be obtained by setting

$$
\varphi_2^* = 0, \quad \varphi_3^* = (1-R)(1+\phi_b), \; \varphi_5^* = (\phi_b - \epsilon B(1+\beta)) R
$$

in our general solution given in Proposition 1. The solution in Galí (2014) corresponds to $\rho = 0$ in (12).

For this solution, the initial value b_0 is indeterminate. Galí (2014) derives all his results for a fixed b_0. From (12) we can see that monetary policy only affects the anticipated component of the old bubble $E_{t-1}b_t$ through the interest rate coefficient ϕ_b. In the case of $\rho = 0$, Galí (2014) shows that a leaning-against-the-wind policy which corresponds to $\phi_b > 0$ generates a larger volatility in the bubble than a policy of benign neglect ($\phi_b = 0$).

Now we consider the solution in the neighborhood of the unstable bubbly steady state.

Proposition 2 *Fix* $U \in \left(0, \bar{U}\right)$. *There is a unique forward-looking linear solution in a neighborhood of the unstable bubbly steady state given by*

$$b_t = (R-1)\frac{\epsilon B + 1}{\chi - \rho}\rho u_{t-1} + \frac{R-1}{R}\left[\frac{\rho}{1+\phi_b}\frac{1+\epsilon B}{\chi - \rho} + 1\right] e_t, \tag{13}$$

where $\chi = R\left(1 + \epsilon B\left(1+\beta\right)\right) > 1$.

In a neighborhood of the unstable bubbly steady state, we have $\chi > 1$. The backward-looking solution in (12) is not stationary. We must solve for b_t forward to obtain the forward-looking solution in (13) so that b_t is stationary. This solution is also called the minimal state variable (MSV) solution in the literature (e.g., Evans and Honkapohja (2001)). In the next section we will focus our analysis on this solution.

Note that if $\rho = 0$ as in Galí (2014), then $b_t = e_t\left(R-1\right)/R$. In this case monetary policy through ϕ_b does not affect bubble dynamics. We thus assume $\rho \in (0,1)$ throughout the paper.

3 Monetary Policy

What is the impact of the monetary policy on bubble dynamics? From equations (6) and (13), we can derive the volatility of the aggregate bubble

$$Var\left(q_t\right) = (R-1)^2\left[R\frac{\epsilon B + 1}{\chi - \rho} - 1\right]^2 \rho^2(1-\rho^2)^{-1}\sigma_e^2 + \left[\frac{(R-1)\,\rho}{1+\phi_b}\frac{1+\epsilon B}{\chi-\rho}\right]^2\sigma_e^2.$$

Thus a leaning-against-the-wind policy (i.e., $\phi_b > 0$) generates a lower volatility of the aggregate bubble than a policy of benign neglect ($\phi_b = 0$), contrary to Galí's result. The volatility is minimized when $\phi_b \to \infty$. Interestingly, when ϕ_b decreases to negative infinity, the bubble volatility also decreases to zero.

We use (13) to compute the volatility of the old bubble

$$Var\left(b_t\right) = \left(\frac{\epsilon B + 1}{\chi - \rho}\rho\right)^2 \frac{(R-1)^2\sigma_e^2}{1-\rho^2} + \left(\frac{R-1}{R}\right)^2\left[\frac{\rho}{1+\phi_b}\frac{1+\epsilon B}{\chi-\rho} + 1\right]^2\sigma_e^2.$$

It is minimized at

$$\phi_b = -\frac{\rho(1+\epsilon B)}{\chi - \rho} - 1 < 0. \tag{14}$$

Figure 1 presents the relation between ϕ_b and the volatilities of the old and aggregate bubbles. We choose the same parameter values as in Galí (2014) by setting $\beta = 1$, $\epsilon = 6$, $U = 0.175$. These values imply $B_s = 0.1$, $B_u = 0.1458$, and $\mathcal{M} = 1.2$. While Galí (2014) studies equilibria around the stable bubbly steady state $B_s = 0.1$, we focus on the solution around the unstable bubbly steady state $B_u = 0.1458$.

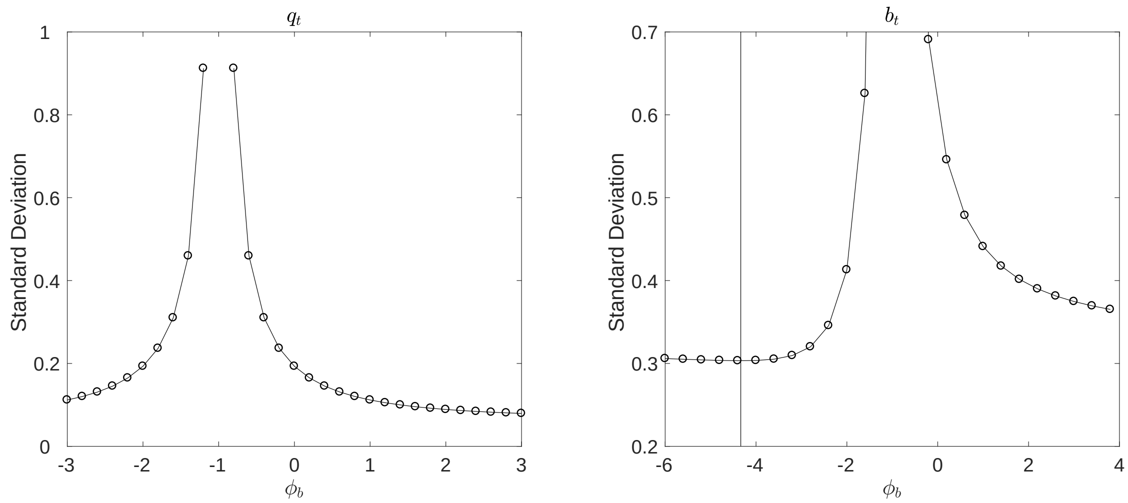

Figure 1: Monetary Policy and Bubble Volatility

Note: This figure plots the standard deviations of the aggregate bubble q_t and old bubble b_t for various coefficients ϕ_b. The vertical line indicates the value of ϕ_b that minimizes the standard deviation of the old bubble. The parameter values are $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, and $\sigma_e^2 = 0.01$. We focus on the unstable bubbly steady state with $B = 0.1458$.

To understand Figure 1, we consider the economy's responses to an exogenous positive bubble shock to u_t. By equations (6) and (7), we see that the old bubble satisfies the asset pricing equation

$$b_t = \frac{1}{R} E_t b_{t+1} - \frac{1}{R} r_t - \frac{(1-R)}{R} u_t. \tag{15}$$

Solving forward shows that the old bubble is equal to the (negative) discounted value of future interest rates and new bubbles. Since $0 < R < 1$, new bubbles $\{u_t\}$ act as negative dividends. An increase in u_t has a direct effect of lowering b_t and an indirect effect through the change in the interest rate r_t. In contrast to Galí (2014), b_t is a jump variable and responds to shocks on impact.

Figure 2 presents the impulse response functions for $\{b_t\}, \{q_t\}, \{r_t\}$, and $\{\pi_t\}$ given a 1% shock to e_0. When monetary policy does not respond to bubbles ($\phi_b = 0$), a positive shock to expand the new bubble u_0 at date 0 crowds out the value of old bubbles b_0 and dampens the aggregate bubble q_0. When $\phi_b > 0$, the central bank will cut the interest rate and hence the fall of the old and aggregate bubbles is mitigated. Thus a leaning-against-the-wind policy lowers the bubble volatility in response to the bubble shock.

When $\phi_b < 0$, the old and aggregate bubbles may rise on impact in response to a positive bubble shock. When the central bank cuts the interest rate to encourage bubbles, this effect may dominate the direct negative effect of the rise in the new bubble on the old bubble as shown in equation (15). As shown in Figure 2, when ϕ_b decreases from -2 to -5, the old and aggregate bubbles are dampened and the fall of interest rate is also mitigated. If bubbles expanded, the central bank would cut the interest rate more, which in turn would encourage bubbles further.

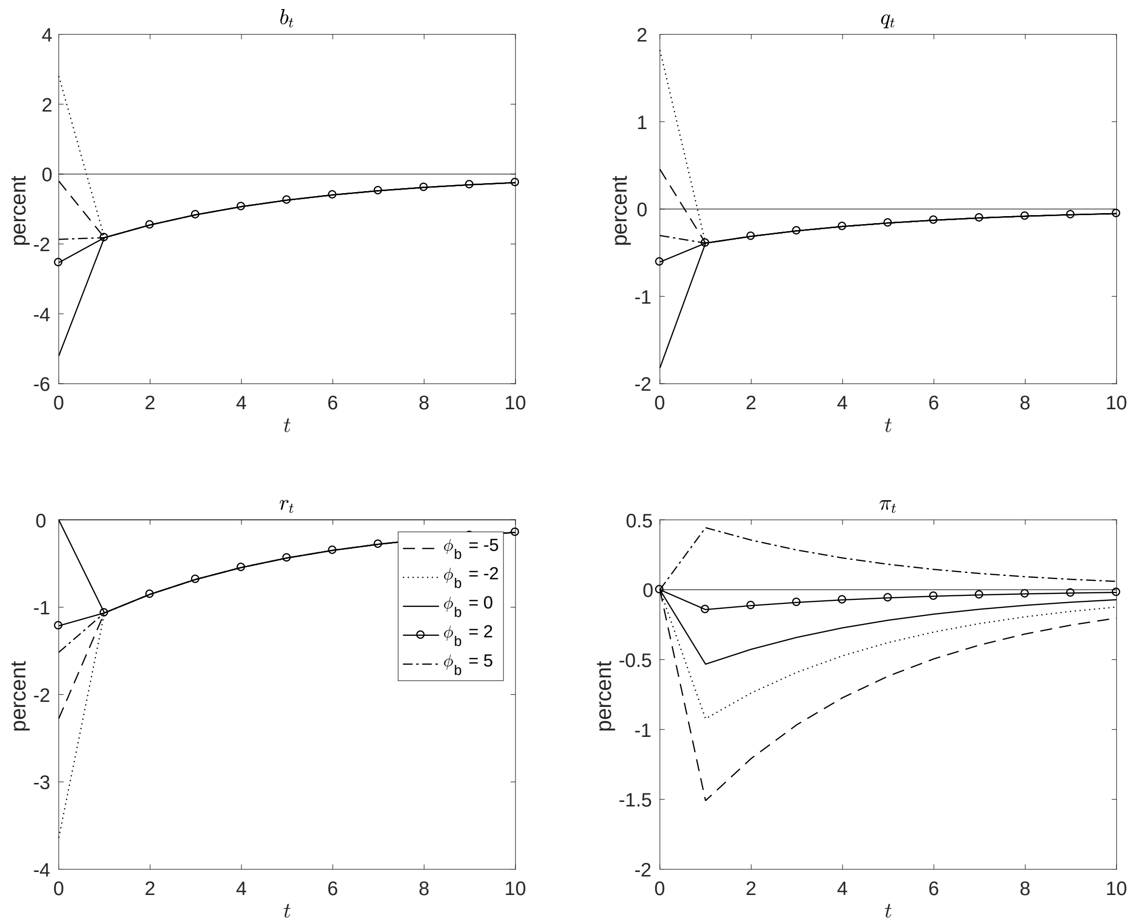

Figure 2: Impulse Responses to A New Bubble Shock

Note: This figure plots the impulse response functions for a one percent positive new bubble shock, in percentage deviation from the steady state. The parameter values are $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, and $\sigma_e^2 = 0.01$. We focus on the unstable bubbly steady state with $B = 0.1458$.

Since firms adjust price one period in advance before shocks are realized, the inflation rate π_t is predetermined. Thus it does not respond to the bubble shock on impact. As shown in Figure 2, it may rise or fall in the second period depending on the value of ϕ_b. In Appendix C we show that the inflation rate around the unstable bubbly steady state is given by

$$\pi_t = \frac{\rho(R-1)[\rho(\epsilon B+1)+(1+\phi_b)(\beta\epsilon BR-\rho)]}{\phi_\pi(\chi-\rho)}u_{t-1}.$$

If $\phi_b = 0$, the inflation rate falls in the second period because $R < 1$ and $\chi > 1$. The central bank can stabilize inflation by two strategies: First, it can set ϕ_π at an arbitrary large value and set ϕ_b at a finite value. Second, it can set ϕ_π at a finite value and set $\phi_b = \rho(\epsilon B+1)/\left(\rho-\beta\epsilon BR\right)-1$.

In Galí's (2014) model, inflation is not a source of welfare losses given synchronized price-setting and an inelastic labor supply. Thus it is not optimal for the central bank to stabilize inflation. To study optimal monetary policy, we follow Galí (2014) to take the unconditional mean of an agent's lifetime utility as a welfare criterion. In a neighborhood of a steady state, we can derive the second-order approximation to the mean:

$$E\left[\ln C_{1,t}+\beta\ln C_{2,t+1}\right] \simeq \ln C_1+\beta\ln C_2-\frac{1}{2}\left(Var\left(c_{1,t}\right)+Var\left(c_{2,t}\right)\right).$$

By the resource constraint $C_{1,t}+C_{2,t}=1$, $Var\left(c_{1,t}\right)$ is proportional to $Var\left(c_{2,t}\right)$. Thus the optimal monetary policy that maximizes welfare will minimize the variance of

$$c_{2,t} = \left(1-\Gamma\right)d_t+\Gamma b_t.$$

In Appendix C we show that

$$d_t = \frac{\chi(R-1)[\phi_b\left(\rho-\epsilon B\beta R\right)-\epsilon B(\beta R+\rho)]}{\beta R^2(1+\phi_b)(\chi-\rho)}e_t.$$

Thus minimizing the volatility of dividends calls for setting

$$\phi_b = \frac{\epsilon B\left(\beta R+\rho\right)}{\epsilon B\beta R-\rho}.$$

However this policy would raise the volatility of the old bubble because it is minimized at a different value given in (14). Thus optimal monetary policy trades off between the volatility of dividends and the volatility of the old bubble.

Note that b_t and d_t are also correlated. In Appendix C we derive that

$$c_{2,t} = \frac{\epsilon B\rho(R-1)}{\chi-\rho}u_{t-1}+\frac{(R-1)\rho(\phi_b-\epsilon B)}{\beta R(1+\phi_b)(\chi-\rho)}e_t,$$

and

$$Var(c_{2,t}) = \left(\frac{\epsilon B\rho(R-1)}{\chi-\rho}\right)^2(1-\rho^2)^{-1}\sigma_e^2+\left[\frac{(R-1)\rho(\phi_b-\epsilon B)}{\beta R(1+\phi_b)(\chi-\rho)}\right]^2\sigma_e^2.$$

From this equation we can show that the optimal coefficient is given by $\phi_b = \epsilon B > 0$. Thus the leaning-against-the-wind policy is optimal. Moreover the optimal coefficient increases with the size of the bubble. Figure 3 illustrates the relation between ϕ_b and $Var\left(c_{2t}\right)$. The welfare loss is minimized at $\phi_b = 0.875$.

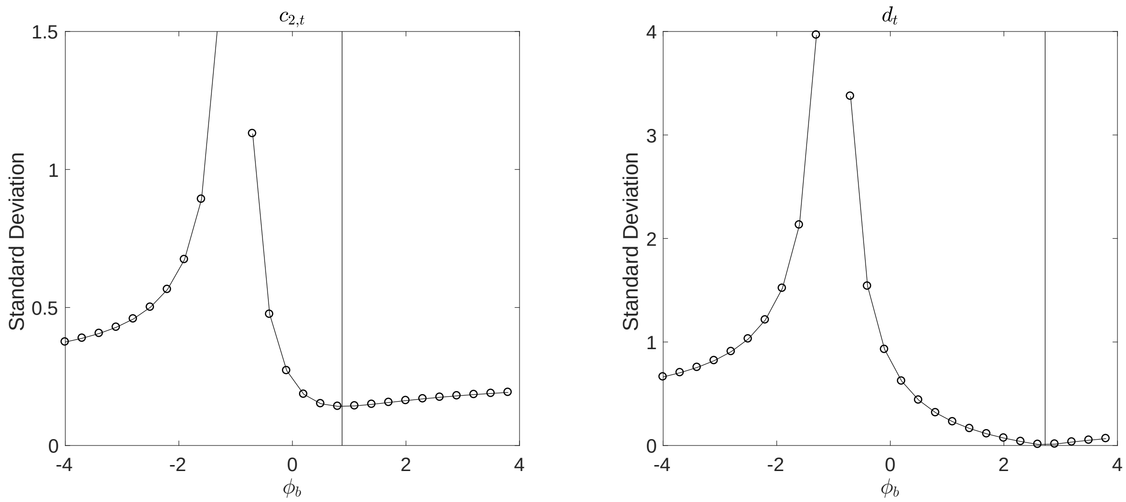

Figure 3: Monetary Policy and Welfare

Note: This figure plots the standard deviations of dividend d_t and consumption of the old $c_{2,t}$ for various values of ϕ_b. The vertical lines indicate the values of ϕ_b that minimize the standard deviation of consumption and dividend respectively. The parameter values are $\beta = 1$, $\epsilon = 6$, $U = 0.175$, $\phi_\pi = 2$, $\rho = 0.8$, and $\sigma_e^2 = 0.01$. We focus on the unstable bubbly steady state with $B = 0.1458$.

4 Learning and Equilibrium Selection

There are multiple (deterministic) steady states and multiple REE solutions in Galí (2014). We will use learning as a selection device to select a particular steady state and a particular REE solution. To understand the basic idea, we consider an economic model with a solution described as a particular parameter vector $\bar{\varphi}$ (e.g., the parameters of an autoregressive process or a steady state). Under adaptive learning agents do not know $\bar{\varphi}$ but estimate it from data using a statistical procedure such as least squares. This leads to estimates φ_t at time t and the question is whether $\varphi_t \to \bar{\varphi}$ as $t \to \infty$. Evans and Honkapohja (2001) show that, for a wide range of economic examples and learning rules, convergence is governed by the corresponding E-stability condition, i.e., the local asymptotic stability of $\bar{\varphi}$ under the differential equation

$$\frac{d\varphi}{d\tau} = T(\varphi) - \varphi, \tag{16}$$

where τ denotes notional or virtual time, $T(\varphi)$ is the mapping from the perceived law of motion (PLM) φ to the implied actual law of motion (ALM) $T(\varphi)$. In the following analysis we will check the E-stability condition.

4.1 Learning a Steady State

We start by the steady states. It is clear that any bubbly steady states Pareto dominates the bubbleless steady state. The following result shows that the unstable bubbly steady state Pareto dominates the stable bubbly steady state for a fixed size of new bubble.

Proposition 3 *For any fixed* $U \in \left(0, \bar{U}\right)$, *the bubbly unstable steady state Pareto dominates the bubbly stable steady state.*

Which steady state is E-stable? We consider the deterministic dynamical system in (1) where we replace B_{t+1} by a forecast B_{t+1}^e. Suppose that the PLM is $B_{t+1} = \Phi$ for an arbitrary Φ. Then the ALM is $B_t = H^{-1}(\Phi, U)$ by (1). The differential equation is given by

$$\frac{d\Phi}{d\tau} = H^{-1}(\Phi, U) - \Phi.$$

Since $0 < \frac{\partial H(B,U)}{\partial B} < 1$ at the stable steady state and $\frac{\partial H(B,U)}{\partial B} > 1$ at the unstable steady state. We immediately obtain the following result.

Proposition 4 *For any fixed* $U \in \left(0, \bar{U}\right)$, *the bubbly unstable steady state is E-stable and the bubbly stable steady state is not E-stable.*

4.2 Learning MSV Solution

In this subsection we study the MSV solution in (13). Suppose the PLM is given by

$$b_t = \varphi_1 u_{t-1} + \varphi_2 e_t.$$

Substitute this PLM into the right-hand side of (10), we can derive the ALM

$$b_t = \tilde{\varphi}_1 u_{t-1} + \tilde{\varphi}_2 e_t,$$

where $(\tilde{\varphi}_1, \tilde{\varphi}_2) = T(\varphi_1, \varphi_2)$ for some mapping T given in Appendix B. Let $\varphi = (\varphi_1, \varphi_2)$. We can analyze the asymptotic stability of the system of differential equations in (16) and derive the following result.

Proposition 5 *The MSV solution in Proposition 2 is E-stable if and only if* $\phi_b > -1$.

In the previous section we have shown that the optimal coefficient ϕ_b is positive for the MSV solution. The preceding proposition shows that the MSV solution under optimal monetary policy is E-stable.

4.3 Learning Sunspot Solution

Now we check the E-stability of the sunspot solutions in Proposition 1. Suppose the PLM is

$$b_t = \varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_2 e_t + \varphi_3 e_{t-1} + \varphi_4 \xi_t + \varphi_5 \xi_{t-1}, \tag{17}$$

Substitute this PLM into the right-hand side of (10), we can derive the ALM

$$b_t = \tilde{\varphi}_0 b_{t-1} + \tilde{\varphi}_1 u_{t-2} + \tilde{\varphi}_2 e_t + \tilde{\varphi}_3 e_{t-1} + \tilde{\varphi}_4 \xi_t + \tilde{\varphi}_5 \xi_{t-1}, \tag{18}$$

where $(\tilde{\varphi}_0, \tilde{\varphi}_1, ..., \tilde{\varphi}_5) = T(\varphi_0, ..., \varphi_5)$ for some mapping T given in Appendix B. Let $\varphi = (\varphi_0, ..., \varphi_5)$. We can analyze the asymptotic stability of the system of differential equations in (16) and derive the following result.

Proposition 6 *For $\phi_b > -1$ the sunspot solution in Proposition 1 is not E-stable.*

Galí (2014) shows that the optimal response coefficient ϕ_b that minimizes the welfare loss is greater than -1. Proposition 6 shows that the equilibrium under this optimal policy is not E-stable.

5 Conclusion

In this paper we have shown that Galí's (2014) counterintuitive results are driven by his choice of a backward-looking sunspot solution around a stable bubbly steady state. His model also features a continuum of bubbly steady states. When deriving the unique forward-looking MSV solution around a bubbly steady state, we obtain results that are consistent with the conventional views. We apply learning as a selection device to select steady state and equilibrium. We find that the unstable bubbly steady state is E-stable and the associated MSV equilibrium is E-stable under optimal monetary policy. But the stable bubbly steady state is not E-stable and the associated sunspot equilibrium is not E-stable under optimal monetary policy. Thus the unstable bubbly steady state and the associated MSV equilibrium should be the focus for policy analysis.

In an infinite-horizon framework without recurrent creation of new bubbles, Miao and Wang (2018) prove that the economy has two steady states. The local equilibrium around the bubbly steady state is unique and the local equilibrium around the the bubbleless steady state is indeterminate of degree one. Miao, Wang, and Xu (2015) and Dong, Miao, and Wang (2017) incorporate recurrent bubbles and show that the economy has a continuum of bubbly states as in Galí (2014). However, they are unable to prove the stability of these steady states analytically due to the complexity of their multi-dimensional equilibrium systems. In contrast to Galí (2014), their numerical results indicate that each bubbly steady state is a saddle point and the local equilibrium around each bubbly steady state is unique. We suspect that the difference in results may be due to the difference in the infinite-horizon and overlapping-generations frameworks. Further theoretical research is needed to understand this issue.

Appendix

A Deriving Equilibrium Bubble Dynamics

Combining (3) to (5) we can obtain

$$\begin{aligned} r_t &= (1-\Gamma)E_t d_{t+1} + \Gamma E_t b_{t+1} + \beta R((1-\Gamma)d_t + \Gamma b_t) \\ &= \Gamma E_t b_{t+1} + \beta R((1-\Gamma)d_t + \Gamma b_t), \end{aligned}$$

where we have used $E_t d_{t+1} = 0$ by (8) in the second equality. Combining the equation above with (6) and (7) yields

$$r_t = \Gamma(r_t + Rb_t + (1-R)u_t) + \beta R((1-\Gamma)d_t + \Gamma b_t).$$

We substitute $\Gamma = \epsilon B/(\epsilon B + 1)$ into the equation above to obtain

$$r_t = \epsilon BR(1+\beta)b_t + \epsilon B(1-R)u_t + \beta R d_t. \tag{A.1}$$

Taking expectations conditional on information at time $t-1$ we obtain

$$E_{t-1}r_t = \epsilon BR(1+\beta)E_{t-1}b_t + \epsilon B(1-R)E_{t-1}u_t, \tag{A.2}$$

where we have used $E_{t-1}d_t = 0$. We use equation (A.2) and interest rate rule (9) to derive

$$\begin{aligned} r_t - E_{t-1}r_t &= \phi_\pi(\pi_t - E_{t-1}\pi_t) + \phi_b(q_t - E_{t-1}q_t) \\ &= \phi_b(q_t - E_{t-1}q_t) \\ &= \phi_b R(b_t - E_{t-1}b_t) + \phi_b(1-R)(u_t - E_{t-1}u_t), \end{aligned} \tag{A.3}$$

where the second equality follows from $\pi_t = E_{t-1}\pi_t$ due to price stickiness and we use (6) to substitute for q_t to derive the third equality. Using (A.2) and (A.3) we derive

$$\begin{aligned} r_t &= r_t - E_{t-1}r_t + E_{t-1}r_t \\ &= \phi_b R b_t + (\epsilon B(1+\beta) - \phi_b)RE_{t-1}b_t + \phi_b(1-R)u_t + (\epsilon B - \phi_b)(1-R)E_{t-1}u_t. \end{aligned}$$

Now we substitute the equation above into (7) and use (6) to derive

$$\begin{aligned} E_t b_{t+1} &= Rb_t + (1-R)u_t \\ &\quad + \phi_b R b_t - (\phi_b - \epsilon B(1+\beta))RE_{t-1}b_t + \phi_b(1-R)u_t - (\phi_b - \epsilon B)(1-R)E_{t-1}u_t \\ &= (\phi_b + 1)Rb_t - (\phi_b - \epsilon B(1+\beta))RE_{t-1}b_t + (\phi_b + 1)(1-R)u_t - (\phi_b - \epsilon B)(1-R)E_{t-1}u_t. \end{aligned}$$

We then obtain (10). Q.E.D.

B Proofs

Proof of Proposition 1: Conjecture that the solution takes the form

$$b_t = \varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_2 e_t + \varphi_3 e_{t-1} + \varphi_4 \xi_t + \varphi_5 \xi_{t-1},$$

where $\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4$, and φ_5 are coefficients to be determined. Substituting this solution into (10) yields

$$\begin{aligned} b_t &= \frac{1}{R(\phi_b+1)}\left[\varphi_0 b_t + \varphi_1 u_{t-1} + \varphi_3 e_t + \varphi_5 \xi_t\right] \\ &+ \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_3 e_{t-1} + \varphi_5 \xi_{t-1}\right) \\ &+ \frac{R-1}{R}\left(\rho u_{t-1} + e_t\right) + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho u_{t-1}. \end{aligned}$$

That is,

$$\begin{aligned} b_t &= \frac{1}{R(\phi_b+1)}[\varphi_0\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_2 e_t + \varphi_3 e_{t-1} + \varphi_4 \xi_t + \varphi_5 \xi_{t-1}\right) \\ &+ \varphi_1\left(\rho u_{t-2} + e_{t-1}\right) + \varphi_3 e_t + \varphi_5 \xi_t] \\ &+ \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\left(\varphi_0 b_{t-1} + \varphi_1 u_{t-2} + \varphi_3 e_{t-1} + \varphi_5 \xi_{t-1}\right) \\ &+ \frac{R-1}{R}\left(\rho^2 u_{t-2} + \rho e_{t-1} + e_t\right) + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\left(\rho^2 u_{t-2} + \rho e_{t-1}\right). \end{aligned}$$

Using the conjectured form for b_t again and matching coefficients, we obtain

$$\varphi_0 = \frac{1}{R(\phi_b+1)}\varphi_0^2 + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_0, \tag{B.1}$$

$$\varphi_1 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_1 + \rho\varphi_1\right) + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_1 + \frac{R-1}{R}\rho^2 + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho^2, \tag{B.2}$$

$$\varphi_2 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_2 + \varphi_3\right) + \frac{R-1}{R}, \tag{B.3}$$

$$\varphi_3 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_3 + \varphi_1\right) + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}\varphi_3 + \frac{R-1}{R}\rho + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b+1)R}\rho, \tag{B.4}$$

$$\varphi_4 = \frac{1}{R(\phi_b+1)}\left(\varphi_0\varphi_4 + \varphi_5\right), \tag{B.5}$$

$$\varphi_5 = \frac{1}{R(\phi_b+1)}\varphi_0\varphi_5 + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b+1}. \tag{B.6}$$

There are two solutions for $\varphi_0 : \varphi_0 = 0$ and

$$\varphi_0 = \chi = R\left(1 + \epsilon B\left(1+\beta\right)\right).$$

In a neighborhood of the stable bubbly steady state, we have $\chi \in (0,1)$. The only stationary solution must corresponds to $\varphi_0 = \chi$ as Galí (2014) points out. We can then solve for the other coefficients:

$$\varphi_1 = (1-R)\,(1+\epsilon B)\,\rho,$$
$$\varphi_2 = \frac{\varphi_3 + (R-1)\,(1+\phi_b)}{R\,(\phi_b+1)-\chi},$$
$$\varphi_4 = \frac{\varphi_5}{R(\phi_b+1)-\chi},$$

and φ_3 and φ_5 are arbitrary numbers. Q.E.D.

Proof of Proposition 2: We take expectations conditional on information at time $t-1$ on both sides of (10) to obtain

$$\begin{aligned} E_{t-1}b_t\left[1-\frac{\phi_b-\epsilon B(1+\beta)}{\phi_b+1}\right] &= \frac{1}{R(\phi_b+1)}E_{t-1}b_{t+1} \\ &+ \left[\frac{R-1}{R}+\frac{(\epsilon B-\phi_b)(R-1)}{(\phi_b+1)R}\right]\rho u_{t-1}. \end{aligned}$$

This implies that

$$E_{t-1}b_t = \frac{1}{R[1+\epsilon B(1+\beta)]}E_{t-1}b_{t+1} - \frac{(1-R)(\epsilon B+1)}{R(1+\epsilon B(1+\beta))}\rho u_{t-1}.$$

By iterating the equation above forward we can derive

$$\begin{aligned} E_{t-1}b_t &= -\frac{(1-R)(\epsilon B+1)}{R(1+\epsilon B(1+\beta))}\left(\frac{1}{1-\rho/R[1+\epsilon B(1+\beta)]}\right)\rho u_{t-1} \\ &= -\frac{(1-R)(\epsilon B+1)}{\chi-\rho}\rho u_{t-1}, \end{aligned}$$

under the condition $\chi \equiv R[1+\epsilon B(1+\beta)] > 1$. Therefore we also have

$$E_t b_{t+1} = -\frac{(1-R)(\epsilon B+1)}{\chi-\rho}\rho u_t = -\frac{(1-R)(\epsilon B+1)}{\chi-\rho}(\rho^2 u_{t-1}+\rho e_t).$$

Substituting the preceding expressions for $E_t b_{t+1}$ and $E_{t-1}b_t$ into (10), we obtain the rational expectations solution in (13). Q.E.D.

Proof of Proposition 3: We use lifetime utility as the welfare criterion. Define the steady state welfare as

$$W_f \equiv \ln(C_1) + \beta\ln(C_2),$$

where C_1 and C_2 denote the steady-state consumption of a consumer in his young and old. In a steady state we have $C_1 = 1/\mathcal{M} - B$ and $C_2 = 1 - 1/\mathcal{M} + B$. Therefore

$$W_f = \ln(\frac{1}{\mathcal{M}} - B) + \beta\ln(1-\frac{1}{\mathcal{M}}+B).$$

We can compute

$$\frac{\partial W_f}{\partial B} = \frac{(\frac{1}{\mathcal{M}}-\frac{1}{1+\beta})-B}{C_1C_2(1+\beta)}.$$

Denote $B^* \equiv 1/\mathcal{M} - 1/(1+\beta)$. Note that $B^* > 0$ under the condition $\mathcal{M} < 1+\beta$. This implies that welfare is increasing with B when $B < B^*$. As shown in Galí (2014) Lemma 1, for any $U \in (0, \bar{U})$ the model has two bubbly steady states. Moreover the stable one B_s is always less than the unstable one B_u. Thus to show the welfare is greater at B_u than at B_s, it suffices to show that $B_u < B^*$.

Since B_u is the larger root of equation $H(B, U) = B$, we have

$$B_u = \frac{-(1 + U - \frac{1+\beta}{\mathcal{M}}) + \sqrt{(1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U}}{2(1+\beta)}.$$

Therefore

$$B_u - B^* = \frac{(1 - U - \frac{1+\beta}{\mathcal{M}}) + \sqrt{(1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U}}{2(1+\beta)}.$$

Note that $1 - U - \frac{1+\beta}{\mathcal{M}} < 0$ by (2). To show $B_u < B^*$, it suffices to show that

$$(1 - U - \frac{1+\beta}{\mathcal{M}})^2 > (1 + U - \frac{1+\beta}{\mathcal{M}})^2 - 4(1+\beta)(1 - \frac{1}{\mathcal{M}})U.$$

This inequality is equivalent to

$$4(1+\beta)U > 4U,$$

which holds true since $U, \beta > 0$. Q.E.D.

Proof of Proposition 4: See the main text.

Proof of Proposition 5: Conjecture the solution takes the form

$$b_t = \varphi_1 u_{t-1} + \varphi_2 e_t.$$

This is the perceived law of motion (PLM). Substitute this PLM into the right hand side of (10) to get the actual law of motion (ALM):

$$\begin{aligned} b_t &= \frac{1}{R(\phi_b + 1)}\varphi_1(\rho u_{t-1} + e_t) + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b + 1}\varphi_1 u_{t-1} \\ &\quad + \frac{R-1}{R}(\rho u_{t-1} + e_t) + \frac{(\epsilon B - \phi_b)(R-1)}{(\phi_b + 1)R}\rho u_{t-1} \\ &= \tilde{\varphi}_1 u_{t-1} + \tilde{\varphi}_2 e_t. \end{aligned}$$

This defines a mapping from the PLM to the ALM, $(\tilde{\varphi}_1, \tilde{\varphi}_2) = T(\varphi_1, \varphi_2)$, where

$$\begin{aligned} \tilde{\varphi}_1 &= \left[\frac{\rho}{R(\phi_b + 1)} + \frac{\phi_b - \epsilon B(1+\beta)}{\phi_b + 1}\right]\varphi_1 + \frac{R-1}{R}\rho\left(\frac{\epsilon B - \phi_b}{\phi_b + 1} + 1\right), \\ \tilde{\varphi}_2 &= \frac{1}{R(\phi_b + 1)}\varphi_1 + \frac{R-1}{R}. \end{aligned}$$

Expectational stability is determined by the following matrix differential equation:

$$\begin{aligned}\frac{d(\varphi_1,\varphi_2)}{d\tau} &= T(\varphi_1,\varphi_2)-(\varphi_1,\varphi_2)\\ &= \begin{bmatrix}\left(\frac{\rho-\chi}{R(\phi_b+1)}\right)\varphi_1+\frac{R-1}{R}\rho\left(\frac{\epsilon B-\phi_b}{\phi_b+1}+1\right)u_{t-1}\\ \frac{1}{R(\phi_b+1)}\varphi_1-\varphi_2+\frac{R-1}{R}\end{bmatrix}.\end{aligned}$$

The Jacobian matrix (evaluated at the REE solution) is given by

$$J=\begin{bmatrix}\frac{\rho-\chi}{R(\phi_b+1)} & 0\\ \frac{1}{R(\phi_b+1)} & -1\end{bmatrix}.$$

The eigenvalues of the Jacobian matrix are

$$\lambda_1=-1,\quad \lambda_2=\frac{\rho-\chi}{R(\phi_b+1)}.$$

E-stability requires that all eigenvalues are less than zero. Since in the unstable steady state $\chi>1$, E-stability requires $\phi_b>-1$. Q.E.D.

Proof of Proposition 6: Suppose that the PLM for b_t is given by (17). Substituting it into (10) yields the ALM

$$b_t=\tilde{\varphi}_0 b_{t-1}+\tilde{\varphi}_1 u_{t-2}+\tilde{\varphi}_2 e_t+\tilde{\varphi}_3 e_{t-1}+\tilde{\varphi}_4\xi_t+\tilde{\varphi}_5\xi_{t-1}, \tag{B.7}$$

where $\tilde{\varphi}_0,...,\tilde{\varphi}_5$ are given by the expressions on the right-hand sides of equations (B.1) through (B.6). Let φ denote the vector of $\varphi_0,...,\varphi_5$ and $T(\varphi)$ denote the vector of $\tilde{\varphi}_0,...,\tilde{\varphi}_5$. We consider the differential equation

$$\frac{d\varphi}{d\tau}=T(\varphi)-\varphi,$$

where $T(\varphi)$ can be explicitly written as

$$\begin{aligned}
\tilde{\varphi}_0 &= \frac{1}{R(\phi_b+1)}\varphi_0^2+\frac{\phi_b-\epsilon B(1+\beta)}{\phi_b+1}\varphi_0,\\
\tilde{\varphi}_1 &= \frac{1}{R(\phi_b+1)}(\varphi_0+\rho)\varphi_1+\frac{\phi_b-\epsilon B(1+\beta)}{\phi_b+1}\varphi_1+\frac{(\epsilon B+1)(R-1)}{(\phi_b+1)R}\rho^2,\\
\tilde{\varphi}_2 &= \frac{1}{R(\phi_b+1)}(\varphi_0\varphi_2+\varphi_3)+\frac{R-1}{R},\\
\tilde{\varphi}_3 &= \frac{1}{R(\phi_b+1)}(\varphi_0\varphi_3+\varphi_1)+\frac{\phi_b-\epsilon B(1+\beta)}{\phi_b+1}\varphi_3+\frac{(\epsilon B+1)(R-1)}{(\phi_b+1)R}\rho,\\
\tilde{\varphi}_4 &= \frac{1}{R(\phi_b+1)}(\varphi_0\varphi_4+\varphi_5),\\
\tilde{\varphi}_5 &= \frac{1}{R(\phi_b+1)}\varphi_0\varphi_5+\frac{\phi_b-\epsilon B(1+\beta)}{\phi_b+1}\varphi_5.
\end{aligned}$$

The Jacobian matrix of $d\varphi/d\tau$ evaluated at the sunspot solution in Proposition 1 ($\varphi_0 = \chi, \varphi_1 = (1-R)(1-\epsilon B), \varphi_2 = \varphi_2^*, \varphi_3 = \varphi_3^*, \varphi_4 = \varphi_4^*, \varphi_5 = \varphi_5^*$) is

$$J = \begin{bmatrix} \frac{\chi}{R(\phi_b+1)} & 0 & 0 & 0 & 0 & 0 \\ \frac{(1-R)(1-\epsilon B)}{R(\phi_b+1)} & \frac{\rho}{R(\phi_b+1)} & 0 & 0 & 0 & 0 \\ \frac{1}{R(\phi_b+1)}\varphi_2^* & 0 & \frac{\chi}{R(\phi_b+1)}-1 & \frac{1}{R(\phi_b+1)} & 0 & 0 \\ \frac{1}{R(\phi_b+1)}\varphi_3^* & \frac{1}{R(\phi_b+1)} & 0 & 0 & 0 & 0 \\ \frac{1}{R(\phi_b+1)}\varphi_4^* & 0 & 0 & 0 & \frac{\chi}{R(\phi_b+1)}-1 & \frac{1}{R(\phi_b+1)} \\ \frac{1}{R(\phi_b+1)}\varphi_5^* & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues are $\frac{\chi}{R(\phi_b+1)}$, $\frac{\rho}{R(\phi_b+1)}$, $\frac{\chi}{R(\phi_b+1)} - 1$, 0, $\frac{\chi}{R(\phi_b+1)} - 1$, and 0. Notice that there are two zero eigenvalues because $\tilde{\varphi}_3 = \varphi_3^*$ and $\tilde{\varphi}_5 = \varphi_5^*$. The other eigenvalues are negative if $\phi_b < -1$. We then complete the proof. Q.E.D.

C Deriving MSV Equilibrium

From Proposition 2 we have the forward-looking MSV solution for the old bubble:

$$b_t = \frac{(R-1)(\epsilon B+1)}{\chi-\rho}\rho u_{t-1} + \frac{R-1}{R}\left[\frac{\rho(\epsilon B+1)}{(\phi_b+1)(\chi-\rho)}+1\right]e_t. \tag{C.1}$$

We use this solution to derive solutions for other variables in the model. By (6) we obtain the solution for q_t:

$$\begin{aligned} q_t &= Rb_t + (1-R)u_t \\ &= (1-R)\left[1-\frac{R(\epsilon B+1)}{\chi-\rho}\right]\rho u_{t-1} + (R-1)\left[\frac{\rho(1+\epsilon B)}{(\phi_b+1)(\chi-\rho)}\right]e_t. \end{aligned} \tag{C.2}$$

By (7) we obtain the solution for r_t:

$$\begin{aligned} r_t &= E_t b_{t+1} - q_t \\ &= (R-1)\left[\frac{(\epsilon B+1)(\rho-R)}{\chi-\rho}+1\right]\rho u_{t-1} + \frac{\phi_b\rho(R-1)(\epsilon B+1)}{(\phi_b+1)(\chi-\rho)}e_t. \end{aligned} \tag{C.3}$$

By (9) we obtain the solution for π_t:

$$\pi_t = \frac{(R-1)[(\epsilon B+1)\rho + (\phi_b+1)(\epsilon BR\beta-\rho)]}{\phi_\pi(\chi-\rho)}\rho u_{t-1}.$$

Substituting (C.3) and (C.1) into (A.1) we obtain the solution for d_t:

$$d_t = \frac{\chi(R-1)[\phi_b\rho - \epsilon B(\beta R(1+\phi_b)+\rho)]}{\beta R^2(1+\phi_b)(\chi-\rho)}e_t.$$

By (5) we obtain the solution for $c_{2,t}$:

$$\begin{aligned} c_{2,t} &= (1-\Gamma)d_t + \Gamma b_t \\ &= \frac{\epsilon B\rho(R-1)}{\chi-\rho}u_{t-1} + \frac{\rho(R-1)(\phi_b-\epsilon B)}{\beta R(1+\phi_b)(\chi-\rho)}e_t. \end{aligned} \tag{C.4}$$

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