Supplemental Appendix to "Robust Contracts in Continuous Time"

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This appendix consist of two sections. In Section A we adopt the Chen-Epstein (2002) recursive multiple-priors utility model to study the robust contracting problem. We compare this case with the robust contracting problem studied in the paper. In Section B we study a model with risk aversion only and compare the solution with our robust contracting solution. We also establish some observational equivalence results.

A Contract with Chen-Epstein Utility

Suppose that the principal's preferences are represented by recursive multiple-priors utility proposed by Chen and Epstein (2002). The contracting problem is given by

$$\max_{(C,\tau,a)} \min_{\{h:|h_t|\leq\kappa\}} E^{Q^h}\left[\int_0^\tau e^{-rt}(dX_t - dC_t) + e^{-r\tau}L\right], \tag{A.1}$$

subject to the incentive constraint and the participation constraint described in the paper.

By the Girsanov theorem, we write the dynamics of W under Q^h as

$$dW_t = \gamma W_t dt - dC_t - \lambda\mu(1-a_t)dt + h_t\phi_t dt + \phi_t dB_t^h. \tag{A.2}$$

We then obtain the HJB equation:

$$rF(W) = \max_{dC\geq 0, \phi\geq\sigma\lambda} \min_{|h|\leq\kappa} \mu + \sigma h - dC + F'(W)(\gamma W - dC + h\phi) + \frac{F''(W)}{2}\phi^2. \tag{A.3}$$

For optimization over ϕ to be well defined, it must be the case that $F''(W) < 0$. Otherwise, F would approach infinity when ϕ approaches infinity.

By an argument similar to that in the paper, we must have $F'(W) \geq -1$. Define $\bar{W}$ as the smallest value such that $F'(W) = -1$. Then the principal pays the agent whenever W_t

hits the boundary $\bar{W}$ and reflects at this point.

Solving for h yields the solution

$$h(F'(W);\phi) = \begin{cases} -\kappa & \text{if } \phi F'(W) + \sigma > 0 \\ \kappa & \text{if } \phi F'(W) + \sigma < 0 \\ \text{any in } [-\kappa, \kappa] & \text{if } \phi F'(W) + \sigma = 0 \end{cases}. \tag{A.4}$$

Substituting this solution into (A.3), we obtain

$$rF(W) = \max_{\phi \geq \sigma\lambda} \ \mu + F'(W)\gamma W + \left(F'(W)\,\phi + \sigma\right) h(F'(W);\phi) + \frac{1}{2}F''(W)\phi^2. \tag{A.5}$$

We consider two cases.

Case 1 $F'(W) \geq 0$

By the incentive constraint $0 < \sigma\lambda \leq \phi$, we can see from (A.4) that $h(F'(W);\phi) = -\kappa$. In this case the HJB equation becomes

$$rF(W) = \max_{\sigma\lambda \leq \phi} \mu + F'(W)\gamma W - \left(F'(W)\,\phi + \sigma\right)\kappa + \frac{1}{2}F''(W)\phi^2.$$

Since F is concave, the first-order condition gives the unconstrained maximizer

$$\phi\,(W) = \frac{F'(W)\kappa}{F''(W)} \leq 0,$$

which violates the incentive constraint. Thus the optimal sensitivity is given by $\phi^*(W) = \sigma\lambda$. That is, the incentive constraint binds. This case happens on the left increasing branch of F for low values of W since F is concave. Intuitively, for low values of W there is a strong concern for liquidation. The optimal contract should expose the agent to minimum uncertainty and hence the optimal sensitivity is such that the incentive constraint just binds. This feature is similar to that in our model with the multiplier preferences.

Case 2 $-1 \leq F'(W) < 0$

Define

$$\tilde{\phi}(W) \equiv -\sigma/F'(W).$$

Since $F'(W) \in [-1, 0)$, we have $\tilde{\phi}(W) \in [\sigma, \infty)$. It follows from $\lambda \in (0,1)$ that $\sigma\lambda < \tilde{\phi}(W)$.

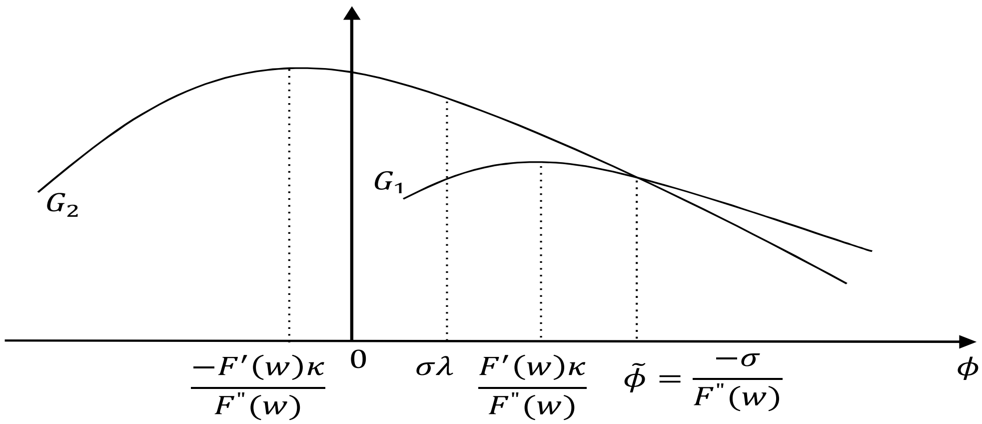

Figure 1: Functions G_1 and G_2 for the case where $\sigma\lambda \leq \frac{F'(W)\kappa}{F''(W)} \leq \tilde{\phi}(W)$.

We can then rewrite the HJB equation (A.5) as

$$rF(W) = \max \left\{ \max_{\sigma\lambda \leq \phi \leq \tilde{\phi}(W)} \underbrace{\mu + F'(W)\gamma W - \left(F'(W)\phi + \sigma\right)\kappa + \frac{1}{2}F''(W)\phi^2}_{G_1(\phi;W)}, \right. \tag{A.6}$$

$$\left. \max_{\tilde{\phi}(W) \leq \phi} \underbrace{\mu + F'(W)\gamma W + \left(F'(W)\phi + \sigma\right)\kappa + \frac{1}{2}F''(W)\phi^2}_{G_2(\phi;W)} \right\}.$$

When $\phi = \tilde{\phi}(W)$, G_1 and G_2 are equal. Since F is concave, $F'(W)\kappa/F''(W) > 0$ achieves the unconstrained maximum for $G_1(\phi; W)$ and $-F'(W)\kappa/F''(W) < 0$ achieves the unconstrained maximum for $G_2(\phi; W)$. Incorporating constraints, we consider three cases.

First, when

$$\sigma\lambda \leq \frac{F'(W)\kappa}{F''(W)} \leq \tilde{\phi}(W),$$

$G_1(\phi^*(W); W)$ is the maximum on the right-hand side of equation (A.6), where the optimal sensitivity is

$$\phi^*(W) = \frac{F'(W)\kappa}{F''(W)},$$

and the worst-case density generator is $h^*(W) = -\kappa$. Figure 1 illustrates the solution.

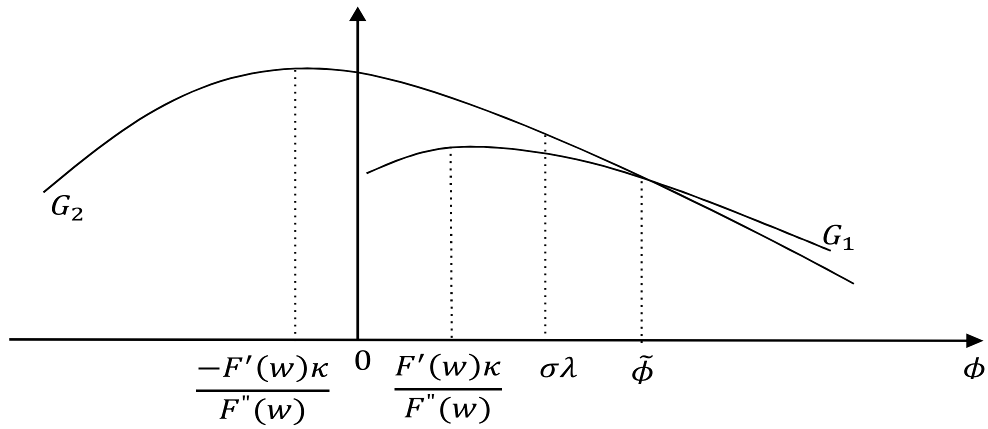

Figure 2: Functions G_1 and G_2 for the case where $\frac{F'(W)\kappa}{F''(W)} \leq \sigma\lambda \leq \tilde{\phi}(W)$.

Second, when

$$\frac{F'(W)\kappa}{F''(W)} \leq \sigma\lambda \leq \tilde{\phi}(W),$$

$G_1(\phi^*(W); W)$ is the maximum on the right-hand side of equation (A.6), where the optimal sensitivity is $\phi^*(W) = \sigma\lambda$ and the worst-case density generator is $h^*(W) = -\kappa$. Figure 2 illustrates the solution.

Third, when

$$\sigma\lambda \leq \tilde{\phi}(W) \leq \frac{F'(W)\kappa}{F''(W)},$$

the optimal sensitivity is $\phi^*(W) = \tilde{\phi}(W)$ and $G_1(\phi^*(W); W) = G_2(\phi^*(W); W)$. In this case the worst-case density generator takes any value in $[-\kappa, \kappa]$ by (A.4). Figure 3 illustrates the solution.

Combining the three cases above, the optimal sensitivity is given by

$$\phi^*(W) = \max\left\{\sigma\lambda, \min\left\{\frac{F'(W)\kappa}{F''(W)}, \tilde{\phi}(W)\right\}\right\}.$$

It is possible that

$$\min\left\{\frac{F'(W)\kappa}{F''(W)}, \tilde{\phi}(W)\right\} > \sigma\lambda$$

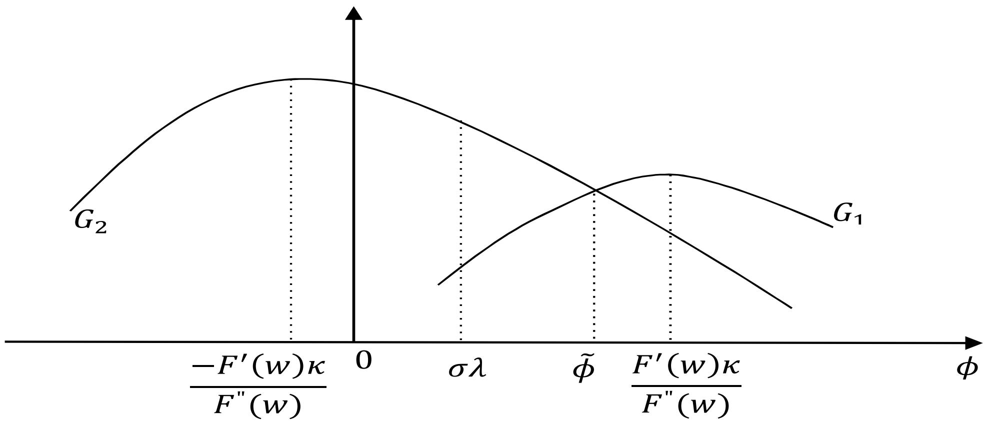

Figure 3: Functions G_1 and G_2 for the case where $\sigma\lambda \leq \tilde{\phi}(W) \leq \frac{F'(W)\kappa}{F''(W)}$.

so that the incentive constraint does not bind. We expect that this case happens when W is sufficiently large. For example, when W reaches the payout boundary $W = \bar{W}$, the HJB equation becomes

$$rF(\bar{W}) = \max\{\max_{\sigma\lambda\leq\phi\leq\sigma} \{\mu - \gamma\bar{W} - (-\phi+\sigma)\,\kappa\}, \max_{\sigma\leq\phi}\{\mu - \gamma\bar{W} + (-\phi+\sigma)\,\kappa\}\}.$$

It follows that the optimal sensitivity is given by $\phi = \sigma > \sigma\lambda$. Thus at the payout boundary the agent bears all the uncertainty. This feature is the same as in our model with the multiplier preferences. Intuitively, when W is sufficiently large, the concern for ambiguity is large and hence the principal wants the agent to share more ambiguity.

In summary, the Chen-Epstein model and the multiplier utility model in continuous time deliver some similar properties of the optimal contract. In particular, the incentive constraint binds when the agent's continuation value is low and does not bind when the agent's continuation value is sufficiently high. The key difference is that the corner solution gives the worst-case belief in the Chen-Epstein model, while the worst-case belief in the multiplier utility model is time varying and state dependent. The time-varying worst-case belief in the multiplier utility model has much more transparent asset pricing implications. Another difference is that the principal's value function is globally concave in the Chen-Epstein model, but this may not be the case in the multiplier utility model. Finally, the multiplier utility model is analytically much more tractable and is numerically much easier to solve.

B Comparison with Risk Aversion

Does risk aversion have the same implications as ambiguity aversion? To address this question, we study a contracting problem with a risk averse principal who has no concern for robustness. Suppose that the principal derives utility from a consumption process (C_t^p) according to time-additive expected utility

$$E^{P^a}\left[\int_0^\infty e^{-rt}u(C_t^p)dt\right],$$

where we take $u(c^p) = -\exp\left(-\alpha c^p\right)/\alpha$ for tractability. Here $\alpha > 0$ represents the CARA parameter. Risk neutrality corresponds to $\alpha = 0$. Since it is generally impossible to have $C_t^p dt + dC_t = dY_t = \mu a_t dt + \sigma dB_t^a$, we suppose that the principal can borrow and save at the interest rate r. Suppose that the agent cannot borrow or save. We use X_t to denote the principal's wealth level and write his budget constraint before liquidation as

$$dX_t = rX_t dt - C_t^p dt - dC_t + \mu a_t dt + \sigma dB_t^a,\ X_0 \text{ given}, \tag{B.1}$$

for $0 \le t < \tau$. At the liquidation time τ, the principal obtains liquidation value L and starts with wealth $X_{\tau-} + L$. The budget constraint after liquidation is given by

$$dX_t = rX_t dt - C_t^p dt,\ X_\tau = X_{\tau-} + L, \tag{B.2}$$

for $t \ge \tau$. The principal selects a contract (C^p, C, τ, a) to solve the following problem:

Problem B.1 *(contract with risk aversion)*

$$\max_{(C^p,C,\tau,a)} \quad E^{P^a}\left[\int_0^\infty e^{-rt}u(C_t^p)dt\right],$$

subject to (B.1), and (B.2), and

$$E^{P^a}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] \ge E^{P^{\hat{a}}}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-\hat{a}_s)ds)\right], \tag{B.3}$$

$$E^{P^a}\left[\int_0^\tau e^{-\gamma s}(dC_s + \lambda\mu(1-a_s)ds)\right] = W_0, \tag{B.4}$$

where $\hat{a}_s \in \{0,1\}$ *and* $W_0 \ge 0$ *is given.*

This particular problem has not been studied in the literature and is of independent interest.[1] In the benchmark model and the model in Section 2, the principal is not allowed to save and consumes the residual profits $dY_t - dC_t$ each time. Since the interest rate and the principal's discount rate are identical, the risk neutral principal is indifferent between spending one dollar now and saving this dollar for consumption tomorrow. Thus allowing saving does not affect the optimal contract except that wealth must be added to the principal's value function without saving to obtain the value function with saving. When the principal is risk averse, the wealth level is a new state variable in addition to the agent's continuation value, making our analysis more complicated. We will show below that due to the lack of wealth effect of the CARA utility, we can simplify our problem to a one-dimensional one.

B. 1 Optimal contract with Agency

Let $V(W_0, X_0)$ denote the principal's value function for Problem B.1 when we vary W_0 and X_0. Suppose that implementing high effort $a_t = 1$ is optimal. Then $V(W, X)$ satisfies the heuristic HJB equation

$$\begin{aligned} rV(W,X) \quad = \quad & \max_{C^p,\ c\geq 0,\ \phi\geq\sigma\lambda} -\frac{1}{\alpha}\exp\left(-\alpha C^p\right) + V_W(W,X)(\gamma W - c) \\ & + V_X(W,X)(rX - C^p - c + \mu) \\ & + V_{WW}(W,X)\frac{\phi^2}{2} + V_{XX}(W,X)\frac{\sigma^2}{2} + V_{XW}(W,X)\sigma\phi. \end{aligned} \tag{B.5}$$

The first-order conditions imply that

$$\exp\left(-\alpha C^p\right) = V_X(W,X),$$

$$V_X(W,X) \geq -V_W(W,X) \text{ with equality when } c > 0,$$

$$\phi = \max\left\{-\frac{V_{XW}(W,X)\sigma}{V_{WW}(W,X)}, \sigma\lambda\right\}.$$

The second-order condition for ϕ is $V_{WW}(W,X) < 0$, i.e., V is concave in W.

Conjecture that the value function takes the form

$$V(W,X) = -\frac{1}{\alpha r}\exp\left(-\alpha r\left[X + H\left(W\right)\right]\right), \tag{B.6}$$

where the function H can be interpreted as the certainty equivalent value to the principal. Substituting this guess into the preceding first-order conditions yields the principal's con-

[1] Biais et al (2007, p. 371) point out that an important future research topic is to introduce a risk averse principal and study the relation between expected stock returns and incentive problems.

sumption policy

$$C^p(W,X) = r\left(X + H(W)\right), \tag{B.7}$$

the optimal sensitivity

$$\phi(W) = \max\left\{\frac{\alpha r\sigma H'(W)}{H''(W) - \alpha r H'(W)^2}, \sigma\lambda\right\}, \tag{B.8}$$

and the payout policy

$$H'(W) \geq -1 \text{ with equality when } c > 0. \tag{B.9}$$

The second-order condition for ϕ becomes

$$H''(W) - \alpha r H'(W)^2 < 0. \tag{B.10}$$

Substituting (B.6), (B.7), (B.8), and (B.9) into (B.5) yields an ODE for $H(W)$,

$$rH(W) = \mu + H'(W)\gamma W + H''(W)\frac{\phi(W)^2}{2} - \alpha r\frac{\left[\phi(W)H'(W) + \sigma\right]^2}{2}. \tag{B.11}$$

We now find boundary conditions for this ODE. First, define a cutoff $\bar{W}$ as the lowest value such that

$$H'\left(\bar{W}\right) = -1. \tag{B.12}$$

For $W \in [0, \bar{W})$, $H'(W) > -1$. Then it is optimal to pay the agent according to $dC = \max\left\{W - \bar{W}, 0\right\}$. By the super-contact condition,

$$H''(\bar{W}) = 0. \tag{B.13}$$

Then equation (B.8) implies that $\phi(\bar{W}) = \sigma > \sigma\lambda$.

When $W = 0$, the project is liquidated and the principal obtains the liquidation value L. Since both the discount rate of the principal and the interest rate equal r, we can show that $C^p(0, X) = r(X + L)$ and $V(0, X) = -\exp\left(-\alpha r(X + L)\right)/\left(\alpha r\right)$ so that

$$H(0) = L. \tag{B.14}$$

Proposition 1 *Consider the contracting problem B.1 with risk aversion. Suppose that im-*

plementing high effort $a_t = 1$ is optimal and

$$L < \frac{\mu}{r} - \frac{\alpha\sigma^2}{2}. \tag{B.15}$$

Suppose that there exists a twice continuously differentiable function $H(W)$ satisfying (B.11) with the boundary conditions (B.12), (B.13), and (B.14) such that condition (B.10) holds and $H'(W) > -1$ on $[0, \bar{W})$. Then the principal's value function is given by (B.6) for $W \in [0, \bar{W}]$, the principal's optimal consumption policy is given by (B.7). The contract delivers the initial value $W \in [0, \bar{W}]$ and the optimal payment C^* given in

$$C_t^* = \int_0^t \mathbf{1}_{\{W_s = \bar{W}\}} dC_s^* \tag{B.16}$$

to the agent whose continuation value (W_t) follows the dynamics

$$dW_t = \gamma W_t dt - dC_t^* + \phi(W_t)\, dB_t^1,\ W_0 = W, \tag{B.17}$$

for $t \in [0, \tau]$, where the optimal sensitivity $\phi(W)$ is given in (B.8). When $W > \bar{W}$, the principal's value function is given by $V(W, X) = -\frac{1}{\alpha r}\exp\left(-\alpha r\left[X + H\left(\bar{W}\right) - \left(W - \bar{W}\right)\right]\right)$. The principal pays $W - \bar{W}$ immediately to the agent and the contract continues with the agent's new initial value $\bar{W}$.

Proof. Given the conjecture in (B.6), we can derive

$$\begin{aligned}
V_X(W,X) &= e^{-\alpha r(X+H(W))},\ \ V_{XX}(W,X) = -\alpha r e^{-\alpha r(X+H(W))}, \\
V_W(W,X) &= H'(W)\, e^{-\alpha r(X+H(W))},\ \ V_{XW}(W,X) = -\alpha r H'(W)\, e^{-\alpha r(X+H(W))}, \\
V_{WW}(W,X) &= \left[H''(W) - \alpha r H'(W)^2\right] e^{-\alpha r(X+H(W))}.
\end{aligned}$$

Substituting these expressions into the HJB equation (B.5), we can derive the optimal policies in the proposition. The proof of the optimality follows a similar argument for Propositions 1-3 in the paper. We omit it here. ■

Condition (B.15) is analogous to condition (17) in the paper and ensures that liquidation is inefficient in the optimal contract with risk aversion. We can give a necessary and sufficient condition for the optimality of implementing high effort analogous to that in Proposition 3 of the paper. For simplicity, we omit this result.

We first observe that when $\alpha = 0$, ODE (B.11) reduces to that in DeMarzo and Sannikov (2006). Furthermore, when H is concave, (B.8) implies $\phi(W) = \sigma\lambda$ and hence the incentive constraint always binds. Since the boundary conditions (B.12), (B.13), and (B.14) are identical to those in DeMarzo and Sannikov (2006), the solution for $H(W)$ and $\bar{W}$ must be identical to theirs too. We next turn to the case of risk aversion with $\alpha > 0$ and compare the solution with that in the case of robustness.

B. 2 Limited Observational Equivalence

When $\alpha r = 1/\theta$, equations (27) and (B.8) are identical and hence the two ODEs (26) and (B.11) are identical. In addition, the boundary conditions are the same. The second-order conditions (15) and (B.10) are also identical. By Propositions 2 in the paper and 1, we have the following result:

Proposition 2 *When $\alpha r = 1/\theta$, the robust contract for Problem 3.1 and the optimal contract with risk aversion for Problem B.1 deliver the same liquidation time and payout policy to the agent. Furthermore, $H(W) = F(W)$, where $F(W)$ is the principal's value function in Problem 3.1.*

Given this result, our previous characterization of the robust contract can be applied here. But the interpretation is different. The tradeoff here is between risk sharing and incentives for the risk averse principal. But the tradeoff in the robust contracting problem is between ambiguity sharing and incentives. In that problem, the principal is risk neutral, but ambiguity averse. The endogenous belief heterogeneity is the driving force for the principal and the agent to share model uncertainty.

Note that Proposition 2 shows only a limited observational equivalence between the robust contract and the contract with risk aversion because the principal's consumption policy and value function are different in these two contracts. In particular, the principal's value function $V(W, X)$ is globally concave in W under assumption (B.10) in Proposition 7, but the value function $F(W)$ in Proposition 2 may not be globally concave. Thus, unlike in the robust contracting problem, public randomization is never optimal in the contracting problem with risk aversion. Moreover, the principal's consumption processes in these two contracts are different. With exponential preferences the principal consumes continuously in time, while with linear preferences the principal only consumes at certain points. Technically, the consumption processes are absolutely continuous in the former case and are singular in the latter case.

The preceding limited observational equivalence has a different nature than the equivalence between robustness and a special class of recursive utility (i.e., risk-sensitive utility) established by Hansen et al (2006). To see this, we consider a discrete-time approximation for intuition. Let the time interval be dt. The time-additive expected utility process (U_t) derived from a consumption process (c_t) satisfies

$$U_t = u(c_t)\,dt + e^{-rdt}E_t[U_{t+dt}],$$

where E_t is the conditional expectation operator with respect to a reference measure P. The function u characterizes both risk aversion and intertemporal substitution. The multiplier utility process (U_t) with a concern for robustness introduced by Hansen and Sargent satisfies the recursion

$$U_t = u(c_t)\,dt + e^{-rdt}\left[\inf_Q E_t^Q[U_{t+dt}] + \theta E_t\Phi\left(\frac{\xi_{t+dt}^Q}{\xi_t^Q}\right)\right],$$

where $\Phi(x) = x\ln x - x + 1$ is the relative entropy index and $\xi_t^Q = dQ/dP|_{\mathcal{F}_t}$. Solving the minimization problem implies that the multiplier utility model is equivalent to the risk-sensitive utility model given by

$$U_t = u(c_t)\,dt - e^{-rdt}\theta\ln E_t\exp\left(\frac{-U_{t+dt}}{\theta}\right).$$

This utility is a special case of recursive utility studied by Epstein and Zin (1989). The parameter $1/\theta$ enhances risk aversion.

In the continuous-time limit as $dt \to 0$, we can represent a utility process by the backward stochastic differential equation

$$dU_t = \mu_t^U dt + \sigma_t^U dB_t,$$

where (B_t) is a standard Brownian motion under P . For the multiplier utility model, the drift μ_t^U satisfies

$$rU_t = u(c_t) + \mu_t^U + \inf_{h_t}\left(\sigma_t^U h_t + \frac{\theta}{2}h_t^2\right) = u(c_t) + \mu_t^U - \frac{\left(\sigma_t^U\right)^2}{2\theta}, \tag{B.18}$$

where the worst-case density is given by $h_t = -\sigma_t^U/\theta$. The expression on the right-hand side of the last equality is the same as that for risk-sensitive utility, which is a special case of the continuous-time model of recursive utility studied by Duffie and Epstein (1992).

We now consider two contracting problems in the "risk neutral" case with $u(c) = c$ by replacing the time-additive expected utility in Problem B.1 with multiplier utility and recursive risk-sensitive utility. Let $V^m(W,X)$ and $V^{rs}(W,X)$ denote the principal's value function in these two problems.

Proposition 3 *The contracting problems B.1 with multiplier utility and risk-sensitive utility are equivalent. They deliver the same liquidation time and payout policy to the agent as in* *the robust contract for Problem 3.1. In addition,* $V^m(W,X) = V^{rs}(W,X) = X + F(W)$, *where* $F(W)$ *is the principal's value function in Problem 3.1.*

Proof. By equation (B.18), the HJB equation for multiplier utility is given by

$$\begin{aligned} rV^m(W,X) \quad = \quad & \max_{C^p,\, c\geq 0,\, \phi\geq\sigma\lambda} C^p + V_W^m(W,X)(\gamma W - c) \\ & + V_X^m(W,X)(rX - C^p - c + \mu) \\ & + V_{WW}^m(W,X)\frac{\phi^2}{2} + V_{XX}^m(W,X)\frac{\sigma^2}{2} + V_{XW}^m(W,X)\sigma\phi \\ & + \min_h \; \left[V_X^m(W,X)\sigma + V_W^m(W,X)\phi\right]h + \frac{\theta}{2}h^2. \end{aligned}$$

The optimal density generator is given by

$$h = -\frac{V_X^m(W,X)\sigma + V_W^m(W,X)\phi}{\theta}.$$

This HJB equation is equivalent to that for risk-sensitive utility after solving for the optimal density. We can easily verify that $V^m(W,X) = V^{rs}(W,X) = X + F(W)$, where $F(W)$ is the value function for Problem 3.1. The optimal policies are also the same. ■

B. 3 Implementation and Asset Pricing with Risk Aversion

Proposition 2 shows that the robust contract and the optimal contract with risk aversion deliver identical liquidation and payout policies when $\alpha r = 1/\theta$. This section will show that the implementation of the two contracts and the asset pricing implications are slightly different. Now the risk averse principal (investors) can put his wealth into two bank accounts. One is the corporate account which holds cash reserves $M_t = W_t/\lambda$ and earns interests at the rate r as in Section 4. Project payoffs are put in this account. The other is the private account with savings $S_t^p = X_t - M_t$ at the interest rate r. There are still debt and equity. The firm pays coupon $[\mu - (\gamma - r)M_t]\,dt$, regular dividends dC_t^*/λ, and special dividends $\left[\sigma - \frac{1}{\lambda}\phi(\lambda M_t)\right]dB_t^1$ (it raises capital through equity issues when this term is negative). The entrepreneur (agent) holds a fraction λ of equity and receives regular dividends dC_t^*. Investors (principal) receive coupon, regular dividends $(1-\lambda)\,dC_t^*/\lambda$, and all special dividends (or inject capital) and put them in the private saving account. Investors finance their consumption spending using this account. The cash reserves M_t follow dynamics as in equation (30). The firm is liquidated when the cash reserves reach zero and pays out special dividends (repurchases equity) or raises capital through equity issues when the cash reserves M_t rise to a level $\hat{W}/\lambda$. It pays regular dividends when the cash reserves M_t hits another higher level $\bar{W}/\lambda$. As in

Section 4, this capital structure is incentive compatible.

The private savings S_t^p follow the dynamics

$$\begin{aligned} dS_t^p &= rS_t^p dt - C^p(\lambda M_t, M_t + S_t^p)\, dt \\ &\quad + [\mu - (\gamma - r) M_t]\, dt + \frac{1-\lambda}{\lambda} dC_t^* + \left[\sigma - \frac{\phi(\lambda M_t)}{\lambda}\right] dB_t^1, \end{aligned}$$

where $S_0^p = X_0 - W_0/\lambda$. The investors' consumption $C^p(\lambda M_t, M_t + S_t^p) = C^p(W_t, X_t)$ achieves their maximized utility in the optimal contract. From the preceding equation, we can see clearly the smoothing role of special dividends. Note that $\sigma - \phi(\lambda M_t)/\lambda < 0$. In good times when $dB_t^1 > 0$, investors inject cash into the firm's cash reserves so that they can receive dividends in bad times when $dB_t^1 < 0$.

We now price debt and equity. The state price in the model with risk averse investors is equal to the intertemporal marginal rate of substitution

$$\pi_t = \pi(t, M_t, S_t^p) = \exp\left(-rt - \alpha\left[C^p(\lambda M_t, S_t^p + M_t) - C^p(\lambda M_0, S_0^p + M_0)\right]\right), \tag{B.19}$$

where $\pi_0 = 1$. Using the state price, we can compute equity value per share as

$$\begin{aligned} S_t &= E_t^{P^1}\left[\int_t^\tau \frac{\pi_s}{\pi_t}\frac{1}{\lambda} dC_s^*\right] + \frac{1}{1-\lambda} E_t^{P^1}\left[\int_t^\tau \frac{\pi_s}{\pi_t}\left(\sigma - \frac{\phi(\lambda M_s)}{\lambda}\right) dB_s^1\right] \\ &= E_t^{P^1}\left[\int_t^\tau \frac{\pi_s}{\pi_t}\frac{1}{\lambda} dC_s^*\right]. \end{aligned}$$

Unlike in the robust contracting problem, special dividends are not priced by the risk averse principal. This is because the principal believes that the events of $dB_s^1 > 0$ and $dB_s^1 < 0$ are equally likely. But the ambiguity averse principal is pessimistic and believes that $dB_s^1 < 0$ is more likely and thus special dividends have a positive price.

We can also compute debt value

$$D_t = E_t^{P^1}\left[\int_t^\tau \frac{\pi_s}{\pi_t}\left[\mu - (\gamma - r) M_s\right] ds + \frac{\pi_\tau}{\pi_t} L\right],$$

and credit yield spread. Due to the lack of wealth effect for CARA utility, the cash reserve level M_t is the only state variable for asset pricing. We still write $S_t = S(M_t)$ and $D_t = D(M_t)$. We will present asset pricing formulas in Appendix B2.

Proposition 4 *For the model with risk aversion, the market price of risk is equal to*

$$\alpha r[H'(\lambda M_t)\phi(\lambda M_t) + \sigma], \tag{B.20}$$

and the local expected equity premium under measure P^1 is

$$\frac{\phi\left(\lambda M_t\right)}{\lambda}\frac{S'\left(M_t\right)}{S\left(M_t\right)}\alpha r[H'(\lambda M_t)\phi(\lambda M_t)+\sigma], \tag{B.21}$$

for $M_t \in \left[0, \bar{W}/\lambda\right]$.

Proof. Applying Ito's Lemma to (B.19) yields

$$\begin{aligned} d\pi_t &= d\pi(t, M_t, S_t^p) = \pi_1(t, M_t, S_t^p)dt + \pi_2(t, M_t, S_t^p)dM_t + \pi_3(t, M_t, S_t^p)dS_t^p \\ &\quad + \frac{1}{2}\pi_{22}(t, M_t, S_t^p)d\left[M, M\right]_t + \frac{1}{2}\pi_{33}(t, M_t, S_t^p)d\left[S^p, S^p\right]_t + \pi_{23}(t, M_t, S_t^p)d\left[M, S^p\right]_t, \end{aligned}$$

where the subscript of π denotes partial derivative and $[X, Y]_t$ denotes the quadratic covariance between any two processes (X_t) and (Y_t). Plugging the dynamics of M_t and S_t^p and using equation (B.11), we can show that

$$-\frac{d\pi_t}{\pi_t} = rdt + \alpha r[H'(\lambda M_t)\phi(\lambda M_t) + \sigma]dB_t^1,$$

where ϕ is given by (B.8). Thus the market price of risk is given by (B.20). Proposition 1 shows that C_t^* makes W_t reflect at a constant boundary $\bar{W}$. This payout policy does not depend on wealth X. It follows that equity value only depends on one state variable M_t. Let $S_t = S\left(M_t\right)$. Since the process (m_t) defined below is a martingale,

$$m_t \equiv \pi_t S_t + \int_0^t \pi_s \frac{1}{\lambda} dC_s^* = E_t\left[\int_0^\tau \pi_s \frac{1}{\lambda} dC_s^*\right],$$

we use Ito's Lemma and set its drift to zero. We then obtain the ODE

$$rS(M) = S'(M)\left[\gamma M - \alpha r(H'(\lambda M)\phi(\lambda M) + \sigma)\frac{\phi(\lambda M)}{\lambda}\right] + S''(M)\frac{\phi(\lambda M)^2}{2\lambda^2}, \tag{B.22}$$

with boundary conditions $S(0) = 0$ and $S'(\bar{W}/\lambda) = 1$.

The local expected equity premium is given by

$$E_t^{P^1}\left[\frac{dS_t}{S_t} + \frac{dC_t^*}{\lambda S_t} - rdt\right].$$

We use (B.22) and Ito's Lemma to compute $dS_t = dS\left(M_t\right)$ and obtain (B.21). ■

When $\alpha r = 1/\theta$, Proposition 2 implies that the market price of risk in (B.20) is the same as $-h^*\left(W_t\right) = -h^*\left(\lambda M_t\right)$, where h^* is given in (20). The latter is the market price of model uncertainty in the model with ambiguity aversion, which comes from the endogenous distortion of beliefs reflected by the worst-case density generator h^*. Because special dividends are not priced in the model with risk aversion, (B.21) is obtained from (31) without the hedge component. Because the hedge component is generally small in our numerical examples, we find that the equity premium in the model with risk aversion is also high for distressed firms with low cash reserves and approaches zero when M approaches $\bar{W}/\lambda$. In Appendix B2 we show that debt value and credit yield spread in the model with risk aversion are the same as those in the model with ambiguity aversion when $\alpha r = 1/\theta$.