Asset Bubbles and Monetary Policy*

Feng Dong† Jianjun Miao‡ Pengfei Wang§

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Abstract

We provide an infinite-horizon model of rational asset bubbles in a dynamic new Keynesian framework. Entrepreneurs are heterogeneous in investment efficiency and face credit constraints. They can trade land as an asset, which also serves as collateral when borrowing from banks with reserve requirements. Land commands a liquidity premium and a land bubble can emerge. Monetary policy can affect the conditions for the existence of a bubble, its steady-state size, and its dynamics including the initial size. The 'leaning against the wind' interest rate policy reduces bubble volatility, but it could also raise inflation volatility. Whether monetary policy should respond to asset bubbles depends on the particular interest rate rule adopted by the central bank and on the type of exogenous shocks hitting the economy.

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†Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai, China. Tel: (+86) 21-52301590. Email: fengdong@sjtu.edu.cn

‡Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215. Tel.: 617-353-6675. Email: miaoj@bu.edu. Homepage: http://people.bu.edu/miaoj.

§Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong. Tel: (+852) 2358 7612. Email: pfwang@ust.hk

"We need to explicitly integrate bubbles, a combination of rational and nonrational intuitive human responses, and other aspects of behavioral economics into our monetary policy models." — Greenspan (2015)

1 Introduction

The booms and busts of asset prices in stock markets and real estate markets around the world have typically been associated with business cycles in the macroeconomy (Borio, Kennedy, and Prowse (1994) and Jordà, Schularick, and Taylor (2015)). The general public, policy makers, and academic researchers often attribute the large movements of asset prices to the emergence and collapse of bubbles because asset price volatility cannot be explained entirely by fundamentals (Shiller (1981)). How do asset bubbles affect the real economy? How does monetary policy affect asset bubbles? Should monetary policy respond to asset bubbles? The goal of our paper is to provide a theoretical framework to address these questions.

These questions have been the subject of a heated debate in the literature. Two views are dominant (Galí (2014)).[1] The first view is that central banks should view price stability and financial stability as highly complementary and mutually consistent objectives. Even if asset prices can amplify and propagate shocks, including asset prices in monetary policy rules may result in only slight gains (Bernanke and Gertler (1999, 2001)). Moreover, asset bubbles driven by non-fundamental shocks are highly unpredictable. Thus monetary policy should not respond to asset prices. The second view is that central banks should act preemptively to prevent bubbles from forming, by raising interest rates or decreasing money supply to appropriate levels.[2] Such a policy, often referred to as "leaning against the wind," may call for a change in the inflation target.

One reason the debate is still ongoing is that the literature has yet to agree on a theoretical framework for understanding the formation of asset bubbles and the mechanism of how asset bubbles interact with the macroeconomy and monetary policy. The debate stems from the model of Bernanke and Gertler (1999) who introduce an irrational bubble to the model of Bernanke, Gertler, and Gilchrist (1999) (henceforth BGG). The former model cannot address the question of how and why a bubble can emerge and burst under rational expectations. We contribute to the literature by providing an infinite-horizon model of rational asset bubbles in a dynamic new Keynesian (DNK) framework.[3] The key feature of our model is that entrepreneurs (or firms) are heterogenous and face credit constraints. In a frictionless Arrow-Debreu economy, rational bubbles cannot emerge

[1] See Gilchrist and Leahy (2002) for a survey of this literature. See Schularick and Taylor (2012) and Brunnermeier and Schnabel (2015) for empirical evidence.

[2] See, e.g., Borio and Lowe (2002), Cecchetti et al. (2000), Issing (2009), ECB (2010), and Blanchard et al. (2012).

[3] By rational bubbles, we mean that they are consistent with individual optimality, rational expectations, and market clearing. See Brunnermeier and Oehmek (2013) for a survey of the literature on rational and irrational bubbles.

and movements of asset prices reflect changes in the underlying economic fundamentals. In this case central banks would not have to worry about asset prices. By contrast, due to credit constraints, an intrinsically useless asset (e.g., land) can provide liquidity and command a liquidity premium.[4] If all agents believe land is valuable, this belief can be self-fulfilling. Land as an asset can raise entrepreneurs' net worth and can also serve as collateral for loans. Efficient entrepreneurs sell land to inefficient ones to finance investment and hence land can be traded at a positive price. A land bubble has an intensive margin effect in that it raises an entrepreneur's net worth and hence his investment. It also has an extensive margin effect in that inefficient entrepreneurs must hold the bubble and will not make investment. The net effect on aggregate investment is typically positive.

To introduce money and monetary policy, we incorporate a banking system with legal restrictions in the sense that banks must meet reserve requirements. Reserve requirements generate a spread between the lending rate and the deposit rate. Households and entrepreneurs can save by making deposits in banks and banks can lend to investing entrepreneurs. The central bank changes the money supply by changing reserves (high-powered money or monetary base). Monetary policy is conducted by following an interest rate rule. In this case money supply is endogenous. To allow monetary policy to have a large impact on the real economy, we introduce monopolistic competition and sticky prices as in Calvo (1983).

Our main results can be summarized as follows. First, monetary policy can affect the conditions for the existence of a bubble. High inflation erodes entrepreneurs' real balance and hence their net worth, generating a large liquidity premium for holding a bubble asset. When the liquidity premium is sufficiently high, a bubble can emerge. Thus raising the inflation target by raising the money supply permanently can fuel a bubble. The higher the inflation target, the more likely a bubble will emerge. On the other hand, if the economy already has a bubble, cutting the inflation target by decreasing the money supply permanently can prick the bubble.

Second, monetary policy can affect the steady-state size and dynamics of an asset bubble including its initial size. In particular, a higher inflation target is associated with a higher steady-state size of the bubble. An expansionary monetary policy by cutting the interest rate or raising the money supply can raise the initial size of the asset bubble, which in turn generates a large amplification effect of monetary policy. Moreover, the coefficients in the interest rate rule affect the dynamics of the asset bubble in response to exogenous shocks. A higher interest rate response to asset bubbles reduces bubble volatility, but may raise inflation volatility.

Third, whether monetary policy should respond to asset bubbles depends on the role of asset bubbles, the particular interest rate rule adopted by the central bank, and the type of exogenous shocks hitting the economy. We consider two types of interest rate rules: (i) a Taylor rule, which

[4] Introducing rents for land will complicate our analysis without changing our key insights. See Miao and Wang (2015a,b), Miao, Wang, and Zha (2014), and Miao, Wang, and Xu (2015) for models of rational bubbles attached to assets with dividends or rents.

responds to inflation and the gap between the actual output and steady-state output, and (ii) an inflation targeting rule, which responds to expected inflation, but not the output gap. We consider two types of shocks: (i) a fundamental TFP shock, and (ii) a non-fundamental sentiment shock to the bubble. We include a weight on the asset bubble in the interest rate rules and search for an optimal weight to maximize household utility. Our calibrated model shows that the central bank should encourage bubbles by cutting interest rates in response to expanding bubbles in all cases. In our model the benefit of an asset bubble is to improve investment efficiency and its cost is to increase business cycle volatility. In response to a positive TFP shock, an efficient real business cycles model implies positive comovements of consumption, investment, output, and labor. But labor drops under the Taylor rule in a DNK model (Galí (1999)). Cutting interest rates when the asset bubble is expanding can raise aggregate demand and hence labor. In response to a positive sentiment shock, the asset bubble expands, but consumption falls on impact under the standard Taylor rule because the nominal and real interest rates rise too much. Cutting interest rates when the asset bubble is expanding can also boost consumption and benefit households.

Under the inflation targeting rule, the optimal coefficients on the asset bubble are generally very small negative numbers. The welfare gains are also quite small, especially for the strong inflation targeting rule considered by Bernanke and Gertler (1999) and Gilchrist and Leahy (2002). As Bernanke and Gertler (1999) argue, to the extent that asset bubbles tend to be positively correlated with movements in output and inflation, policies based on these two variables subsume most of the gains from reacting to asset bubbles. In fact, in response to a TFP shock or a sentiment shock, the simple strong inflation targeting rule can generate the right comovements of macroeconomic quantities, giving very small welfare gains from reacting to asset bubbles.

2 Basic Intuition and Related Literature

Our model features a standard aggregate supply block as in the DNK literature. The key part of our model is about aggregate demand and asset pricing equations. In the case without aggregate uncertainty, our model implies that deposits (or bonds) and land satisfy the following two asset pricing equations

$$1 = SDF_{t+1}\frac{R_t}{\Pi_{t+1}}\left(1+LIQ_{t+1}\right), \tag{1}$$

$$p_t^h = SDF_{t+1}p_{t+1}^h\left(1+LIQ_{t+1}^h\right), \tag{2}$$

where SDF_{t+1}, R_t, and Π_{t+1} denote the stochastic discount factor (SDF), the nominal interest rate, and the inflation rate between periods t and $t+1$, respectively, p_t^h denotes the land price, and LIQ_{t+1} and LIQ_{t+1}^h are the liquidity premiums for the deposits and land, respectively.[5] The

[5] Equations (1) and (2) follow from (24) and (42) with $\delta_h = 0$.

liquidity premium comes from credit constraints.

Land is intrinsically useless in the sense that it does not pay any dividend. Its fundamental value is zero as the equilibrium with $p_t^h = 0$ for all t satisfies (2). There may exist a bubbly equilibrium in which $p_t^h > 0$ for all t. In a steady state with an infinitely lived representative agent, the SDF is equal to his subjective discount factor $\beta \in (0, 1)$. For a bubble $p_t^h = p_{t+1}^h > 0$ to exist in the steady state, equation (2) becomes

$$1 = \beta \left(1 + LIQ^h\right).$$

This equation cannot hold in a frictionless model without liquidity premium $LIQ^h = 0$, but it can hold if there is a positive liquidity premium $LIQ^h > 0$.[6] The liquidity premium provides a benefit for holding a bubble asset, even though the bubble asset does not provide any dividends, such that discounting by β cannot eliminate its value in the steady state. We have elaborated on this point in our previous studies (e.g., Miao and Wang (2012, 2014, 2015a,b), Miao, Wang, and Xu (2015), Miao, Wang, and Xu (2016), and Miao, Wang, and Zhou (2015, 2016)). Our analysis of the conditions for the existence of a bubble revolves around the existence of a solution to the preceding equation and how monetary policy affects LIQ^h.

In the special case of $LIQ_{t+1} = LIQ_{t+1}^h$, equations (1) and (2) imply that the growth rate of the bubble is equal to the real interest rate

$$p_t^h = \frac{p_{t+1}^h}{R_t/\Pi_{t+1}}. \tag{3}$$

This equation also holds in overlapping generations (OLG) models in which a bubble can exist without a liquidity premium because the SDF is not equal to β in the steady state of OLG models. In particular, equations (1) and (3) imply that $SDF = R/\Pi = 1$ in the steady state. The existence conditions are then about whether there is a solution to this equation.

Our model is consistent with the conventional wisdom that monetary policy can have an impact on asset price bubbles and that it can fuel or prick a bubble. Galí (2014) challenges this wisdom using an OLG model. The OLG framework naturally incorporates household heterogeneity and incomplete market participation, which can allow a bubble to emerge without any other frictions (Samuelson (1958) and Tirole (1985)). Our infinite-horizon model features financial frictions and has the advantage that it can be tractably integrated into the dynamic stochastic general equilibrium framework or the DNK framework and hence has the potential to be quantified (see Ikeda (2013) and Miao, Wang, Xu (2015)). After all, asset bubbles are quantitative observations and a theory of bubbles would be vacuous if it cannot be quantified.

Galí (2014) sets up an elegant simple model in which equilibrium dynamics can be summarized by a unidimensional system based on (3). He shows that the conditions for the existence of bubbles

[6] More technically, the transversality condition will rule out bubbles without liquidity premium. This condition is not needed in OLG models with finitely lived agents.

are independent of monetary policy. He also shows that his unidimensional equilibrium system has a continuum of stable bubbly steady states and a continuum of unstable bubbly steady states. He focuses on a stable bubbly steady state and assumes that the bubble p_t^h is a backward-looking predetermined variable. In this case the initial size of the bubble p_0^h is exogenously given and independent of monetary policy. Equation (3) then implies that a larger real interest rate generates a higher growth rate of the bubble. By contrast, the asset bubble in our model is a nonpredetermined forward-looking variable like any asset prices. Our equilibrium system is multi-dimensional and also features a continuum of bubbly steady states. But any bubbly steady state is a local saddle point.[7] In response to an exogenous shock, the initial size of the asset bubble jumps and provides a powerful amplification mechanism. In our model monetary policy affects the initial size of the bubble and its dynamics through interest rates and liquidity premium.

Like Galí (2014), we find that the "leaning against the wind" policy may not be optimal, but for a different reason. Our model shows that this policy can lower bubble volatility at the expense of raising inflation volatility. Galí's model shows that this policy may raise bubble volatility and reduce dividend volatility depending on the size of the bubble. Inflation does not cause a welfare loss in his model.

Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), Martin and Ventura (2012, 2015), and Asriyan et al. (2016) introduce credit constraints to OLG models of asset bubbles. Only Asriyan et al. (2016) introduce money and monetary policy. None of these papers derives asset pricing equations like (1) and (2). The role of the liquidity premium is emphasized by Kiyotaki and Moore (2008) in their infinite-horizon model with credit constraints. They focus on fiat money, which is also a pure bubble. Our model borrows some insights from theirs but differs from theirs in many ways: (i) the bubble asset in our model also serves as collateral, (ii) we have a DNK framework with sticky prices, and (iii) we have a banking system with reserve requirements. The reserve requirements in our model are critical for the non-superneutrality of money and for monetary policy to affect the conditions for the existence of a bubble.

Our paper is also related to infinite-horizon real models of rational bubbles with credit constraints (Santos and Woodford (1997), Kocherlakota (2009), Wang and Wen (2012), Miao and Wang (2012, 2014, 2015a, 2015b), Hirano and Yanagawa (2013), Miao, Wang, and Zha (2014), Miao, Wang, and Xu (2015), Miao, Wang, and Xu (2016), Miao, Wang and Zhou (2015, 2016), and Aoki and Nikolov (2015)).[8] Ikeda (2013) introduces monetary policy and wage rigidities to the model of Miao, Wang and Xu (2015). Based on Bayesian estimations, he shows that the optimal monetary policy calls for monetary tightening to restrain the boom at the cost of greater inflation

[7]This is a numerical result because our equilibrium system is too complicated to permit a theoretical result. We have obtained this result numerically in Miao, Wang, and Xu (2015) and formally proved this result theoretically in simpler models of Miao and Wang (2012, 2015a, 2015b) and Miao, Wang and Zhou (2015).

[8]See Miao (2014) for a recent survey.

volatility.

More broadly, our model is related to the large literature on the relationship between monetary policy and asset prices (see Gilchrist and Leahy (2002) for a survey). Our model shares many insights with Kiyotaki and Moore (1997), BGG (1999), and Bernanke and Gertler (1999, 2001), in which the financial accelerator, bank lending, net worth, and balance sheet channels are important transmission and propagation mechanisms. Our result that a strong inflation targeting rule does not call for the need to respond to asset bubbles because the welfare gain is too small confirms the early finding of Bernanke and Gertler (1999), albeit asset bubbles are rational in our model.

3 The Model

Consider an infinite-horizon economy consisting of households, firms, retailers, financial intermediaries (banks), and a government (monetary authority). Following BGG (1999), we assume that retailers are monopolistically competitive and their role is to introduce nominal price rigidities.

3.1 Households

There is a continuum of identical households of measure unity. The representative household is an extended family consisting of workers, entrepreneurs, and bankers. Each entrepreneur runs a firm and workers supply labor to firms. Bankers are identical and each banker manages a bank. The family and firms can save by making deposits in banks which in turn extend loans to borrowers. Entrepreneurs, bankers, and retailers hand in their dividends to households who are shareholders. Each household chooses consumption $\{C_t\}$, labor supply $\{N_t\}$, and deposits $\{S_{a,t+1}\}$ to maximize utility

$$\max_{\{C_t, S_{a,t+1}, N_t\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \psi N_t\right), \tag{4}$$

subject to

$$C_t + \frac{S_{a,t+1}}{P_t} = w_t N_t + D_t + \frac{R_{t-1} S_{at}}{P_t} + X_t, \tag{5}$$

where β is the subjective discount factor, w_t is the real wage, P_t is the price level, D_t is the total dividends from bankers, entrepreneurs, and retailers, R_{t-1} is the nominal interest rate (deposit rate) between periods $t-1$ and t, and X_t denotes the sum of money transfers and lump-sum taxes/transfers from the government. Suppose that the household cannot borrow so that $S_{a,t+1} \geq 0$.

The first-order conditions imply that

$$w_t \;=\; \frac{\psi}{\Lambda_t}, \tag{6}$$

$$1 \;\geq\; E_t\left[\frac{\beta \Lambda_{t+1}}{\Lambda_t} \frac{R_t}{\Pi_{t+1}}\right], \text{ with equality when } S_{at+1} > 0, \tag{7}$$

where $\Lambda_t = 1/C_t$ denotes the household marginal utility and $\Pi_{t+1} = P_{t+1}/P_t$ denotes the (gross) inflation rate between periods t and $t+1$.

3.2 Banks

Bankers are identical of measure unity. At each time t, the representative bank receives total deposits S_{t+1} and chooses loans L_{t+1} and reserves M_{t+1}. For simplicity, suppose that there is no bank net worth and that there is no interest on reserves. Reserves are often called high-powered money or monetary base. We do not consider currency in circulation or inter-bank loan markets in this model.

The balance sheet equation is given by

$$L_{t+1} + M_{t+1} = S_{t+1}. \tag{8}$$

The bank is also subject to the reserve requirement

$$M_{t+1} \geq \lambda S_{t+1}, \tag{9}$$

where $\lambda \in (0,1)$ denotes the required reserve ratio. The bank's objective is to maximize profits

$$\max_{M_{t+1}, L_{t+1}} E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} \left(\frac{L_{t+1} R_{lt} + M_{t+1} - S_{t+1} R_t}{P_{t+1}} \right), \tag{10}$$

subject to (8) and (9), where R_{lt} denotes the loan rate between periods t and $t+1$.

We can show that, as long as $R_{lt} > R_t$, the constraint (9) binds so that

$$M_{t+1} = \lambda S_{t+1}, \tag{11}$$

and

$$R_{lt} = \frac{R_t - \lambda}{1 - \lambda}. \tag{12}$$

This equation implies that the equilibrium deposit rate must satisfy $R_t > 1$ in order for $R_{lt} > R_t$ to hold. It also shows that the bank makes zero profit and that the lending rate R_{lt} increases with the deposit rate in order for the bank to break even because reserves do not bear any interest.

3.3 Firms

Each entrepreneur $j \in [0,1]$ runs a firm that combines labor N_{jt} and capital K_{jt} to produce an intermediate (wholesale) good j according to the technology

$$Y_{jt} = A_t K_{jt}^{\alpha} N_{jt}^{1-\alpha}, \ \alpha \in (0,1),$$

where A_t denotes TFP, which follows an AR(1) process

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{at},$$

where $\{\varepsilon_{at}\}$ is a white noise process. The entrepreneur sells wholesale goods to retailers at the real price p_{wt}. The static profit maximization problem yields

$$R_{kt}K_{jt} = \max_{N_{jt}} \; p_{wt}A_tK_{jt}^{\alpha}N_{jt}^{1-\alpha} - w_tN_{jt},$$

where

$$R_{kt} = \alpha\left(\frac{1-\alpha}{w_t}\right)^{\frac{1-\alpha}{\alpha}}\left(p_{wt}A_t\right)^{\frac{1}{\alpha}}. \tag{13}$$

and the first-order condition gives labor demand

$$w_t = (1-\alpha)\,p_{wt}A_tK_{jt}^{\alpha}N_{jt}^{-\alpha}. \tag{14}$$

At the beginning of period t, the entrepreneur faces idiosyncratic investment-specific shock ε_{jt} and makes investment I_{jt} to increase his capital stock so that the law of motion for capital follows

$$K_{jt+1} = (1-\delta)K_{jt} + \varepsilon_{jt}I_{jt}, \tag{15}$$

where $\delta \in (0,1)$ represents the depreciation rate. Suppose that the cumulative distribution function of ε_{jt} is F and the density function is f on $[\varepsilon_{\min}, \varepsilon_{\max}] \subset [0,\infty)$ and ε_{jt} is independently and identically distributed across firms and over time. Assume that there is no insurance market against the idiosyncratic investment-specific shock and that investment is irreversible at the firm level so that $I_{jt} \geq 0$.

The entrepreneur chooses to save $S_{jt+1} \geq 0$ in banks at the deposit rate R_t and borrow $L_{jt+1} \geq 0$ at the lending rate R_{lt}. He is endowed with $\delta_h \in (0,1)$ units of an intrinsically useless bubble asset (e.g., land or commercial real estate) at the beginning of each period t. Its nominal price is denoted by $P_{t|t}^{h}$, which is nonnegative by free disposal. In each period a fraction δ_h of each vintage of the bubble assets is assumed to lose its value, due to depreciation for example. This implies that the total amount of bubble assets outstanding remains constant and is normalized to one. Our modeling of such recurrent bubbles is related to Martin and Ventura (2012), Galí (2014), and Miao, Wang, and Xu (2015). The purpose is to introduce a non-fundamental sentiment shock that drives movements of the asset bubbles.

Entrepreneurs can trade each vintage of land, but land trading is illiquid. Following Kiyotaki and Moore (2008), we impose the resaleability constraint

$$H_{j,t+1|t-k} \geq \omega\left(1-\delta_h\right)H_{j,t|t-k} \geq 0, \quad k = 0,1,2,..., \tag{16}$$

where $\omega \in (0,1)$, $H_{j,t+1|t-k}$ represents the period t choice of land endowed in period $t-k$, and $H_{j,t|t-k}$ represents the period $t-1$ choice of land endowed in period $t-k$. By convention, we set $H_{j,t|t} = \delta_h/\left(1-\delta_h\right)$. Constraint (16) says that the entrepreneur can sell at most a fraction $(1-\omega)$ of his undepreciated land each period due to market illiquidity. In addition, he cannot short sell land.

A key assumption is that entrepreneurs face borrowing constraints due to imperfect contract enforcement. We impose

$$0 \leq L_{jt+1} \leq \theta\omega \left[(1-\delta_h) \sum_{k=0}^{\infty} P_{t|t-k}^h H_{j,t|t-k} \right] + \mu P_t K_{jt}, \tag{17}$$

where $\theta \in (0,1)$, $\mu \in (0,1)$, and $P_{t|t-k}^h$ denotes the period t nominal price of land at vintage $t-k$. The interpretation of the constraint above is that the entrepreneur's borrowing is limited by the collateral value of a fraction $\theta \in (0,1)$ of his existing nontraded undepreciated land plus newly endowed land and a fraction μ of his existing capital.[9]

We can write the flow-of-funds constraints as

$$\begin{aligned} & D_{jt} + I_{jt} + \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k} + \frac{1}{P_t}\left(S_{jt+1} + L_{jt} R_{lt-1}\right) \\ = \; & R_{kt} K_{jt} + \frac{1}{P_t}\left(S_{jt} R_{t-1} + L_{jt+1}\right) + (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k}, \end{aligned} \tag{18}$$

where D_{jt} denotes real dividends and $p_{t|t-k}^h = P_{t|t-k}^h / P_t$ denotes the real land price. Suppose that equity finance is so costly that the firm does not issue any new equity.[10] Thus we impose

$$D_{jt} \geq 0. \tag{19}$$

The entrepreneur's objective is to maximize the discounted present value of dividends. We can write his decision problem using dynamic programming

$$\begin{aligned} & V_t\left(K_{jt}, S_{jt}, L_{jt}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \qquad (20) \\ = & \max_{\{I_{jt}, S_{jt+1}, L_{jt+1}, H_{jt+1}\}} D_{jt} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}\left(K_{jt+1}, S_{jt+1}, L_{jt+1}, \left\{H_{j,t+1|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt+1}\right), \end{aligned}$$

subject to (15), (16), (17), (18), and (19), where we have used the household's intertemporal marginal rate of substitution as the stochastic discount factor. Here $V_t(\cdot)$ denotes the value function.

Define Tobin's (marginal) Q as

$$q_t^k \equiv \frac{\partial}{\partial K_{jt+1}} E_t \frac{\beta \Lambda_{t+1}}{\Lambda_t} V_{t+1}\left(K_{jt+1}, S_{jt+1}, L_{jt+1}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt+1}\right).$$

The following proposition characterizes the entrepreneur's optimal decisions.

[9] Unlike Kiyotaki and Moore (1997), we do not use future values of land and capital as collateral. Using future values as collateral will complicate algebra significantly without changing our key insights. See Caballero and Krishnamurthy (2006), Miao and Wang (2015b), and Miao, Wang and Zhou (2015) for related discussions.

[10] Our key insights will not change as long as new equity issues are sufficiently limited (see Miao and Wang (2005b) and Miao, Wang, and Xu (2015)).

Proposition 1 *In an equilibrium with $R_{lt} > R_t > 1$, there exist two cut-off values ε_t^* and ε_t^{**} such that the decision rules are given by $I_{jt} = 0$ if $\varepsilon_{jt} < \varepsilon_t^*$ and*

$$\begin{aligned} I_{jt} &= R_{kt}K_{jt} + \frac{1}{P_t}\left(S_{jt}R_{t-1} - S_{jt+1} + L_{jt+1} - L_{jt}R_{lt-1}\right) \\ &\quad + (1-\omega)(1-\delta_h)\sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} \text{ if } \varepsilon_{jt} \geq \varepsilon_t^*, \end{aligned}$$

$$S_{jt+1} = \begin{cases} 0, & \text{if } \varepsilon_{jt} \geq \varepsilon_t^* \\ \text{indeterminate}, & \text{otherwise} \end{cases},$$

$$\frac{L_{jt+1}}{P_t} = \begin{cases} 0, & \text{if } \varepsilon_{jt} < \varepsilon_t^{**} \\ \theta\omega(1-\delta_h)\sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} + \mu K_{jt}, & \text{otherwise} \end{cases},$$

$$H_{j,t+1|t-k} = \begin{cases} \omega(1-\delta_h)H_{j,t|t-k}, & \text{if } \varepsilon_{jt} \geq \varepsilon_t^* \\ \text{indeterminate}, & \text{otherwise} \end{cases},$$

where the cut-off values satisfy

$$\varepsilon_t^* = \frac{1}{q_t^k}, \tag{21}$$

$$\varepsilon_t^{**} = \frac{R_{lt}}{R_t}\varepsilon_t^*, \tag{22}$$

and q_t^k, R_t, and $p_{t|t-k}^h$ satisfy[11]

$$\begin{aligned} q_t^k &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} R_{kt+1}\left[1 + \int_{\varepsilon_{t+1}^*}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - 1\right) dF(\varepsilon)\right] \\ &\quad + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1}^k (1-\delta) + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\mu \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - \frac{R_{lt}}{R_t}\right) dF(\varepsilon), \end{aligned} \tag{23}$$

$$1 = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\frac{R_t}{\Pi_{t+1}}\left[1 + \int_{\varepsilon_{t+1}^*}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - 1\right) dF(\varepsilon)\right], \tag{24}$$

$$\begin{aligned} p_{t|t-k}^h &= (1-\delta_h)\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1|t-k}^h \left[1 + (1-\omega)\int_{\varepsilon_{t+1}^*}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - 1\right) dF(\varepsilon)\right] \\ &\quad + (1-\delta_h)\beta\theta\omega E_t \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1|t-k}^h \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - \frac{R_{lt}}{R_t}\right) dF(\varepsilon). \end{aligned} \tag{25}$$

This proposition shows that there are two cutoff values ε_t^* and ε_t^{**} such that the firm makes investment if and only if $\varepsilon_{jt} > \varepsilon_t^* = 1/q_t^k$. This is consistent with Tobin's rule. Making one unit of investment costs one unit of consumption goods, but yields ε_{jt} units of capital, and hence the marginal benefit is $q_t^k \varepsilon_{jt}$. When the marginal benefit is higher than the marginal cost, the firm

[11] The usual transversality conditions must also hold. Moreover, for ease of exposition, we assume that parameter values are such that ε_t^* and ε_t^{**} are in the interior of $(\varepsilon_{\min}, \varepsilon_{\max})$.

invests. When investing, it sells land as much as possible to finance investment. It does not save, but borrows from banks if and only if investment is sufficiently profitable (i.e., investment efficiency exceeds the cutoff $\varepsilon_t^{**} > \varepsilon_t^*$), since the lending rate R_{lt} is higher than the deposit rate R_t. The ratio of these two cutoffs $\varepsilon_t^{**}/\varepsilon_t^*$ is equal to the ratio of the lending rate to the deposit rate, which is interpreted as the external finance premium as in BGG (1999). When the firm decides to borrow, it does so up to the credit limit.

Equations (23), (24), and (25) are the asset pricing equations for capital, deposits, and land of vintage $t-k$. In addition to the usual terms in these equations, two integral terms deserve discussion. Both terms represent liquidity premium due to financial frictions. The first term represents the liquidity premium from internal funds, savings (deposits), and land. Since the entrepreneur sells a fraction $1-\omega$ of land when his investment efficiency exceeds ε_{t+1}^*, this fraction appears in (25). The second term represents the liquidity premium from collateral. One unit of (real) land value can allow the firm to borrow $\theta\omega$ units from banks. Each unit borrowed in period t must be repaid at a external finance premium R_{lt}/R_t in period $t+1$. The marginal benefit is $q_{t+1}^k\varepsilon$ in period $t+1$. Thus the preceding equation gives the expected profits from one unit of borrowing. The multiplicative factor $(1-\delta_h)$ represents the undepreciated value because a fraction δ_h of land loses its value. The multiplicative factor $\theta\omega$ appears in equation (25) because only $\theta\omega$ units can be borrowed using one unit of land value as collateral. Similarly, the factor μ appears in equation (23) because only μ dollars can be borrowed using one dollar of capital as collateral.

3.4 Retailers

Retailers are monopolistically competitive. In each period t they buy intermediate goods from entrepreneurs at the real price p_{wt} and sell good j at the nominal price P_{jt}. Intermediate goods are transformed into final goods according to the CES aggregator

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1. \tag{26}$$

Thus retailers face demand given by

$$Y_{jt} = \left(\frac{P_{jt}}{P_t}\right)^{-\sigma} Y_t, \tag{27}$$

where the price index is given by

$$P_t \equiv \left[\int_0^1 P_{jt}{}^{1-\sigma} di\right]^{\frac{1}{1-\sigma}}. \tag{28}$$

To introduce price stickiness, we assume that each retailer is free to change its price in any period only with probability $1-\xi$, following Calvo (1983). Following Erceg, Henderson, and Levin (2000), we also assume that whenever the retailer is not allowed to reset its price, its price is automatically increased at the steady-state inflation rate. The retailer selling good j chooses the nominal price P_t^* in period t to maximize the discounted present value of profits

$$\max_{P_t^*} \sum_{k=0}^{\infty} \xi^k E_t \left[\frac{\beta^k \Lambda_{t+k}}{\Lambda_t} \left((1+\tau) \frac{\Pi^k P_t^*}{P_{t+k}} - p_{w,t+k} \right) Y_{jt+k}^* \right], \tag{29}$$

subject to the demand curve

$$Y_{jt+k}^* = \left(\frac{\Pi^k P_t^*}{P_{t+k}} \right)^{-\sigma} Y_{t+k}, \tag{30}$$

where τ denotes the output subsidy and Π denotes the steady-state inflation target. We use the household intertemporal marginal rate of substitution as the stochastic discount factor because retailers must hand in all profits to households who are the shareholders.

The first-order condition gives the pricing rule

$$P_t^* = \frac{1}{1+\tau} \frac{\sigma}{\sigma - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} p_{w,t+k} P_{t+k}^{\sigma} Y_{t+k} (\Pi^k)^{-\sigma}}{E_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} P_{t+k}^{\sigma-1} (\Pi^k)^{1-\sigma} Y_{t+k}}. \tag{31}$$

We set $1+\tau = \sigma/(\sigma-1)$ to completely remove the distortion due to monopolistic competition. Let $p_t^* = P_t^*/P_t$. We can then write the pricing rule in a recursive form as follows

$$p_t^* = \frac{\Gamma_t^a}{\Gamma_t^b}, \tag{32}$$

where

$$\Gamma_t^a = \Lambda_t p_{wt} Y_t + \beta\xi E_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma} \Gamma_{t+1}^a, \tag{33}$$

$$\Gamma_t^b = \Lambda_t Y_t + \beta\xi E_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma-1} \Gamma_{t+1}^b. \tag{34}$$

The aggregate price level satisfies

$$P_t = \left[\xi \left(\Pi P_{t-1} \right)^{1-\sigma} + (1-\xi) \left(P_t^* \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}},$$

or

$$1 = \left[\xi \left(\frac{\Pi}{\Pi_t} \right)^{1-\sigma} + (1-\xi) \, p_t^{*1-\sigma} \right]^{\frac{1}{1-\sigma}}. \tag{35}$$

3.5 Monetary Policy

To close the model, we consider two types of interest rate rules for monetary policy. First, the central bank sets the nominal deposit rate in response to the current inflation, output, and asset prices:

$$\ln R_t = \ln R + \phi_\pi \ln(\Pi_t/\Pi) + \phi_y \ln(Y_t/Y) + \phi_p \ln\left(p_t^h/p^h \right) + v_t, \tag{36}$$

where p_t^h denotes the aggregate land price index to be defined later, Π is the target inflation rate, and R, Y, and p^h denote the steady-state levels of the nominal deposit rate, output, and land price. Assume that $\{v_t\}$ follows an AR(1) process

$$v_t = \rho_v v_{t-1} + \varepsilon_{vt},$$

where $\rho_v \in (0,1)$ and $\{\varepsilon_{vt}\}$ is a white noise process. In the DNK literature one often focuses on the zero-inflation steady state by setting $\Pi = 1$. But we will study the case with $\Pi > 1$ because a positive inflation rate is important for the emergence of a bubble.

Second, we consider the inflation targeting rule following Bernanke and Getter (1999) and Gilchrist and Leahy (2002):

$$\ln R_t = \ln R + \phi_\pi E_t \ln (\Pi_{t+1}/\Pi) + \phi_p \ln \left(p_t^h/p^h\right) + v_t. \tag{37}$$

As a baseline rule, we set $\phi_p = 0$ in (36) and (37) and call the former the Taylor rule. In Section 6 we will search for the optimal value of ϕ_p to maximize household utility with full commitment.

The interest rate rule policy implies that money supply is endogenous. Let reserves follow the dynamics

$$M_{t+1} = M_t \exp(g_t), \tag{38}$$

where g_t is the endogenous exponential growth rate of money. Assume that the increased money is transferred to households in a lump-sum manner. If one assumes that g_t is exogenous, then the nominal interest rate will be endogenous. In our analysis below we will follow the tradition of the DNK framework to adopt the interest rate rule.

3.6 Equilibrium System

The market-clearing conditions for bank loans, deposits, and land are given by

$$\int L_{jt}dj = L_t, \int S_{jt}dj + S_{at} = S_t, \int H_{j,t+1|t-k}dj = \delta_h (1-\delta_h)^k, \ k = 0,1,2... \tag{39}$$

for all t. Equation (7) and (24) imply that $S_{a,t+1} = 0$ because the deposit rate is too low.[12] Assuming $S_{a0} = 0$, we have $S_{at} = 0$ for all $t \geq 0$.

Define aggregate capital, aggregate investment, and aggregate labor as $K_t = \int K_{jt}dj$, $I_t = \int I_{jt}dj$, and $N_t = \int N_{jt}dj$. By the labor demand condition (14), we can show that the capital-labor ratio is independent of j and hence we have

$$w_t = (1-\alpha) p_{wt} A_t K_t^\alpha N_t^{-\alpha}. \tag{40}$$

Plugging (40) into (13), we can show that R_{kt} is equal to the marginal revenue product of capital

$$R_{kt} = \alpha p_{wt} A_t K_t^{\alpha-1} N_t^{1-\alpha}. \tag{41}$$

[12]Similarly, households will not hold any land by equation (25), even though they are allowed to trade land.

Define an index for pre-existing bubbles as

$$p_t^E = \sum_{k=1}^{\infty} p_{t|t-k}^h \left(1-\delta_h\right)^k \delta_h.$$

The size of the total new bubble is denoted by $p_t^N = \delta_h p_{t|t}^h$. Define the economy's aggregate bubble as $p_t^h = p_t^E + p_t^N$. Using these definitions, we can rewrite equation (25) as

$$\begin{aligned} p_t^h &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}^E \left[1 + (1-\omega) \int_{\varepsilon_{t+1}^*}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - 1\right) dF\left(\varepsilon\right)\right] \\ &\quad + \beta\theta\omega E_t \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}^E \int_{\varepsilon_{t+1}^{**}}^{\varepsilon_{\max}} \left(q_{t+1}^k \varepsilon - \frac{R_{lt}}{R_t}\right) dF\left(\varepsilon\right). \end{aligned} \tag{42}$$

This equation is analogous to equation (14) in Galí (2014). Note that p_{t+1}^E appears on the right-hand side of (42) because all bubbles today become old tomorrow.

By Proposition 1, firms with $\varepsilon_{jt} \geq \varepsilon_t^{**}$ take loans from banks up to the borrowing limit in (17). Aggregating the borrowing limits from all lending banks yields

$$\frac{L_{t+1}}{P_t} = \left(\theta\omega p_t^h + \mu K_t\right)\left(1 - F\left(\varepsilon_t^{**}\right)\right).$$

Using (8) and (11), we can rewrite the preceding equation as

$$m_{t+1} = \frac{\lambda}{1-\lambda}\left(\theta\omega p_t^h + \mu K_t\right)\left(1 - F\left(\varepsilon_t^{**}\right)\right), \tag{43}$$

where we have used $m_{t+1} \equiv M_{t+1}/P_t$ to denote the real balance of reserves. This equation gives money demand. Equation (38) gives the money supply

$$m_{t+1} = \frac{m_t}{\Pi_t} \exp\left(g_t\right). \tag{44}$$

By Proposition 1, equations (8), (11), (12), and the market-clearing conditions above, we can show that

$$I_t = \left(R_{kt} K_t + \frac{m_t}{\Pi_t} + (1-\omega) p_t^h\right)\left(1 - F\left(\varepsilon_t^*\right)\right) + \left(\theta\omega p_t^h + \mu K_t\right)\left(1 - F\left(\varepsilon_t^{**}\right)\right), \tag{45}$$

and

$$\begin{aligned} K_{t+1} &= (1-\delta) K_t + \left(R_{kt} K_t + \frac{m_t}{\Pi_t} + (1-\omega) p_t^h\right) \int_{\varepsilon_t^*}^{\varepsilon_{\max}} \varepsilon dF\left(\varepsilon\right) \\ &\quad + \left(\theta\omega p_t^h + \mu K_t\right) \int_{\varepsilon_t^{**}}^{\varepsilon_{\max}} \varepsilon dF\left(\varepsilon\right). \end{aligned} \tag{46}$$

Equation (45) shows that aggregate investment is financed by internal funds $R_{kt}K_t$, the real balance m_t/Π_t (or returns on net savings), land sales $(1-\omega)p_t^h$, and bank loans $\theta\omega p_t^h + \mu K_t$. The two equations above show that asset bubbles have intensive and extensive margin effects on aggregate investment and capital accumulation. The two cutoffs ε_t^* and ε_t^{**} reflect the extensive margin effect. The term $1 - F(\varepsilon_t^*)$ represents the mass of firms that sell land to finance investment and the term $1 - F(\varepsilon_t^{**})$ represents the mass of firms that take loans to finance investment.

Aggregating (27) over all firms yields

$$Y_t = \frac{A_t}{\Delta_t} K_t^\alpha N_t^{1-\alpha}, \tag{47}$$

where

$$\Delta_t = \int \left(\frac{P_{jt}}{P_t} \right)^{-\sigma} dj$$

is the price dispersion, which satisfies the recursive equation

$$\Delta_t = (1 - \xi) p_t^{*-\sigma} + \xi \left(\frac{\Pi}{\Pi_t} \right)^{-\sigma} \Delta_{t-1}. \tag{48}$$

The resource constraint is given by

$$C_t + I_t = Y_t. \tag{49}$$

In summary, we have shown that the equilibrium system in a neighborhood of the steady state with $R_t > 1$ consists of 21 equations (6), (12), (21), (22), (23), (24), (32), (33), (34), (35), (40), (41), (42), (43), (44), (45), (46), (47), (48), and (49), plus a monetary policy equation (36) or (37), for 22 variables $\{w_t, \varepsilon_t^*, \varepsilon_t^{**}, q_t^k, R_t, R_{lt}, p_t^*, \Gamma_t^a, \Gamma_t^b, \Pi_t, p_{wt}, N_t, R_{kt}, p_t^h, p_t^E, m_t, g_t, I_t, K_t, Y_t, \Delta_t, C_t\}$, where $\Lambda_t = 1/C_t$. Here K_t, m_t, and Δ_t are endogenously predetermined variables. The usual transversality conditions must also hold.

As in Galí (2014), the sizes of pre-existing bubbles and new bubbles cannot be independently determined in equilibrium. That is, there could exist a continuum of bubbly equilibria. Galí then assumes that new bubbles follow an exogenous IID process, which is interpreted as a bubble shock. For ease of computations, we follow Miao, Wang, and Xu (2015) and assume that new bubbles account for a stochastic fraction of aggregate bubbles so that $p_t^E = s_t p_t^h$, where $s_t \in (0, 1)$. Suppose that

$$\ln s_t = (1 - \rho_s) \ln s + \rho_s \ln s_{t-1} + \varepsilon_{st},$$

where $\rho_s \in (0, 1)$, $s \in (0, 1)$ is the non-stochastic steady-state value of s_t, and $\{\varepsilon_{st}\}$ is a white noise process.[13] We can interpret s_t as an exogenously given bubble shock or sentiment shock that drives the movements of aggregate bubbles.

4 Steady-State Analysis

There are two types of steady state equilibria. In the bubbleless steady state, the land price is equal to zero. In a bubbly steady state, the land price is positive. We remove the time subscript

[13]We need the shock to be small ε_{st} so that s_t is between 0 and 1.

from a variable to indicate its steady state value. Whenever necessary, we also use a variable with subscript f or b to indicate its bubbleless or bubbly steady state value, respectively. In both types of steady states we have $p^* = p_w = \Delta = 1$ and the inflation rate is equal to the growth rate of money, $\Pi = \exp(g)$. Due to the full price indexation assumed earlier, both the flexible price equilibrium and the sticky price equilibrium have the same steady states.

We first derive the equations that hold in both the bubbleless and bubbly steady states. By equation (24), we can derive the nominal deposit rate as

$$R = \frac{\Pi \beta^{-1}}{1 + \int_{\varepsilon^*}^{\varepsilon_{\max}} \left(\frac{\varepsilon}{\varepsilon^*} - 1 \right) dF(\varepsilon)} \equiv R(\varepsilon^*). \tag{50}$$

It is straightforward to show that $R(\varepsilon^*)$ increases with ε^* and $R/\Pi < 1/\beta$. Since the steady-state real interest rate R/Π in both the bubbly and bubbleless steady states is too low, households will not hold land (even if they are allowed to trade land) and will not save in a neighborhood of either steady state so that $S_{a,t+1} = 0$.

By (12) and (22), we have

$$\varepsilon^{**} = \frac{1 - \lambda / R(\varepsilon^*)}{1 - \lambda} \varepsilon^* \equiv \varepsilon(\varepsilon^*). \tag{51}$$

We can easily check that ε^{**} increases with ε^*. Using (23) and (51), we can derive

$$R_k = \frac{1}{\varepsilon^*} \frac{\frac{1}{\beta} - 1 + \delta - \mu \int_{\varepsilon^{**}}^{\varepsilon_{\max}} (\varepsilon - \varepsilon^{**}) \, dF(\varepsilon)}{1 + \int_{\varepsilon^*}^{\varepsilon_{\max}} \left(\frac{\varepsilon}{\varepsilon^*} - 1 \right) dF(\varepsilon)} \equiv R_k(\varepsilon^*). \tag{52}$$

We will impose assumptions below such that $R > 1$ so that $R_l > R$ and $\varepsilon^{**} > \varepsilon^*$. The critical step of establishing the existence of steady state equilibria is to show the existence of a cutoff ε^*. Once this cutoff is obtained, other steady-state values can be easily derived.

4.1 Bubbleless Steady State

In a bubbleless steady state $p^h = 0$ and hence we can ignore equation (25). To show the existence of a bubbleless steady state with $R_f > 1$, we impose the following:

Assumption 1 *Let $\beta E[\varepsilon] > \Pi \varepsilon_{\min}$ and $\Pi > \beta$.*

The first inequality holds for any distribution with $\varepsilon_{\min} = 0$ and the second is needed because it is implied by (50) for $R > 1$ in the steady state.

Lemma 1 *Under assumption 1, there exists a unique $\underline{\varepsilon} \in (\varepsilon_{\min}, \varepsilon_{\max})$ such that*

$$1 + \int_{\underline{\varepsilon}}^{\varepsilon_{\max}} \left(\frac{\varepsilon}{\underline{\varepsilon}} - 1 \right) dF(\varepsilon) = \Pi \beta^{-1}.$$

This lemma implies that $R(\underline{\varepsilon}) = 1$. Since $R(\varepsilon^*)$ increases with ε^*, we need to find an $\varepsilon^* > \underline{\varepsilon}$ so that $R = R(\varepsilon^*) > 1$ in the steady state.

Proposition 2 *Suppose that assumption 1 holds and μ satisfies the assumptions in Appendix A. Then there exists a unique bubbleless steady state with $R_f = R\left(\varepsilon_f^*\right) > 1$, where $\varepsilon_f^* \in (\underline{\varepsilon}, \varepsilon_{\max})$ is the unique solution to the equation*

$$\delta = \left(R_k(\varepsilon^*) + \frac{\lambda\mu}{1-\lambda}\frac{1-F(\varepsilon(\varepsilon^*))}{\Pi}\right)\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon), \tag{53}$$

where $\varepsilon(\varepsilon^*)$ is given in (51).

Equation (53) is derived from the steady-state version of equation (46). The left-hand side of (53) is the depreciation rate and the right-hand side is the ratio of investment in efficiency units to the capital stock. Investment (in consumption units) is financed by internal funds $R_k(\varepsilon^*)K$ and the real balance of reserves (i.e., savings minus loan repayments) by equation (43)

$$\frac{m}{\Pi} = \frac{\lambda\mu K}{1-\lambda}\frac{1-F(\varepsilon(\varepsilon^*))}{\Pi}. \tag{54}$$

In addition, firms with efficiency levels higher than $\varepsilon(\varepsilon^*)$ borrow from banks. Since land has no value in a bubbleless equilibrium, firms can only borrow against a fraction μ of their capital. This explains the last term in (53).

After determining the investment threshold ε_f^*, we then derive the other threshold $\varepsilon_f^{**} = \varepsilon(\varepsilon_f^*)$ using (51). The other equilibrium variables can also be easily determined as shown in the proof of Proposition 3 in Appendix A. From the analysis above, our model clearly implies the non-superneutrality of money in the sense that the steady-state levels of the capital-labor ratio, consumption, and output all depend on the inflation rate.

Proposition 3 *Suppose that the assumptions in Proposition 2 hold. Then in the bubbleless steady state ε_f^*, K_f/N_f, w_f, Y_f/N_f, and C_f all decrease with Π.*

The intuition is that the inflation rate Π affects entrepreneurs' net worth and hence their investment behavior. In particular, it affects the investment cutoff ε_f^* according to (53). The impact on ε_f^* in turn affects the mass of investing entrepreneurs and the liquidity premium and hence the real economy. Note that the assumption of the reserve requirement $\lambda \in (0,1)$ is crucial for the non-superneutrality of money and also for inflation to affect the bubbly steady state analyzed in the next subsection.

4.2 Bubbly Steady State

We now turn to the bubbly steady state. As in Galí (2014), there could exist a continuum of bubbly steady states because the sizes of pre-existing bubbles and new bubbles cannot be independently determined. Here we will take the fraction s of pre-existing bubbles as given. The following proposition studies the existence issue.

Proposition 4 *Suppose that the assumptions in Proposition 2 hold so that there exists a bubbleless steady state with $R_f = R\left(\varepsilon_f^*\right) > 1$, where ε_f^* is the investment threshold. Then there exists a unique bubbly steady state with $R_b = R\left(\varepsilon_b^*\right) > R_f > 1$, where the cutoff $\varepsilon_b^* \in \left(\varepsilon_f^*, \varepsilon_{\max}\right)$ is the unique solution to the equation*

$$\beta s\left(1+(1-\omega)\int_{\varepsilon^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon^*}-1\right)dF\left(\varepsilon\right)\right)+\beta s\theta\omega\int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon^*}-\frac{\varepsilon\left(\varepsilon^*\right)}{\varepsilon^*}\right)dF\left(\varepsilon\right)=1, \tag{55}$$

if and only if

$$\beta s\left(1+(1-\omega)\int_{\varepsilon_f^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_f^*}-1\right)dF\left(\varepsilon\right)\right)+\beta s\theta\omega\int_{\varepsilon(\varepsilon_f^*)}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_f^*}-\frac{\varepsilon(\varepsilon_f^*)}{\varepsilon_f^*}\right)dF\left(\varepsilon\right)>1, \tag{56}$$

where $\varepsilon\left(\varepsilon^*\right)$ is given in (51).

Equation (55) follows from the steady-state version of the asset pricing equation (42) for land when $p_t^E = s_t p_t^h > 0$ is constant over time. The interpretation of condition (56) is as follows: The left-hand side of (56) represents the benefit of buying one unit of aggregate land today and the right-hand side represents the associated cost. In addition to the resale value of land, land commands a liquidity premium due to credit constraints. The liquidity premium has two components. First, the entrepreneur can sell a fraction $(1-\omega)$ of land to finance investment if his investment efficiency exceeds ε_f^*. Second, when his investment efficiency exceeds $\varepsilon_f^{**} = \varepsilon(\varepsilon_f^*)$, he can borrow from banks using a fraction θ of his unsold land as collateral. This explains the two integral terms in (56). When the benefit exceeds the cost in the bubbleless steady state, entrepreneurs have incentives to trade land at a positive price, thereby creating a bubble.

Once the cutoff ε_b^* is determined, we can follow a similar procedure to that described in the previous subsection to derive other steady state variables. The details can be found in the proof of Proposition 4 in Appendix A.

How do we relate our existence condition (56) to the traditional condition that the bubbleless steady-state interest rate must be lower than the rate of economic growth (Tirole (1985))? Unlike the OLG model of Tirole (1985), our model features credit constraints in the infinite-horizon DNK framework. We have the following result.

Proposition 5 *Suppose that the assumptions in Proposition 2 hold so that there exists a bubbleless steady state with $R_f = R\left(\varepsilon_f^*\right) > 1$, where ε_f^* is the investment threshold. A necessary condition for the existence of a bubbly steady state with $R_b > 1$ is*

$$\Pi > \max\left\{\beta+\frac{1/s-\beta}{1-\omega+\theta\omega}, R_f\right\} > 1.$$

This proposition shows that for a bubbly steady state to exist, the net inflation rate must be positive and sufficiently high and the gross real interest rate R_f/Π must be less than the gross growth rate of the economy (which is 1). But these conditions are not sufficient as shown in the previous proposition.

Does monetary policy affect the existence condition for a bubbly steady state?

Proposition 6 *The higher the inflation rate, the more likely a bubbly steady state can exist.*

The intuition behind this proposition is as follows. As shown in Proposition 3, an increase in the rate of inflation reduces an entrepreneur's real balance and hence his net worth and investment. To maintain the aggregate investment rate at the same level as the depreciation rate in the bubbleless steady state, there must be more firms to make investment. Thus the investment threshold must fall or Tobin's Q must rise, causing the liquidity premium to rise in the bubbleless steady state. This raises the benefit of trading land so that condition (56) is more likely to be satisfied for higher inflation.

As mentioned earlier, the assumption of the reserve requirement is critical. Without this assumption (i.e., $\lambda = 0$), we would have the superneutrality of money and inflation (or monetary policy) would have no effect on the condition for the existence of a bubble.

It is possible that (56) does not hold for low inflation so that only a bubbleless equilibrium can exist initially. The central bank can conduct an expansionary monetary policy by increasing money supply such that inflation rises to a permanently higher level. At this higher level, condition (56) holds so that a bubble can emerge. On the other hand, suppose that the economy is initially in the bubbly steady state. When the central bank reduces money supply such that inflation decreases to a permanently lower level at which condition (56) fails to hold, the asset bubble will collapse. This result is consistent with the conventional wisdom that an expansionary monetary policy can fuel a bubble and a contractionary monetary policy can prick a bubble. The key intuition comes from the balance sheet channel. Inflation caused by expansionary monetary policy reduces an entrepreneur's net worth and raises the liquidity premium, thereby raising his demand for the bubble asset.

We now conduct a comparative statics analysis based on the parameter values given in Section 5.1. We study the impact of the inflation target on the bubbly steady state when (56) holds. Figure 1 presents the result, which shows that higher inflation has a negative effect on the economy in the long run. Higher inflation raises the size of the asset bubble, but lowers output, consumption, capital, and labor. The intuition is that higher inflation lowers the real balance and hence entrepreneurs' net worth, leading to lower investment. On the other hand, the credit-constrained entrepreneurs will demand more bubble assets to finance investment. In the meantime, higher inflation is associated with higher money supply, which can support a larger size of bubbles and inflate asset bubbles. As discussed earlier, a larger size of the asset bubble has both intensive and extensive margin effects. It allows entrepreneurs to borrow more to make investment. But as the borrowing cutoff ε_t^{**} rises (see Figure 1), fewer firms will borrow and invest. The net effect leads to reduced aggregate investment and capital accumulation.

5 Dynamic Responses to Shocks

In this section we study the dynamic responses of the model economy to various shocks. We shall focus on a bubbly steady state and local dynamics around the bubbly steady state. To provide quantitative experiments, we need to assign parameter values.

5.1 Calibration

We calibrate our model at quarterly frequency so that the bubbly steady state is roughly consistent with the long-run behavior of the US economy. We assume that the investment-efficiency shock follows a Pareto distribution with $F(\varepsilon) = 1 - (\varepsilon/\varepsilon_{\min})^{-\frac{1}{\eta}}$. We set $\varepsilon_{\min} = 1 - \eta$ so that the unconditional mean is 1. As is standard in the business cycle literature, we set $\alpha = 0.33$, $\beta = 0.99$, and $\delta = 0.025$. We choose $\eta = 0.33$ to match the aggregate investment-to-output ratio of 0.2 as in the data. We choose the utility weight on labor $\psi = 3.36$ so that the average number of hours worked in the bubbly steady state equals 25% of the total time endowment. We set $\Pi = 1.01$ so that the steady state annual inflation rate is 4%, which is consistent with the average inflation rate during the period between 1975 and 2016. We set $\lambda = 0.1$ so that the required reserve ratio is 10%. As in the DNK literature (e.g., BGG (1999) and Gilchrist and Leahy (2002)), we set $\xi = 0.75$ and $\sigma = 11$, implying that the duration of price adjustments is four quarters and the steady-state markup is $\sigma/(1-\sigma) = 1.1$.

We set $\omega = 0.25$, which is in line with the ratio of inventory to the sum of existing home sales and inventory, according to the National Association of Retailer. A large literature using mortgage data finds that the loan-to-value ratio is typically between 70% and 80%. Thus we set $\theta = 0.75$, meaning that entrepreneurs can borrow against 75% of the value of their existing real estate. We set the pledgeability parameter for capital $\mu = 0.2$, which is consistent with the estimates reported in Liu, Wang, and Zha (2013) and Miao, Wang, and Xu (2015). We use residential land to proxy the bubble asset in our model and use the land price data in Liu, Wang, and Zha (2013) to calibrate the sentiment shock. We find that the ratio of the land value to output in the data is 2.6. As a baseline estimate, we suppose that about 30% of this ratio is attributed to bubbles. We then set $s = 0.995$ such that $p^h/Y = 0.84$ in the bubbly steady state. Our calibrated parameter values imply that the bubbly steady-state quarterly gross nominal deposit rate R_b is 1.0039 and the bubbleless steady state quarterly nominal gross deposit rate R_f is 1.0003. All conditions in Proposition 4 are satisfied.

Now we assign parameter values for the shocks and the interest rate rules. As in the DNK literature (e.g., Galí (2008)), we set $\phi_\pi = 1.5$, $\phi_y = 0.5/4$, and $\phi_p = 0$ in the Taylor rule (36). For the inflation targeting rule (37), we set $\phi_p = 0$, $\phi_\pi = 1.1$ (weak inflation targeting), and $\phi_\pi = 2$ (strong inflation targeting), as in Bernanke and Gertler (1999) and Gilchrist and Leahy (2002). As baseline values, we set the persistence and volatility parameters for the monetary policy shock and technology shock as $\rho_v = 0.5$, $\sigma_v = 0.0025$, $\rho_a = 0.9$, and $\sigma_a = 0.01$ (Galí (2008)). We choose the persistence and volatility parameters for the sentiment shock to match the persistence of 0.8177 and the volatility of 0.0945 for the HP-filtered land price data in Liu, Wang, and Zha (2013). We find that $\rho_s = 0.8177$ and $\sigma_s = 0.01$.

Similar to Galí (2014), our model implies a continuum of bubbly steady states indexed by s. In his unidimensional equilibrium system, Galí (2014) shows that there are two types of bubbly steady states: one is stable and the other is unstable. We are unable to derive this result theoretically for our multi-dimensional equilibrium system. However, we have verified numerically that the bubbly steady state is a saddle point for many values of s. Galí (2014) restricts his analysis to the stable steady state and views the bubble as a backward-looking predetermined variable that converges to the steady state starting at any initial value. He also introduces a sunspot shock to drive the dynamics of the bubble. By contrast, when the bubbly steady state is a saddle point, the bubble is a forward-looking nonpredetermined variable and its initial value must be solved endogenously.

5.2 Technology Shocks

We start by analyzing the impact of a positive technology shock that raises ε_{at} by 1% initially. Figure 2 plots the impulse responses of some key variables and shows that consumption, investment, and output all rise, whereas inflation and nominal interest rate both decline on impact. The positive comovement of consumption, investment, and output is easy to understand as in the real business cycles literature. The negative responses of inflation and nominal interest rate follow the usual intuition in the DNK literature (e.g., Woodford (2003) and Galí (2008)). To see this, we log-linearize the equilibrium system around the non-stochastic bubbly steady state and derive the following New Keynesian Phillips curve[14]

$$\hat{\Pi}_t = \frac{1}{\xi}(1-\xi)(1-\beta\xi)\hat{p}_{wt} + \beta E_t \hat{\Pi}_{t+1},$$

where the marginal cost $\hat{p}_{wt}$ (the relative price of wholesale goods) satisfies

$$\hat{p}_{wt} = \alpha \hat{R}_{kt} + (1-\alpha)\,\hat{w}_t - \hat{A}_t.$$

Following a positive technology shock, the marginal cost goes down or the markup (the inverse of the marginal cost) goes up. Thus inflation declines. The central bank's interest rate rule calls for a

[14] We use hatted variable to denote the log deviation from the deterministic steady state. Appendix B presents the complete log-linearized equilibrium system.

decline of the nominal interest rate by increasing the money supply. Hours worked also decline due to price rigidities under the Taylor rule and the weak inflation targeting rule. Firms that do not adjust prices to meet their output demand have to reduce hours when facing a positive technology shock (see, e.g., Galí (1999, 2008)). This negative effect can dominate depending on the parameter values in preferences, technology, and the interest rate rule. For the strong inflation targeting rule, labor actually rises because the large fall of interest rates stimulates aggregate demand. This monetary policy rule causes consumption, investment, and output to have the largest rise on impact.

A special feature of our model is that the land price responds even though land has no fundamental payoffs. To see the intuition, we can derive the log-linearized equation for the land price

$$\begin{aligned} \hat{p}_t^h &= E_t\hat{p}_{t+1}^h + \hat{C}_t - E_t\hat{C}_{t+1} + \hat{s}_t - s\beta\theta\omega\left[1 - F\left(\varepsilon^{**}\right)\right]\frac{\varepsilon^{**}}{\varepsilon^*}E_t\hat{\varepsilon}_{t+1}^{**} \\ &\quad - s\beta\left[(1-\omega)\int_{\varepsilon^*}^{\varepsilon_{\max}}\frac{\varepsilon}{\varepsilon^*}dF\left(\varepsilon\right) + \theta\omega\frac{1}{\varepsilon^*}\int_{\varepsilon^{**}}^{\varepsilon_{\max}}\left(\varepsilon - \varepsilon^{**}\right)dF\left(\varepsilon\right)\right]E_t\hat{\varepsilon}_{t+1}^*. \end{aligned} \tag{57}$$

Three factors affect the land price: the SDF $\hat{C}_t - E_t\hat{C}_{t+1}$, the sentiment shock $\hat{s}_t$, and the liquidity premium. The liquidity premium has two components as reflected by the last two terms in the equation above. Figure 2 shows that the two cutoffs $\hat{\varepsilon}_t^*$ and $\hat{\varepsilon}_t^{**}$ fall on impact in response to a positive technology shock. Thus the liquidity premium rises. The SDF also rises because consumption growth declines. Both components cause the land price to rise on impact. Note that the land bubble p_t^h or $\hat{p}_t^h$ is a forward-looking variable and its initial value must be endogenously determined. Our discussions on the movements of the land bubble follow the conventional wisdom on any asset prices.

The fall of ε_t^* is due to the rise of the capital price q_t^k as $\varepsilon_t^* = 1/q_t^k$, which in turn is driven by the rise in the marginal revenue product of capital R_{kt} following a positive technology shock. The fall of ε_t^{**} is due to the fall of the lending rate R_{lt}. When the nominal deposit rate R_t falls, R_{lt} must fall by (12), which is derived from the zero-profit condition of banks. Thus real bank lending L_{t+1}/P_t rises to enable more investment. Note that money supply (g_t) increases in the short run in order to support the lower nominal deposit rate and higher bank lending.

5.3 Monetary Policy Shocks

Next we consider the impact of an expansionary monetary policy shock when v_t or ε_{vt} drops by 25 basis points initially in (36) and (37). This corresponds to a one percentage point drop of the annual nominal rate on impact holding other variables fixed. Figure 3 displays the impulse responses. We find that the (quarterly) nominal interest rate R_t initially drops by less than 25 basis points because the interest rate endogenously responds to changes in output and inflation by the Taylor rule (36) and to expected inflation by the inflation targeting rule (37).

The traditional transmission mechanism of monetary policy is through the interest rate channel. When prices are sticky, the real rate falls and thus consumption rises following an expansionary monetary policy shock. This can be seen from the log-linearized equation of (24):

$$\hat{C}_t = E_t\hat{C}_{t+1} - (\hat{R}_t - E_t\hat{\Pi}_{t+1}) + \frac{R\beta}{\Pi}\int_{\varepsilon^*}^{\varepsilon_{\max}} \frac{\varepsilon}{\varepsilon^*} dF(\varepsilon)\, E_t\hat{\varepsilon}^*_{t+1}. \tag{58}$$

This is analogous to the dynamic IS curve in the traditional DNK model, with an additional liquidity premium term. The fall of the real rate also causes the increase in investment and hence output. Thus monetary policy is not neutral in the DNK framework.

In addition to this interest rate channel, our model features a lending channel and an asset price channel. An expansionary monetary policy causes money supply to increase in the short run and hence bank lending rises. The lending rate R_{lt} falls with the deposit rate R_t. This in turn raises investment. This is similar to the lending channel discussed in BGG (1999). A unique feature of our model is that asset prices rise for the following three reasons. First, the increased bank lending calls for more collateral from firms. This raises the land price since a fixed fraction of land is used as collateral. Second, the SDF rises on impact because consumption rises initially before gradually declining. Third, both components of the liquidity premium rise in response to an expansionary monetary policy shock. As Figure 3 shows, both cutoffs ε_t^* and ε_t^{**} fall on impact. The intuition is the following. Since consumption, investment, and output rise as discussed earlier, labor demand must also rise on impact as capital is predetermined. The real wage rate rises by the labor supply condition (6). By the labor demand condition (40), the marginal cost or the relative price of wholesale goods p_{wt} must go up, causing inflation to rise. Moreover, it follows from equation (41) that the marginal revenue product of capital R_{kt} must rise, pushing up the capital price q_t^k. As a result, the liquidity premium rises or the investment cutoff ε_t^* falls, and hence the lending cutoff ε_t^{**} falls with R_t by (12) and (22).

The increased asset price raises entrepreneurs' net worth and collateral value, allowing them to finance more investment. Both the intensive and extensive margin effects work in the same direction so that the asset price channel provides a large amplification effect on investment. As shown in Figure 3, investment rises by about 1-2% on impact in response to a cut of 25 basis points in the nominal deposit rate.

Our result that an expansionary monetary policy by cutting interest rates drives asset bubbles on impact is different from that in Galí (2014). Galí focuses on the stable steady state in his unidimensional equilibrium system and assumes that the asset bubble is predetermined and hence its initial size is independent of monetary policy. By contrast, the asset bubble in our paper is a forward-looking nonpredetermined variable so that both its initial size and growth rate are determined by the real interest rate. The initial jump of the asset bubble provides an important amplification mechanism for monetary policy.

5.4 Sentiment Shocks

Suppose that agents suddenly become more optimistic in that there is a positive sentiment shock that raises ε_{st} by a standard deviation initially. As Figure 4 shows, this shock immediately raises the land price by the asset pricing equation, thereby raising entrepreneurs' net worth and collateral value. Thus bank lending and investment rise, causing output to follow suit. Labor must rise on impact because output rises and capital is predetermined. The increased bank lending causes money supply and inflation to go up. The initial rise of inflation is quantitatively small. The nominal deposit rate must rise due to the interest rate rules. Since prices are sticky, the real deposit rate rises and hence households have an incentive to save instead of consuming. For the Taylor rule in (36), the real interest rate rises so much so that consumption falls on impact. Thus there is no comovement between consumption and investment in response to a positive sentiment shock. In Miao and Wang (2015b) and Miao, Wang, and Xu (2015) we show that introducing endogenous capital utilization can amplify the initial impact on output and hence allow consumption to rise. For the inflation targeting rule in (37), the nominal interest rate does not respond to output directly so that the rise of the real interest rate is smaller, causing consumption to rise on impact. This effect is stronger for the strong inflation targeting rule.

Increased investment generates more capital accumulation and hence the capital price q_t^k falls on impact. Consequently the investment cutoff $\varepsilon_t^* = 1/q_t^k$ rises as does the lending cutoff ε_t^{**}. This implies that fewer firms make investment. But this negative extensive margin effect is dominated by the positive intensive margin effect so that aggregate investment rises. The net effect on investment is relatively small: a 5.5% increase in the land price only raises aggregate investment by about 0.5% on impact. In the meantime, the liquidity premium falls on impact, but this negative effect on the land price is dominated by the direct effect of the positive sentiment shock.

Note that Figure 4 shows that the asset bubble contracts over time, but the real interest rate is still positive. This is because the initial sentiment shock dies out over time so that the path of the asset bubble follows that of the shock by (57). The initial jump of the asset bubble roughly reflects the cumulative effect of the initial sentiment shock, $0.01/(1-0.8177) = 5.5\%$.

In summary, the non-fundamental sentiment shock can drive large movements of the asset bubble due to its forward-looking nature. But it has a small positive impact on the real economy and generates mild inflation. The main reason is that the opposite intensive and extensive margin effects partially offset each other and mitigate the impact of a sentiment shock on aggregate demand and the real economy.

Monetary Policy Rules	Sentiment Shock		TFP shock	
	weight ϕ_p	gain Ω (%)	weight ϕ_p	gain Ω (%)
$\hat{R}_t = 1.5\hat{\Pi}_t + 0.125\hat{Y}_t + \phi_p \hat{p}_{ht}$	-0.36	60	-0.28	0.34
$\hat{R}_t = 1.1E_t\hat{\Pi}_{t+1} + \phi_p \hat{p}_{ht}$	-0.03	0.51	-0.03	0.38
$\hat{R}_t = 2E_t\hat{\Pi}_{t+1} + \phi_p \hat{p}_{ht}$	-0.02	0.02	-0.02	0.01

Table 1: Optimal weights on asset bubbles and welfare gains.

6 Should Monetary Policy Respond to Asset Bubbles?

In the previous section we have shown how asset bubbles and other macroeconomic variables respond to various exogenous shocks when monetary policy does not respond to asset bubbles. In this section we address the question whether monetary policy should respond to asset bubbles.

We consider the Taylor rule in (36) by fixing the parameters $\phi_\pi = 1.5$ and $\phi_y = 0.125$. We also consider the weak inflation targeting rule and strong inflation targeting rule by fixing $\phi_\pi = 1.1$ and $\phi_\pi = 2$ in (37). We turn off the monetary policy shock by setting $v_t = 0$. We search for the weight ϕ_p that maximizes the unconditional mean of household utility in (4) in response to each of the two shocks to sentiment and TFP.[15] Table 1 presents the optimal weights ϕ_p on asset bubbles and the welfare gains from optimally responding to asset bubbles.

The welfare gains are computed as follows. Let $\{C_t\}$ and $\{N_t\}$ denote the equilibrium consumption and labor processes for $\phi_p = 0$. Let V and V^* denote the unconditional means of household utility when monetary policy does not respond and when it does respond to asset bubbles, respectively. Then the welfare gains Ω in terms of the increase in consumption satisfies the equation

$$E \sum_{t=0}^{\infty} \beta^t \left[\ln(1+\Omega)C_t - \psi N_t\right] = V^*.$$

Solving yields $\Omega = \exp\left((1-\beta)\left(V^* - V\right)\right) - 1$.

Table 1 shows that the welfare gain is equal to a 60% increase in consumption for monetary policy to respond to asset bubbles under the Taylor rule in (36), when only the sentiment shock hits the economy. The weight on asset bubbles is equal to -0.36, meaning that the central bank under the Taylor rule should cut the nominal interest rate by about 1.44% per annum when the asset bubble expands by 1% from the steady state. The welfare gains are small for all other cases and close to zero for the strong inflation targeting rule.

Figure 5 presents the unconditional expected household utility as a function of the weight ϕ_p under the Taylor-type rule in (36) and the weak inflation targeting rule in (37) with $\phi_\pi = 1.1$. The top left panel shows that the expected utility first decreases sharply with ϕ_p when only the

[15] We use the second-order approximation method to compute the welfare gains implemented by Dynare (Adjemian et al (2011)).

sentiment shock hits. The expected utility is maximized at the smallest $\phi_p = -0.36$ such that the Blanchard-Kahn condition is satisfied. The top right panel shows that the expected utility is a hump-shaped function of ϕ_p and the Blanchard-Kahn condition is violated for ϕ_p smaller than -0.35. The bottom two panels show that the expected utility decreases with ϕ_p and is maximized at the smallest $\phi_p = -0.03$ such that the Blanchard-Kahn condition is satisfied.

Figure 6 presents the impulse responses to a positive TFP shock for three cases under the interest rate rule in (36): (i) monetary policy optimally responds to asset bubbles; (ii) monetary policy does not respond to asset bubbles; and (iii) prices are flexible with $\phi_p = 0$. In the flexible price equilibrium, consumption, investment, output, and labor all comove in response to a positive technology shock. But when prices are sticky and $\phi_p = 0$, labor falls on impact. To correct this inefficiency, monetary policy should respond to the expanding asset bubble by setting $\phi_p = -0.28$ to cut the nominal interest rate . In this way the asset price can increase more so that bank lending rises more, mitigating the fall of inflation. The Taylor-type interest rate rule also calls for cutting the nominal interest rate in response to a decline of inflation. Thus the real interest rate falls. In the meantime, the asset price and bank lending channels cause investment and output to rise more. This allows consumption to rise more and labor to also rise instead of falling. Overall, the impulse responses of consumption, investment, output, and labor under the optimal simple rule are closer to those in the flexible price equilibrium.

Figure 7 presents impulse responses to a positive sentiment shock under the interest rate rule in (36). We find that consumption, investment, output, and labor all increase in response to a positive sentiment shock under the optimal simple rule ($\phi_p = -0.36$). Without responding to the asset bubble, consumption falls because the real interest rate rises (also see Figure 4). Moreover, the impact of the sentiment shock on the real economy and inflation is very small for both the sticky price and flexible price equilibria. Under the optimal simple rule, the central bank should cut the nominal interest rate to reduce the real interest rate and raise consumption, thereby raising household utility. In the meantime, the asset bubble rises more on impact. As Figures 6 and 7 show, our model is different from Galí's (2014) in which the asset price does not move on impact. In our model the initial rise of the bubble improves allocation efficiency by raising aggregate investment and output.

Figure 8 shows that, under the Taylor-type rule (36), inflation volatility increases with ϕ_p conditional on the TFP shock and is a U-shaped function of ϕ_p conditional on the sentiment shock. For both cases, bubble volatility decreases with ϕ_p. Thus the 'leaning against the wind' interest rate policy will reduce bubble volatility, but at the possible cost of raising inflation volatility. This is in sharp contrast to Galí's (2014) result that such a policy may raise bubble volatility conditional on the sentiment shock. The intuition is that a larger ϕ_p reduces the initial response of the bubble to either a TFP shock or a sentiment shock as the bubble is a forward-looking variable.[16] This reduces aggregate demand more and causes a larger drop of inflation from the target in response to a positive TFP shock, so that inflation volatility is higher conditional on the TFP shock. In response to a positive sentiment shock, a larger negative ϕ_p lowers nominal interest rates more and raises aggregate demand more, thereby generating a larger rise of inflation relative to the target. But a larger positive ϕ_p raises nominal interest rates more and reduces aggregate demand more, thereby generating a larger drop of inflation relative to the target. In both cases inflation volatility is higher conditional on the sentiment shock.

Will the preceding results change if the central bank adopts the inflation targeting rule in (37)? Table 1 shows that the optimal weights on the asset bubble are generally small under the inflation targeting rule. The welfare gains are negligible for the strong inflation targeting rule. Under the inflation targeting rule, the nominal interest rate does not respond to output changes and hence is more accommodative to economic growth. Thus consumption can rise in response to a positive sentiment shock due to the small change in the real interest rate and labor can rise in response to a positive TFP shock due to the fall of the real interest rate, as shown in Figures 2 and 4. This means that the distortion caused by price stickiness is less severe, especially for the strong inflation targeting rule. Thus the welfare gains are very small for the strong inflation targeting rule.[17] Our results confirm the early findings of Bernanke and Gertler (1999).

7 Discussions

In our model deposits are money (or M1) and households do not hold money in equilibrium because the return is too low around the steady state. Moreover, there is no cash in our model. We can adopt several approaches in monetary economics (e.g., Chari, Christiano, and Eichenbaum (1995), Einarsson and Marquis (2001), and Galí (2014)) to allow households to hold money including deposits and cash in equilibrium. For example, we could introduce cash-in-advance constraints and deposit-in-advance constraints. We could also simply introduce cash or deposits in the utility function because they provide some services other than "storage of wealth."

For simplicity, we ignore cash and introduce deposits in the utility function as follows

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \psi N_t + \chi \ln \left(\frac{S_{a,t+1}}{P_t} \right) \right), \tag{59}$$

where $\chi > 0$. In this case aggregate investment satisfies

$$I_t = \left(R_{kt} K_t + \frac{m_t}{\Pi_t} - \frac{s_{at} R_{t-1}}{\Pi_t} + (1-\omega) p_t^h \right) (1 - F(\varepsilon_t^*)) + \left(\theta \omega p_t^h + \mu K_t \right) (1 - F(\varepsilon_t^{**})), \tag{60}$$

[16] For space limitation, we have not reported the figures for the impact of ϕ_p on the impulse response functions. Such results may be foreshadowed from Figures 6 and 7.

[17] Regarding the impact of ϕ_p on the inflation and bubble volatilities under the inflation targeting rule (37), we find a figure similar to Figure 8 and hence will not report it here.

where $s_{at+1} \equiv S_{a,t+1}/P_t$ denotes household real deposits. Since banks have to pay interest on household deposits, the funds available for aggregate investment is reduced by $s_{at}R_{t-1}/\Pi_t$ relative to equation (45). The conditions for the existence of bubbleless and bubbly steady states are too complicated to permit a complete characterization. For example, one has to subtract $s_a R/\Pi$ in the first parenthesis of equation (53). Thus the bubbleless steady-state cutoff becomes smaller. But we need additional equations to pin down this cutoff because s_a is endogenous.

For the bubbly steady state, we can still prove that there is a unique solution for the bubbly steady-state cutoff in equation (55) if and only if condition (56) holds as in Proposition 4. But this is not sufficient to pin down the bubbly steady state for the reason discussed above. We need numerical methods to solve the model. We use numerical examples to illustrate the intuition by varying $\chi > 0$ and fixing other parameter values as in Section 5.1. We find that the bubble size is larger when $\chi > 0$ is larger. In this case, households prefer to save more in banks, banks can extend more loans, and repayments to households are also higher. Inefficient firms have more funds to buy bubble assets and efficient firms can sell bubble assets at a higher price to repay loans. The increased bubble size is used partly to repay household deposits so that real investment, consumption, and output barely change.

We also find that impulse responses are qualitatively similar for $\chi > 0$ and $\chi = 0$. The key difference is that household saving $s_{a,t+1}$ responds to shocks. In particular, $s_{a,t+1}$ falls in response to a positive TFP shock or an expansionary monetary policy shock because the real interest rate falls. But $s_{a,t+1}$ rises in response to a sentiment shock because the real interest rate rises. The movements of $s_{a,t+1}$ can amplify the shocks.

For another extension, it is straightforward to introduce long-run growth to our model as in Tirole (1985), Galí (2014), Miao and Wang (2015b), and Miao, Wang, and Xu (2015). With a positive economic growth rate, the bubbly and bubbleless steady-state real interest rates can be positive. Since such an analysis is standard, we leave out the details.

In our model asset bubbles improves investment efficiency, but generate excessive volatility. Striking a balance between these benefit and cost, optimal monetary policy encourages asset bubbles, even if the welfare gains could be small. One can identify other sources of benefits and costs of asset bubbles. For example, Miao and Wang (2014) and Miao, Wang, and Zhou (2015) show that asset bubbles can generate resource misallocation. Ikeda and Phan (2016) show that risk-shifting can lead to welfare-reducing bubbles. Introducing these costs to our model may generate a different type of optimal monetary policy. For example, in results available upon request, we find that "the leaning against the wind" policy is optimal based on the model of Miao, Wang, and Zhou (2015).

8 Conclusion

We have presented an infinite-horizon model of rational asset bubbles in the DNK framework. We have studied how monetary policy affects asset bubbles in the steady state and their dynamics in response to fundamental and non-fundamental shocks. We have also studied whether central banks should respond to asset bubbles. Our model can deepen our understanding of the relation between monetary policy and asset bubbles. Many of our results are different from those in Galí's (2014) OLG model. We believe that modeling the interaction between financial frictions and asset bubbles and the role of the bubbles are critical for understanding the differences. Moreover, whether asset bubbles are treated as a forward-looking or backward-looking variable is important for understanding their dynamics. Some of our results confirm the early findings of Bernanke and Gertler (1999, 2001) and provide a theoretical foundation for the conventional wisdom on the role of monetary policy in managing asset bubbles. In terms of future research, it would be interesting to study the issue of liquidity traps. The present paper focuses on local dynamics around a bubbly steady state. Dong, Miao, Wang and Xu (2016) show that the economy can enter a liquidity trap with interest rates at the zero lower bound after bubbles collapse. It would also be interesting to enrich our model by confronting it with data along the lines of Ikeda (2013) and Miao, Wang, and Xu (2015). Finally, it is important to conduct empirical work on the relation between monetary policy and asset bubbles along the lines of Galí and Gambetti (2015) and Brunnermeier and Schnabel (2015).

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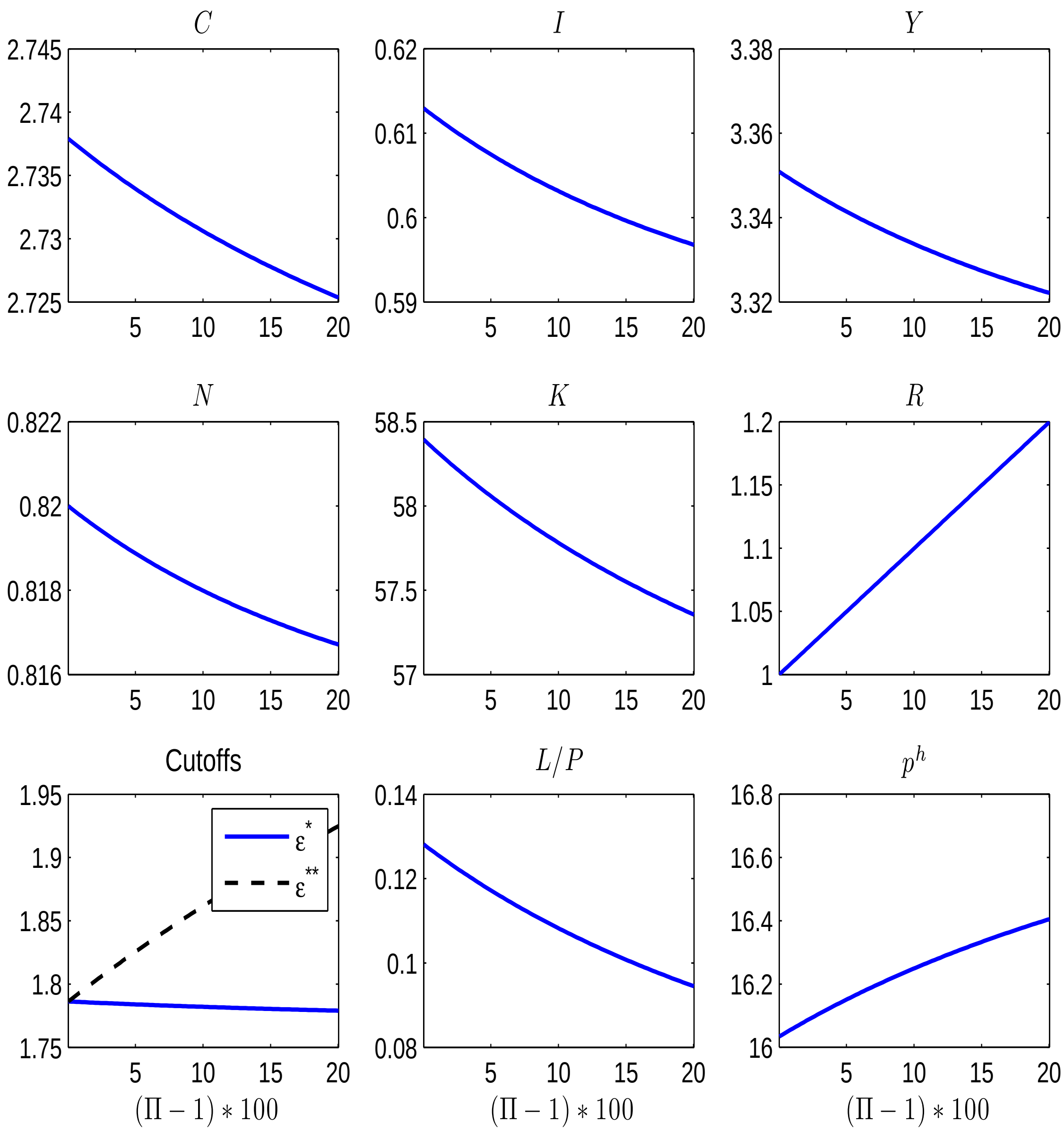

Figure 1: Comparative statics for the bubbly steady state. Parameter values are given in Section 5.1.

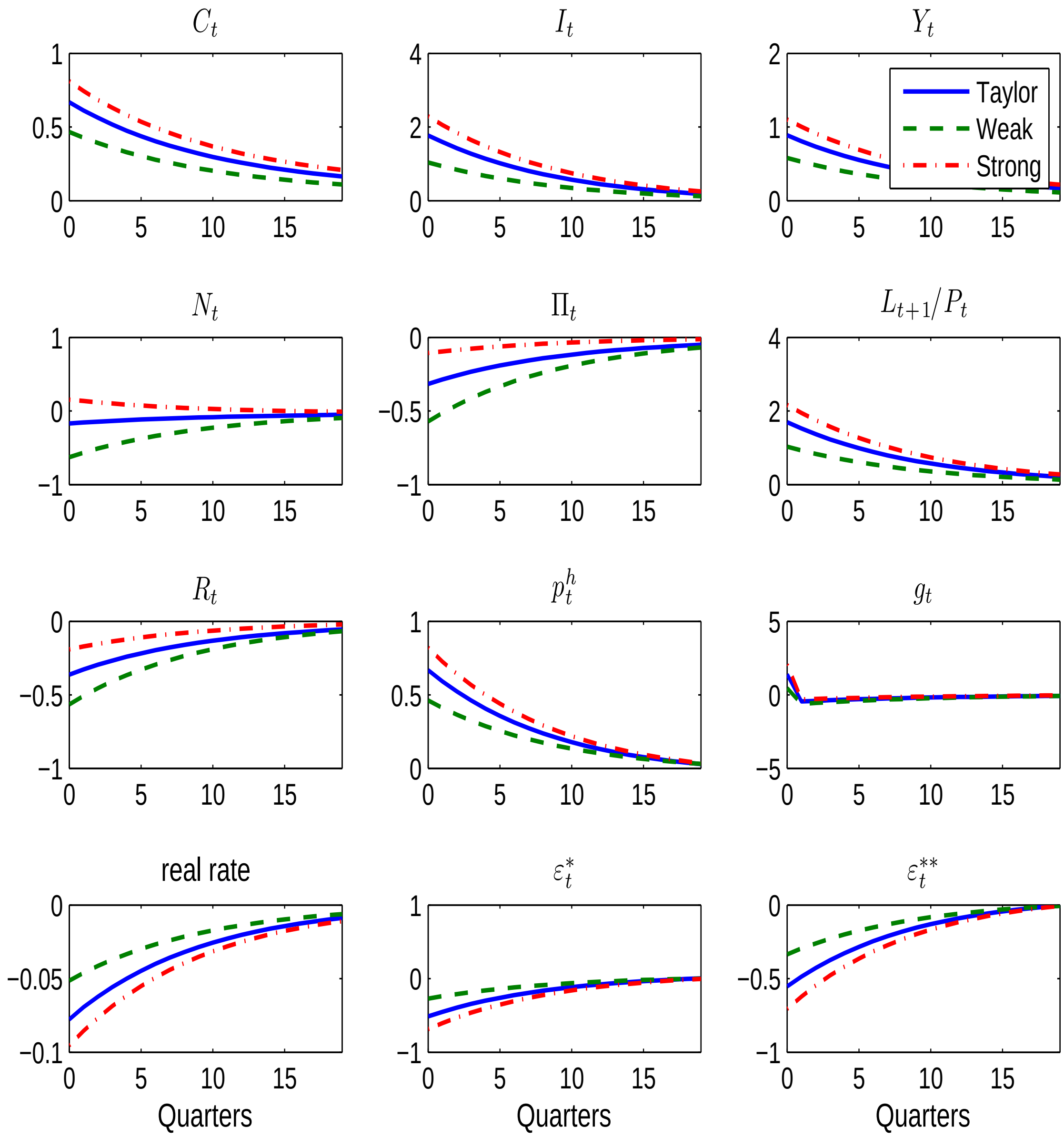

Figure 2: Impulse responses to a positive 1% technology shock. All vertical axes are measured in percentage. The solid, dashed, and dot dashed lines represent responses under the Taylor rule in (36), the weak inflation targeting rule in (37) with $\phi_\pi = 1.1$, and the strong inflation targeting rule in (37) with $\phi_\pi = 2$. In all these rules, $\phi_p = 0$.

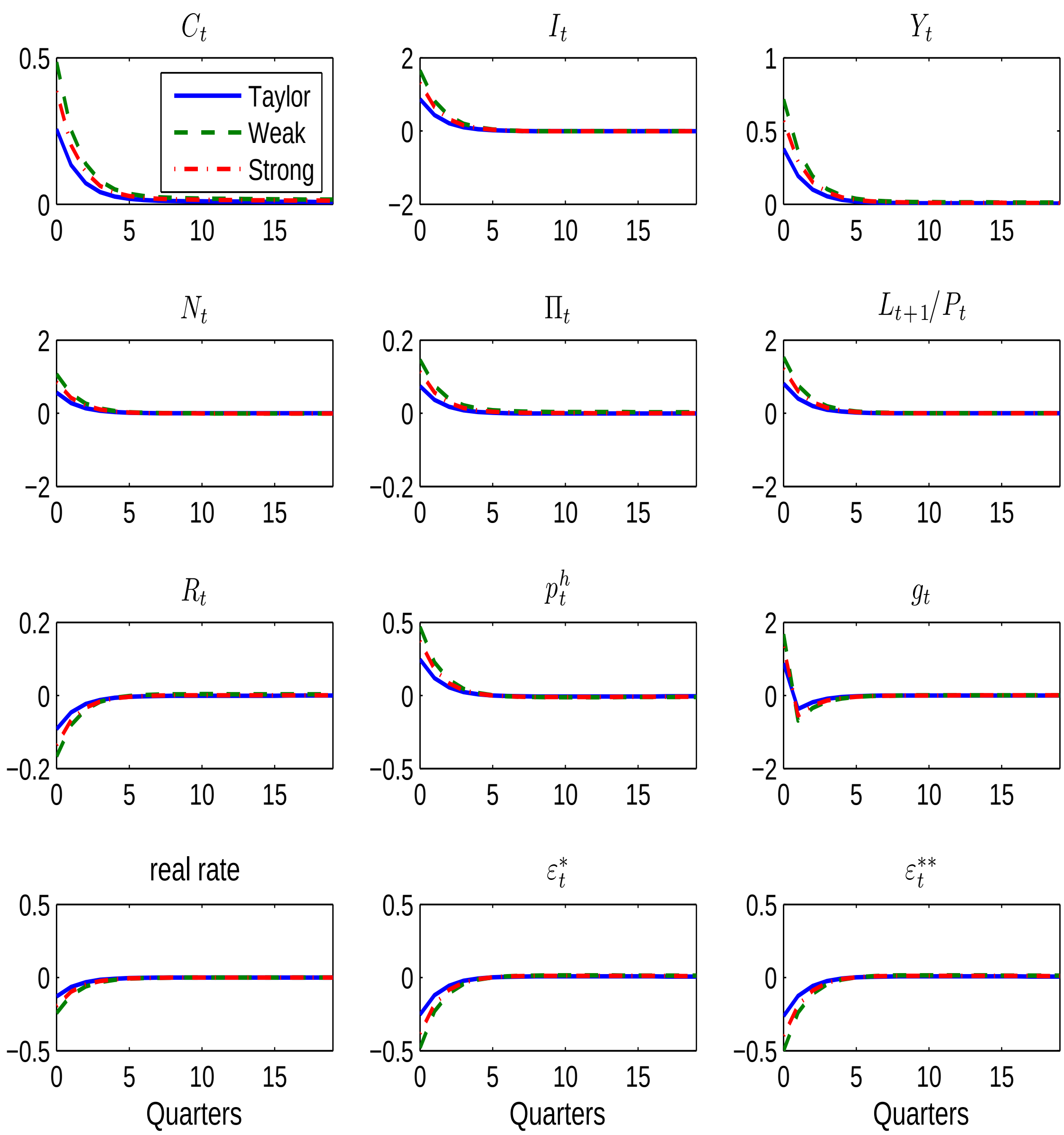

Figure 3: Impulse responses to a negative 0.25% shock to the nominal interest rate. All vertical axes are measured in percentage. The solid, dashed, and dot dashed lines represent responses under the Taylor rule in (36), the weak inflation targeting rule in (37) with $\phi_\pi = 1.1$, and the strong inflation targeting rule in (37) with $\phi_\pi = 2$. In all these rules, $\phi_p = 0$.

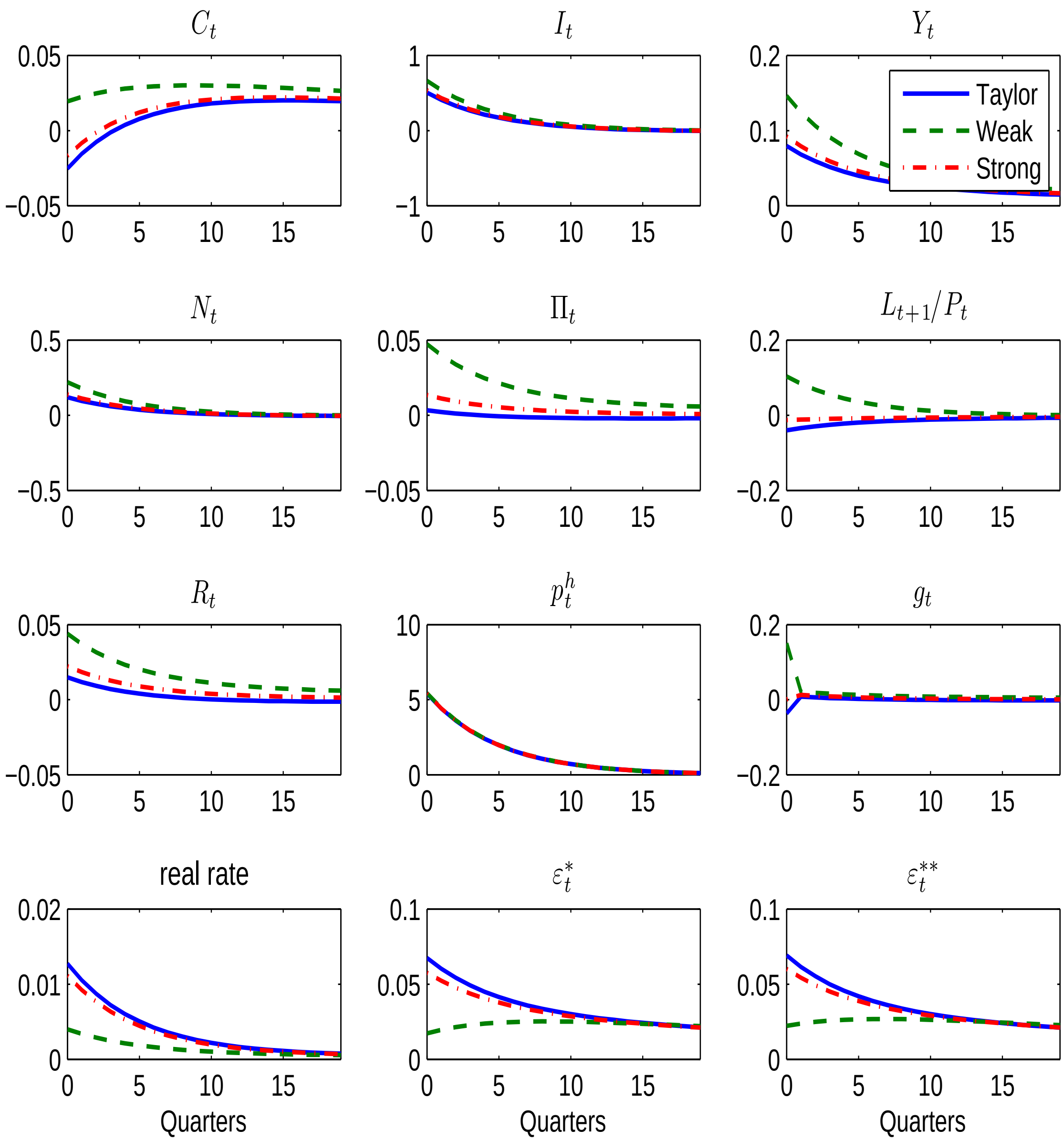

Figure 4: Impulse responses to a positive 1% shock to the sentiment. All vertical axes are measured in percentage. The solid, dashed, and dot dashed lines represent responses under the Taylor rule in (36), the weak inflation targeting rule in (37) with $\phi_\pi = 1.1$, and the strong inflation targeting rule in (37) with $\phi_\pi = 2$. In all these rules, $\phi_p = 0$.

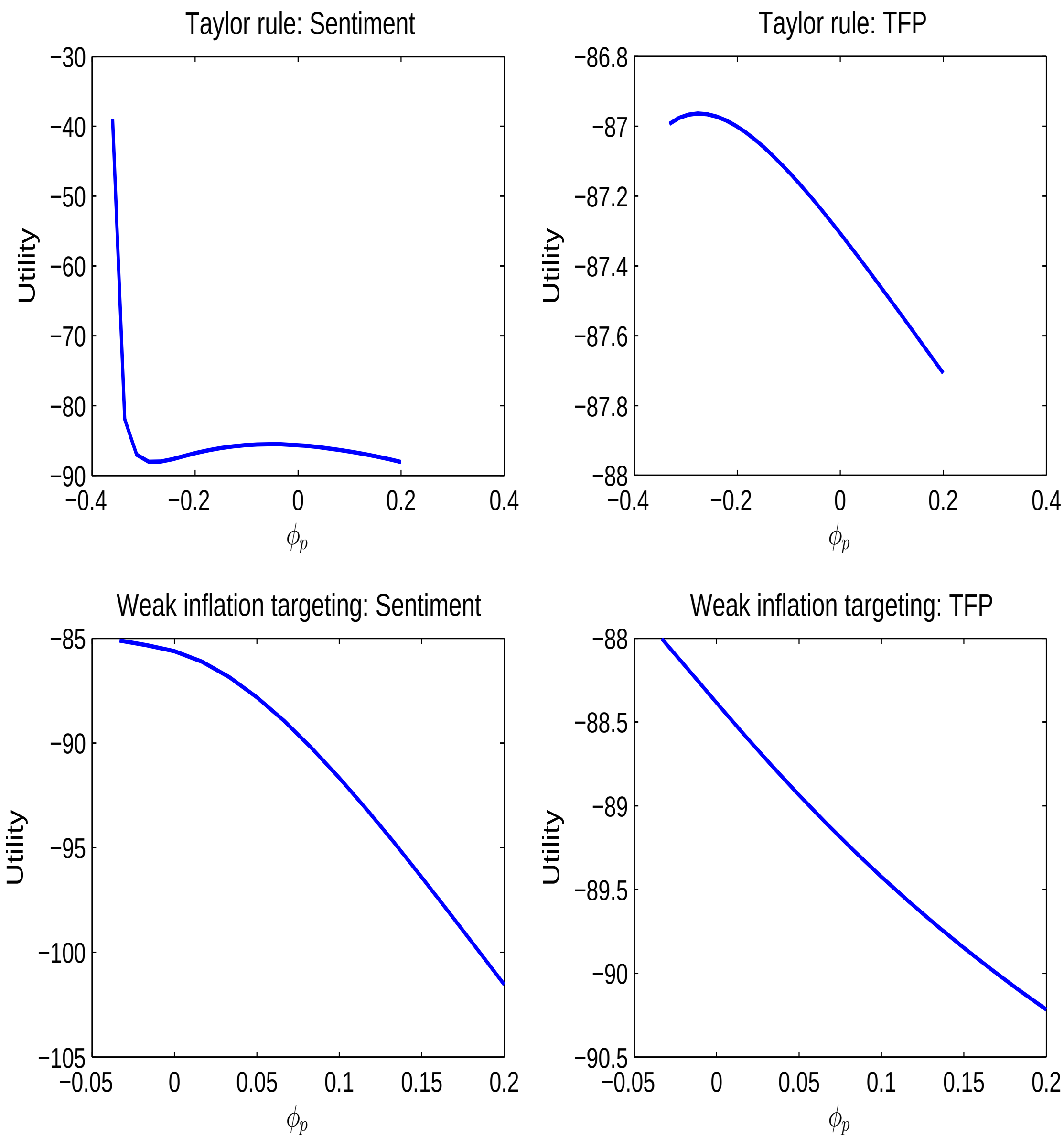

Figure 5: Unconditional expected household utility as a function of the weight ϕ_p on the asset bubble when only sentiment shocks or TFP shocks are turned on. The top two panels are for the interest rate rule in (36) with $\phi_\pi = 1.5$ and $\phi_y = 0.125$. The bottom two panels are for the weak inflation targeting rule in (37) with $\phi_\pi = 1.1$.

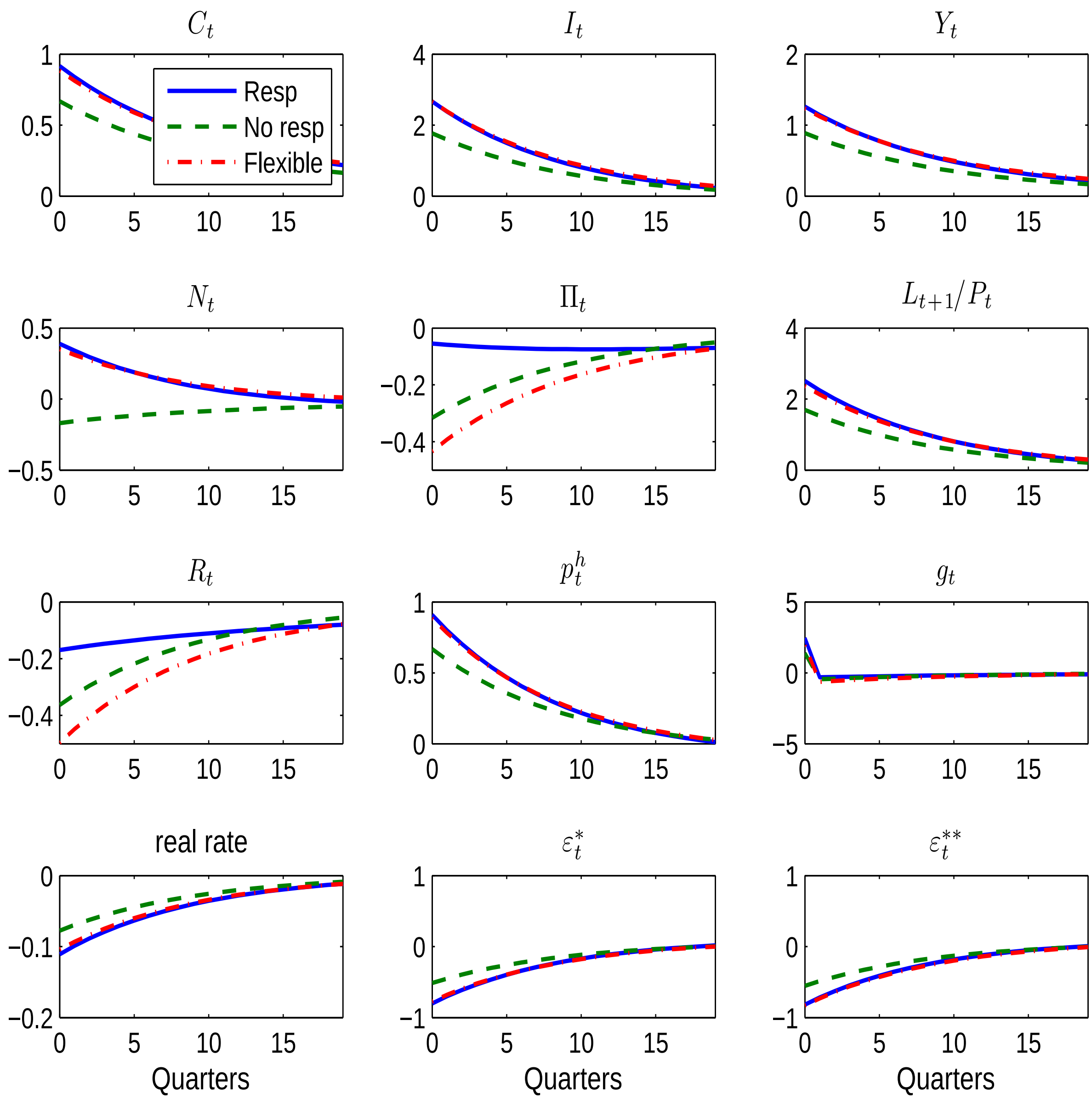

Figure 6: Impulse responses to a positive 1% TFP shock under the interest rate rule in (36). All vertical axes are measured in percentage. The solid lines represent the case with the optimal weight on asset bubbles given in Table 1. The dashed lines represent the case with $\phi_p = 0$. The dot dashed lines represent the responses in the flexible price equilibrium

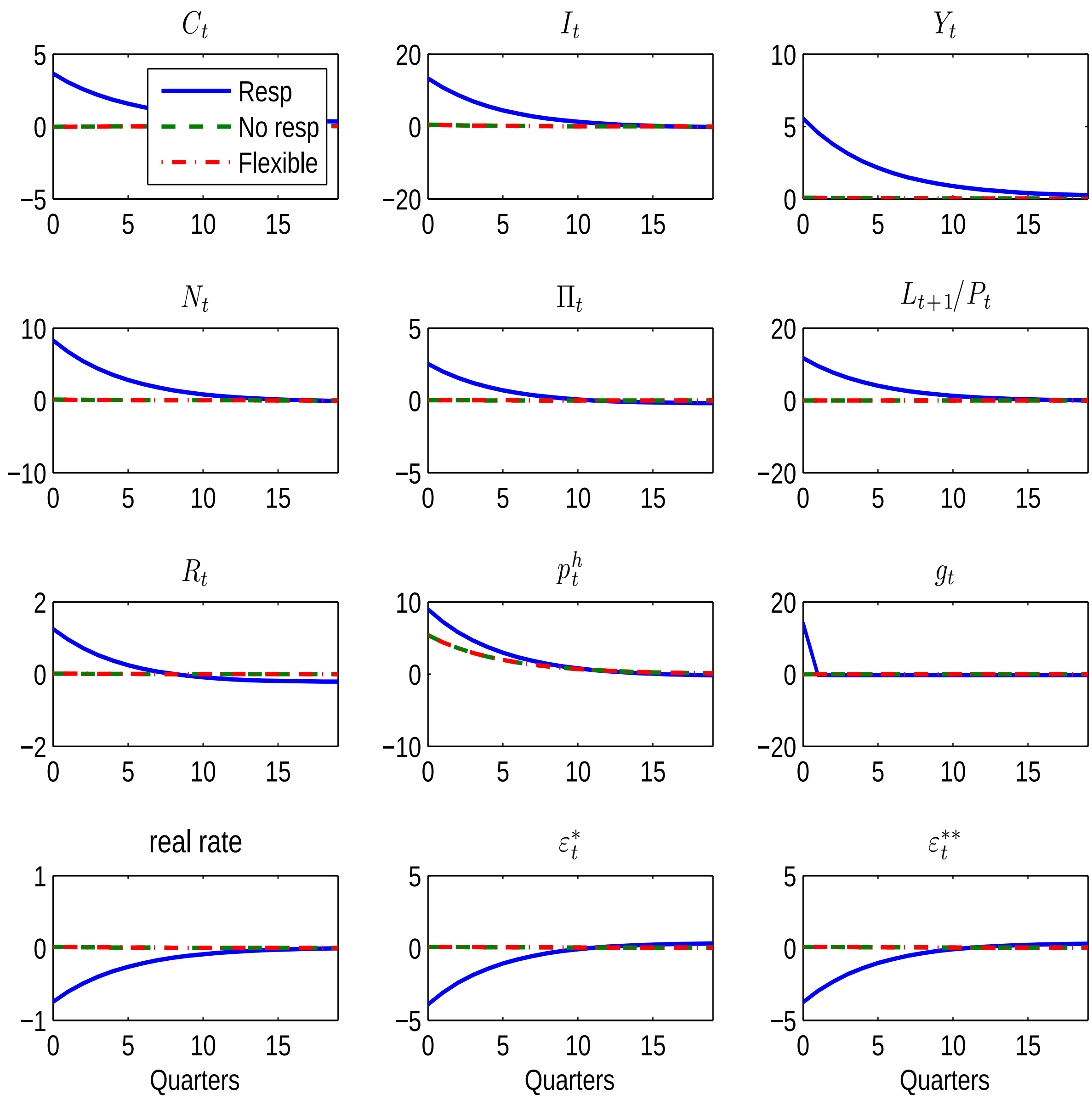

Figure 7: Impulse responses to a positive 1% sentiment shock under the interest rate rule in (36). All vertical axes are measured in percentage. The solid lines represent the case with the optimal weight on asset bubbles given in Table 1. The dashed lines represent the case with $\phi_p = 0$. The dot dashed lines represent the responses in the flexible price equilibrium

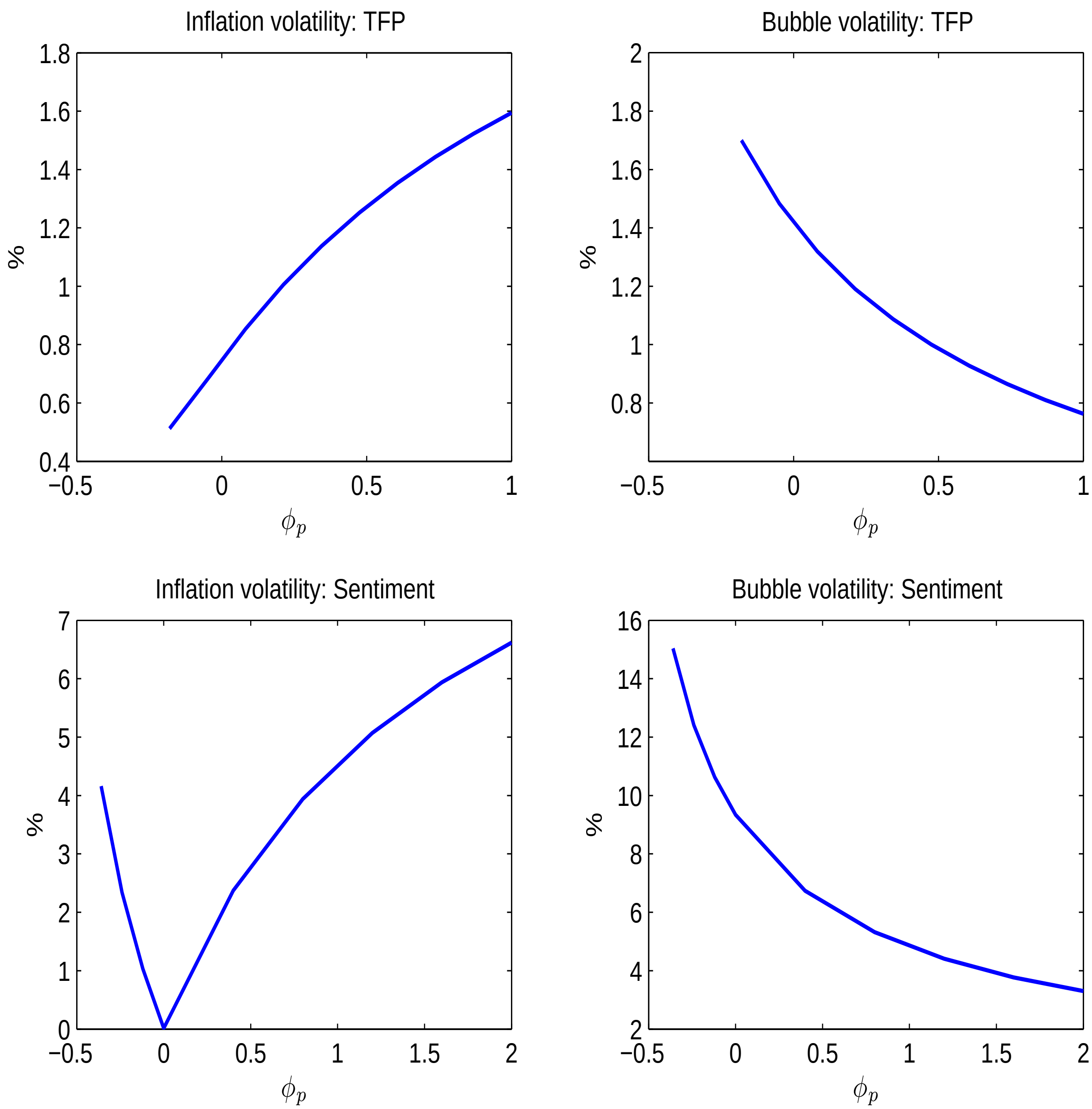

Figure 8: The inflation and output volatilities as functions of ϕ_p under the Taylor-type rule in (36) when only sentiment shocks or TFP shocks are turned on.

Appendix

A Proofs

Proof of Proposition 1: We conjecture that the value function takes the following form

$$
\begin{aligned}
& V_t\left(K_{jt}, S_{jt}, L_{jt}, \{H_{j,t|t-k}\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \qquad \text{(A.1)}\\
= \quad & \phi_t^k(\varepsilon_{jt}) K_{jt} + \phi_t^s(\varepsilon_{jt}) S_{jt} - \phi_t^l(\varepsilon_{jt}) L_{jt} + \sum_{k=0}^{\infty} \phi_{t|t-k}^h(\varepsilon_{jt}) H_{j,t|t-k},
\end{aligned}
$$

where $\phi_t^i(\varepsilon_{jt})$, $i \in \{k, s, l, h\}$, satisfy

$$
\begin{aligned}
q_t^k &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^k(\varepsilon)\, dF(\varepsilon), && \text{(A.2)}\\
\frac{1}{P_t} &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^s(\varepsilon)\, dF(\varepsilon), && \text{(A.3)}\\
\frac{R_{lt}}{R_t}\frac{1}{P_t} &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^l(\varepsilon)\, dF(\varepsilon), && \text{(A.4)}\\
p_{t|t-k}^h &= \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1|t-k}^h(\varepsilon)\, dF(\varepsilon). && \text{(A.5)}
\end{aligned}
$$

By convention, we set $H_{j,t|t} = \frac{\delta_h}{1-\delta_h}$.

Substituting (15), (18), and the above conjecture into the Bellman equation (20), we obtain

$$
\begin{aligned}
& V_t\left(K_{jt}, S_{jt}, L_{jt}, \{H_{j,t|t-k}\}_{k=0}^{\infty}, \varepsilon_{jt}\right)\\
= \quad & \max_{I_{jt}, S_{jt+1}, L_{jt+1}, H_{jt+1}} R_{kt}K_{jt} - I_{jt} + \frac{1}{P_t}\left(S_{jt}R_{t-1} - S_{jt+1} + L_{jt+1} - L_{jt}R_{lt-1}\right)\\
& + (1-\delta_h)\sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} - \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k}\\
& + q_t^k\left((1-\delta)K_{jt} + \varepsilon_{jt}I_{jt}\right) + \frac{1}{P_t}S_{jt+1} - \frac{R_{lt}}{R_t}\frac{1}{P_t}L_{jt+1} + \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k}\\
= \quad & \max_{I_{jt}, S_{jt+1}, L_{jt+1}} \left(R_{kt} + q_t^k(1-\delta)\right)K_{jt} + \frac{R_{t-1}}{P_t}S_{jt} - \frac{R_{lt-1}}{P_t}L_{jt}\\
& + (1-\delta_h)\sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} + \left(q_t^k\varepsilon_{jt} - 1\right)I_{jt} - \frac{1}{P_t}\left(\frac{R_{lt}}{R_t} - 1\right)L_{jt+1}. \qquad \text{(A.6)}
\end{aligned}
$$

Since $I_{jt} \geq 0$ and $D_{jt} \geq 0$, it follows from (A.6) that $I_{jt} = 0$ if $\varepsilon_{jt} < 1/q_t^k \equiv \varepsilon_t^*$, but the firm makes as much investment as possible so that $D_{jt} = 0$ if $\varepsilon_{jt} > \varepsilon_t^*$. Therefore, when $\varepsilon_{jt} \geq 1/q_t^k \equiv \varepsilon_t^*$, it follows from (18) that

$$
\begin{aligned}
I_{jt} \quad = \quad & R_{kt}K_{jt} + \frac{S_{jt}R_{t-1} - S_{jt+1} + L_{jt+1} - L_{jt}R_{lt-1}}{P_t} \qquad \text{(A.7)}\\
& + (1-\delta_h)\sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} - \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k}.
\end{aligned}
$$

Using this investment rule, we can simplify (A.6) for $\varepsilon_{jt} < \varepsilon_t^*$ as

$$\begin{aligned} & V_t\left(K_{jt}, S_{jt}, L_{jt}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \\ = & \max_{L_{jt+1}} \left(R_{kt} + q_t^k (1-\delta)\right) K_{jt} + \frac{R_{t-1}}{P_t} S_{jt} - \frac{R_{lt-1}}{P_t} L_{jt} \\ & + (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} - \frac{1}{P_t}\left(\frac{R_{lt}}{R_t} - 1\right) L_{jt+1}. \end{aligned} \tag{A.8}$$

Given $R_{lt} > R_t$, we have $L_{jt+1} = 0$. Therefore, when $\varepsilon_{jt} < \varepsilon_t^*$, equation (A.8) becomes

$$\begin{aligned} & V_t\left(K_{jt}, S_{jt}, L_{jt}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \qquad (A.9) \\ = & \left(R_{kt} + q_t^k (1-\delta)\right) K_{jt} + \frac{R_{t-1}}{P_t} S_{jt} - \frac{R_{lt-1}}{P_t} L_{jt} + (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k}. \end{aligned}$$

Matching coefficients in (A.1) and (A.9), we have

$$\phi_t^k(\varepsilon_{jt}) = R_{kt} + q_t^k (1-\delta), \ \phi_{t|t-k}^h(\varepsilon_{jt}) = (1-\delta_h) p_{t|t-k}^h, \tag{A.10}$$

$$\phi_t^s(\varepsilon_{jt}) = \frac{R_{t-1}}{P_t}, \ \phi_t^l(\varepsilon_{jt}) = \frac{R_{lt-1}}{P_t}, \tag{A.11}$$

for $\varepsilon_{jt} < \varepsilon_t^*$.

Next consider the case of $\varepsilon_{jt} \geq \varepsilon_t^* = 1/q_t^k$. Substituting (A.7) into (A.6) yields

$$\begin{aligned} & V_t\left(K_{jt}, S_{jt}, L_{jt}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \\ = & \max_{S_{jt+1}, L_{jt+1}, H_{jt+1}} \left(R_{kt} + q_t^k (1-\delta)\right) K_{jt} + \frac{R_{t-1}}{P_t} S_{jt} - \frac{R_{lt-1}}{P_t} L_{jt} + (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} \\ & + \left(q_t^k \varepsilon_{jt} - 1\right)\left(R_{kt} K_{jt} + \frac{S_{jt} R_{t-1} - S_{jt+1} + L_{jt+1} - L_{jt} R_{lt-1}}{P_t}\right) - \frac{1}{P_t}\left(\frac{R_{lt}}{R_t} - 1\right) L_{jt+1} \\ & + \left(q_t^k \varepsilon_{jt} - 1\right)\left[(1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} - \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k}\right] \\ = & \max_{S_{jt+1}, L_{jt+1}, H_{jt+1}} \left(R_{kt} + \left(q_t^k \varepsilon_{jt} - 1\right) R_{kt} + q_t^k (1-\delta)\right) K_{jt} + \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{t-1}}{P_t} S_{jt} \\ & - \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{lt-1}}{P_t} L_{jt} + \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right)(1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} \\ & - \left(q_t^k \varepsilon_{jt} - 1\right) S_{jt+1} - \frac{1}{P_t}\left(\frac{R_{lt}}{R_t} - q_t^k \varepsilon_{jt}\right) L_{jt+1} - \left(q_t^k \varepsilon_{jt} - 1\right) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t+1|t-k}. \end{aligned} \tag{A.12}$$

Thus we must have $S_{jt+1} = 0$ and

$$H_{j,t+1|t-k} = \omega (1-\delta_h) H_{j,t|t-k}, \quad k = 0, 1, 2, ... \tag{A.13}$$

Moreover, using (17) gives

$$\frac{L_{jt+1}}{P_t} = \begin{cases} 0, & \text{if } \varepsilon_{jt} < \varepsilon_t^{**} \\ \theta\omega (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} + \mu K_{jt}, & \text{otherwise} \end{cases}, \tag{A.14}$$

where we define

$$\varepsilon_t^{**} \equiv \frac{R_{lt}}{R_t q_t^k} = \frac{R_{lt}}{R_t}\varepsilon_t^*. \tag{A.15}$$

Since $R_{lt} > R_t$, $\varepsilon_t^{**} > \varepsilon_t^*$.

Using the preceding decision rules, we can derive that

$$\begin{aligned}
& V_t\left(K_{jt}, S_{jt}, L_{jt}, \left\{H_{j,t|t-k}\right\}_{k=0}^{\infty}, \varepsilon_{jt}\right) \\
= & \left(R_{kt} + \left(q_t^k \varepsilon_{jt} - 1\right) R_{kt} + q_t^k (1-\delta)\right) K_{jt} + \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{t-1}}{P_t} S_{jt} \\
& - \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{lt-1}}{P_t} L_{jt} + \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k} \\
& + \frac{q_t^k}{P_t}\left(\theta\omega P_t^h H_{jt} + P_t \mu K_{jt}\right) \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\} \\
& - \left(q_t^k \varepsilon_{jt} - 1\right) \omega (1-\delta_h) \sum_{k=0}^{\infty} p_{t|t-k}^h H_{j,t|t-k},
\end{aligned} \tag{A.16}$$

for $\varepsilon_{jt} \geq \varepsilon_t^*$.

Matching coefficients in (A.1) and (A.16) yields

$$\begin{aligned}
\phi_t^k(\varepsilon_{jt}) &= R_{kt} + \left(q_t^k \varepsilon_{jt} - 1\right) R_{kt} + q_t^k (1-\delta) + \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\} q_t^k \mu \\
&= \left(1 + \left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right) R_{kt} + q_t^k (1-\delta) + \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\} q_t^k \mu,
\end{aligned}$$

$$\phi_t^s(\varepsilon_{jt}) = \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{t-1}}{P_t} = \left(1 + \left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right) \frac{R_{t-1}}{P_t},$$

$$\phi_t^l(\varepsilon_{jt}) = \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) \frac{R_{lt-1}}{P_t} = \left(1 + \left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right) \frac{R_{lt-1}}{P_t},$$

$$\begin{aligned}
\frac{\phi_t^h(\varepsilon_{jt})}{1-\delta_h} &= \left(1 + \left(q_t^k \varepsilon_{jt} - 1\right)\right) p_{t|t-k}^h + \theta\omega q_t^k p_{t|t-k}^h \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\} - \left(q_t^k \varepsilon_{jt} - 1\right) p_{t|t-k}^h \omega \\
&= \left(1 + (1-\omega)\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1\right)\right) p_{t|t-k}^h + \theta\omega q_t^k p_{t|t-k}^h \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\},
\end{aligned}$$

for $\varepsilon_{jt} \geq \varepsilon_t^*$.

Combining the above two cases, we deduce that, for any $\varepsilon_{jt} \in (\varepsilon_{\min}, \varepsilon_{\max})$, we have

$$\phi_t^k(\varepsilon_{jt}) = \left(1 + \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right) R_{kt} + q_t^k (1-\delta) + \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\} q_t^k \mu,$$

$$\phi_t^s(\varepsilon_{jt}) = \left(1 + \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right) \frac{R_{t-1}}{P_t},$$

$$\phi_t^l(\varepsilon_{jt}) = \left(1 + \max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right) \frac{R_{lt-1}}{P_t},$$

$$\frac{\phi_t^h(\varepsilon_{jt})}{1-\delta_h} = \left(1 + (1-\omega)\max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*} - 1, 0\right)\right) p_{t|t-k}^h + \theta\omega q_t^k p_{t|t-k}^h \max\left\{\varepsilon_{jt} - \varepsilon_t^{**}, 0\right\}.$$

Substituting the preceding four equations into (A.2)-(A.5) yields (23), (24), (25) in the proposition. Q.E.D.

Proof of Lemma 1: Define the function

$$G(z) = \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon, z) dF(\varepsilon) - \frac{\Pi}{\beta} z = 0, \; z \in [\varepsilon_{\min}, \varepsilon_{\max}].$$

Note that $G'(z) = F(z) - \frac{\Pi}{\beta} < 0$, $G(\varepsilon_{\min}) = E[\varepsilon] - \frac{\Pi}{\beta}\varepsilon_{\min} > 0$, and $G(\varepsilon_{\max}) = \varepsilon_{\max} - \frac{\Pi}{\beta}\varepsilon_{\max} < 0$ by assumption 1. The intermediate value theorem ensures that there exists a unique solution for $\underline{\varepsilon} \in (\varepsilon_{\min}, \varepsilon_{\max})$ in the equation in the lemma. Q.E.D.

Proof of Proposition 2: In any steady state, we can use (12) and (50) to derive that

$$\varepsilon^{**} = \frac{R_l}{R}\varepsilon^* = \frac{1 - \frac{\lambda}{R(\varepsilon^*)}}{1-\lambda}\varepsilon^* = \frac{\varepsilon^* - \frac{\lambda\beta}{\Pi}\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon)}{1-\lambda}. \tag{A.17}$$

By assumption 1 and $\lambda \in (0,1)$, we can show that

$$\frac{d\varepsilon^{**}}{d\varepsilon^*} = \frac{1 - \frac{\lambda\beta}{\Pi}F(\varepsilon^*)}{1-\lambda} > 0.$$

Thus $\varepsilon^{**} = \varepsilon(\varepsilon^*)$ increases with ε^*. We need the following result.

Lemma 2 *For a sufficiently small μ, $R_k(\varepsilon^*)$ decreases with ε^* for $\varepsilon^* \in (\varepsilon_{\min}, \varepsilon_{\max})$. A sufficient condition is*

$$0 < \mu < \frac{\frac{1}{\beta} - 1 + \delta}{\max\limits_{\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_{\max}]} \frac{1 - F(\varepsilon(\varepsilon^*))}{F(\varepsilon^*)} \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon) \frac{1 - \frac{\lambda\beta}{\Pi}F(\varepsilon^*)}{1-\lambda} + \int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}} (\varepsilon - \varepsilon(\varepsilon^*)) dF(\varepsilon)}. \tag{A.18}$$

Proof. Using (52), we can compute that

$$\begin{aligned}
&\frac{dR_k(\varepsilon^*)}{d\varepsilon^*} \\
= \; & \frac{\mu[1 - F(\varepsilon^{**})]\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon) \frac{d\varepsilon^{**}}{d\varepsilon^*} - \left[\frac{1}{\beta} - 1 + \delta - \mu\int_{\varepsilon^{**}}^{\varepsilon_{\max}} (\varepsilon - \varepsilon^{**}) dF(\varepsilon)\right] F(\varepsilon^*)}{\left[\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon)\right]^2} \\
= \; & \frac{\mu(1 - F(\varepsilon^{**}))\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon) \frac{1 - \frac{\lambda\beta}{\Pi}F(\varepsilon^*)}{1-\lambda} - \left[\frac{1}{\beta} - 1 + \delta - \mu\int_{\varepsilon^{**}}^{\varepsilon_{\max}} (\varepsilon - \varepsilon^{**}) dF(\varepsilon)\right] F(\varepsilon^*)}{\left[\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon)\right]^2}.
\end{aligned}$$

If $\mu = 0$,

$$\frac{dR_k(\varepsilon^*)}{d\varepsilon^*} = -\frac{\frac{1}{\beta} - 1 + \delta}{\left[\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon^*, \varepsilon) dF(\varepsilon)\right]^2} < 0.$$

Since $R_k(\varepsilon^*)$ is a continuous function, the above inequality is also true for a sufficiently small μ. ■

Define the function

$$H(\varepsilon^*) = \left(R_k(\varepsilon^*) + \frac{\lambda\mu}{1-\lambda}\frac{1 - F(\varepsilon(\varepsilon^*))}{\Pi}\right)\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon), \tag{A.19}$$

for $\varepsilon^* \in [\underline{\varepsilon}, \varepsilon_{\max}]$. Since $\varepsilon(\varepsilon^*)$ increases with ε^*, it follows from the preceding lemma that $H(\varepsilon^*)$ decreases with ε^*. Note that $H(\varepsilon_{\max}) = 0$ and

$$H(\underline{\varepsilon}) = \left(R_k(\underline{\varepsilon}) + \frac{\lambda\mu}{1-\lambda}\frac{1-F(\varepsilon(\underline{\varepsilon}))}{\Pi}\right)\int_{\underline{\varepsilon}}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon(\underline{\varepsilon})}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) > \delta,$$

given the following assumption

$$\mu > \frac{\delta - R_k(\underline{\varepsilon})\int_{\underline{\varepsilon}}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\frac{\lambda}{1-\lambda}\frac{1-F(\varepsilon(\underline{\varepsilon}))}{\Pi}\int_{\underline{\varepsilon}}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \int_{\varepsilon(\underline{\varepsilon})}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}. \tag{A.20}$$

By the intermediate value theorem, there exists a unique solution for $\varepsilon_f^* \in (\underline{\varepsilon}, \varepsilon_{\max})$ in the equation $H(\varepsilon^*) = \delta$. Since $\varepsilon_f^* > \underline{\varepsilon}$, it follows Lemma 1 and (50) that $R_f = R\left(\varepsilon_f^*\right) > R(\underline{\varepsilon}) = 1$. Finally, the additional assumptions on μ not explicitly specified in the proposition are (A.18) and (A.20). Q.E.D.

Proof of Proposition 3: Equation (52) gives $R_{kf} = R_k(\varepsilon_f^*)$. Since $R_{kf} = \alpha Y_f/K_f$, we can determine Y_f/K_f. We now use the following procedure to derive other bubbleless steady-state variables. Equation (54) determines m_f/K_f. Equation (45) gives

$$\frac{I_f}{K_f} = \left(R_{kf} + \frac{m_f}{\Pi K_f}\right)\left[1 - F(\varepsilon_f^*)\right] + \mu\left[1 - F(\varepsilon_f^{**})\right]. \tag{A.21}$$

Thus we can derive $I_f/Y_f = (I_f/K_f)/(Y_f/K_f)$, and hence $C_f/Y_f = 1 - I_f/Y_f$.

Using (6), we can derive that

$$\psi = \frac{w_f}{C_f} = (1-\alpha)\frac{Y_f}{N_f}\frac{1}{C_f},$$

and thus

$$N_f = \frac{1-\alpha}{\psi}\frac{Y_f}{C_f}.$$

Now we use $R_{kf} = \alpha A K_f^{\alpha-1} N_f^{1-\alpha}$ to solve for K_f and then use the ratios derived above to determine Y_f, m_f, I_f, and C_f.

Consider the function H defined in (A.19). It decreases with Π. Since the solution for ε_f^* is given by the intersection of the downward sloping curve $H(\varepsilon^*)$ and the line δ, ε_f^* decreases with Π. Since $R_{kf} = \alpha K_f^{\alpha-1} N_f^{1-\alpha} = \alpha Y_f/K_f = R_k\left(\varepsilon_f^*\right)$, it follows from Lemma 2 that K_f/N_f decreases with Π, but Y_f/K_f increases with Π. The rest of results are straightforward to prove and hence are omitted. Q.E.D.

Proof of Proposition 4: We first prove sufficiency. Suppose that (56) holds. Define the function

$$G(\varepsilon^*) = \beta s\left(1 + (1-\omega)\int_{\varepsilon^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon^*} - 1\right)dF(\varepsilon)\right) + \beta s\theta\omega\int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon^*} - \frac{\varepsilon(\varepsilon^*)}{\varepsilon^*}\right)dF(\varepsilon) \tag{A.22}$$

for $\varepsilon^* \in [\varepsilon_f^*, \varepsilon_{\max})$. Since $R(\varepsilon^*)$ increases with ε^*, it follows from (51) that $\varepsilon(\varepsilon^*)$ and $\varepsilon(\varepsilon^*)/\varepsilon^*$ increase with ε^*. Thus $G(\varepsilon^*)$ decreases with ε^*. Moreover, $G(\varepsilon_{\max}) = \beta < 1$ and $G\left(\varepsilon_f^*\right) < 1$ by (56). By the intermediate value theorem, there exists a unique solution $\varepsilon_b^* \in (\varepsilon_f^*, \varepsilon_{\max}^*)$ to (55).

Since $R_f = R\left(\varepsilon_f^*\right) > 1$ and $R(\varepsilon^*)$ increases with ε^*, we have $R(\varepsilon_b^*) > R\left(\varepsilon_f^*\right) > 1$. Once the bubbly steady-state cutoff ε_b^* is determined, other bubbly steady-state values can be easily solved.

In particular, we use equation (51) to derive $\varepsilon_b^{**} = \varepsilon(\varepsilon_b^*)$. Equation (52) gives $R_{kb} = R_k(\varepsilon_b^*)$. Since $R_{kb} = \alpha Y_b/K_b$, we can determine Y_b/K_b. The steady-state version of equation (46) implies that

$$\frac{\delta}{\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)} = \left(R_k(\varepsilon_b^*) + \frac{m_b}{K_b}\frac{1}{\Pi} + (1-\omega)\frac{p^h}{K_b}\right) + \left(\theta\omega\frac{p^h}{K_b} + \mu\right)\frac{\int_{\varepsilon(\varepsilon_b^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}, \tag{A.23}$$

where we use equation (43) to derive

$$\frac{m_b}{K_b} = \frac{\lambda}{1-\lambda}\left(\theta\omega\frac{p^h}{K_b} + \mu\right)(1 - F(\varepsilon_b^{**})).$$

Using these two equations, we can solve for p^h/K_b and m_b/K_b,

$$\frac{p_h}{K_b} = \frac{\delta - \left[R_k(\varepsilon_b^*) + \frac{\mu}{\Pi}\frac{\lambda}{1-\lambda}(1 - F(\varepsilon(\varepsilon_b^*)))\right]\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - \mu\int_{\varepsilon(\varepsilon_b^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\frac{\theta\omega}{\Pi}\frac{\lambda}{1-\lambda}\left[1 - F\left(\varepsilon\left(\varepsilon_b^*\right)\right)\right]\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + (1-\omega)\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \theta\omega\int_{\varepsilon(\varepsilon_b^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}.$$

We can show that the numerator on the right-hand side of the equation above is an increasing function of ε_b^* when μ is sufficiently small. Since that expression is equal to zero when ε_b^* is replaced with ε_f^* (see (53)), it follows from $\varepsilon_b^* > \varepsilon_f^*$ that $p_h/K_b > 0$.

Using (45), we can solve for I_b/K_b and hence $I_b/Y_b = (I_b/K_b)/(Y_b/K_b)$. It follows from the resources constraint that $C_b/Y_b = 1 - I_b/Y_b$. Using (6) and $w_b = (1-\alpha)Y_b/N_b$, we can show that

$$N_b = \frac{1-\alpha}{\psi}\frac{Y_b}{C_b}.$$

Using $R_{kb} = \alpha K_b^{\alpha-1}N_b^{1-\alpha}$, we can solve for K_b. We can then determine other equilibrium variables using the ratios derived above.

We next prove necessity. Suppose that a bubbly steady state exists. In a bubbly steady state $p^h > 0$ and in a bubbleless steady state $p^h = 0$. Since (A.23) holds for both bubbly and bubbleless steady states, we have

$$\begin{aligned}
\delta &= \left[R_k(\varepsilon_f^*) + \frac{\lambda}{1-\lambda}\frac{\mu}{\Pi}\left(1 - F\left(\varepsilon(\varepsilon_f^*)\right)\right)\right]\int_{\varepsilon_f^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon\left(\varepsilon_f^*\right)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) \\
&> \left[R_k(\varepsilon_b^*) + \frac{\lambda}{1-\lambda}\frac{\mu}{\Pi}(1 - F(\varepsilon(\varepsilon_b^*)))\right]\int_{\varepsilon_b^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon\left(\varepsilon_b^*\right)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon).
\end{aligned}$$

When μ is sufficient small, Lemma 2 implies that $R_k(\varepsilon^*)$ decreases with ε^*. We have also shown that $\varepsilon(\varepsilon^*)$ increases with ε^*. Thus we deduce that

$$\left[R_k(\varepsilon^*) + \frac{\lambda}{1-\lambda}\frac{\mu}{\Pi}(1 - F(\varepsilon(\varepsilon^*)))\right]\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\int_{\varepsilon(\varepsilon^*)}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)$$

decreases with ε^*. The preceding inequality then implies that $\varepsilon_b^* > \varepsilon_f^*$. We have shown that $G(\varepsilon^*)$ defined in (A.22) decreases with ε^*. Since equation (42) implies that $G(\varepsilon_b^*) = 1$ in a bubbly steady state, it follows from $\varepsilon_b^* > \varepsilon_f^*$ that $G\left(\varepsilon_f^*\right) > 1$. Q.E.D.

Proof of Proposition 5: In the bubbly steady state, $G\left(\varepsilon_b^*\right)=1$ where G is defined in (A.22). Since

$$\frac{\varepsilon_b^{**}}{\varepsilon_b^*}=\frac{\varepsilon\left(\varepsilon_b^*\right)}{\varepsilon_b^*}=\frac{R_{lb}}{R_b}>1,$$

we have

$$\int_{\varepsilon\left(\varepsilon_b^*\right)}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-\frac{\varepsilon\left(\varepsilon_b^*\right)}{\varepsilon_b^*}\right)dF\left(\varepsilon\right)<\int_{\varepsilon_b^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-1\right)dF\left(\varepsilon\right).$$

Thus

$$\begin{aligned}&\beta s\left(1+(1-\omega)\int_{\varepsilon_b^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-1\right)dF\left(\varepsilon\right)\right)+\beta s\theta\omega\int_{\varepsilon_b^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-1\right)dF\left(\varepsilon\right)\\ >\ & G\left(\varepsilon_b^*\right)=1.\end{aligned}$$

This implies that

$$\int_{\varepsilon_b^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-1\right)dF\left(\varepsilon\right)>\frac{\frac{1}{\beta s}-1}{1-\omega+\theta\omega}. \tag{A.24}$$

By (50), we have

$$1<R\left(\varepsilon_b^*\right)=\frac{\Pi}{\beta\left(1+\int_{\varepsilon_b^*}^{\varepsilon_{\max}}\left(\frac{\varepsilon}{\varepsilon_b^*}-1\right)dF\left(\varepsilon\right)\right)}<\frac{1}{\beta}\frac{\Pi}{1+\frac{1/(\beta s)-1}{1-\omega+\theta\omega}}.$$

Thus

$$\Pi>\beta+\frac{1/s-\beta}{1-\omega+\theta\omega}>1.$$

Moreover, since $\varepsilon_b^*>\varepsilon_f^*$,

$$\Pi>\frac{1}{\beta}\frac{\Pi}{1+\frac{1/(\beta s)-1}{1-\omega+\theta\omega}}>R(\varepsilon_b^*)>R(\varepsilon_f^*).$$

We then obtain the desired result. Q.E.D.

Proof of Proposition 6: In the proof of Proposition 5 we have shown that ε_f^* decreases with Π. Since the left-hand side of (56) is a decreasing function of ε_f^*, condition (56) is more likely to hold for a higher Π. Q.E.D.

B Log-linearized System

The log-linearized equilibrium system can be described by 17 equations for 17 variables $\{\hat{C}_t$, $\hat{I}_t$, $\hat{K}_t$, $\hat{Y}_t$, $\hat{N}_t$, $\hat{\Pi}_t$, $\hat{p}_{wt}$, $\hat{w}_t$, $\hat{p}_{ht}$, $\hat{q}_t^k$, $\hat{R}_{kt}$, $\hat{\varepsilon}_t^*$, $\hat{\varepsilon}_t^{**}$, $\hat{R}_{lt}$, $\hat{m}_t$, $\hat{g}_t$, $\hat{R}_t\}$.

- Aggregate demand

$$\hat{C}_t=E_t\hat{C}_{t+1}-(\hat{R}_t-E_t\hat{\Pi}_{t+1})+\frac{R\beta}{\Pi}\int_{\varepsilon^*}^{\varepsilon_{\max}}\frac{\varepsilon}{\varepsilon^*}dF\left(\varepsilon\right)E_t\hat{\varepsilon}_{t+1}^*, \tag{B.1}$$

$$\begin{aligned}\hat{I}_t\ \ =\ \ &\frac{R_kK}{I}\left(1-F\left(\varepsilon^*\right)\right)(\hat{R}_{kt}+\hat{K}_t)+\frac{m}{\Pi}\left(1-F\left(\varepsilon^*\right)\right)(\hat{m}_t-\hat{\Pi}_t)\\&+\left(1-F\left(\varepsilon^{**}\right)\right)\frac{\mu K}{I}\hat{K}_t+\left[\left(1-F\left(\varepsilon^*\right)\right)(1-\omega)+\left(1-F\left(\varepsilon^{**}\right)\right)\theta\omega\right]\frac{p_h}{I}\hat{p}_{ht}\\&-\left[R_kK+\frac{m}{\Pi}+(1-\omega)p_h\right]\frac{f(\varepsilon^*)\varepsilon^*}{I}\hat{\varepsilon}_t^*-(\theta\omega p_h+\mu K)\frac{f(\varepsilon^{**})\varepsilon^{**}}{I}\hat{\varepsilon}_t^{**},\end{aligned} \tag{B.2}$$

$$\begin{aligned}\hat{K}_{t+1} &= (1-\delta)\hat{K}_t + R_k\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\left(\hat{K}_t+\hat{R}_{kt}\right) \\ &+\mu\int_{\varepsilon^{**}}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\hat{K}_t + \frac{m}{\Pi K}\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\left(\hat{m}_t-\hat{\Pi}_t\right) \\ &+\left[(1-\omega)\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)+\theta\omega\int_{\varepsilon^{**}}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\right]\frac{p^h}{K}\hat{p}_t^h \\ &-\left[R_k+\frac{m}{\Pi K}+(1-\omega)\frac{p^h}{K}\right](\varepsilon^*)^2 f(\varepsilon^*)\hat{\varepsilon}_t^* - (\theta\omega\frac{p^h}{K}+\mu)(\varepsilon^{**})^2 f(\varepsilon^{**})\hat{\varepsilon}_t^{**},\end{aligned} \tag{B.3}$$

$$\hat{Y}_t = \frac{C}{Y}\hat{C}_t + \frac{I}{Y}\hat{I}_t. \tag{B.4}$$

- Aggregate supply

$$\hat{\Pi}_t = \frac{1}{\xi}(1-\xi)(1-\beta\xi)\hat{p}_{wt} + \beta E_t\hat{\Pi}_{t+1}, \tag{B.5}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{N}_t, \tag{B.6}$$

$$\hat{w}_t = \hat{p}_{wt} + \hat{A}_t + \alpha\hat{K}_t - \alpha\hat{N}_t, \tag{B.7}$$

$$\hat{C}_t = \hat{w}_t. \tag{B.8}$$

- Asset prices

$$\begin{aligned}\hat{p}_t^h &= E_t\hat{p}_{t+1}^h + \hat{C}_t - E_t\hat{C}_{t+1} + \hat{s}_t - s\beta\theta\omega\left[1-F(\varepsilon^{**})\right]\frac{\varepsilon^{**}}{\varepsilon^*}E_t\hat{\varepsilon}_{t+1}^{**} \\ &-s\beta\left[(1-\omega)\int_{\varepsilon^*}^{\varepsilon_{\max}}\frac{\varepsilon}{\varepsilon^*}dF(\varepsilon)+\theta\omega\frac{1}{\varepsilon^*}\int_{\varepsilon^{**}}^{\varepsilon_{\max}}(\varepsilon-\varepsilon^{**})dF(\varepsilon)\right]E_t\hat{\varepsilon}_{t+1}^*,\end{aligned} \tag{B.9}$$

$$\begin{aligned}\hat{q}_t^k &= \hat{C}_t - E_t\hat{C}_{t+1} + \frac{R_k\Pi}{q^k R}E_t\hat{R}_{kt+1} + \beta(1-\delta)E_t\hat{q}_{t+1}^k - \beta\mu\varepsilon^{**}[1-F(\varepsilon^{**})]E_t\hat{\varepsilon}_{t+1}^{**} \\ &-\beta\left[\mu\int_{\varepsilon^{**}}^{\varepsilon_{\max}}(\varepsilon-\varepsilon^{**})dF(\varepsilon)+R_k\int_{\varepsilon^*}^{\varepsilon_{\max}}\varepsilon dF(\varepsilon)\right]E_t\hat{\varepsilon}_{t+1}^*,\end{aligned} \tag{B.10}$$

$$\hat{R}_{kt} = \hat{p}_{wt} + \hat{A}_t + (\alpha-1)\hat{K}_t + (1-\alpha)\hat{N}_t. \tag{B.11}$$

- Lending rate and two cutoffs

$$\hat{R}_{lt} - \hat{R}_t = \hat{\varepsilon}_t^{**} - \hat{\varepsilon}_t^*, \tag{B.12}$$

$$\hat{\varepsilon}_t^* = -\hat{q}_t^k, \tag{B.13}$$

$$\hat{\varepsilon}_t^{**} = \hat{\varepsilon}_t^* + \frac{\lambda/R}{1-\lambda/R}\hat{R}_t. \tag{B.14}$$

- Money demand and supply

$$\hat{m}_{t+1} = \frac{\theta\omega p_h}{\theta\omega p_h+\mu K}\hat{p}_{ht} + \frac{\mu K}{\theta\omega p_h+\mu K}\hat{k}_t - \frac{\varepsilon^{**}f(\varepsilon^{**})}{1-F(\varepsilon^{**})}\hat{\varepsilon}_t^{**}, \tag{B.15}$$

$$\hat{m}_{t+1} = \hat{m}_t - \hat{\Pi}_t + \hat{g}_t. \tag{B.16}$$

- Interest rate rule

$$\hat{R}_t = \phi_\pi \hat{\Pi}_t + \phi_y \hat{Y}_t + \phi_p \hat{p}_{ht} + \hat{v}_t, \tag{B.17}$$

or

$$\hat{R}_t = \phi_\pi E_t \hat{\Pi}_{t+1} + \phi_p \hat{p}_{ht} + \hat{v}_t, \tag{B.18}$$

- Exogenous shocks

$$\begin{aligned}
\hat{A}_t &= \rho_a \hat{A}_{t-1} + \varepsilon_{at}, \\
\hat{v}_t &= \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \\
\hat{s}_t &= \rho_s \hat{s}_{t-1} + \varepsilon_{st}.
\end{aligned}$$