



Economic growth under money illusion[☆]

Jianjun Miao^{a,b,*}, Danyang Xie^{c,1}

^a Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA

^b CEMA, Central University of Finance and Economics, and AFR, Zhejiang University, China

^c Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

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ABSTRACT

Empirical and experimental evidence documents that money illusion is persistent and widespread. This paper incorporates money illusion into a stochastic continuous-time monetary model of endogenous growth. We model an agent's money illusion behavior by assuming that he maximizes nonstandard utility derived from both nominal and real quantities. Money illusion affects an agent's perception of the growth and riskiness of real wealth and distorts his consumption/savings decisions. It influences long-run growth via this channel. We show that the welfare cost of money illusion is negligible, whereas its impact on long-run growth is noticeable even if the degree of money illusion is low.

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1. Introduction

The term money illusion refers to the phenomenon where people confuse nominal with real magnitudes. It is widely believed that this term was coined by Irving Fisher who devoted an entire book to the subject (Fisher, 1928). The presence of money illusion has frequently been invoked to account for the short-run non-neutrality of money by Keynesian economists and by some quantity theorists such as Fisher.² However, money illusion is often regarded as irrational and costly to decision makers, and hence economists have resisted to use money illusion in formal analysis, with the possible exception of Akerlof and Yellen (1985a,b). Interests on money illusion have been revived by a growing body of empirical and experimental evidence since the mid-1990s, which prompted theoretical studies by Akerlof et al. (1996, 2000) on the Phillips curve, Piazzesi and Schneider (2008) on housing market, and by Basak and Yan (2010) on investor behavior.

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* Corresponding author at: Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, USA. Tel.: +1 617 353 6675.

E-mail addresses: miaoj@bu.edu (J. Miao), dxie@ust.hk (D. Xie).

¹ Tel.: +852 2358 7615.

² According to Keynesian economics (Keynes, 1936; Leontief, 1936), workers suffer from money illusion. The labor supply depends on the nominal wage rate whereas the demand depends on the real wage. A rise in the price level will raise the equilibrium level of employment.

In this paper, we extend the preference-based formulation of money illusion by Basak and Yan (2010) and incorporate this extension into an endogenous growth model. We study how the presence of money illusion affects the relationship between inflation and long-run economic growth. Endogenous growth is introduced by adopting the one-sector AK framework of Jones and Manuelli (1990) and Rebelo (1991). Money is introduced via the money-in-the-utility (MIU) function framework of Sidrauski (1967a,b).³ We show that money illusion affects an agent's perception of the growth and riskiness of real wealth and hence distorts his consumption/savings decisions. Intuitively, real wealth includes real money balances. A money-illusioned agent values both real and nominal terms and hence both expected inflation and inflation uncertainty affect his decision rules.

Our model is tractable and allows us to derive closed-form solutions. Yet, it is rich enough for calibration and quantitative assessments. We establish that the welfare cost of money illusion is second order, whereas the impact on long-run economic growth is first order in terms of the parameter that captures the degree of money illusion. When calibrating our model to the U.S. annual data from 1960 to 2006, we find that a small degree of money illusion, in the sense that the representative agent puts 5% weight on nominal quantities in utility evaluation, results in a negligible welfare loss of 0.06% of real income, whereas it lowers the rate of economic growth by a noticeable 0.11 percentage point. This result complements the one in Basak and Yan (2010),⁴ and is reminiscent of the Keynesian proposition that small deviations from rationality have small welfare losses, but can have a significant impact on economic outcomes (Akerlof, 2002; Akerlof and Yellen, 1985b; Mankiw, 1985).

We show that in theory, the monetary authority can choose a growth rate of the money supply to eliminate the cost of money illusion by correcting the distortions on consumption/savings decisions. This monetary policy implements a specific nonzero expected inflation rate or a constant nominal interest rate such that the distortions arising from the agent's misperception of the growth and riskiness of real wealth offset each other.

One may argue that money illusion should not persist in the long run as agents can learn. However, as argued by Shafir et al. (1997), money illusion arises in large part because it is considerably easier and more natural for individuals to think in nominal rather than in real terms. This tendency is likely to persist despite economists' attempts to educate the public. Akerlof et al. (2000) use a variety of psychological evidence to argue that high inflation, not the passage of time, may dissipate money illusion. High inflation is salient so that people may take into account the difference between nominal and real values. Our quantitative results provide support for this psychological argument. We find that the welfare cost of money illusion is small for low inflation. It rises nonlinearly with the expected inflation rate and becomes large for high inflation.

Our model can shed light on the growth and inflation relationship.⁵ While some researchers find evidence for a negative relationship (e.g., Barro, 1996; Chari et al., 1995), other empirical studies show that this relationship is not robust (e.g., Bullard and Keating, 1995; Sarel, 1996; Bruno and Easterly, 1998; Dotsey and Sarte, 2000; Fischer et al., 2002; Khan and Senhadji, 2001). We prove that this relationship depends on the representative agent's risk attitudes or the elasticity of intertemporal substitution.⁶ In particular, growth and inflation are negatively (positively) related if the degree of relative risk aversion is greater (less) than unity. They are independent if the degree of relative risk aversion is equal to unity.

The rest of the paper is organized as follows. In Section 2, we survey the related literature and provide justification to our formulation. In Section 3, we study a MIU model of endogenous growth in the presence of money illusion. In Section 4, we analyze the quantitative effects of money illusion and discuss the monetary policy that eliminates the welfare cost of money illusion. We conclude in Section 5. Proofs and technical details are relegated to appendices.

2. Related literature

Our paper is related to two strands of literature. First, it is related to the literature on money illusion, which dates back to the early 20th century.⁷ Fisher (1928, p. 4) defines money illusion as "the failure to perceive that the dollar, or any other unit of money, expands or shrinks in value." Leontief (1936) defines that there is no money illusion if demand and supply functions are homogeneous of degree zero in all nominal prices. This is what Leontief (1936) called the "homogeneity postulate." Beginning with Haberler (1941, p. 460) other writers have used the term money illusion as synonymous with a violation of this homogeneity postulate. Patinkin (1965) objects this use on the grounds that it fails to take into account the real balance effect. Patinkin (1965, p. 22) defines that "an individual will be said to be suffering from such an illusion if his excess-demand functions for commodities do not depend [...] solely on relative prices and real wealth."

In a static model, the absence of money illusion in Patinkin's sense is equivalent to the assumption of rational behavior, in the following sense. Let an agent's demand functions $x_i^*(p_1, \dots, p_n, W)$ for goods $i = 1, \dots, n$, together with his money

³ For robustness, we also analyze a model with a cash-in-advance (CIA) constraint on consumption purchases. We find identical results (see Appendix B).

⁴ Basak and Yan (2010) demonstrate that the welfare loss of money illusion to an investor is small for typical environments, while its impact on equilibrium can be considerable.

⁵ We note that both inflation and economic growth are endogenous. Their relationship essentially refers to the relationship between money growth and output growth since inflation is determined by the money growth.

⁶ Under our power utility specification, the degree of relative risk aversion is equal to the inverse of the elasticity of intertemporal substitution.

⁷ See Howitt (1987) for an early survey of the literature on money illusion.

demand function $M^*(p_1, \dots, p_n, W)$, be defined as the maximizers of the utility function $U(x_1, \dots, x_n; M, p_1, \dots, p_n)$ subject to the budget constraint: $p_1x_1 + \dots + p_nx_n + M = W$, where W is initial nominal wealth. The utility function includes money M and the nominal prices p_i because money is assumed to yield unspecified services whose value depends on prices.⁸ The function U is said to be illusion-free if it is homogeneous of degree zero in (M, p_1, \dots, p_n) . It is easily verified that x_i^* 's are illusion-free in Patinkin's sense if and only if they can be derived from an illusion-free U (see [Dusansky and Kalman, 1974](#); [Howitt and Patinkin, 1980](#)). Using the revealed preference theory in a static setting, [Loffler \(2001\)](#) provides an axiomatic foundation for a utility function to deliver a demand function that violates the homogeneity of degree zero property, and hence exhibits money illusion. He proves that this utility function depends on nominal wealth.

Our modeling of money illusion follows the above early literature and extends it to a dynamic setting. Our modeling is closely related to that of [Basak and Yan \(2010\)](#) and includes theirs as a special case. There are alternative modeling approaches in the recent literature. [Akerlof et al. \(2000\)](#) assume that the productivity of a firm depends on the nominal wage paid by the firm relative to a reference nominal wage. An illusioned firm's reference nominal wage underestimates inflation. In an efficiency wage model, [Shafir et al. \(1997\)](#) model money illusion by assuming that an illusioned worker's effort depends not only on real wage, but also on the ratio of the current nominal wage to the previous nominal wage. In a two-period model, [Cohen et al. \(2005\)](#) assume that (i) a money-illusioned agent maximizes expected utility over nominal wealth instead of real wealth, and (ii) this agent believes that the nominal growth of corporate assets do not depend on inflation. [Piazzesi and Schneider \(2008\)](#) assume that a money-illusioned agent has a standard utility function over real consumption, but this agent mistakenly believes that an asset's nominal payoffs do not depend on inflation.

Second, our paper is related to the literature on the relationship between growth and inflation. Most studies in this literature are empirical and document mixed evidence on this relationship. Standard theoretical models cannot explain this evidence. As mentioned in the introduction, [Barro \(1996\)](#) and [Chari et al. \(1995\)](#) report a negative relationship. [Chari et al. \(1995\)](#) use four types of endogenous growth models (the one-sector AK model, the two-sector AK model, the human capital model in [Lucas \(1988\)](#), and the externality model in [Romer, 1986](#)) to explain this negative relationship. They argue that the standard models' problem may be due to their standard narrow assumption that all money is held by the public for making transactions. When the models are adjusted to also assume that banks are required to hold money, the models do a much better job. [Jones and Manuelli \(1995\)](#) study two endogenous growth model (the AK model and [Lucas, 1988](#) model) in which a demand for money is introduced via cash-in-advance (CIA) constraints. They show that the relationship between inflation and growth depends on whether inflation affects investment decisions.

Some empirical studies show that the negative relationship between growth and inflation is not robust (e.g., [Bullard and Keating, 1995](#); [Sarel, 1996](#); [Bruno and Easterly, 1998](#); [Dotsey and Sarte, 2000](#); [Fischer et al., 2002](#); [Khan and Senhadji, 2001](#)). [Bullard and Keating \(1995\)](#) show that a permanent shock to inflation is not associated with a permanent movement in the level of real output for most countries in their sample. [Sarel \(1996\)](#) finds evidence of a significant structural break in the function that relates economic growth to inflation. [Dotsey and Sarte \(2000\)](#) provide an endogenous growth AK model with CIA constraints and monetary policy uncertainty. They show that inflation and growth are negatively related in the long run, but are positively related in the short run.

Our paper contributes to the above two strands of literature. To the best of our knowledge, our paper is the first study that explores the impact of money illusion on the relationship between growth and inflation.

3. The model

We first introduce the model environment and then analyze the agent's optimization problem. Finally, we characterize the monetary competitive equilibrium.

3.1. Environment

We consider a monetary economy consisting of a representative agent and a monetary authority. Time is continuous and the horizon is infinite. To generate long-run growth endogenously, we adopt a simple one-sector AK model ([Jones and Manuelli, 1990](#); [Rebelo, 1991](#)). This model is analytically tractable and is widely adopted in the literature on the growth and inflation relationship (e.g., [Dotsey and Sarte, 2000](#); [Chari et al., 1995](#)). Unlike the standard deterministic AK model, we introduce uncertainty into the production technology as in [Eaton \(1981\)](#) to see how an uncertain environment may amplify the impact of money illusion. At each point in time, the agent decides how much to consume and how much of his wealth to invest in productive capital, nominal bonds, and money. The nominal bonds yield a constant nominal interest rate R and are in zero net supply. The nominal interest rate R is endogenous and will be verified to be constant in equilibrium.

Preferences. To have a role of money, we adopt [Sidrauski \(1967a,b\)](#) formulation of the money-in-the-utility (MIU) function. Specifically, the agent derives utility from consumption and money. Unlike the standard formulation, we assume that the agent suffers from money illusion. To capture money illusion, we adopt the following time-additive expected

⁸ There are other reasons why preferences depend on prices. For example, [Pollak \(1977\)](#) and [Veblen \(1899\)](#) argue that people may judge quality by price or a higher price enhances the snob appeal of a good.

utility function:

$$E \left\{ \int_0^\infty e^{-\rho t} U(c_t, M_t, P_t) dt \right\}, \tag{1}$$

where (c_t) , (M_t) , and (P_t) are consumption, money, and price processes, respectively. As in Section 2, we define that U is illusion-free if it satisfies homogeneity of degree zero in M and P . Consequently, to have money illusion, this homogeneity of degree zero property must be violated.

Because we analyze long-run growth, we need the utility function to satisfy certain homogeneity property in consumption. This consideration leads us to take a specific CES functional form

$$U(c, M, P) = \frac{1}{1-\gamma} (\alpha [c^{1-\theta} (Pc)^\theta]^{1-\varphi} + (1-\alpha) [(M/P)^{1-\theta} M^\theta]^{1-\varphi})^{(1-\gamma)/(1-\varphi)}, \tag{2}$$

where $\gamma > 0$ is the risk aversion parameter, $1/\varphi > 0$ the elasticity of substitution between consumption and money, and $\alpha \in (0, 1)$ represents the relative weight of consumption and money. The most important parameter is $\theta \in [0, 1]$ which represents the degree of money illusion. The case with $\theta = 0$ is illusion-free and the case with $\theta = 1$ means complete money illusion. The interpretation is that the agent values both real and nominal terms, with weight θ being put on nominal terms. This interpretation is consistent with the psychological interpretation of money illusion in Shafir et al. (1997). They argue that “people are generally aware that there is a difference between real and nominal values, but because at a single point in time, or over a short period, money is a salient and natural unit, people often think of transactions in predominantly nominal terms. Consequently, the evaluation of transactions often represents a mixture of nominal and real assessments, which gives rise to money illusion.” Note that one should not interpret our utility model as agents like high price. Instead, the utility function states that an illusioned agent derives utility from his misperceived mixture of real and nominal consumption, $c^{1-\theta} (Pc)^\theta$, as well as real and nominal money balances, $(M/P)^{1-\theta} M^\theta$.

The CES formulation above nests several well-known cases. The case with $\gamma = 1$ corresponds to the logarithmic utility function, and the case with $\varphi = 1$ corresponds to the following:

$$U(c, M, P) = \frac{1}{1-\gamma} (c^\alpha (M/P)^{1-\alpha})^{1-\gamma} P^{\theta(1-\gamma)}. \tag{3}$$

For $\theta = 0$, this function is used by Fischer (1979) when he studies the transitional path of capital accumulation and by Lucas (2000) in a study of the welfare cost of inflation. When we study the CIA model in Appendix B, money does not enter the utility function. We thus adopt the following specification:

$$U(c, M, P) = \frac{[c^{1-\theta} (Pc)^\theta]^{1-\gamma}}{1-\gamma}. \tag{4}$$

This specification is first adopted by Basak and Yan (2010) in a pure-exchange economy model. Our specification in (2) is novel and generalizes that in Basak and Yan (2010).

Technology. The agent operates a technology with constant returns to scale. In particular, with k_t units of real capital at time t , the technology yields stochastic output

$$dy_t = Ak_t dt + k_t \sigma_k dz_t, \tag{5}$$

during period dt , where $A > 0, \sigma_k > 0$ and (z_t) is a standard Brownian motion (i.e., dz_t is temporally independent normally distributed random variable with zero mean and variance dt). Here dz_t represents a supply shock. For simplicity, we assume that capital does not depreciate. This technological specification is crucial for obtaining a tractable solution in dynamic settings under uncertainty. It is adopted by Eaton (1981), Turnovsky (1993) and Rebelo and Xie (1999), among others.

Monetary authority. The monetary authority sets a simple money growth rule such that the money supply, \bar{M}_t , satisfies

$$d\bar{M}_t = \mu \bar{M}_t dt, \tag{6}$$

where $\mu > 0$ is the constant money growth rate. Any money supply policy must be implemented by a policy of fiscal transfers, open market operations, or both. As a starting point, we assume that the agent is given real lump-sum transfers v_t such that

$$v_t = \mu \frac{\bar{M}_t}{P_t}. \tag{7}$$

Alternative assumptions about the uses of growth of the money supply may lead to different conclusions about the relationship between inflation and growth. For example, using the growth of the money supply to subsidize capital formation or reduce other distortional taxes may stimulate growth.

Equilibrium. A monetary competitive equilibrium consists of stochastic processes (c_t) , (k_t) , (B_t) , (M_t) , (P_t) , and R such that: (i) the agent maximizes utility given by (1) and (2) subject to the budget constraint,

$$dk_t + \frac{dB_t}{P_t} + \frac{dM_t}{P_t} = (Ak_t - c_t) dt + k_t \sigma_k dz_t + \frac{RB_t}{P_t} dt + v_t dt, \tag{8}$$

where k_0 is given, P_t denotes the date t price level, B_t and M_t denote the date t bond and money holdings, and v_t denotes the date t real value of the government transfers. If $v_t < 0$, it is interpreted as lump-sum taxes; (ii) all markets clear in that $B_t = 0$, $\bar{M}_t = M_t$ for any t , and

$$dk_t = (Ak_t - c_t) dt + k_t \sigma_k dz_t, \quad (9)$$

and (iii) monetary policy satisfies (6) and (7).

3.2. Consumption/savings decision

To solve for a monetary competitive equilibrium, we first solve the agent's decision problem by dynamic programming taking prices and transfers as given. To this end, we must specify the dynamics of the price level P_t and the transfers v_t . We conjecture that the equilibrium law of motion for P_t follows the following geometric Brownian motion process:

$$\frac{dP_t}{P_t} = \pi dt - \sigma_P dz_t, \quad P_0 = 1, \quad \sigma_P > 0, \quad (10)$$

where dz_t comes from the only uncertainty in the model, the supply shock. In the conjectured Eq. (10), the expected inflation rate $(1/dt)E[dP_t/P_t]$ is equal to a constant π and the unexpected inflation rate is represented by $-\sigma_P dz_t$. We will derive π and σ_P and verify that the conjecture in (10) is correct in equilibrium later. The negative sign on the right-hand side of (10) captures the intuition that a positive supply shock reduces prices.

It follows from Ito's Lemma that the price of money, $1/P_t$, also follows a geometric Brownian motion:

$$d\left(\frac{1}{P_t}\right) = -\frac{1}{P_t^2} dP_t + \frac{1}{P_t^3} (dP_t)^2 = \left(\frac{\sigma_P^2}{P_t} - \frac{\pi}{P_t}\right) dt + \frac{\sigma_P}{P_t} dz_t. \quad (11)$$

Define the agent's real wealth level net of his entitlement for the future stream of lump-sum transfers as $w_t = k_t + B_t/P_t + M_t/P_t$. Then, by definition,

$$dw_t = dk_t + d\left(\frac{B_t}{P_t}\right) + d\left(\frac{M_t}{P_t}\right). \quad (12)$$

It follows from Ito's Lemma that $d(M_t/P_t)$ and $d(B_t/P_t)$ can be written as:

$$d\left(\frac{M_t}{P_t}\right) = \frac{dM_t}{P_t} + M_t d\left(\frac{1}{P_t}\right), \quad (13)$$

and

$$d\left(\frac{B_t}{P_t}\right) = \frac{dB_t}{P_t} + B_t d\left(\frac{1}{P_t}\right). \quad (14)$$

We can now use (11), (13), and (14) to rewrite the agent's budget constraint (8) as

$$dw_t = dk_t + \frac{dB_t}{P_t} + \frac{dM_t}{P_t} + (B_t + M_t) d\left(\frac{1}{P_t}\right) = (Ak_t - c_t) dt + k_t \sigma_k dz_t + \frac{RB_t}{P_t} dt + v_t dt + (w_t - k_t)[(\sigma_P^2 - \pi) dt + \sigma_P dz_t]. \quad (15)$$

Let $k_t = \phi_t w_t$ and $B_t/P_t = \psi_t w_t$, where ϕ_t and ψ_t are to be determined. Thus, $M_t/P_t = (1 - \phi_t - \psi_t)w_t$. We can then rewrite the budget constraint (15) as

$$dw_t = (w_t[A\phi_t + R\psi_t + (1 - \phi_t)(\sigma_P^2 - \pi)] + v_t - c_t) dt + [\phi_t \sigma_k + (1 - \phi_t)\sigma_P] w_t dz_t. \quad (16)$$

We conjecture that in equilibrium ϕ_t takes some constant value ϕ^* given in (24) below. Since in equilibrium $B_t = 0$ and $\psi_t = 0$, we thus have

$$\frac{M_t}{P_t} = w_t(1 - \phi^*). \quad (17)$$

This equation is analogous to the money demand in the quantity theory of money since wealth is proportional to aggregate income in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion coefficients in (17), we obtain

$$\phi^* \sigma_k + (1 - \phi^*) \sigma_P = \sigma_P.$$

Thus, $\sigma_P = \sigma_k$. Namely, beyond the expected inflation, the price changes negatively and proportionally with the supply shock one-for-one. This results from the fact that (i) the production function is linear in k_t and (ii) the real money demand is proportional to wealth and hence to capital.

We next turn to the dynamics of transfers v_t . The transfers depend on the equilibrium holdings of money, which in turn depend on the aggregate wealth level in the economy, denoted by \bar{w} . Therefore, it is important to derive the dynamics of aggregate wealth. We conjecture, as in Rebelo and Xie (1999), that aggregate real wealth follows the diffusion process:

$$d\bar{w}_t = f(\bar{w}_t) dt + h(\bar{w}_t) dz_t, \quad (18)$$

where f and h are functions to be determined such that it coincides with the representative agent's wealth level, $\bar{w}_t = w_t$. In this case, the lump-sum transfer satisfies

$$v_t = \mu \bar{M}_t / P_t = \mu \bar{w}_t (1 - \phi^*). \tag{19}$$

We are now ready to solve the agent's dynamic programming problem, given the three laws of motion (10), (15), and (18) for P_t , w_t , and \bar{w}_t , respectively. It is natural that the agent's wealth level and the price level are state variables. When solving his decision problem, the agent takes the lump-sum transfers as given. Thus, he must take into account the law of motion for the aggregate wealth level. This implies that the aggregate wealth level should be an additional state variable. Exploring the homogeneity property of the utility function, we conjecture that the value function takes the following form:

$$J(w, \bar{w}, P) = b \frac{(w + \beta \bar{w})^{1-\gamma}}{1-\gamma} P^{\theta(1-\gamma)}, \tag{20}$$

where b and β are constants to be determined.

By the standard dynamic programming theory, the value function satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\begin{aligned} 0 = \max_{\phi, \psi} & U(c, M, P) - \rho J(w, \bar{w}, P) + J_1(w, \bar{w}, P) \{w[A\phi + R\psi + (1-\phi)(\sigma_k^2 - \pi)] + v - c\} \\ & + J_2(w, \bar{w}, P) f(\bar{w}) + J_3(w, \bar{w}, P) P\pi + \frac{1}{2} J_{11}(w, \bar{w}, P) \sigma_k^2 w^2 + \frac{1}{2} J_{22}(w, \bar{w}, P) [h(\bar{w})]^2 + \frac{1}{2} J_{33}(w, \bar{w}, P) [P\sigma_k]^2 \\ & + J_{12}(w, \bar{w}, P) h(\bar{w}) w \sigma_k - J_{13}(w, \bar{w}, P) w P \sigma_k^2 - J_{23}(w, \bar{w}, P) h(\bar{w}) P \sigma_k, \end{aligned} \tag{21}$$

where U is given by (2). Solving this HJB equation, we obtain

Proposition 1. Suppose

$$\eta \equiv \frac{\rho - (1-\gamma)A}{\gamma} + \frac{1}{2}(1-\gamma)\sigma_k^2 + \frac{\theta(\gamma-1)}{\gamma} \left[\pi + \frac{\sigma_k^2}{2} ((\gamma-1)(2-\theta)-1) \right] > 0. \tag{22}$$

Let the price level, the aggregate wealth level, and the transfers satisfy Eqs. (10), (18), and (19), respectively, with $\sigma_p = \sigma_k$,

$$f(\bar{w}) = (A - \eta)\bar{w}, \quad h(\bar{w}) = \sigma_k \bar{w}, \tag{23}$$

and

$$\phi^* = 1 - \frac{\eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\phi} - \mu}. \tag{24}$$

Then the value function is given by (20) where b and β are given by

$$b = \eta^{-\gamma} \left(\frac{R}{1-\alpha}\right)^{\gamma-1} \left(\alpha \left(\frac{\alpha R}{1-\alpha}\right)^{(1-\phi)/\phi} + (1-\alpha) \right)^{(1-\gamma)\phi/(1-\phi)}, \tag{25}$$

$$\beta = \mu \left[R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\phi} - \mu \right]^{-1}. \tag{26}$$

In addition, the nominal interest rate satisfies

$$R = A + \pi - \sigma_k^2 = A + \pi - \sigma_p^2, \tag{27}$$

the optimal consumption rule is given by

$$c = \frac{\left(\frac{\alpha R}{1-\alpha}\right)^{1/\phi} \eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\phi}} (w + \beta \bar{w}), \tag{28}$$

and the optimal money demand rule is given by

$$\frac{M}{P} = \frac{\eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\phi}} (w + \beta \bar{w}). \tag{29}$$

Eq. (27) is a modified Fisher equation, which corrects for uncertainty. It shows that the nominal interest rate is equal to the expected real interest rate, A , plus the expected inflation rate, minus the variance of the inflation rate, σ_p^2 . This equation follows from a simple no arbitrage argument. In the presence of inflation uncertainty, the nominal interest rate is not generally equal to the expected real interest rate plus the expected inflation rate. This has to do with the fact that, in general, $E(1/P) \neq 1/E(P)$ when inflation is stochastic (see, for example, Fischer, 1975).

Importantly, Proposition 1 verifies that the conjectured value function (20) is correct. Proposition 1 also characterizes the agent's decision rules. To discuss these decision rules, we first analyze the special case with $\theta = 0$. This case characterizes the behavior of a fully rational agent without money illusion. In this case, as is well known from the portfolio choice theory

(e.g., Merton, 1969), the fully rational agent consumes a constant fraction of his total wealth and allocates another constant fraction of his total wealth as cash. This total wealth, $w + \beta \bar{w}$, includes the discounted present value of transfers and wealth from holding capital, bonds, and cash. The two constant fractions in (28) and (29) depend on the nominal interest rate R , which is the opportunity cost of holding money. They also depend on the term η , which represents the marginal propensity to consume out of wealth in a consumption/savings model without money (e.g., Merton, 1969). As shown in (22), the return A to the investment and the variance of the return σ_k^2 are important determinants of η . Their effects on consumption through η depend on the degree of risk aversion γ , which determines the relative strength of substitution and wealth effects. In particular, the wealth effect dominates when $\gamma > 1$ so that an increase in A or a decrease in σ_k^2 raises current consumption. When $\gamma < 1$, the substitution effect dominates so that the opposite result holds. In the borderline case with $\gamma = 1$, the two effects cancel out so that changes in investment opportunities do not influence consumption.

We next turn to the behavior of a money illusioned agent with $\theta > 0$. Proposition 1 shows that the consumption and money demands exhibit similar functional forms to that for a fully rational agent with $\theta = 0$. The key difference is that there is a new term in η as shown in Eq. (22)

$$\frac{\theta(\gamma-1)}{\gamma} \pi + \frac{\theta(\gamma-1)}{\gamma} \frac{\sigma_k^2}{2} ((\gamma-1)(2-\theta)-1).$$

This term reflects the effect of money illusion and consists of two components. The first component is related to the expected inflation rate and depends on the degrees of risk aversion and money illusion. It is present even in the deterministic case with $\sigma_k = 0$. The intuition is as follows. By definition, real wealth includes real money balances since $w_t = k_t + B_t/P_t + M_t/P_t$. A money-illusioned agent values both real and nominal terms. He misperceives the growth of real wealth and makes consumption and savings decisions based on both real and nominal terms. As a result, expected inflation affects his decision rules. The second component captures precautionary motive to guard against inflation uncertainty. The latter component depends on the inflation volatility, $\sigma_p = \sigma_k$, and the degrees of risk aversion and money illusion. In sum, money illusion distorts the agent's perception of the growth and riskiness of real wealth. The overall distortion is positively related to $\theta(\gamma-1)/\gamma$.

In the deterministic case where $\sigma_k = 0$, the inflation rate is also deterministic since $\sigma_p = \sigma_k = 0$. In this case, the expected inflation rate is equal to the actual inflation rate. Money illusion distorts the agent's perception of the growth of real wealth only. Eq. (22) reveals that for $\gamma > 1$, higher expected inflation induces the money illusioned agent to consume more. The intuition is that a higher expected inflation rate π implies a higher nominal wealth growth rate, and a money-illusioned agent misperceives his rapidly growing nominal wealth as real growth. This perception has two opposite wealth and substitution effects. When $\gamma > 1$, the wealth effect dominates so that the agent consumes more in response to an increased expected inflation rate. By contrast, when $\gamma < 1$, the substitution effect dominates so that the agent consumes less. When $\gamma = 1$, the wealth and substitution effects cancel out so that the agent does not respond to the change in the expected inflation rate. A similar analysis applies to the effect of expected inflation on the money demand.

3.3. Equilibrium

After analyzing the agent's decision problem, we are ready to solve for the equilibrium in which the price level, the aggregate wealth level, and the transfers satisfy (10), (18), and (19) with the restrictions given in Proposition 1. The key step is to solve for the expected inflation rate π . Using the market-clearing conditions, we have

Proposition 2. Suppose condition (22) holds for π given by

$$\pi = \frac{\gamma}{\theta + (1-\theta)\gamma} \left\{ \mu - \frac{A-\rho}{\gamma} + \frac{1}{2}(3-\gamma)\sigma_k^2 + \frac{\theta(\gamma-1)\sigma_k^2}{2\gamma} ((\gamma-1)(2-\theta)-1) \right\}. \tag{30}$$

Then it is the equilibrium expected inflation rate. The equilibrium laws of motion for the price level, the aggregate wealth level, and the transfers satisfy Eqs. (10), (18), and (19), respectively, where $\sigma_p = \sigma_k$ and f and h are given in (23). The equilibrium consumption level and capital stock satisfy $c = \eta k$ and

$$c = \left(\frac{\alpha R}{1-\alpha} \right)^{1/\varphi} (1-\phi^*)w, \tag{31}$$

$$M/P = (1-\phi^*)w, \quad k = \phi^*w,$$

where ϕ^* is given by (24). In addition, the volatility of the growth of capital is equal to σ_k and the expected growth rate of capital is given by

$$g = A - \eta. \tag{32}$$

Proposition 2 verifies our conjecture of the constant expected inflation, π , given that the money growth rate, μ , is constant. In addition, when condition (22) holds, Proposition 2 also verifies other conditions in Proposition 1. As a result, Propositions 1 and 2 together fully characterizes the equilibrium for our MIU model. Because the equilibrium consumption and capital are proportional to real wealth, consumption and real wealth also grow at the rate $g = A - \eta$.

As in Proposition 1, we first discuss properties of the equilibrium without money illusion ($\theta = 0$). In this case, η takes the value

$$\eta^0 \equiv \frac{\rho - (1-\gamma)A}{\gamma} + \frac{1}{2}(1-\gamma)\sigma_k^2. \tag{33}$$

As a result, Eq. (32) implies that the expected growth rate of capital is given by

$$g^0 \equiv \frac{A-\rho}{\gamma} + \frac{1}{2}(\gamma-1)\sigma_k^2. \tag{34}$$

Thus, monetary policy does not influence economic growth and real allocation. Its effect is to change inflation as revealed by Eq. (30). This result is consistent with other studies in the absence of money illusion ($\theta = 0$) (e.g., Chang and Lai, 2000; Marquis and Reffett, 1991; Rebelo and Xie, 1999; Wang and Yip, 1992).

One may wonder whether the neutrality of money remains true under money illusion if consumption and the real money balances are separable in the utility function. Our CES specification in (2) gives rise to separability when $\gamma = \varphi$

$$U(c, M, P) = \frac{1}{1-\varphi} (\alpha [c^{1-\theta} (Pc)^\theta]^{1-\varphi} + (1-\alpha) [(M/P)^{1-\theta} M^\theta]^{1-\varphi}), \quad \varphi > 0.$$

Plugging in $\gamma = \varphi$ in (22), we see from Propositions 1 and 2 that the growth rate of capital is given by

$$g = A - \left\{ \frac{\rho - (1-\varphi)A}{\varphi} + \frac{1}{2}(1-\varphi)\sigma_k^2 + \frac{\theta(\varphi-1)}{\varphi} \left[\pi + \frac{\sigma_k^2}{2} ((\varphi-1)(2-\theta)-1) \right] \right\}, \tag{35}$$

which depends on the expected inflation rate π as long as $\varphi \neq 1$. Thus, money is non-neutral in this case. When $\gamma = \varphi = 1$, the utility function is log-linear and completely separable in c , P , and M . In this case, money illusion does not affect consumption decision and money is neutral even in the presence of money illusion.

We now discuss properties of the equilibrium with money illusion ($\theta > 0$).⁹ In this case, the expected inflation rate enters η , and thus influences real allocations and economic growth. This effect depends on the risk aversion parameter γ . In particular, when $\gamma > 1$, there is a negative relation between growth and inflation. Intuitively, an increase in the expected inflation rate makes the money-illusioned agent to consume more and save less as shown in Proposition 1. Thus, it lowers economic growth. The opposite result holds for $\gamma < 1$. In the borderline case $\gamma = 1$, inflation does not affect economic growth. Our results are related to the empirical studies on the inflation and growth relationship. While some researchers find evidence for a negative relationship (e.g., Barro, 1996; Chari et al., 1995), other empirical studies show that this relationship is not robust (e.g., Bullard and Keating, 1995; Sarel, 1996; Bruno and Easterly, 1998; Dotsey and Sarte, 2000; Fischer et al., 2002; Khan and Senhadji, 2001). Our results provide a partial rationale for this nonrobustness.

In the deterministic case with $\sigma_k = 0$, it follows from (35) that the rate of economic growth is given by:

$$g = A - \frac{\rho - (1-\gamma)A}{\gamma} - \frac{\theta(\gamma-1)}{\gamma} \pi. \tag{36}$$

Comparing (35) and (36), we can see that without uncertainty, the effect of money illusion is to distort the agent's perception of the growth of real wealth only. In the presence of uncertainty with $\sigma_k > 0$, money illusion has an additional effect which distorts the agent's perception of the riskiness of his real wealth. As a result, output volatility also affects economic growth and this effects depends on the degrees of money illusion and risk aversion.

4. Quantitative effects of money illusion

As argued by Akerlof (2002), “a major contribution of behavioral macroeconomics is to demonstrate that, under sensible behavioral assumption, monetary policy does affect real outcomes just as Keynesian economics long asserted. Cognitive psychology pictures decision makers as ‘intuitive scientists’ who summarize information and make choices based on simplified mental frames.” One critique often made by the neoclassical synthesis is that irrational behavior is costly to decision makers and hence implausible. We will demonstrate that money illusion is not only commonplace but also sensible: the welfare losses from decision rules under money illusion are extremely small, but the effects on economic growth are quite large.

4.1. Welfare cost

Following Lucas (2000), we define the welfare cost $\Delta(\theta, \pi)$ of money illusion to be the percentage income compensation needed to leave the agent indifferent between $\theta = 0$ and $\theta > 0$ when the expected inflation rate is π . Formally, consider an illusion-free agent with the standard utility function $U^0(c, M, P)$ given by (2) in the MIU model, where $\theta = 0$. If he mistakenly follows the decision rules derived from the equilibrium with money illusion characterized in Propositions 1 and 2, then his

⁹ Note that when $\gamma = \varphi$ in (2), the utility function is additively separable over money and consumption under money illusion. Our analysis carries over to this special case.

utility value is given by

$$V(k_0) = E \left\{ \int_0^\infty e^{-\rho t} U^0(c_t, M_t, P_t) dt \right\}, \tag{37}$$

where k_0 is the initial capital holdings, $c_t = \eta k_t$ and $M_t/P_t = k_t(1 - \phi^*)/\phi^*$ given in Proposition 2, with $\theta > 0$. The indirect utility function from the equilibrium consumption plan without money illusion is given by

$$V^0(k_0) = E \left\{ \int_0^\infty e^{-\rho t} U^0(c_t^0, M_t^0, P_t^0) dt \right\},$$

where $c_t^0 = \eta^0 k_t$, and $M_t^0/P_t^0 = k_t(1 - \phi^*)/\phi^*$, with $\theta = 0$. Then the welfare cost $\Delta(\theta, \pi)$ is defined as the solution to the following equation:

$$V(k_0(1 + \Delta(\theta, \pi))) = V^0(k_0). \tag{38}$$

Notice that this equation actually defines $\Delta(\theta, \pi)$ as percentage capital compensation. Since in equilibrium, capital, output and wealth are all proportional to each other, we may also interpret $\Delta(\theta, \pi)$ as percentage output or wealth compensation.

We shall emphasize that we define $\Delta(\theta, \pi)$ as a function of the expected inflation rate π , which is endogenous in equilibrium. It is determined by the money growth rate μ as shown in Eq. (30). Since the inflation rate is observable and commonly used in practice, we will focus on it directly, but have in mind that there is an underlying money growth rate determining this expected inflation rate in equilibrium.

The following proposition gives a closed-form solution for $\Delta(\theta, \pi)$.

Proposition 3. Suppose¹⁰

$$\rho - (1 - \gamma)(A - \eta(\theta, \pi) - \gamma\sigma_k^2/2) > 0. \tag{39}$$

Then the welfare cost of money illusion is given by

$$\Delta(\theta, \pi) = \left(\frac{\rho - (1 - \gamma)(A - \eta(\theta, \pi) - \gamma\sigma_k^2/2)}{\rho - (1 - \gamma)(A - \eta(0, \pi) - \gamma\sigma_k^2/2)} \right)^{1/(1-\gamma)} \frac{\eta(0, \pi)}{\eta(\theta, \pi)} - 1, \tag{40}$$

where we define $\eta(\cdot)$ in (22) as a function of $\theta \in [0, 1]$ and π . In addition, π is given by Eq. (30).

Using this proposition, we can apply the Taylor expansion theorem to show that the welfare cost of money illusion is second order in θ .

Proposition 4. Suppose conditions (22) and (39) hold. Then holding π fixed,

$$\Delta(\theta, \pi) = \Delta_{11}(0, \pi)\theta^2 + o(\theta^2), \quad \theta \in [0, 1],$$

where $o(\theta^2)$ represents terms that have a higher order than θ^2 .

The intuition behind this proposition follows from the implications of the Envelope Theorem as discussed in Akerlof and Yellen (1985b). In our model, without money illusion, its welfare cost is zero. Thus, $\theta = 0$ minimizes $\Delta(\theta, \pi)$. As a result, a small degree of money illusion or a small positive value of $\theta \in [0, 1]$ has a second-order effect on the welfare cost $\Delta(\theta, \pi)$. Does a small degree of money illusion have a significant effect on economic equilibria? We can see immediately from Eq. (22) and Proposition 2 that the growth rate g of output contains a first order term of θ . Thus, a small degree of money illusion has a first-order effect on economic growth.

We now provide some quantitative estimates by calibrating model parameters. We first calibrate a standard AK growth model without money. The parameters are γ, ρ, A , and σ_k . As is standard in the macroeconomics and finance literature, we set $\gamma = 2$, and $\rho = -\ln(0.98)$. The choice of ρ ensures that the subjective discount factor in a discrete time model is equal to 0.98. To calibrate A and σ_k , we use the US annual constant price GDP data and population data from 1960 to 2006 as provided by CEIC Data Company Ltd. Simple calculation shows that the average per capita real GDP growth rate is 0.0223 and the standard deviation of the per capita real GDP growth is 0.0204, which are consistent with the numbers in Ramey and Ramey (1995) for an earlier sample.¹¹ We thus set $g^0 = 0.0223$ and $\sigma_k = 0.0204$. We then use Eq. (34) to solve for $A = 0.0644$.

To gauge the welfare cost $\Delta(\theta, \pi)$ in (40) and the growth effect of money illusion, we need to assign a value of the expected inflation rate. We find the average annual inflation rate is 0.0425 from 1960 to 2006 in the US, as documented in the 2007 Economic Report of the President. Fig. 1 plots the welfare cost $\Delta(\theta, \pi)$ and the percentage point change in the growth rate,

$$g - g^0 = \eta(0, \pi) - \eta(\theta, \pi),$$

as a function of $\theta \in [0, 1]$ for $\pi = 0.0425$. Notice that under our calibration money illusion always lowers the expected growth rate of output since $\gamma > 1$. Fig. 1 reveals that the welfare cost of money illusion $\Delta(\theta, \pi)$ is generally small compared

¹⁰ Condition (39) ensures that the indirect utility function $V(k_0)$ is finite. In standard models with $\theta = 0$, conditions (22) and (39) are equivalent.

¹¹ Ramey and Ramey (1995) document that the average real per capita GDP growth rate in the USA between 1962 and 1985 was 0.0214 and the standard deviation of the per capita real GDP growth was 0.0259.

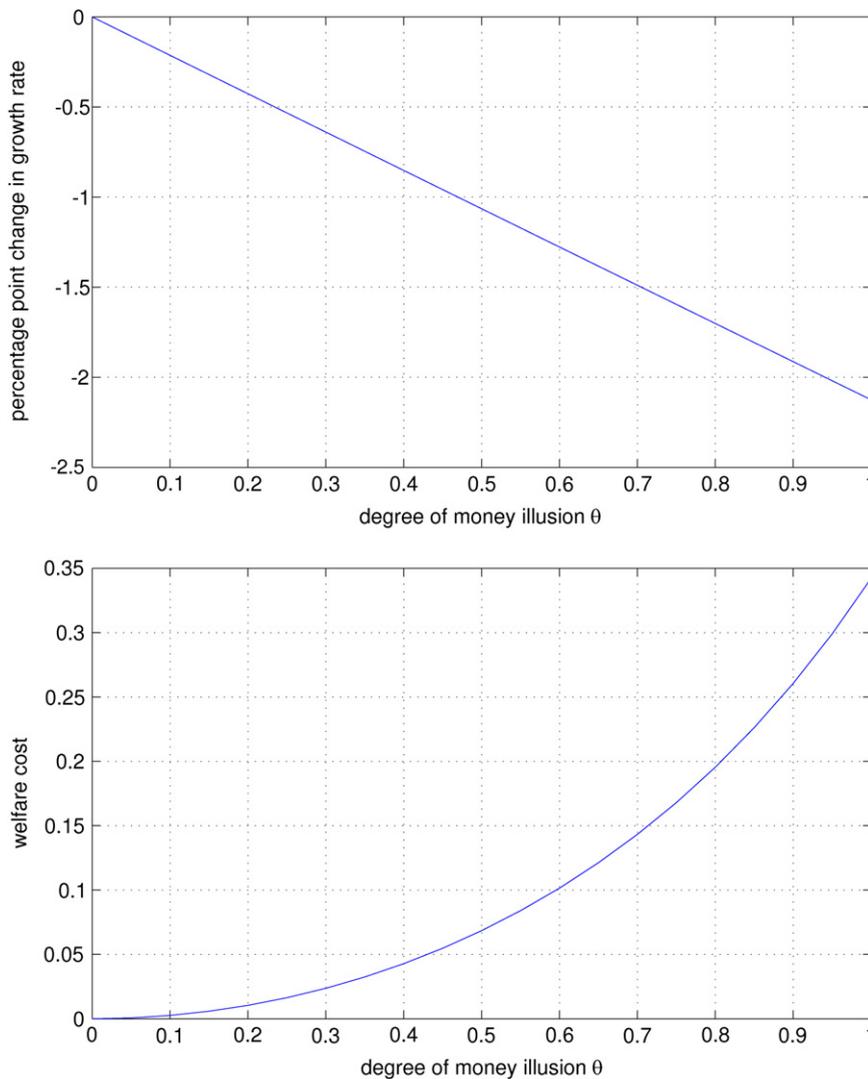


Fig. 1. The welfare cost and growth effects of money illusion. We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, $\sigma_k = 0.0204$, and $\pi = 0.0425$.

to the change in the rate of economic growth. When θ is large, both numbers are large. In the extreme case with complete money illusion $\theta = 1$, we find $\Delta(\theta, \pi) = 34.22\%$ and the rate of economic growth decrease by 2.13 percentage point. For small values of θ , the welfare cost is extremely small, but the change in growth rate can still be non-negligible. For example, when $\theta = 0.05$, the welfare cost $\Delta(\theta, \pi) = 0.06\%$, which is almost negligible, but the growth rate decreases by a noticeable 0.11 percentage point relative to the growth rate without money illusion. As another example, when $\theta = 0.1$, $\Delta(\theta, \pi) = 0.26\%$, but the growth rate decreases by 0.21 percentage point. Are these growth effects significant? As an illustration, consider the difference in real GDP between two economies that are otherwise the same. But the representative agent in one economy has money illusion with degree $\theta = 0.05$ and the agent in the other economy has no money illusion $\theta = 0$. Suppose both economies start in 1960 with the same level of income, their income would differ by about 5% in 2007. If the agent in the first economy has $\theta = 0.1$, then the income difference in 2007 would be 10%.

The quantitative effects of money illusion depends crucially on the expected inflation rate. Fig. 2 plots the effects of money illusion on welfare costs and economic growth for $\pi = 0.02, 0.05, 0.10$, and 0.20 and for small degrees of money illusion $\theta \in [0, 0.2]$. This figure reveals that the welfare cost and the growth effect of money illusion are very small for low values of inflation. However, they rise nonlinearly with the expected inflation rate. In particular, they are extremely large for high values of the expected inflation rate, even when the degree of money illusion is relatively small. For example, when $\pi = 0.2$ and $\theta = 0.1$, the welfare cost is about 6% of the per capita wealth and the rate of economic growth decreases by about 1 percentage point. This is understandable because in an environment with high expected inflation, a money-illusioned agent mistakes his rapidly growing nominal wealth as real growth and hence consumes more (wealth effect dominates the substitution effect if $\gamma > 1$), lowering the rate of economic growth significantly.

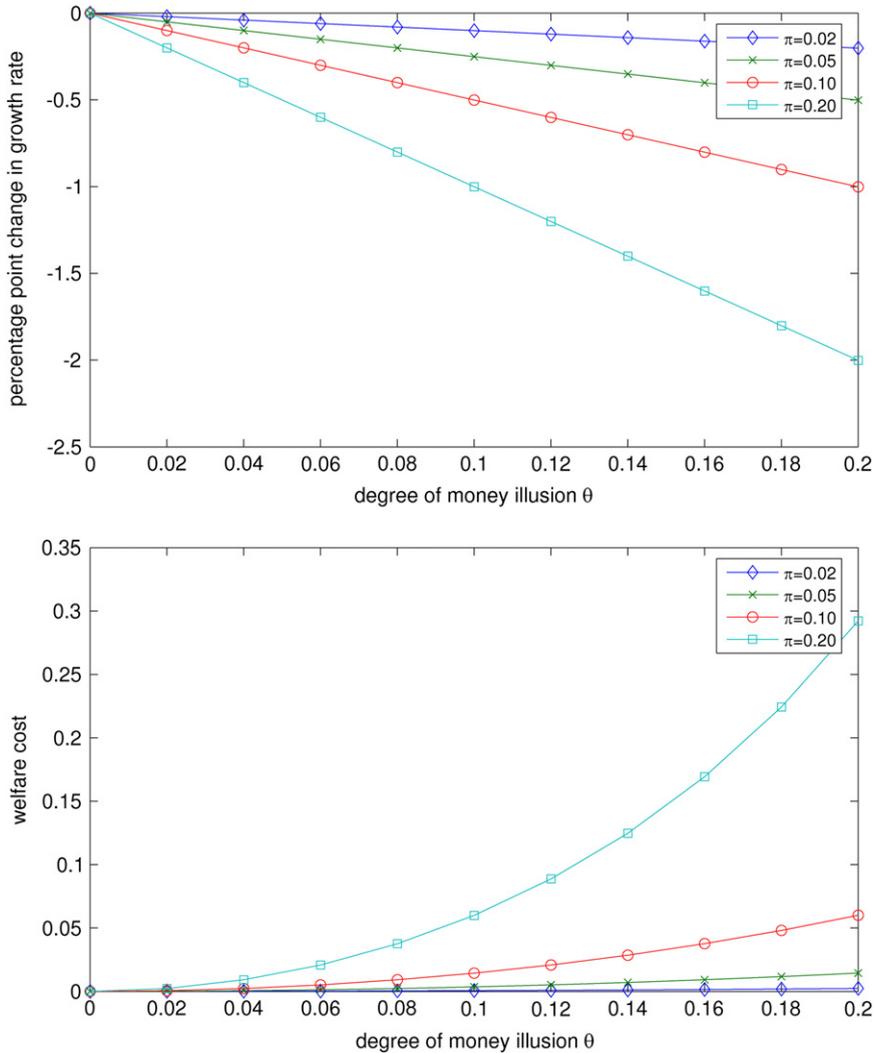


Fig. 2. Effects of money illusion for different expected inflation rates. We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.

An important parameter in our model is the coefficient of relative risk aversion γ . We now consider its effect. Propositions 1 and 2 show that the economic effects may be very different for $\gamma < 1$ and $\gamma > 1$. Since the case of $\gamma > 1$ is empirically more plausible, we conduct experiments with $\gamma > 1$. Fig. 3 plots the welfare cost and the percentage point change in the growth rate for $\gamma = 1.5, 2, 4, 6$ and $\theta \in [0, 0.2]$. This figure reveals that the quantitative effects of money illusion is generally small for low risk aversion. These effects increase nonlinearly with risk aversion. When the degree of risk aversion is high, the economic effects of money illusion is significant. For example, when $\gamma = 6$ and $\theta = 0.05$, the welfare cost is 0.4% of income, which is small. But the growth rate decreases by a significant 0.18 percentage point. When $\theta = 0.1$, both numbers are large in that the welfare cost is 1.6% of income, and the growth rate decreases by 0.37 percentage point.

4.2. Eliminating the cost of money illusion

Is there a monetary policy that minimizes the welfare cost of money illusion? In fact, for any θ , the monetary authority can choose a money growth rate μ to achieve an expected inflation rate π in order to make the welfare cost $\Delta(\theta, \pi)$ equal to zero. Formally, by Eqs. (22) and (40) we only need to set

$$\pi = \frac{\sigma_k^2}{2}(1 + (\gamma - 1)(\theta - 2)) \quad \text{for } \gamma \neq 1. \tag{41}$$

We can then use Eq. (30) to determine μ . One may intuitively argue that the monetary authority should set the expected inflation rate to zero in order to remove money illusion. This is true in our model when there is no uncertainty. That is, when $\sigma_k = 0$, (41) implies that $\pi = 0$. With uncertainty, even though the expected inflation rate is zero, the welfare cost of money illusion is positive because inflation is volatile and the variability of inflation distorts the money illusioned agent's

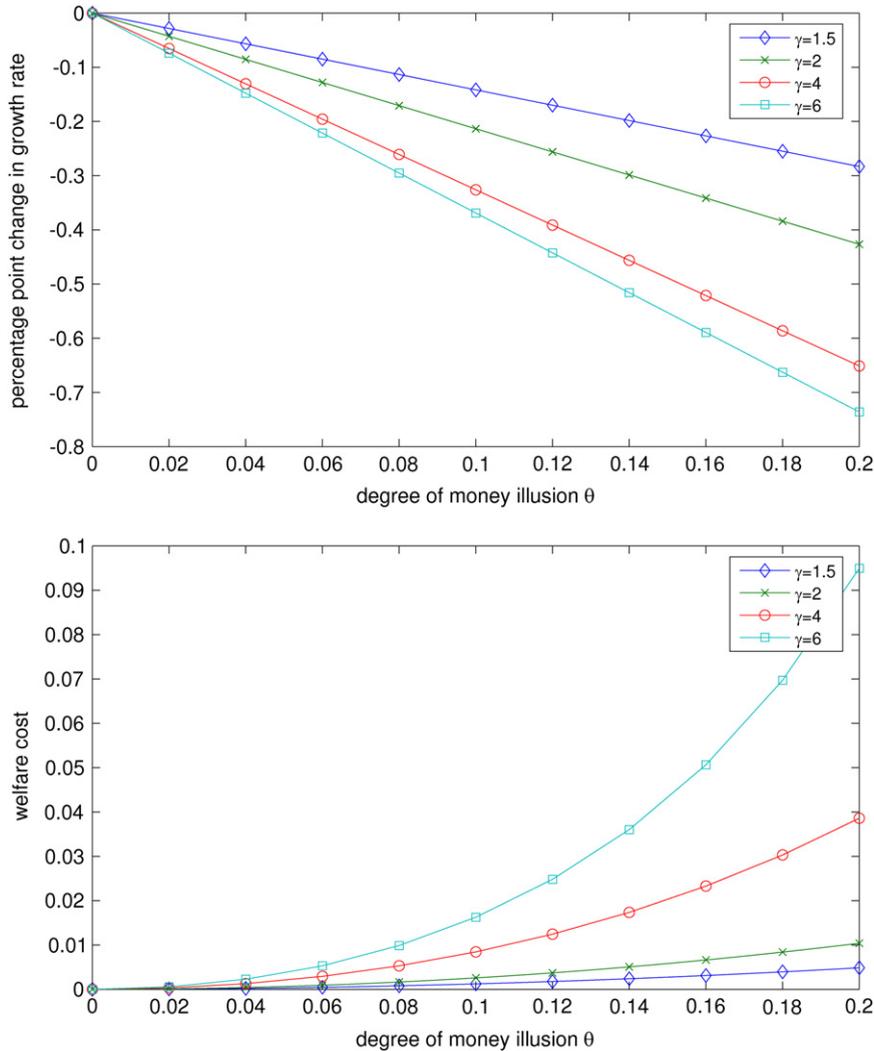


Fig. 3. Effects of money illusion for different degrees of risk aversion. We set parameter values: $A=0.0644$, $\rho = -\ln(0.98)$, $\sigma_k = 0.0204$, and $\pi = 0.0425$.

perception of the riskiness of real wealth. To eliminate the welfare cost of money illusion, the monetary authority should set a policy such that the distortions on the growth and riskiness of real wealth offset each other, as shown in Eq. (41).

This equation also implies that the cost-minimizing inflation rate π increases with θ for $\gamma > 1$, and vice versa for $\gamma < 1$. The intuition follows from the utility function in (2). For this utility function, marginal utility of consumption decreases with θ if and only if $\gamma > 1$. Thus, when $\gamma > 1$, an agent with a higher degree θ of money illusion tends to consume less. To correct for this cost, the monetary authority should change money supply to raise the expected inflation rate if π is positive (or lower the expected deflation rate if π is negative) to make the money-illudioned agent to feel wealthier and thus to consume more. Since $R = A + \pi - \sigma_k^2$, the nominal interest rate follows an identical monotonic relation with θ . Note that Eq. (41) implies that the cost-minimizing inflation rate may be negative depending on the values of γ and θ . Fig. 4 plots the cost-minimizing inflation rate and nominal interest rate for $\theta \in [0, 1]$. This figure reveals that the cost-minimizing inflation rates are negative and small, and the cost-minimizing nominal interest rates are positive. In addition, both rates do not vary much with θ due to the small value of volatility σ_k under our benchmark calibration (see Eq. (41)).

We now consider a sensitivity analysis for different values of the risk aversion parameter. Fig. 5 plots the cost-minimizing interest rates and expected inflation rate. This figure reveals several features. First, the cost-minimizing expected inflation rates are generally small and close to zero. In addition, for most values of γ , they take negative values. For low values of γ , e.g., $\gamma = 1.5$, the cost minimizing expected inflation rates are positive. Second, both the cost-minimizing nominal interest rates and expected inflation rates are not sensitive to changes in risk aversion.

We shall emphasize that the cost-minimizing nominal interest rate in our model is generally not equal to zero, as postulated by the Friedman rule. In the standard literature of optimal monetary policy, the objective of optimal monetary policy is to

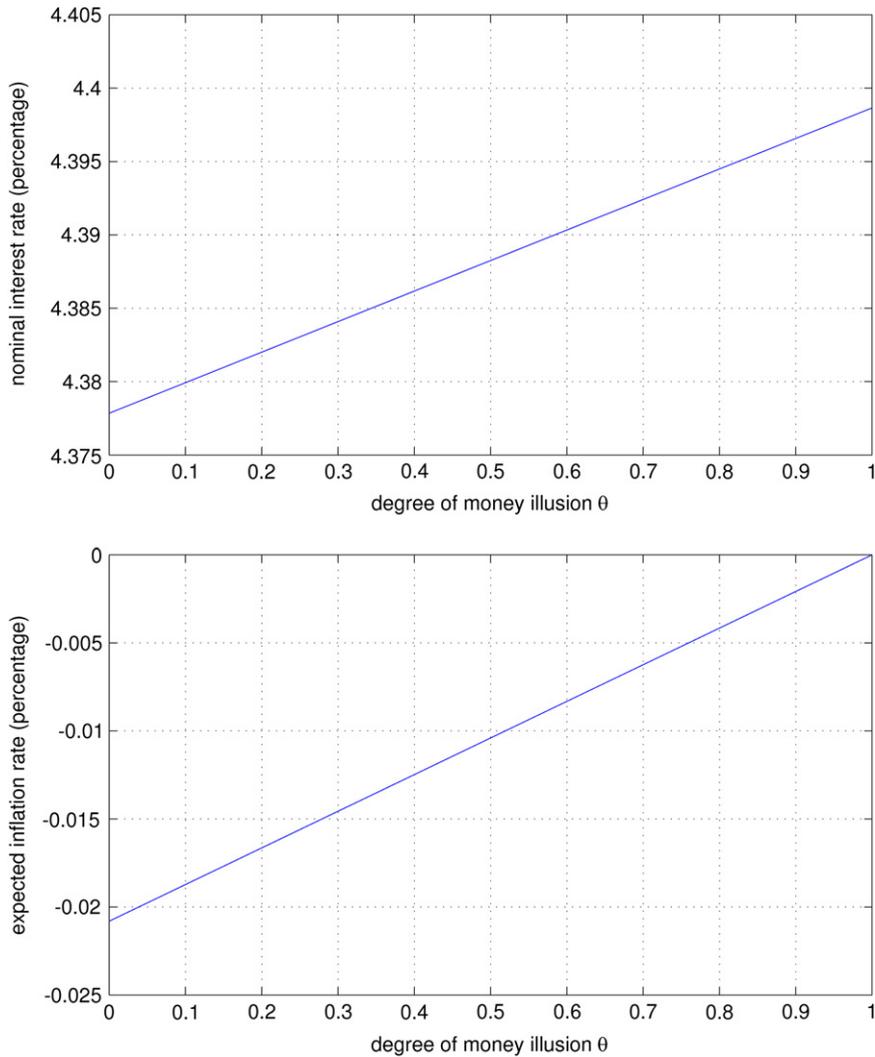


Fig. 4. Cost-minimizing nominal interest rates and expected inflation rates. We set parameter values: $\gamma = 2$, $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.

maximize individuals' utility. In our model, the money illusioned agent is irrational and we argue that the monetary authority should not maximize the utility function in (2) or (4). Instead, if the monetary authority maximizes $V(k_0)$ given in (37) with a standard $U^0(c, M, P)$, then this optimal monetary policy coincides with our cost-minimizing policy.

We should also point out that the monetary policy that minimizes the welfare cost of money illusion requires the monetary authority to know the degree of money illusion θ , which is hard to estimate. In addition, this policy requires that the expected inflation rate be equal to a targeted inflation rate given by Eq. (41). As a result, this policy may be hard to implement in practice. That said, policymakers should be aware of the possibility of a money-illusioned general public. It is conceivable that poor countries may have a higher distortion from money illusion due to lack of education and thus the targeted inflation rate should be higher in those countries if γ is believed to be greater than unity.

5. Conclusion

In this paper, we have presented a tractable stochastic continuous-time monetary model of endogenous growth in which the representative agent suffers from money illusion. Our paper provides a first study on the relationship between money illusion and inflation as related to economic growth and policy. Our analysis has demonstrated that money illusion distorts an agent's perception of the growth and riskiness of real wealth and thus his consumption/savings choice. Its impact on long-run growth is via this channel. We have shown that the welfare cost of money illusion is second order, whereas its impact on long-run growth is first order in terms of the degree of money illusion. Our model has implications

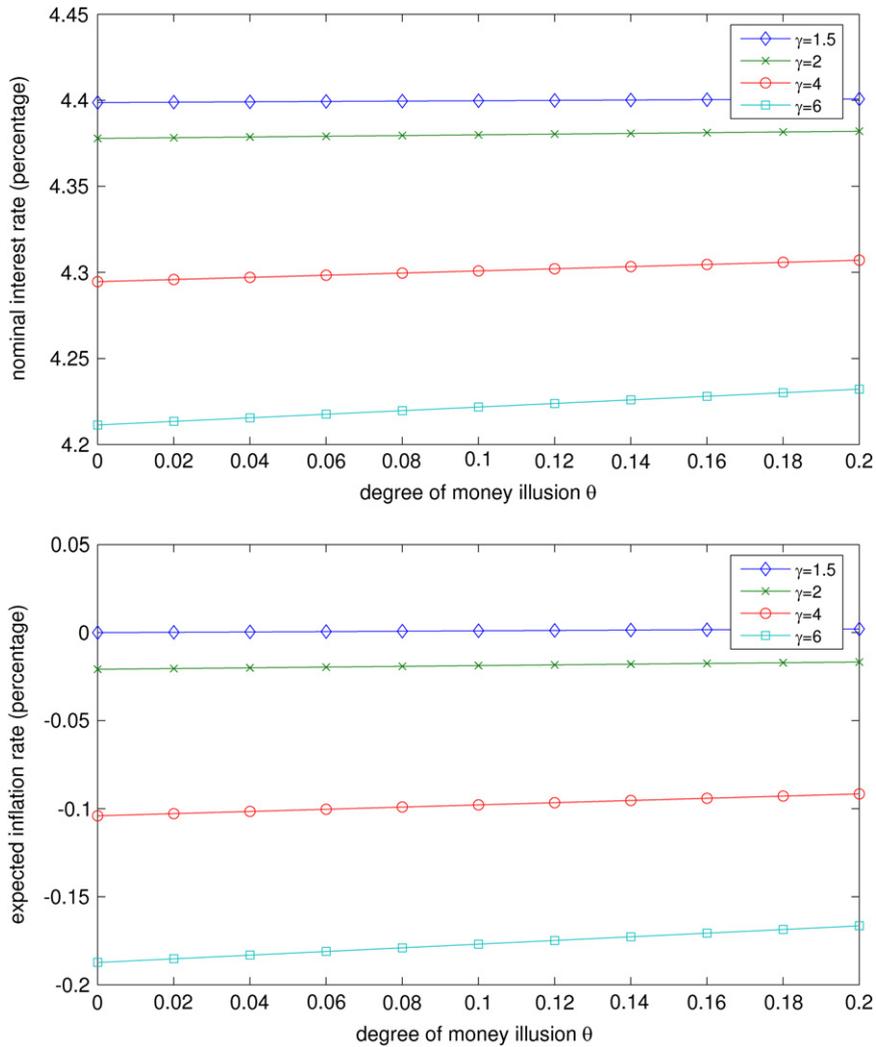


Fig. 5. Effects of risk aversion on the cost-minimizing nominal interest rates and expected inflation rates. We set parameter values: $A = 0.0644$, $\rho = -\ln(0.98)$, and $\sigma_k = 0.0204$.

for the empirical relationship between growth and inflation. Our results suggest this relationship crucially depends on the risk aversion parameter.

Our model can be varied in a number of dimensions. First, although we have shown in theory that the monetary authority can choose a growth rate of the money supply to eliminate the cost of money illusion by correcting the distortions on consumption/savings decisions, more needs to be done to study what procedures are required for an effective implementation. Second, we may introduce labor-leisure choice and consider a type of the Lucas (1988) human-capital model. Third, we may separate risk aversion and intertemporal substitution in the utility specification as in Epstein and Zin (1989). We can then analyze how they interact with money illusion in an agent’s consumption/savings decision problem. Finally, we may use our modeling of money illusion to study price and wage rigidities in a multi-agent or multi-firm model. We can then examine business cycle implications.

Appendix A. Proofs

Proof of Proposition 1. By definition,

$$M/P = (1 - \phi - \psi)w. \tag{A.1}$$

Substituting this equation, the utility function (2), and the conjectured value function into the HJB equation (21), we cancel out $P^{\theta(1-\gamma)}$ and obtain

$$\begin{aligned}
 0 = & \max_{\phi, \psi} \frac{1}{1-\gamma} (\alpha c^{1-\varphi} + (1-\alpha)[(1-\phi-\psi)w]^{1-\varphi})^{(1-\gamma)/(1-\varphi)} - \rho b \frac{(w+\beta\bar{w})^{1-\gamma}}{1-\gamma} \\
 & + b(w+\beta\bar{w})^{-\gamma} (w[A\phi + R\psi + (1-\phi)(\sigma_k^2 - \pi)] + v - c) \\
 & + \beta b(w+\beta\bar{w})^{-\gamma} f(\bar{w}) + b\theta(w+\beta\bar{w})^{1-\gamma} \pi - \frac{\gamma}{2} b(w+\beta\bar{w})^{-\gamma-1} \sigma_k^2 w^2 \\
 & - \frac{\gamma}{2} \beta^2 b(w+\beta\bar{w})^{-\gamma-1} [h(\bar{w})]^2 + \frac{1}{2} \theta(\theta(1-\gamma)-1) b(w+\beta\bar{w})^{1-\gamma} \sigma_k^2 \\
 & - \gamma \beta b(w+\beta\bar{w})^{-\gamma-1} h(\bar{w}) w \sigma_k - \theta(1-\gamma) b(w+\beta\bar{w})^{-\gamma} w \sigma_k^2 - \theta(1-\gamma) b \beta (w+\beta\bar{w})^{-\gamma} h(\bar{w}) \sigma_k.
 \end{aligned} \tag{A.2}$$

Taking the first-order conditions with respect to c, ϕ , and ψ yields

$$\alpha c^{-\varphi} (\alpha c^{1-\varphi} + (1-\alpha)[(1-\phi-\psi)w]^{1-\varphi})^{(1-\gamma)/(1-\varphi)-1} = b(w+\beta\bar{w})^{-\gamma}, \tag{A.3}$$

$$(1-\alpha)[(1-\phi-\psi)w]^{-\varphi} (\alpha c^{1-\varphi} + (1-\alpha)[(1-\phi-\psi)w]^{1-\varphi})^{(1-\gamma)/(1-\varphi)-1} = b(w+\beta\bar{w})^{-\gamma} (A + \pi - \sigma_k^2), \tag{A.4}$$

$$(1-\alpha)[(1-\phi-\psi)w]^{-\varphi} (\alpha c^{1-\varphi} + (1-\alpha)[(1-\phi-\psi)w]^{1-\varphi})^{(1-\gamma)/(1-\varphi)-1} = b(w+\beta\bar{w})^{-\gamma} R. \tag{A.5}$$

Eq. (27) follows from (A.4) and (A.5). Using (A.3) and (A.5), we derive

$$c = \left(\frac{\alpha R}{1-\alpha} \right)^{1/\varphi} (1-\phi-\psi)w. \tag{A.6}$$

We substitute (A.6) back into (A.5) to derive

$$(1-\phi-\psi)w = (w+\beta\bar{w}) b^{-1/\gamma} \left(\frac{R}{1-\alpha} \right)^{-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + (1-\alpha) \right)^{(\varphi-\gamma)/(1-\varphi)\gamma}. \tag{A.7}$$

Substituting (A.6), (23) and (27) into (A.2) yields

$$\begin{aligned}
 0 = & \frac{1}{1-\gamma} \left[\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + 1 - \alpha \right]^{(1-\gamma)/(1-\varphi)} [(1-\phi-\psi)w]^{1-\gamma} - \frac{\rho b(w+\bar{w})^{1-\gamma}}{1-\gamma} \\
 & + b(w+\beta\bar{w})^{-\gamma} \left[wA - w(1-\phi-\psi) \left(R + \left(\frac{\alpha R}{1-\alpha} \right)^{1/\varphi} \right) + v \right] \\
 & + \beta b(w+\beta\bar{w})^{-\gamma} \bar{w}(A-\eta) + \theta b(w+\beta\bar{w})^{1-\gamma} \pi \\
 & - \frac{\gamma}{2} b(w+\beta\bar{w})^{-\gamma-1} \sigma_k^2 w^2 - \frac{\gamma}{2} \beta^2 b(w+\beta\bar{w})^{-\gamma-1} \sigma_k^2 \bar{w}^2 - \gamma \beta b(w+\beta\bar{w})^{-\gamma-1} \sigma_k^2 \bar{w} w \\
 & + \frac{1}{2} \theta(\theta(1-\gamma)-1) b(w+\beta\bar{w})^{1-\gamma} \sigma_k^2 - \theta(1-\gamma) b(w+\beta\bar{w})^{1-\gamma} \sigma_k^2.
 \end{aligned} \tag{A.8}$$

Using (A.7), and substituting the expression for v in (19), we can simplify this equation to derive

$$\begin{aligned}
 0 = & \frac{(w+\beta\bar{w}) b^{1-1/\gamma}}{1-\gamma} \left(\frac{R}{1-\alpha} \right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + (1-\alpha) \right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} - \frac{\rho b(w+\bar{w})}{1-\gamma} \\
 & + b w A - (w+\beta\bar{w}) b^{1-1/\gamma} \left(\frac{R}{1-\alpha} \right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + (1-\alpha) \right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} \\
 & + \frac{b \bar{w} \mu \eta}{R + \left(\frac{\alpha R}{1-\alpha} \right)^{1/\varphi} - \mu} + \beta b \bar{w} (A-\eta) - \frac{\gamma}{2} b(w+\beta\bar{w}) \sigma_k^2 + \theta b(w+\beta\bar{w}) \left[\pi + \frac{1}{2} \sigma_k^2 ((\theta-2)(1-\gamma)-1) \right].
 \end{aligned} \tag{A.9}$$

Since (A.9) holds for all w and \bar{w} , we set the coefficients of w to zero to obtain

$$\begin{aligned}
 0 = & \frac{1}{1-\gamma} b^{1-1/\gamma} \left(\frac{R}{1-\alpha} \right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + (1-\alpha) \right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} - \frac{\rho b}{1-\gamma} \\
 & + b \left[A - b^{-1/\gamma} \left(\frac{R}{1-\alpha} \right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha} \right)^{(1-\varphi)/\varphi} + (1-\alpha) \right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} \right] - \frac{\gamma}{2} b \sigma_k^2 \\
 & + b \theta \left[\pi + \frac{1}{2} \sigma_k^2 ((\theta-2)(1-\gamma)-1) \right].
 \end{aligned}$$

Using the definition of η in (22) and simplifying the preceding equation, we obtain Eq. (25). Similarly, we set the coefficients of \bar{w} to zero to derive

$$0 = \frac{b^{1-1/\gamma} \beta}{1-\gamma} \left(\frac{R}{1-\alpha}\right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha}\right)^{(1-\varphi)/\varphi} + (1-\alpha)\right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} - \frac{\rho b \beta}{1-\gamma} \\ - b^{1-1/\gamma} \beta \left(\frac{R}{1-\alpha}\right)^{1-1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha}\right)^{(1-\varphi)/\varphi} + (1-\alpha)\right)^{(1-\gamma)\varphi/(1-\varphi)\gamma} + \frac{b \mu \eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} - \mu} + \beta b(A-\eta) - \frac{\gamma}{2} b \beta \sigma_k^2 \\ + b \theta \beta \left[\pi + \frac{1}{2} \sigma_k^2 ((\theta-2)(1-\gamma)-1) \right].$$

Using the preceding two equations, we can derive Eq. (26). Using Eqs. (A.6), (A.7), and (25), we can derive the consumption rule (28). Using (A.1), (A.7), and (25), we can derive the money demand (29). Finally, we require $\eta > 0$ so that both consumption and money holdings are positive. \square

Proof of Proposition 2. In equilibrium, bond holdings are zero so that $\psi = 0$. In addition, the representative agent's wealth is equal to aggregate wealth so that $w = \bar{w}$. Eq. (A.7) then implies that ϕ takes a constant value ϕ^* in equilibrium, which satisfies

$$1 - \phi^* = (bR)^{-1/\gamma} (1 + \beta)(1 - \alpha)^{1/\gamma} \left(\alpha \left(\frac{\alpha R}{1-\alpha}\right)^{(1-\varphi)/\varphi} + (1-\alpha)\right)^{(\varphi-\gamma)/(1-\varphi)\gamma}.$$

Substituting the expressions for b and β in Eqs. (25) and (26), we obtain Eq. (24), confirming our conjecture in Proposition 1.

We next verify the dynamics of transfers. Eq. (A.1) implies that Eq. (17) holds in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion terms yield $\sigma_P = \sigma_k$. In addition, the equilibrium lump-sum transfer satisfies

$$v = \mu \bar{M} / P = \mu \bar{w} (1 - \phi^*). \tag{A.10}$$

Since $w = \bar{w}$ in equilibrium, \bar{w} and w must follow the same diffusion process. Substituting (A.6), (A.10), $\phi = \phi^*$, and $\psi = 0$ into the drift of the agent's wealth dynamics (16), we obtain

$$f(\bar{w}) = \bar{w} \left[A \phi^* + (1 - \phi^*) \left(\sigma_k^2 - \pi - \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} \right) \right] + v = \bar{w} \left[A - (1 - \phi^*) \left(R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} \right) \right] + \mu \bar{w} (1 - \phi^*) \\ = \bar{w} \left[A - (1 - \phi^*) \left(R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} - \mu \right) \right] = \bar{w} (A - \eta),$$

where the last equality follows from the definition of ϕ^* in (24). Matching the diffusion terms of the wealth dynamics yields $h(\bar{w}) = \sigma_k \bar{w}$. Thus, we have verified (23).

We now determine the equilibrium inflation rate. Applying Ito's Lemma to the expressions on the two sides of the equation,

$$\frac{M}{P} = (1 - \phi^*) \bar{w},$$

and matching the drift terms, we obtain

$$\mu - \pi + \sigma_k^2 = f(\bar{w}) / \bar{w} = A - \eta.$$

By the definition of η in (22), we can solve for the expected inflation rate given by (30), which verifies that the expected inflation rate, π , is constant when money grows at a constant rate, μ . By (27), we obtain

$$R - \mu = \eta. \tag{A.11}$$

In equilibrium, Eq. (A.6) implies that

$$\frac{c}{w} = \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} (1 - \phi^*) = \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} \frac{\eta}{R + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi} - \mu} = \frac{\eta}{\eta + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}} \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi},$$

where we have substituted Eqs. (24) and (A.11). Similarly, we can derive

$$k = \phi^* w = \frac{\left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}}{\eta + \left(\frac{\alpha R}{1-\alpha}\right)^{1/\varphi}} w.$$

Thus, we obtain $c/k = \eta$. Using the resource constraint (9), we obtain the expected growth rate of capital in Eq. (32). Since output, consumption, capital and wealth are proportional to each other, they all have the same expected growth rate. \square

Proof of Proposition 3. By Proposition 2,

$$\frac{M_t/P_t}{k_t} = \frac{1-\phi^*}{\phi^*} = \frac{c_t}{k_t} \left(\frac{\alpha R}{1-\alpha}\right)^{-\varphi} = \eta(\theta, \pi) \left(\frac{\alpha R}{1-\alpha}\right)^{-\varphi}.$$

Since

$$U^0(c, M, P) = \frac{1}{1-\gamma} (\alpha c^{1-\varphi} + (1-\alpha)(M/P)^{1-\varphi})^{(1-\gamma)/(1-\varphi)},$$

we can compute

$$\begin{aligned} V(k_0) &= E \left\{ \int_0^\infty e^{-\rho t} \frac{1}{1-\gamma} (\alpha c_t^{1-\varphi} + (1-\alpha)(M_t/P_t)^{1-\varphi})^{(1-\gamma)/(1-\varphi)} dt \right\} \\ &= \frac{\eta(\theta, \pi)^{1-\gamma}}{1-\gamma} E \left\{ \int_0^\infty e^{-\rho t} \left(\alpha k_t^{1-\varphi} + (1-\alpha) \left(\frac{\alpha R}{1-\alpha}\right)^{-(1-\varphi)/\varphi} k_t^{1-\varphi} \right)^{(1-\gamma)/(1-\varphi)} dt \right\} \\ &= \frac{\eta(\theta, \pi)^{1-\gamma}}{1-\gamma} E \left\{ \int_0^\infty e^{-\rho t} k_t^{1-\gamma} dt \right\} \left(\alpha + (1-\alpha) \left(\frac{\alpha R}{1-\alpha}\right)^{-(1-\varphi)/\varphi} \right)^{(1-\gamma)/(1-\varphi)}. \end{aligned}$$

Thus,

$$V(k_0) = \frac{\eta(\theta, \pi)^{1-\gamma} k_0^{1-\gamma}}{1-\gamma} \frac{\left(\alpha + (1-\alpha) \left(\frac{\alpha R}{1-\alpha}\right)^{-(1-\varphi)/\varphi} \right)^{(1-\gamma)/(1-\varphi)}}{\rho - (1-\gamma)(A - \eta(\theta, \pi) - \gamma \sigma_k^2/2)}.$$

We finally use the definition (38) to derive (40). □

Proof of Proposition 4. We take logarithm on both sides of Eq. (40) to derive

$$\ln[1 + \Delta(\theta, \pi)] = \frac{1}{1-\gamma} \ln(\rho - (1-\gamma)(A - \eta(\theta, \pi) - \gamma \sigma_k^2/2)) - \ln(\eta(\theta, \pi)) + I,$$

where I is some term independent of θ . Holding π fixed and differentiating both sides with respect to θ , we obtain

$$\frac{1}{1 + \Delta(\theta, \pi)} \Delta_1(\theta, \pi) = \frac{\eta_1(\theta, \pi)}{[\rho - (1-\gamma)(A - \eta(\theta, \pi) - \gamma \sigma_k^2/2)]} - \frac{\eta_1(\theta, \pi)}{\eta(\theta, \pi)}.$$

Given the expression of $\eta(\theta, \pi)$ below

$$\eta(\theta, \pi) = \frac{\rho - (1-\gamma)A}{\gamma} + \frac{1}{2}(1-\gamma)\sigma_k^2 + \frac{\theta(\gamma-1)}{\gamma} \left[\pi + \frac{\sigma_k^2}{2}((\gamma-1)(2-\theta)-1) \right],$$

it is straightforward to verify that

$$\rho - (1-\gamma)(A - \eta(0, \pi) - \gamma \sigma_k^2/2) = \eta(0, \pi).$$

Thus,

$$\frac{1}{1 + \Delta(\theta, \pi)} \Delta_1(0, \pi) = \eta_1(0, \pi) \left[\frac{1}{[\rho - (1-\gamma)(A - \eta(0, \pi) - \gamma \sigma_k^2/2)]} - \frac{1}{\eta(0, \pi)} \right] = 0.$$

This implies that $\Delta_1(0, \pi) = 0$. In addition, it is straightforward to see that $\Delta(0, \pi) = 0$. Thus, $\Delta(\theta, \pi)$ is second order of θ . □

Appendix B. A cash-in-advance model

In Section 3, we introduce a role of money by assuming the agent derives utility from holding money directly. This way of modeling suffers from the criticism of creating positive value for money by simply assuming the problem away. We value money because it is useful in facilitating transactions. To incorporate this role, we study a cash-in-advance (CIA) model in this appendix. We will show that both the MIU and CIA models yield identical results.

Instead of assuming money in the utility function as in (2), we assume that the agent has the utility function (4). This utility function also captures money illusion. We assume consumption purchases must be financed by cash so that the following cash-in-advance constraint must hold

$$P_t c_t \leq M_t. \tag{B.1}$$

The rest of the model follows the same structure as that in Section 3. We also use a similar procedure to solve for a monetary competitive equilibrium. In particular, we conjecture the price dynamics follow Eq. (10). Define the real wealth level $w_t = k_t + B_t/P_t + M_t/P_t$. Let $k_t = \phi_t w_t$ and $B_t/P_t = \psi_t w_t$, where ϕ_t and ψ_t are to be determined. Using (8), we derive the law of motion for the wealth level (16). We conjecture that in equilibrium ϕ_t takes a constant value ϕ^* given in

(B.2) below. As a result, Eq. (17) holds so that $\sigma_p = \sigma_k$. We conjecture that the equilibrium aggregate real wealth satisfies (18) and the lump-sum transfer is given by (19).

As in Section 3, given the above conjecture, we first study the agent's decision problem. We conjecture that the value function $J(w, \bar{w}, P)$ takes the form in (20), where b and β are constants to be determined. By the standard dynamic programming theory, this value function satisfies the HJB equation (21), where $U(c, M, P)$ is given by (4). Solving (21), we obtain.

Proposition 5. Suppose assumption (22) holds. Let the price level, the aggregate wealth level, and the transfers satisfy Eqs. (10), (18), and (19), respectively, where $\sigma_p = \sigma_k$, $f(\bar{w})$ and $h(\bar{w})$ satisfy (23) and

$$\phi^* = 1 - \frac{\eta}{R+1-\mu}. \tag{B.2}$$

Then the value function is given by (20) where b and β are given by

$$b = \eta^{-\gamma} (1+R)^{\gamma-1}, \tag{B.3}$$

$$\beta = \mu(R+1-\mu)^{-1}. \tag{B.4}$$

In addition, the nominal interest rate R satisfies (27), and the optimal consumption and money demand rules are given by

$$c = \frac{M}{P} = \frac{\eta}{R+1} (w + \beta \bar{w}). \tag{B.5}$$

Proof. At optimum the CIA constraint (B.1) binds so that

$$c = M/P = (1-\phi-\psi)w, \tag{B.6}$$

where the last equality follows from (A.1). Substituting this equation and the conjectured value function into the HJB equation (21), we cancel out $P^{\theta(1-\gamma)}$ and obtain

$$\begin{aligned} 0 = \max_{\phi, \psi} & \left[\frac{(1-\phi-\psi)w^{1-\gamma}}{1-\gamma} - \rho b \frac{(w+\beta\bar{w})^{1-\gamma}}{1-\gamma} + b(w+\beta\bar{w})^{-\gamma} (w[A\phi + R\psi + (1-\phi)(\sigma_k^2 - \pi) - (1-\phi-\psi)] + v) \right. \\ & + \beta b(w+\beta\bar{w})^{-\gamma} f(\bar{w}) + b\theta(w+\beta\bar{w})^{1-\gamma} \pi - \frac{\gamma}{2} b(w+\beta\bar{w})^{-\gamma-1} \sigma_k^2 w^2 \\ & - \frac{\gamma}{2} \beta^2 b(w+\beta\bar{w})^{-\gamma-1} [h(\bar{w})]^2 + \frac{1}{2} \theta(\theta(1-\gamma)-1) b(w+\beta\bar{w})^{1-\gamma} \sigma_k^2 - \gamma \beta b(w+\beta\bar{w})^{-\gamma-1} h(\bar{w}) w \sigma_k \\ & \left. - \theta(1-\gamma) b(w+\beta\bar{w})^{-\gamma} w \sigma_k^2 - \theta(1-\gamma) b \beta (w+\beta\bar{w})^{-\gamma} h(\bar{w}) \sigma_k \right]. \end{aligned} \tag{B.7}$$

Taking first-order conditions yields

$$(1-\phi-\psi)w = (w+\beta\bar{w})[b(A+1+\pi-\sigma_k^2)]^{-1/\gamma}, \tag{B.8}$$

$$(1-\phi-\psi)w = (w+\beta\bar{w})[b(R+1)]^{-1/\gamma}. \tag{B.9}$$

From these two equations, we obtain Eq. (27).

Substituting Eqs. (B.9), (19), and (B.2) into (B.7), we cancel out $(w+\beta\bar{w})^{-\gamma}$ and obtain

$$\begin{aligned} 0 = & \frac{(w+\beta\bar{w})[b(R+1)]^{1-1/\gamma}}{1-\gamma} - \frac{\rho b(w+\beta\bar{w})}{1-\gamma} + b \left[Aw - (w+\beta\bar{w})(R+1)[b(R+1)]^{-1/\gamma} + \frac{\mu\bar{w}}{R+1-\mu} \right] \\ & + \beta b \bar{w} (A-\eta) - \frac{\gamma}{2} b(w+\beta\bar{w}) \sigma_k^2 + b\theta(w+\beta\bar{w}) \left[\pi + \frac{\sigma_k^2}{2} ((1-\gamma)(\theta-2)-1) \right]. \end{aligned} \tag{B.10}$$

Matching the coefficients of w yields

$$0 = \frac{[b(R+1)]^{1-1/\gamma}}{1-\gamma} - \frac{\rho b}{1-\gamma} + b(A - [b(R+1)]^{1-1/\gamma}) - \frac{1}{2} \gamma \sigma_k^2 b + b\theta \left[\pi + \frac{\sigma_k^2}{2} ((1-\gamma)(\theta-2)-1) \right]. \tag{B.11}$$

Using this equation and the definition of η in (22), we obtain (B.3). Matching the coefficients of \bar{w} yields

$$0 = \frac{[b(R+1)]^{1-1/\gamma} \beta}{1-\gamma} - \frac{\rho b \beta}{1-\gamma} - (w+\beta\bar{w}) \beta [b(R+1)]^{1-1/\gamma} + \frac{b\mu}{R+1-\mu} - \frac{\gamma}{2} b \beta \sigma_k^2 + b\theta \beta \left[\pi + \frac{\sigma_k^2}{2} ((1-\gamma)(\theta-2)-1) \right]. \tag{B.12}$$

Using Eqs. (B.11) and (B.12), we obtain Eq. (B.4). By (B.6), (B.9), and (B.3), we obtain (B.5). Finally, we require $\eta > 0$ to have positive consumption. \square

This proposition verifies that the conjectured value function in (20) is correct. In addition, this proposition characterizes a money illusioned agent's optimal consumption and money holdings policies. The equality of consumption and real money balances in (B.5) reflects the fact that the CIA constraint binds. As in Proposition 1, the agent consumes a constant fraction of his total wealth and holds an identical fraction of his total wealth as cash. This constant fraction depends on the interest rate R and the expected inflation rate π via η . The latter dependence reflects money illusion. The effect of inflation

on optimal consumption and money holdings and its intuition are identical to those in the MIU model analyzed in the previous section. So we omit the discussion here.

We now solve for the equilibrium in which the price level, the aggregate wealth level, and the transfers satisfy (10), (18), and (19) with the restrictions given in Proposition 3. In addition, we must solve for the expected inflation rate π .

Proposition 6. *Suppose condition (22) holds. Then in equilibrium the expected inflation rate π satisfies (30). In addition, the equilibrium consumption level and capital stock satisfy $c = \eta k$,*

$$c = M/P = (1 - \phi^*)w, \quad \text{and} \quad k = \phi^*w,$$

where ϕ^* is given by (B.2), and the expected growth rate g of capital is given by (32).

Proof. In equilibrium, $\psi = 0$ and $w = \bar{w}$. We then use Eqs. (B.9), (B.3), and (B.4) to derive Eq. (B.2). We next verify the dynamics of transfers. Eq. (A.1) implies that Eq. (17) holds in equilibrium. Applying Ito's Lemma to this equation and matching the diffusion terms yield $\sigma_P = \sigma_k$. In addition, the equilibrium lump-sum transfer satisfies

$$v = \mu M/P = \mu \bar{w}(1 - \phi^*). \quad (\text{B.13})$$

Since $w = \bar{w}$ in equilibrium, \bar{w} and w must follow the same diffusion process. Substituting (B.6), (A.10), $\phi = \phi^*$, and $\psi = 0$ into the drift of the agent's wealth dynamics (16), we obtain

$$f(\bar{w}) = \bar{w}[A\phi^* + (1 - \phi^*)(\sigma_k^2 - \pi - 1)] + v = \bar{w}[A - (1 - \phi^*)(R + 1)] + \mu \bar{w}(1 - \phi^*) = \bar{w}[A - (1 - \phi^*)(R + 1 - \mu)] = \bar{w}(A - \eta),$$

where the last equality follows from the definition of ϕ^* in (B.2). Matching the diffusion terms of the wealth dynamics yields $h(\bar{w}) = \sigma_k \bar{w}$. Thus, we have verified (23).

We now determine the equilibrium inflation rate. Applying Ito's Lemma to the expressions on the two sides of the equation,

$$\frac{M}{P} = (1 - \phi^*)\bar{w},$$

and matching the drift terms, we obtain

$$\mu - \pi + \sigma_k^2 = f(\bar{w})/\bar{w} = A - \eta.$$

By the definition of η in (22), we can solve for the expected inflation rate given by (30). By (27), we obtain (A.11).

We next solve for the real allocations. In equilibrium, Eq. (B.6) implies that

$$c = (1 - \phi^*)w = \frac{\eta w}{R + 1 - \mu} = \frac{\eta w}{\eta + 1},$$

where we have substituted equation (B.2) and (A.11). Similarly, we can derive

$$k = \phi^*w = \frac{w}{\eta + 1}.$$

Thus, we obtain $c/k = \eta$. Using the resource constraint (9), we obtain the expected growth rate of capital in Eq. (32). Since output, consumption, capital and wealth are proportional to each other, they all have the same expected growth rate. \square

This proposition demonstrates that both the CIA and MIU models give an identical equilibrium consumption-capital ratio η and an identical growth rate g of capital. In particular, for a fully rational agent without money illusion, monetary policy has no real effects. It only affects the price level and inflation. With money illusion, monetary policy has real effects and generates nonzero correlation between inflation and economic growth if the degree of risk aversion is not equal to unity as in Section 3.

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