Fiscal and Monetary Policy Interactions in a Model with Low Interest Rates

By Jianjun Miao and Dongling Su*

We provide a new Keynesian model where entrepreneurs face uninsurable idiosyncratic investment risk and credit constraints. Government bonds provide liquidity services. Multiple steady states with positive values of public debt can be supported for a given permanent deficit-to-output ratio. The steady-state interest rates are lower than the economic growth rate and public debt contains a bubble component. We analyze the determinacy regions of policy parameter space and find that a large set of monetary and fiscal policy parameters can achieve debt and inflation stability given persistent fiscal deficits both away from and at the zero interest rate lower bound.
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Our paper is motivated by two empirical observations as shown in Figure 1.[1] First, since 1980, nominal interest rates on US government bonds have steadily declined. They are lower than the US nominal GDP growth rates on average over 1950-2018 and also in each of the recent 10 years. According to current forecasts of GDP growth, this is expected to remain the case for the foreseeable future. Second, the US government has experienced fiscal deficits for many years, especially since early 2000s. The average primary-deficits-to-GDP ratio over 1950-2019 is 0.48%. Moreover, public debt has risen since mid 1970. While it dropped in late 1990s, it started to rise again since 2000, reaching a peak of 70% of GDP in 2019. Similar patterns of declining safe rates and rising public debt for many other countries are documented by Rachel and Smith (2015) and Reinhart and Rogoff (2010).

Low interest rates and high public debt pose serious challenges to policy makers and academic researchers. In this case the fiscal theory of the price level (FTPL) fails to

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[1]In Panel A, the data of nominal GDP growth rates and tax adjusted safe rates over 1950-2018 are taken from Blanchard (2019*a*). The data of nominal returns on the entire portfolio of US government bonds over 1950-2017 is taken from Hall et al. (2018*a*). Because the data are quite volatile, we follow Rachel and Smith (2015) and plot the moving averages over the past 5 years in Panel A. The data of the market value of publicly held federal debt in Panel B over 1950-2019 is taken from Hall et al. (2018*a*). Since we consider the budget constraint of the consolidated government, we exclude the public debt held by government institutions such as the central bank. The surplus-to-GDP data for Panel C is imputed from budget identity and taken from Cochrane (2019*a*).

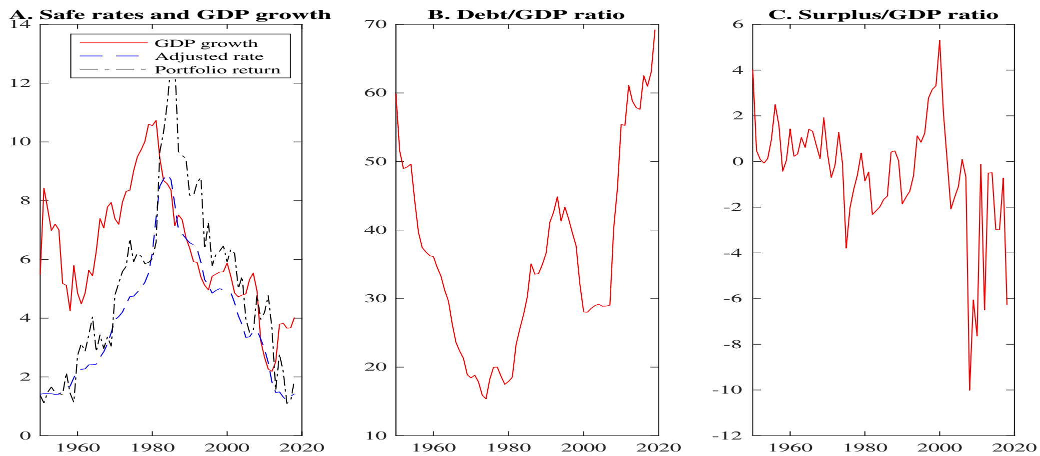

Figure 1. : Nominal Safe rates and GDP growth, public-debt/GDP ratios, and primary-surpluses/GDP ratios. All vertical axes are in percentage.

work according to the traditional revaluation channel because public debt priced as the present value of future surpluses may not be well defined. In this paper we address the following positive questions: What are the implications of low interest rates for public debt policy? Can permanent primary deficits be sustained in the long run? What coordination of monetary and fiscal policy is needed to provide a nominal anchor and price stability? How does the economy respond to fiscal and monetary policy shocks?

To address these questions, we build a dynamic new Keynesian (DNK) model with financial frictions. Our critical assumption is that entrepreneurs face credit constraints and uninsurable idiosyncratic investment shocks (Kiyotaki and Moore (1997, 2019)). They can only trade one-period riskfree private and government bonds. The two types of bonds are perfect substitutes except that they are issued by different suppliers. Public bonds have a crowding-in effect in addition to the usual crowding-out effect. In particular, public bonds provide liquidity services because they can raise owners' net worth and relax credit constraints. When the investment shock is sufficiently high, productive entrepreneurs sell bonds to finance real investment. Unproductive entrepreneurs are willing to buy bonds despite their low returns for precautionary reasons, because unproductive entrepreneurs anticipate that they may become productive in the future and need to finance real investment using bonds. The low interest rate on the bonds can support a positive value of government bonds even though these bonds are unbacked by taxes or even when they are rolled over to finance principal and interest payments as well as primary deficits.

We characterize the (nonstochastic) steady states of our detrended equilibrium system. We show that if the long-run surplus is positive, then there is a unique steady state in which the real interest rate is higher than the economic growth rate and the real value of public debt is equal to the present value of future surpluses discounted by the real interest rate. Low interest rates are possible in the steady state only when the government runs permanent primary deficits or when there is zero deficit/surplus. There are multiple steady states for a given long-run primary-deficit-to-output ratio, if it is not too high. In this case all steady-state interest rates are lower than the economic growth rate and all steady-state real values of public debt are positive. If the long-run surplus/deficit is zero, then there are exactly two steady states. In one steady state, public debt has no value; and in the other, public debt has a positive value, which is a pure bubble.

The multiplicity of steady states is generated by the debt Laffer curve that gives a non-monotonic relation between the total interest expense and the real interest rate. Such non-monotonicity is due to the positive relation between the interest rate and public debt. In our model with financial frictions, an increase in public debt reduces the liquidity premium and raises the real interest rate.

Under a reasonable calibration when the long-run deficit-to-GDP ratio is set at 0.445%, our model delivers exactly two steady states, one of which is associated with a lower interest rate, a lower output level, and a lower debt-to-GDP ratio (labeled steady state L), and the other is labeled steady state H. The debt-to-GDP ratio in steady state L is targeted at the average 35.9% of the US data over 1950-2019. We find that, for the real version of our calibrated model, there is a unique bounded equilibrium around steady state L and there is no bounded equilibrium around steady state H.[2] By contrast, for our monetary model, the results are very different because the price level (inflation) together with real variables must be determined in equilibrium.

We follow Leeper's (1991) approach by specifying feedback rules for monetary and fiscal policy and study what policy rules can produce a unique locally stable solution for both inflation and public debt. According to Leeper (1991), monetary policy is active if the interest rate rule satisfies the Taylor principle; otherwise, it is passive. Fiscal policy is passive if the government can raise enough taxes (primary surplus) to stabilize debt dynamics when public debt rises; otherwise it is active. The critical value for the fiscal policy response parameter is the steady-state interest expense.[3] Leeper (1991) argues that an active policy must be combined with a passive policy to achieve equilibrium determinacy. An active monetary policy and passive fiscal policy mix corresponds to the conventional case (regime M). An active fiscal policy and passive monetary policy mix (regime F) is associated with the FTPL.

Relative to many studies in the literature, we find the following novel results in addition to the above steady-state analysis:

1) There are three regions of the policy parameter space for each of the two steady states. These regions categorize local equilibrium determinacy around each steady state. The first region generates explosive solutions, the second region generates multiple bounded equilibria, and the third region generates a unique stable equilibrium. These regions are different for different steady states. Moreover, both active and passive monetary policies can achieve equilibrium determinacy, even if fiscal policy is passive.

[2] As in the literature including our monetary model, we treat the real public debt as predetermined. The equilibrium determinacy is different if the real debt is treated as a non-predetermined variable (see Kaas (2016)).

[3] Woodford (2003, p.312) calls a passive fiscal policy rule locally Ricardian and an active rule otherwise.

As the real interest rate is positively related to public debt due to liquidity premium, a passive fiscal policy cannot stabilize debt alone unless a passive monetary policy allows surprise inflation to revalue public debt.

2) The government can select a particular steady state by specifying a fiscal target (debt and tax level) as that steady-state value. Then the deterministic detrended equilibrium system will converge to the intended steady state along a saddle path. Thus a complete specification of fiscal policy must include both fiscal target and policy response coefficient.[4]

3) An active monetary policy can be combined with a debt rollover fiscal policy to stabilize debt and inflation in a world with low interest rates. This corresponds to regime M in which monetary policy controls inflation and fiscal policy stabilizes public debt. In this regime, tax cuts or an increase in government transfer or spending financed by debt can pay for itself when the interest rate is lower than the economic growth rate. But the stimulative effect is small because of the crowding-out effect of persistent debt.

4) The standard FTPL views public debt as the present value of future surpluses, which may explode when the interest rate is lower than the economic growth rate. We decompose the real value of public debt into a fundamental component and a bubble component in Lemma 1. The fundamental component is equal to the present value of future surpluses/deficits and the bubble component is equal to the present value of liquidity services provided by the government bonds. Both components are discounted by the household stochastic discount factor (SDF), i.e., intertemporal marginal rate of substitution, and are both finite because the implied discount rate is asymptotically higher than the economic growth rate.

5) According to the standard FTPL, the transmission of shocks relies on the revaluation of public debt as the present value. Our decomposition of debt value complements the standard FTPL under low interest rates. In this case a shock to either nominal interest rate or primary surplus affects the value of public debt through both its fundamental and bubble components. In regime F the fiscal policies discussed earlier have a much larger stimulative effect and can generate persistent high inflation.

6) In a liquidity trap with an occasionally binding zero lower bound (ZLB) on nominal interest rates generated by a credit crunch, regime F dominates regime M in terms of welfare. The debt rollover fiscal policy combined with either an active or passive monetary policy is not optimal. By contrast, a more "irresponsible" fiscal policy, that cuts net taxes as the debt level rises, combined with a pegged nominal interest rate policy is optimal within a set of policy rules discussed earlier because this policy can generate future higher inflation and stimulate aggregate demand.

[4]Benhabib, Schmitt-Grohe and Uribe (2002) show that multiple steady states can exist for the interest-rate rule under the zero lower bound on nominal interest rates. They discuss several fiscal and monetary policies that can select a unique equilibrium around the intended steady state.

Our results have some implications for the US and Japan experiences. First, Clarida, Gali and Gertler (2000) and Lubik and Schorfheide (2004) document evidence that the Fed interest rate policy was passive prior to 1980 and then it became active during the Volcker–Greenspan era. According to Leeper (1991), to ensure price determinacy, fiscal policy must be active prior to 1980 and must shift to be passive after 1980. Our model shows that both active and passive monetary policies can achieve price determinacy even if fiscal policy remains passive. Second, Japan has mostly run primary deficits since 1960s and with no primary surpluses in sight, but inflation has not risen much. This seems inconsistent with the FTPL as the present value of future deficits discounted by the low real interest rate may explode to negative infinity. Our model shows that public debt contains a bubble component,[5] which must be included in its valuation when the interest rate is lower than the economic growth rate. Once taking into account this component, a large set of monetary and fiscal policy response parameters in either regime M or regime F can achieve stable debt and inflation dynamics given persistent fiscal deficits.

Related literature. Our paper is related to three strands of the literature. First, our paper is closely related to the recent literature on the implications of low interest rates for monetary and fiscal policies. Bullard and Russell (1999), Chalk (2000), Blanchard (2019*a*), and Brumm et al. (2021) study public debt policy based on the overlapping generations (OLG) model of Diamond (1965). In a dynamically inefficient economy, the government can rollover public debt at a low interest rate or run Ponzi schemes to support permanent deficits. Kaas (2016) studies similar questions in a model with infinitely-lived agents, in which entrepreneurs are subject to uninsurable idiosyncratic productivity risk and credit constraints. Reis (2021) studies similar questions in a continuous-time setup. Brunnermeier, Merkel and Sannikov (2020*a*) also study similar questions in a continuous-time model with uninsurable idiosyncratic capital return risk, but without credit constraints. Like us, Brunnermeier, Merkel and Sannikov (2020*a*) and Sims (2020) emphasize the importance of the positive relation between public debt and the real interest rate to generate a debt Laffer curve. Unlike our paper, all these papers do not study the interactions of monetary and fiscal policies.

Reis (2021) takes inflation as given and analyzes how inflation affects the fiscal space and fiscal capacity. In a representative agent model with distortionary taxes, Sims (2019) shows that when the low interest rate on debt arises from its providing liquidity services, zero fiscal cost is equivalent to finance through seigniorage, which is generally optimal. The interest rate in his model is always higher than the economic growth rate.

Bassetto and Cui (2018) revisit the implications of the FTPL with low interest rates which are generated by sources such as dynamic inefficiency, liquidity premium of public debt, or its favorable risk profile. Like us, they show that the interest-rate-peg policy discussed in Woodford (1995, 2001) may not pin down a unique equilibrium price level. Brunnermeier, Merkel and Sannikov (2020*b*) also revisit the FTPL under the interest-rate-peg policy and show that a particular fiscal policy can pin down a unique equilibrium price

[5]See Jiang et al. (2019*a*) for evidence.

level. Like us, they emphasize that public debt contains a bubble component when interest rates are low. Both papers derive multiple steady states. Unlike these two papers, our paper considers general feedback rules for monetary and fiscal policies following Leeper (1991) (with the interest-rate-peg policy as a special case), analyzes determinacy regions of policy parameter space, and studies dynamic responses of the economy to monetary and fiscal policy shocks.

Similar to these papers and the early paper by Woodford (1990), we show that there are multiple steady states with low interest rates generated by the liquidity premium under incomplete markets and credit constraints and that public debt contains a bubble component under persistent primary deficits. The liquidity premium of public debt can also be generated in models with monetary search frictions (e.g., Berentsen and Waller (2018), Bassetto and Cui (2018), and Dominguez and Gomis-Porqueras (2019)). In these models monetary policy is nonneutral in the long run. By contrast, we adopt the DNK framework that is more amenable to quantitative analysis and Bayesian estimation (e.g., Lubik and Schorfheide (2004)).

Second, our paper is related to the literature on the fiscal and monetary policy interactions surveyed by Canzoneri, Cumby and Diba (2010) and Leeper and Leith (2016), and particularly related to the FTPL developed primarily by Leeper (1991), Woodford (1994, 1995), Sims (1994), and Cochrane (1998).[6] This literature is too large for us to cite all relevant papers. We mention three closely related papers by Cui (2016), Canzoneri et al. (2011), and Billi and Walsh (2021). Cui (2016) studies a DNK model based on Kiyotaki and Moore (2019) with endogenous fluctuations in liquidity. Also using a DNK model, Canzoneri et al. (2011) argue that government bonds provide liquidity services and are imperfect substitutes for money. Both papers feature a unique steady state and derive three regions of policy parameter space similar to ours. Using a standard DNK model, Billi and Walsh (2021) show that an "irresponsible" fiscal policy combined with a passive monetary policy can stabilize inflation and debt and improve welfare in a liquidity trap.[7] All these three papers do not study the case when interest rates are lower than the economic growth rate.

Third, our paper is related to the literature on asset bubbles surveyed by Miao (2014) and Martin and Ventura (2018). Asset bubbles can emerge in either dynamically inefficient OLG models (Tirole (1985)) or in models with infinitely-lived agents facing financial frictions or under incomplete markets. Our model is based on Miao and Wang (2012), Miao, Wang and Zhou (2015), Hirano and Yanagawa (2017), Miao and Wang (2018), Kiyotaki and Moore (2019), Dong, Miao and Wang (2020), and Biswas, Hanson and Phan (2020), in which credit constraints are important for the emergence of a bubble. Unlike these papers, we focus on the interactions of monetary and fiscal policies under low interest rates, which pose new issues and generate new insights absent from these papers.

[6]Bassetto (2002) revisits the FTPL in a game theoretic framework.

[7]There is a large literature on monetary policy given the ZLB constraints. Early important papers include Eggertsson and Woodford (2003, 2006) and Eggertsson (2006). See Billi and Walsh (2021) for additional references.

I. Model

In this section we present a cashless DNK model with financial frictions and consider an infinite-horizon economy consisting of households, firms, retailers, and a government (fiscal/monetary authority).

A. Households

There is a continuum of identical households of measure unity. The representative household is an extended family consisting of workers, entrepreneurs, and retailers. Each entrepreneur runs a firm and workers supply labor to firms. The family and firms can trade one-period riskfree nominal private and government bonds. The two types of bonds are perfect substitutes except that the private bonds are in zero net supply and the government bonds are issued by the government only. Entrepreneurs and retailers hand in their dividends to households who are shareholders.

Each household chooses consumption $\{C_t\}$, labor supply $\{N_t\}$, and real bond holdings $\{D_{ht}\}$ to maximize utility

$$\max_{\{C_t, D_{ht}, N_t\}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \psi N_t\right)\right], \tag{1}$$

subject to

$$C_t + D_{ht} = W_t N_t + \Upsilon_t + \frac{R_{t-1}}{\Pi_t} D_{ht-1} - T_t, \tag{2}$$

where $\beta \in (0,1)$ is the subjective discount factor, W_t is the real wage, R_{t-1} is the nominal interest rate between periods $t-1$ and t, $\Pi_t = P_t/P_{t-1}$ is the gross inflation rate, Υ_t denotes total real dividends from entrepreneurs and retailers, and T_t denotes net real lump-sum taxes. Here P_t denotes the aggregate price level in period t. Assume that households cannot borrow so that $D_{ht} \geq 0$ for all t. To flesh out our insights in a simplest possible way, we follow much of the literature (e.g., Leeper (1991), Canzoneri, Cumby and Diba (2010), and Leeper and Leith (2016)) and do not consider distortionary taxes.

The first-order conditions imply that

$$W_t = \frac{\psi}{\Lambda_t}, \tag{3}$$

$$1 \geq \mathbb{E}_t \beta \frac{\Lambda_{t+1}}{\Lambda_t} \frac{R_t}{\Pi_{t+1}}, \text{ with equality when } D_{ht} > 0, \tag{4}$$

where $\Lambda_t = 1/C_t$ denotes the household marginal utility. We will show later that the household will not hold any bonds (i.e., $D_{ht} = 0$) in an equilibrium around the neighborhood of a steady state, because the real return on the bonds is too low.

B. Entrepreneurs

Each entrepreneur $j \in [0,1]$ runs a firm that combines labor N_{jt} and capital K_{jt-1} to produce an intermediate (wholesale) good j in period t according to the production function

$$Y_{jt} = K_{jt-1}^{\alpha} \left(A_t N_{jt}\right)^{1-\alpha}, \ \alpha \in (0,1),$$

where A_t denotes the labor-augmenting technology that grows at the rate g. For simplicity we assume that A_t is deterministic with $A_0 = 1$.

The entrepreneur sells wholesale goods to retailers at the real price p_{wt}. The static profit maximization problem yields

$$R_{kt} K_{jt-1} = \max_{N_{jt}} \ p_{wt} K_{jt-1}^{\alpha} \left(A_t N_{jt}\right)^{1-\alpha} - W_t N_{jt},$$

where we can show that

$$R_{kt} = \alpha \left(\frac{(1-\alpha) A_t}{W_t}\right)^{\frac{1-\alpha}{\alpha}} p_{wt}^{\frac{1}{\alpha}}, \tag{5}$$

and the first-order condition gives labor demand

$$W_t = (1-\alpha) p_{wt} A_t K_{jt-1}^{\alpha} \left(A_t N_{jt}\right)^{-\alpha}. \tag{6}$$

Here we may interpret R_{kt} as the capital return, which is also equal to the social marginal product of capital as will be shown later.

At the beginning of period t, the entrepreneur faces idiosyncratic investment-specific shock ε_{jt} and makes investment I_{jt} to increase his capital stock so that the law of motion for capital follows

$$K_{jt} = (1-\delta) K_{jt-1} + \varepsilon_{jt} I_{jt}, \tag{7}$$

where $\delta \in (0,1)$ represents the depreciation rate. Suppose that the cumulative distribution function of ε_{jt} is F and the density function is f on $[\varepsilon_{\min}, \varepsilon_{\max}] \subset (0,\infty)$ and ε_{jt} is independently and identically distributed across firms and over time. Assume that there is no insurance market against the idiosyncratic investment-specific shock and that investment is irreversible at the firm level so that $I_{jt} \geq 0$.

Entrepreneur j can hold B_{jt} units of private bonds and $D_{jt} \geq 0$ units of government bonds in terms of the consumption good. His flow-of-funds constraints are given by

$$C_{jt} + I_{jt} + B_{jt} + D_{jt} = R_{kt} K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}, \tag{8}$$

where C_{jt} denotes real dividends. Entrepreneur j can use his capital as collateral to borrow and faces the following borrowing constraint due to imperfect contract enforcement:[8]

$$B_{jt} \geq -\mu K_{jt-1}, \;\; \mu \in [0, 1). \tag{9}$$

The parameter μ reflects the degree of financial frictions and will play an important role in our model. Suppose that equity finance is so costly that the firm does not issue any new equity.[9] Thus we impose the constraint

$$C_{jt} \geq 0. \tag{10}$$

The entrepreneur's objective is to maximize the discounted present value of dividends. We can write his decision problem using dynamic programming

$$V_t \left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) = \max_{\{I_{jt}, D_{jt}, B_{jt}\}} C_{jt} + \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1} \left(K_{jt}, B_{jt}, D_{jt}, \varepsilon_{jt+1}\right), \tag{11}$$

subject to (7), (8), (9), and (10), where we have used the household's intertemporal marginal rate of substitution as the SDF because all firms are owned by households. Here $V_t(\cdot)$ denotes the value function.

Define Tobin's (marginal) Q as

$$q_t^k = \frac{\partial}{\partial K_{jt}} \mathbb{E}_t \beta \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1} \left(K_{jt}, B_{jt}, D_{jt}, \varepsilon_{jt+1}\right).$$

The following proposition characterizes entrepreneur j's optimal decisions. Its proof, together with all proofs of other theoretical results, is given in Online Appendix A.

PROPOSITION 1: *Suppose that $\varepsilon_t^* \equiv 1/q_t^k \in (\varepsilon_{\min}, \varepsilon_{\max})$ in an equilibrium. Then, for $\varepsilon_{jt} \geq \varepsilon_t^*$, we have $B_{jt} = -\mu K_{jt-1}$, $D_{jt} = 0$,*

$$I_{jt} = \left(R_{kt} + \mu\right) K_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}; \tag{12}$$

and for $\varepsilon_{jt} < \varepsilon_t^*$, we have $I_{jt} = 0$, but B_{jt} and D_{jt} are indeterminate. Moreover, q_t^k and

[8] Unlike Kiyotaki and Moore (1997), we do not use future capital as collateral. Using future capital as collateral will complicate algebra significantly without changing our key insights. See Caballero and Krishnamurthy (2006), Miao and Wang (2018), and Miao, Wang and Zhou (2015) for related discussions.

[9] Our key insights will not change as long as new equity issues are sufficiently limited (see Miao and Wang (2018) and Miao, Wang and Xu (2015)).

R_t satisfy

$$q_t^k = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} R_{kt+1} \left(1 + q_{t+1}^l\right) + \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1}^k (1-\delta) + \beta\mu \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1}^l, \tag{13}$$

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{R_t}{\Pi_{t+1}} \left(1 + q_{t+1}^l\right), \tag{14}$$

where

$$q_t^l \equiv \int_{\varepsilon_t^*}^{\varepsilon_{\max}} \left(q_t^k \varepsilon - 1\right) dF(\varepsilon). \tag{15}$$

The transversality condition holds

$$\lim_{i\to\infty} \mathbb{E}_t \frac{\beta^i \Lambda_{t+i}}{\Lambda_t} \left(q_{t+i}^k K_{jt+i} + B_{jt+i} + D_{jt+i}\right) = 0. \tag{16}$$

This proposition shows that there is an investment cutoff ε_t^* such that entrepreneur j makes real investment if $\varepsilon_{jt} \geq \varepsilon_t^*$. The cutoff ε_t^* is equal to the inverse of Tobin's Q when the investment profit is exactly zero. The entrepreneur uses his internal funds $R_{kt}K_{jt-1}$, private debt μK_{jt-1}, and the principal and interest value of government bonds $R_{t-1}D_{jt-1}/\Pi_t$ to finance investment expenditures as shown in (12). If $\varepsilon_{jt} < \varepsilon_t^*$, he does not make real investment and buys bonds from other productive entrepreneurs. Because entrepreneurs are effectively risk neutral as shown in (11), they are indifferent between specific levels of bond holdings. Only aggregate level is determined in equilibrium by market clearing. Thus the interest rates on private and public bonds are the same and satisfy the asset pricing equation (14). Unlike the standard equation without financial frictions, there is a liquidity premium term q_{t+1}^l in (14).

Intuitively, both private and government bonds raise entrepreneurs' net worth and help them relax credit constraints. Purchasing one dollar of bonds today not only generates a benefit of the principal plus interest tomorrow, but also allows a productive, credit-constrained entrepreneur with $\varepsilon_{t+1}^j > \varepsilon_{t+1}^*$ to finance investment so that he makes $q_{t+1}^k \varepsilon_{t+1}^j - 1$ dollars of profits tomorrow. The integral term in (15) gives the expected profits generated by holding bonds.

Define the real interest rate as

$$R_t^r = \left\{\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \left(1 + q_{t+1}^l\right)\right\}^{-1}. \tag{17}$$

Then the real interest rate is negatively related to the liquidity premium and is not equal to the inverse of the household SDF $\beta\Lambda_{t+1}/\Lambda_t$ in the deterministic case. Due to the liquidity premium, the real interest rate can be lower than the economic growth rate, but the inverse of the household SDF is greater than the economic growth rate. This property is important for the valuation of public debt studied in Section I.F.

Equation (13) is an asset-pricing equation for Tobin's Q. Unlike the standard equation without financial frictions, the liquidity premium term also appears in (13) because capital return raises an entrepreneur's net worth and also because capital is used as collateral in our model.

The transversality condition (16) says that each entrepreneur does not leave any positive value of assets in the long run. It is necessary for optimality. We will show later that this condition cannot rule out a bubble in public debt.

C. Retailers

Retailers are monopolistically competitive and their role is to introduce nominal price rigidities. In each period t they buy intermediate goods from entrepreneurs at the real price p_{wt} and sell intermediate good j at the nominal price P_{jt}. Intermediate goods are transformed into final goods according to a CES aggregator.

Assuming staggered pricing of retailers as in Calvo (1983) and full indexation, we can show that the inflation rate Π_t satisfies

$$1 = \left[\xi \left(\frac{\Pi}{\Pi_t} \right)^{1-\sigma} + (1-\xi)\, p_t^{*1-\sigma} \right]^{\frac{1}{1-\sigma}}, \tag{18}$$

where $1-\xi$ represents the probability of price adjustments, p_t^* is the re-optimized relative price, σ is the elasticity of substitution among intermediate goods, and Π is the trend inflation rate. We can also show that p_t^* satisfies

$$p_t^* = \frac{\sigma}{\sigma-1} \frac{\Gamma_t^a}{\Gamma_t^b}, \tag{19}$$

where

$$\Gamma_t^a = \Lambda_t p_{wt} Y_t + \beta\xi\mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma} \Gamma_{t+1}^a, \tag{20}$$

$$\Gamma_t^b = \Lambda_t Y_t + \beta\xi\mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma-1} \Gamma_{t+1}^b. \tag{21}$$

Equations (18), (19), (20), and (21) constitute the standard new Keynesian block and can be used to derive the linearized Phillips Curve. Further details on the microfoundation can be found in Online Appendix F.

D. Monetary and Fiscal Policies

In each period t, the government issues one-period riskfree nominal debt $(P_t D_t)$, where D_t denotes real debt. The government budget constraint is given by

$$G_t + \frac{R_{t-1} D_{t-1}}{\Pi_t} = T_t + D_t, \quad t \geq 0, \tag{22}$$

where G_t denotes real government spending and $R_{-1} P_{-1} D_{-1} > 0$ is given. Let $S_t \equiv T_t - G_t$ denote the real primary surplus. Then we rewrite (22) as

$$\frac{R_{t-1} D_{t-1}}{\Pi_t} = S_t + D_t. \tag{23}$$

Following Leeper (1991), suppose that the government may adjust real lump-sum taxes in response to the real value of public debt. Because our model features long-run growth, we consider detrended policy rule as in Cui (2016):

$$\tau_t / y = \tau / y + \phi_d (d_{t-1} - d)/y + z_{\tau,t}, \tag{24}$$

where $\tau_t = T_t / A_t$, $d_t = D_t / A_t$, and $y_t = Y_t / A_t$, and the variables τ, d, and y are the corresponding steady-state values. The parameter ϕ_d describes the strength of fiscal adjustment. The variable $z_{\tau,t}$ follows an AR(1) process

$$z_{\tau,t} = \rho_\tau z_{\tau,t-1} + \epsilon_{\tau,t},$$

where $|\rho_\tau| < 1$ and $\epsilon_{\tau,t}$ is a normal white noise with mean zero and variance σ_τ^2.

Assume that the detrended government spending $G_{at} = G_t / A_t$ follows the following AR(1) process

$$\ln G_{at} - \ln G_a = \rho_G \left(\ln G_{a,t-1} - \ln G_a \right) + \epsilon_{Gt}, \tag{25}$$

where G_a is the steady-state value of G_{at}, $|\rho_G| < 1$, and ϵ_{Gt} is a normal white noise with mean zero and variance σ_G^2.

The monetary authority sets the nominal interest rate as a function of the current inflation rate:

$$R_t = R \left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \exp(z_{mt}), \tag{26}$$

where R and Π denote the nominal interest rate and the inflation rate targets (steady-state values). The variable z_{mt} follows an AR(1) process

$$z_{mt} = \rho_m z_{mt-1} + \epsilon_{mt},$$

where $|\rho_m| < 1$ and ϵ_{mt} is a normal white noise with mean zero and variance σ_m^2. The parameter ϕ_π describes the strength of the interest rate adjustment in response to inflation. Assume that all shocks in the model are independent of each other.

E. *Equilibrium*

Equations (4) and (14) suggest that the interest rate is too low for the household to hold any government bonds so that $D_{ht} = 0$ in equilibrium. Thus the market-clearing conditions for private and government bonds are given by

$$\int B_{jt} dj = 0, \ \int D_{jt} dj = D_t.$$

Define aggregate investment, aggregate labor, and aggregate capital as $I_t = \int I_{jt} dj$, $N_t = \int N_{jt} dj$, and $K_t = \int K_{jt} dj$. The labor demand condition (6) implies that all firms have the same capital-labor ratio and thus we have

$$W_t = (1-\alpha) p_{wt} A_t K_{t-1}^{\alpha} \left(A_t N_t \right)^{-\alpha}. \tag{27}$$

Using (27) to eliminate W_t in (5), we can show that the capital return is equal to the social marginal product of capital

$$R_{kt} = \alpha p_{wt} K_{t-1}^{\alpha-1} \left(A_t N_t \right)^{1-\alpha}. \tag{28}$$

By Proposition 1 and the market-clearing conditions described above, we obtain aggregate investment as follows

$$I_t = \left((\mu + R_{kt}) K_{t-1} + \frac{R_{t-1}}{\Pi_t} D_{t-1} \right) (1 - F(\varepsilon_t^*)). \tag{29}$$

Intuitively, aggregate investment is financed by private debt μK_{t-1} and the net worth of high productivity entrepreneurs with $\varepsilon_{jt} \geq \varepsilon_t^*$. The latter consists of capital return $R_{kt} K_{t-1}$ and the real value of government bonds $R_{t-1} D_{t-1} / \Pi_t$. Equation (29) shows that public debt has a crowding-in effect because it raises entrepreneurs' net worth. Public debt also has a crowding-out effect because unproductive firms must hold public debt and do not make capital investment. In particular, ε_t^* increases with public debt so that the measure of investing firms $(1 - F(\varepsilon_t^*))$ decreases with public debt.

We can also derive the aggregate capital stock from (7) as

$$K_t = (1-\delta)K_{t-1} + I_t \frac{\int_{\epsilon_t^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{1-F(\varepsilon_t^*)}, \tag{30}$$

where the last fraction term represents the average efficiency of investment. Aggregate output is given by

$$Y_t = \frac{1}{\Delta_t} K_{t-1}^{\alpha} \left(A_t N_t\right)^{1-\alpha}, \tag{31}$$

where the price dispersion $\Delta_t = \int (P_{jt}/P_t)^{-\sigma} dj$ satisfies the following recursive condition

$$\Delta_t = (1-\xi) p_t^{*-\sigma} + \xi \left(\frac{\Pi}{\Pi_t}\right)^{-\sigma} \Delta_{t-1}, \tag{32}$$

with Δ_{-1} being exogenously given. The aggregate resource constraint is given by

$$C_t + I_t + G_t = Y_t. \tag{33}$$

Given initial conditions for predetermined variables $\{K_{t-1}, R_{t-1}, D_{t-1}, \Delta_{t-1}\}$ and a monetary and fiscal monetary policy mix with exogenous policy shocks $\{z_{mt}, z_{\tau,t}, \epsilon_{Gt}\}$, a competitive equilibrium consists of 20 variables $\{N_t$, q_t^k, R_t, q_t^l, p_t^*, Γ_t^a, Γ_t^b, Π_t, D_t, T_t, p_{wt}, W_t, R_{kt}, I_t, K_t, Y_t, Δ_t, $C_t, \Lambda_t, \varepsilon_t^*\}$ satisfying a system of 20 equations (3), (13), (14), (15), (18), (19), (20), (21), (23), (24), (26), (27), (28), (29), (30), (31), (32), (33), $\Lambda_t = 1/C_t$, and $\varepsilon_t^* = 1/q_t^k \in (\varepsilon_{\min}, \varepsilon_{\max})$.

F. Public Debt Valuation

In this subsection we briefly review the basic idea of the FTPL and discuss how to modify it when the interest rate is lower than the economic growth rate and/or when the government runs persistent primary deficits.

Following Cochrane (1998, 2020), we can use (14) to rewrite the government budget constraint (23) as

$$\frac{R_{t-1}D_{t-1}}{\Pi_t} = S_t + \frac{R_t D_t}{R_t} = S_t + \mathbb{E}_t \frac{\beta \Lambda_{t+1}}{\Lambda_t}\left(1 + q_{t+1}^l\right)\frac{R_t D_t}{\Pi_{t+1}}. \tag{34}$$

To best understand the standard FTPL when $S_t > 0$ and $R_t^r > 1+g$ for all t, we consider the deterministic case and solve (34) forward to derive the asset-pricing equation:

$$\frac{R_{t-1}D_{t-1}}{\Pi_t} = \sum_{i=0}^{\infty} \frac{S_{t+i}}{R_t^r R_{t+1}^r \cdots R_{t+i-1}^r} + \lim_{T\to\infty} \frac{R_{t+T+1}^r D_{t+T+1}}{R_t^r R_{t+1}^r \cdots R_{t+T}^r}, \tag{35}$$

where we have used the deterministic version of (17). Since the primary surplus S_{t+i} grows at the economic growth rate g in the long-run, the first term (present value) on the right-hand side of equation (35) is finite and the second term (bubble) vanishes.

However, if $R_t^r < 1+g$ or $S_t \leq 0$ for all t, then both the present value and the bubble term in (35) explode, making this equation hard to interpret. Using (34), we provide another decomposition of debt value:

LEMMA 1: *The real (maturity) value of public debt satisfies*

$$\frac{D_{t-1}R_{t-1}}{\Pi_t} = \mathbb{E}_t \sum_{i=0}^{\infty} \frac{\beta^i \Lambda_{t+i}}{\Lambda_t} S_{t+i} + \mathbb{E}_t \sum_{i=0}^{\infty} \frac{\beta^{i+1}\Lambda_{t+i+1}}{\Lambda_t} q_{t+i+1}^l \frac{D_{t+i}R_{t+i}}{\Pi_{t+i+1}}. \tag{36}$$

This lemma shows that public debt value can be decomposed into two components. The first component represents the fundamental value of public debt defined as the present value of current and future real primary surplus/deficit. The second component is the present value of liquidity service provided by public debt. This value depends on the agent's beliefs about future value of public debt because the flow payoff $q_{t+i+1}^l D_{t+i}R_{t+i}/\Pi_{t+i+1}$ depends on debt value itself. As a result, we may interpret it as a bubble component. This interpretation is especially useful when the government runs persistent deficits. Then the fundamental value is nonpositive. In particular, when surplus/deficit is always zero, $S_t = 0$ for all t, then the value of public debt is completely supported by the speculative beliefs about future value.

The decomposition in (36) is not unique depending on different choices of the SDF. Our choice of the household SDF is natural and admits intuitive interpretations. The implied long-run household discount rate is $(1+g)/\beta - 1$ given log utility, which is higher than the economic growth rate g. But the long-run real interest rate can be lower than the economic growth rate as shown in Section II. Thus using the household SDF can ensure both components in (36) have finite values even when interest rates are lower than the economic growth rate and/or when there are persistent deficits. Reis (2021) uses the marginal product of capital as the discount rate (which is higher than the economic growth rate) and derives an equation similar to (36) in continuous time.

We have already used the transversality condition (16) to derive (36) and it cannot rule out the bubble component as long as a liquidity premium exists. In standard frictionless model without liquidity premium $\left(q_t^l = 0 \text{ for all } t\right)$, the real value of public debt is equal to its fundamental value, which may explode, when the interest rate is lower than the economic growth rate and/or when the government runs persistent primary deficits (Bassetto and Cui (2018)). In the stochastic case, Bassetto and Cui (2018) and Cochrane (2020) present examples to show that low interest rates (less than the economic growth rate) and persistent fiscal deficits on average can generate a finite fundamental value of public debt when agents are sufficiently risk averse or when risk is sufficiently high.[10] Then the usual FTPL can ensure price determinacy.

While the conventional monetary regime treats equation (36) as a constraint (implying that fiscal policy needs to adjust when the present value of future surpluses differs from the real value of debt), the FTPL views it as an equilibrium condition. Specifically, given a positive predetermined nominal value of debt $(P_{t-1}D_{t-1}R_{t-1})$, the current price P_t will adjust to ensure equation (36) holds. A shock to the current primary surplus does not have to lead to changes in future primary surpluses; instead, the price level can adjust to restore equality (36).

We argue that the usual FTPL fails or is incomplete when public debt contains a bubble component. In this case the bubble component in (36) must be taken into account in the valuation of public debt when applying the FTPL (see Brunnermeier, Merkel and Sannikov (2020a,b) for a similar point). For example, when primary surplus $S_t = 0$ for all t, equation (36) becomes

$$\frac{R_{t-1}D_{t-1}}{\Pi_t} = D_t = \mathbb{E}_t \sum_{i=0}^{\infty} \frac{\beta^{i+1}\Lambda_{t+i+1}}{\Lambda_t} q^l_{t+i+1} \frac{D_{t+i}R_{t+i}}{\Pi_{t+i+1}}. \tag{37}$$

The fundamental value of debt is zero, but debt can be rolled over and has a finite positive value as shown in the next section.

II. Steady States

Our model features long-run balanced growth. To study steady states and the dynamics around the balanced growth path, we first detrend the equilibrium system by using the transformation of $x_t = X_t/A_t$ for any variable $X_t \in \{K_t,\, D_t,\, S_t,\, Y_t,\, W_t,\, C_t,\, I_t,\, G_t\}$. For marginal utility, the variable $\lambda_t = A_t\Lambda_t$ has no trend. The capital return R_{kt} has no trend. The detrended system is provided in Online Appendix B. Shutting down all aggregate shocks while keeping idiosyncratic shocks, we focus on the steady states in which the primary surplus to output ratio s/y is an exogenous constant over time. We use variables without time subscripts to denote their steady-state values and present the steady-state system in Online Appendix C.

A. Investment Cutoff

The critical step to solve for a steady state is to derive the investment cutoff ε^*. Once it is determined, other variables can be easily derived as shown in Online Appendix C. Since monetary policy is neutral in the steady state, we only need to solve for real variables. We

[10] In the example of Bassetto and Cui (2018), the government must have fewer deficits or levy more taxes in bad times to maintain low interest rates. This implication seems counterfactual.

first derive the real interest rate $R^r \equiv R/\Pi$ and the capital return R_k. It follows from (14) and $q^k = 1/\varepsilon^*$ that

$$R^r = \frac{(1+g)/\beta}{1+\int_{\varepsilon^*}^{\varepsilon_{\max}} (\varepsilon/\varepsilon^*-1)\, dF(\varepsilon)} \equiv R^r(\varepsilon^*). \tag{38}$$

The term $\int_{\varepsilon^*}^{\varepsilon_{\max}} (\varepsilon/\varepsilon^*-1)\, dF(\varepsilon)$ in the denominator equals the steady-state liquidity premium q^l in (15). Clearly, R^r is less than $(1+g)/\beta$ in the standard DNK model due to financial frictions. It can be checked that $R^r(\varepsilon^*)$ increases with ε^*. Intuitively, as the investment cutoff ε^* increases, more efficient firms make investment, Tobin's Q ($q^k = 1/\varepsilon^*$) declines, the liquidity premium q^l declines, and the real interest rate R^r rises.

Using (13) and $q^k = 1/\varepsilon^*$, we obtain

$$R_k = \frac{\left[(1+g)/\beta-(1-\delta)\right]/\varepsilon^* - \mu \int_{\varepsilon^*}^{\varepsilon_{\max}} (\varepsilon/\varepsilon^*-1)\, dF(\varepsilon)}{1+\int_{\varepsilon^*}^{\varepsilon_{\max}} (\varepsilon/\varepsilon^*-1)\, dF(\varepsilon)} \equiv R_k(\varepsilon^*). \tag{39}$$

The shape of $R_k(\varepsilon^*)$ plays an important role in characterizing the steady-state equilibria.

LEMMA 2: *For any $\mu > 0$, $R_k(\varepsilon^*)$ has a unique maximum at $\varepsilon_k \in (\varepsilon_{\min}, \varepsilon_{\max})$, which satisfies*

$$\mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g)-1+\delta)F(\varepsilon_k) = 0.$$

Moreover, $\partial R_k(\varepsilon^*)/\partial\varepsilon^* > 0$ for $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_k]$ and $\partial R_k(\varepsilon^*)/\partial\varepsilon^* < 0$ for $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$. If $\mu = 0$, we have $\varepsilon_k = \varepsilon_{\min}$ and $\partial R_k(\varepsilon^*)/\partial\varepsilon^* < 0$ for $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_{\max}]$.

This lemma shows that R_k is not monotonic in ε^*. This is because an increase of ε^* has two opposing effects: It reduces the liquidity premium q^l in the denominator of (39) and hence raises R_k for a similar intuition discussed earlier. But it also reduces Tobin's Q and hence reduces the return to capital R_k. Lemma 2 shows that the second effect on the return to capital R_k dominates the first liquidity premium effect for small Tobin's Q or high ε^*, but the opposite is true for low ε^*.

Next, dividing (30) by $k_t \equiv K_t/A_t$ and using (29), we can derive the steady-state real value of government liabilities relative to capital as

$$\frac{R^r d}{k} = \frac{(g+\delta)}{\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)} - (\mu + R_k(\varepsilon^*)) \equiv \Phi(\varepsilon^*). \tag{40}$$

The function $\Phi(\varepsilon^*)$ represents the maturity value of public debt (including both the principal and interest) relative to capital. We can verify that $\Phi(\varepsilon^*)$ increases with ε^* on $[\varepsilon_k, \varepsilon_{\max}]$, but may not be monotonic on $[\varepsilon_{\min}, \varepsilon_{\max}]$. Intuitively, the steady-state public debt to capital ratio $\Phi(\varepsilon^*)$ is equal to the aggregate investment ratio $(g+\delta)/\int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)$ net of the part financed by internal funds and external private borrowing $\mu + R_k(\varepsilon^*)$. The aggregate investment ratio increases with ε^* because more investment is needed to accumulate the same amount of capital in the steady state when there are fewer highly efficient investing firms. Moreover, the capital return $R_k(\varepsilon^*)$ decreases with ε^* only on $[\varepsilon_k, \varepsilon_{\max}]$ by Lemma 2.

LEMMA 3: *There exists a unique solution $\varepsilon^* = \varepsilon_l \in (\varepsilon_k, \varepsilon_{\max})$ to the equation $\Phi(\varepsilon^*) = 0$. For a sufficiently small $\mu \geq 0$, we have $\Phi(\varepsilon^*) < 0$ on $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_k]$.*

This lemma shows that there is a unique cutoff ε_l such that the steady-state value $R^r d$ is equal to zero by (40). Since $R^r d \geq 0$, Lemma 3 and (40) show that any steady-state cutoff ε^* must satisfy $\varepsilon^* \geq \varepsilon_l > \varepsilon_k$ if $\mu \geq 0$ is sufficiently small. Otherwise, public debt has a negative value by (40). Throughout our analysis, we will maintain this assumption. Intuitively, if μ is too high, investing firms can use external private debt instead of public bonds to finance investment. Given a small μ, for a sufficient small ε^*, the capital return $R_k(\varepsilon^*)$ is so high that investing firms can use internal funds instead of public bonds to finance investment, even though the liquidity premium is high.

Now we derive the steady-state version of the government budget constraint (23) as

$$\left(\frac{R^r}{1+g} - 1\right)\frac{d}{y} = \frac{s}{y}. \tag{41}$$

The left side of equation (41) represents the interest payment of public debt relative to output and the right side represents the primary-surplus-to-output ratio. Rewrite the expression on the left side as

$$\left(\frac{R^r}{1+g} - 1\right)\frac{d}{y} = \frac{R^r - (1+g)}{R^r}\frac{R^r d}{k}\frac{k}{y}\frac{1}{1+g}. \tag{42}$$

It follows from (28) and (31) that $k/y = \alpha(1+g)p_w/R_k$, where $p_w = 1 - 1/\sigma$. Substituting this expression into (42) and using (38) and (40), we can rewrite (41) as

$$\Psi(\varepsilon^*) \equiv \frac{R^r(\varepsilon^*) - (1+g)}{R^r(\varepsilon^*)}\frac{\alpha p_w}{R_k(\varepsilon^*)}\Phi(\varepsilon^*) = \frac{s}{y}. \tag{43}$$

For any exogenously given steady-state surplus-to-output ratio s/y, equation (43) determines the steady state cutoff ε^*.

Figure 2 illustrates the functions $R^r(\varepsilon^*)$, $R_k(\varepsilon^*)$, $\Phi(\varepsilon^*)$, and $\Psi(\varepsilon^*)$ for $\varepsilon_{\max} = \infty$. As shown in Panel B, a steady-state investment cutoff is determined by the crossing point of the curve $\Psi(\varepsilon^*)$ and the horizontal line s/y by assuming that s/y is exogenously given.[11] The steady-state $R^r d/k$ can be read from the curve $\Phi(\varepsilon^*)$ in Panel D. This curve crosses the horizontal axis at ε_l. For $R^r d/k \geq 0$, any steady-state cutoff must be higher than ε_l. In

[11] If we follow Reis (2021) to assume that s/d is exogenously given, we would also obtain a unique steady state as in his paper.

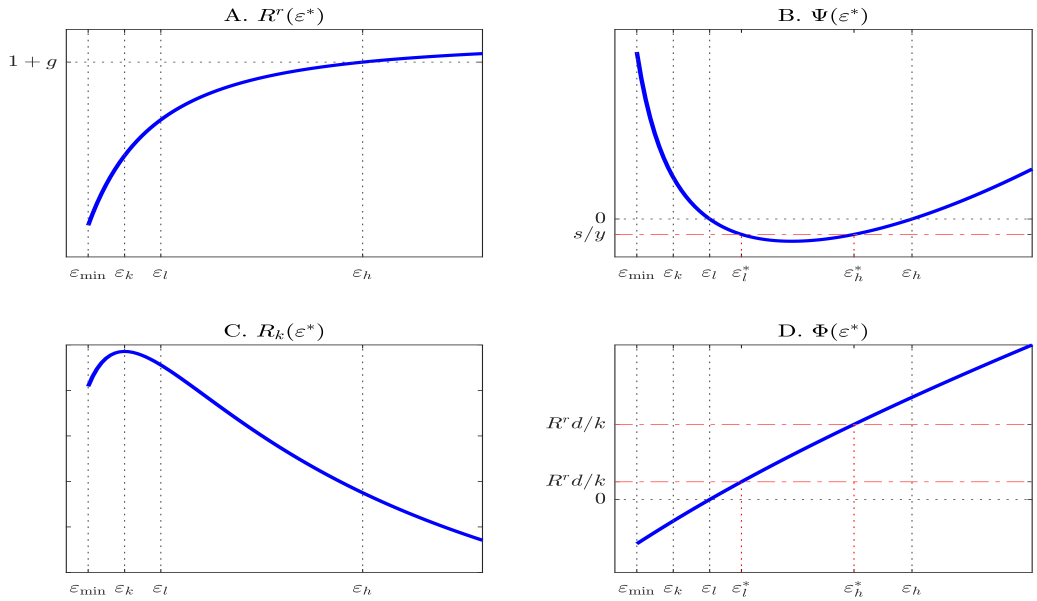

Figure 2. : Functions $R^r(\varepsilon^*), R_k(\varepsilon^*), \Phi(\varepsilon^*), \Psi(\varepsilon^*)$, and determination of steady state.

the next three subsections we consider three cases depending on the signs of s/y. We will show that there may be multiple steady states because $\Psi(\varepsilon^*)$ is not a monotonic function.

B. Government Debt as a Pure Bubble

We first consider the case in which $S_t = 0$ for all t. Then public debt is an unbacked asset like a pure bubble (Diamond (1965) and Tirole (1985)). Its fundamental value is zero. There exists a steady state in which the detrended debt has no value, i.e., $d = 0$. There may exist another steady state in which detrended debt has a finite positive value ($d > 0$) due to liquidity premium supported by self-fulfilling beliefs. The following proposition establishes the condition under which unbacked public debt can be rolled over indefinitely.

PROPOSITION 2: *Suppose that $\mu \geq 0$ is sufficiently small and the steady-state primary-surplus-to-output ratio is fixed at $s/y = 0$. Then there always exists a steady state in which the investment cutoff is ε_l given in Lemma 3, $d = 0$, and $R^r = R^r(\varepsilon_l)$. This is the unique steady state if $R^r(\varepsilon_l) > 1+g$. If $R^r(\varepsilon_l) < 1+g$, then there also exists another steady state in which the investment cutoff $\varepsilon_h \in (\varepsilon_l, \varepsilon_{\max})$ is the unique solution to the equation $R^r(\varepsilon_h) = 1+g$, the real interest rate is $R^r = 1+g$, and the real value of government liabilities relative to capital is given by $R^r d/k = \Phi(\varepsilon_h) > 0$.*

The condition in this proposition is similar to that in Tirole (1985), Miao and Wang (2018), and Dong, Miao and Wang (2020); that is, the real interest rate $R^r(\varepsilon_l)$ in the bubbleless steady state must be lower than the economic growth rate. Because the steady-state real interest rate in the standard model without financial frictions is equal to $(1+g)/\beta > 1+g$, unbacked public debt cannot be valued or rolled over. Figure 2 presents the case of $R^r(\varepsilon_l) < 1+g$. A notable feature is that $\Psi(\varepsilon^*)$ is U-shaped and crosses the horizontal axis at two points ε_l and ε_h. Panels A and C of Figure 2 show that the real interest rate R^r increases with ε^* and the capital return R_k decreases with ε^* for $\varepsilon^* > \varepsilon_k$, where ε_k is defined in Lemma 2. Thus the capital stock in the bubbly steady state is higher than in the bubbleless steady state due to the crowding-in effect of public debt.

The critical assumption to generate a low interest rate is the presence of financial frictions. The following proposition establishes the impact of the parameter μ, which captures the tightness of borrowing constraints or the degree of financial frictions.

PROPOSITION 3: *As μ decreases, both ε_l and $R^r = R^r(\varepsilon_l)$ decrease.*

This proposition shows that as the borrowing constraints are tighter, the real interest rate becomes lower and hence the unbacked public debt is more likely to be valued and rolled over. When the real interest rate is sufficiently low such that $R^r(\varepsilon_l) < 1+g$, another steady state emerges in which the interest rate is equal to the economic growth rate, $R^r(\varepsilon_h) = 1+g$. In this case, the steady-state version of equation (14) becomes

$$1 = \beta\left(1+q^l\right), \quad q^l = \int_{\varepsilon_h}^{\varepsilon_{\max}} \left(\frac{\varepsilon}{\varepsilon_h} - 1\right) dF(\varepsilon).$$

Thus the liquidity premium satisfies $q_l = 1/\beta - 1$. Without the liquidity premium, the above equation cannot hold for $\beta \in (0,1)$. Since $R^r(\varepsilon_h) = 1+g$ and public debt D_t grows at the rate $1+g$ in the steady state, the transversality condition cannot rule out a bubble in the steady state. Formally, the steady state version of (37) gives

$$\frac{R_{t-1}D_{t-1}}{\Pi_t} = \sum_{i=0}^{\infty} \frac{\beta^{i+1}\Lambda_{t+i+1}}{\Lambda_t} q^l_{t+i+1} \frac{D_{t+i}R_{t+i}}{\Pi_{t+i+1}} = \frac{D_t R^r(\varepsilon_h)}{1+g}\frac{\beta q^l}{1-\beta} = D_t \ > 0, \tag{44}$$

where we have used the following deterministic steady-state properties

$$\frac{\Lambda_{t+i}}{\Lambda_t} = \frac{C_t}{C_{t+i}} = (1+g)^{-i}.$$

In our model government bonds are net worth of entrepreneurs and help them overcome borrowing constraints. They are willing to trade government bonds to insure against their idiosyncratic investment shocks. Thus government bonds command liquidity premium and can be valued even though they are not backed by any fiscal surplus.

C. Government Debt Backed by Fiscal Surplus

In this subsection we study the case in which there is fiscal surplus in the steady state, i.e., $s/y > 0$. Then public debt is backed by fiscal surplus. In this case we have the standard result.

PROPOSITION 4: *Suppose that $\mu \geq 0$ is sufficiently small and the steady-state primary-surplus-to-output ratio is fixed at $s/y > 0$. Then there exists a unique steady state in which $R^r = R^r(\varepsilon_p) > 1+g$, where $\varepsilon_p \in (\varepsilon_{\min}, \varepsilon_{\max})$ is the unique solution to equation (43). The real value of government liabilities relative to capital is given by $R^r d/k = \Phi(\varepsilon_p) > 0$.*

Because $R^r > 1+g$ and public debt grows at the economic growth rate $1+g$ in the steady state, the bubble component in (35) in the steady state is equal to zero. Thus the real value of public debt is entirely determined by the present value in (35) discounted by R^r.

To prove Proposition 4. We consider two cases. First, if $R^r(\varepsilon_l) > 1+g$, then we can show that there $\Psi(\varepsilon^*)$ is positive and increases with ε^* on $(\varepsilon_l, \varepsilon_{\max}]$. Thus there is a unique steady state cutoff ε_p such that (43) holds for $s/y > 0$ and $R^r(\varepsilon_p) > R^r(\varepsilon_p) = 1+g$.

Second, Figure 2 shows the case of $R^r(\varepsilon_l) < 1+g$. The curve $\Psi(\varepsilon^*)$ crosses the horizontal line with $s/y > 0$. In Panel D we ignore the crossing point in the region $[\varepsilon_{\min}, \varepsilon_l]$ as the implied $R^r d/y < 0$. The crossing point ε_p must be in the region $[\varepsilon_h, \varepsilon_{\max}]$. Then we also have $R^r(\varepsilon_p) > R^r(\varepsilon_h) = 1+g$.

D. Sustainability of Fiscal Deficits

Can a permanent fiscal deficit be sustained in the long run? What is the maximum sustainable primary deficit in the long run? The following proposition addresses these questions.

PROPOSITION 5: *Suppose that μ is sufficiently small and that $R^r(\varepsilon_l) < 1+g$. For any given $s/y \in (-\underline{s}, 0)$, where*

$$-\underline{s} = \min_{\varepsilon^* \in [\varepsilon_l, \varepsilon_h]} \Psi(\varepsilon^*) < 0,$$

there exist (at least) two steady states with $R^r(\varepsilon_l) < R^r(\varepsilon_l^*) < R^r(\varepsilon_h^*) < R^r(\varepsilon_h) = 1+g$, where $\varepsilon_l < \varepsilon_l^* < \varepsilon_h^* < \varepsilon_h$ and both ε_l^* and ε_h^* solve equation (43). The real value of government liabilities relative to capital is given by $R^r d/k = \Phi(\varepsilon_l^*)$ and $R^r d/k = \Phi(\varepsilon_h^*)$, respectively. If $s/y < -\underline{s}$, then there does not exist a steady state.

The critical condition in this proposition is the same as that in Proposition 2; that is, the steady-state real interest rate on the unbacked public debt must be lower than the economic growth rate. This condition can support not only a steady state with a positive value of the unbacked public debt as in Proposition 2, but also at least two other steady states in which primary deficits last forever. Figure 2 illustrates the case with exactly two steady states. In these steady states the real interest rates are less than the economic growth rate.

The multiplicity is due to the non-monotonicity of $\Psi(\varepsilon^*)$, which is similar to a tax Laffer curve.[12] Intuitively, for $s/y < 0$ and $R^r < 1+g$, the government effectively taxes households

$$-\Psi(\varepsilon^*) = \left(\frac{1}{R^r} - \frac{1}{1+g}\right)\frac{R^r d}{y}$$

to cover primary deficits $-s/y > 0$ by equations (41) and (43). An increase in the real interest rate R^r reduces the "tax rate" $1/R^r - 1/(1+g)$, but it may raise the "tax base" $R^r d$ due to the liquidity premium. In particular, an increase in R^r reduces the liquidity premium q^l by (14), thereby reducing the capital price q^k. Then aggregate capital demand rises. The credit-constrained entrepreneurs need more public debt value $R^r d$ to raise their net worth to finance investment. Thus total "taxes" $-\Psi(\varepsilon^*)$ may first increase with R^r and later decrease with R^r. This implies that there may exist multiple interest rates such that $\Psi(\varepsilon^*) = s/y$ holds. Equivalently, there may exist multiple cutoffs ε^* such that this equation holds because the interest rate R^r increases monotonically with ε^*.

Now we use (36) to show that the detrended real value of debt in the deterministic steady-state satisfies

$$d(1+g)^{t-1}R^r = \frac{s(1+g)^t}{1-\beta} + \frac{\beta q^l}{1+g}\frac{d(1+g)^t R^r}{1-\beta}, \tag{45}$$

where the first and second terms on the right-hand side of equation (45) represent the fundamental and bubble components, respectively. The fundamental value is negative given $s < 0$. By (14), we have $\left(1+q^l\right)\beta = (1+g)/R^r$. It follows from (45) and Proposition 5 that

$$d = \frac{s(1+g)}{R^r - (1+g)} > 0.$$

Thus a positive value of the bubble component offsets the negative fundamental value to ensure a positive real value of public debt. The government can keep rolling over debt to finance fiscal deficits and repay debt at an interest rate lower than the economic growth rate. As discussed in Section II, entrepreneurs are willing to hold government bonds because these bonds provide self-insurance against idiosyncratic investment shocks. A positive value of public debt can be supported in equilibrium.

If the deficit-to-output ratio $|s|/y$ is too high, then the government may issue too much bonds that exceed the demand capacity of entrepreneurs. As a result an equilibrium does not exist and primary deficits cannot be sustained.

Brunnermeier, Merkel and Sannikov (2020*a*,*b*) offer a different interpretation by solving (23) forward discounted by the real interest rate. We then obtain the following equation

[12] This curve is similar to but different from the seigniorage Laffer curve in the monetary economics literature.

in the deterministic steady state:

$$d = \lim_{T\to\infty}\left[\sum_{k=0}^{T}\left(\frac{1+g}{R^r}\right)^{k+1}s + \left(\frac{1+g}{R^r}\right)^{T+1}d\right]. \tag{46}$$

They interpret the first term on the right-hand side of (46) as the fundamental value and the second term as the bubble component. If $R^r < 1+g$, then the fundamental value explodes to negative infinity for $s < 0$. However, the bubble component explodes to positive infinity. The sum of these two components can be a finite positive value. This interpretation may be less intuitive when used to discuss the transmission mechanism of shocks.

III. Quantitative Implications

In this section we study quantitative implications of our model by calibrating our model to the US data at quarterly frequency. We are interested in the following questions: What is the maximum sustainable level of the primary deficit? What is the implied value of public debt? What is the stability of the steady states and the local determinacy of equilibria around a steady state? What are the dynamic responses of the economy to a monetary or fiscal policy shock?

A. Calibration

Our model can generate multiple steady states depending on parameter values. To study the possibility of sustaining permanent primary deficits, we calibrate our model such that the conditions in Proposition 5 hold. Moreover, we calibrate our model at quarterly frequency such that the steady state with the lowest interest rate (steady state L) matches some long-run moments in the US quarterly data over 1950-2019. We choose this steady state because there has been a secular decline in interest rates across almost all advanced economies.[13]

The calibrated parameters are listed in Table 1. We first set $\alpha = 0.33$ and $\beta = 0.99$ as in the standard macroeconomics literature. Set $g = 0.0315/4$ so that the annual real GDP growth rate is 3.15% as in the data. Set $\Pi = 1 + 0.0309/4$ to be consistent with the average annual inflation rate of 3.09% during the period 1950-2019. We calibrate the steady-state G_t/Y_t to match the long-run average 11.1% in the data.[14] Set the long-run surplus-to-output ratio s/y to -0.445%, which ensures the debt-to-GDP ratio $(d/(4y))$ in steady state L to match the long-run average 35.9% in the data. Set $\xi = 0.75$ and $\sigma = 11$ so that the duration of price adjustments is four quarters and the steady-state

[13] Kaas (2016) also calibrates his model such that the steady state with a lower interest rate matches the data. Figure 3 Panel D shows that a reduction of the deficit/output ratio leads to a decrease in the debt/output ratio around steady state L, which is consistent with the conventional wisdom. But steady state H has a perverse comparative statics implication.

[14] The data of government spending to GDP ratio is taken from Jiang et al. (2019*a*). Government spending is defined as discretionary spending plus the domestic net transfer payments before interest payments.

Table 1—: Calibrated Parameters at Quarterly Frequency

Parameter	Values	Description	Target
α	0.33	Capital elasticity	Capital income share
g	0.79%	Labor efficiency growth	GDP growth
$\Pi - 1$	0.77%	Inflation target	Inflation rate
G/Y	11.1%	Government spending share	Government spending to GDP ratio
s/y	−0.445%	Primary surplus to output ratio	Public debt to GDP ratio
ξ	0.75	Price adjustment probability	Duration of price adjustments
σ	11	Goods elasticity of substitution	Price markup
β	0.992	Discount factor	
δ	1.217%	Depreciation rate	Equity return
μ	0.197	Capital pledgeability	Real interest rate
η	0.358	Pareto shape	Investment-to-GDP ratio
$\epsilon_{\min}$	0.642	Pareto scale	Normalization $\mathbb{E}[\varepsilon] = 1$
ψ	3.75	Labor disutility	Number of hours worked

markup is $\sigma/(\sigma - 1) = 1.1$, consistent with the DNK literature. We choose ψ such that the steady-state labor is equal to 0.25 as in the business cycles literature.

It remains to calibrate three parameters δ, η, and μ. We adopt the Pareto distribution for the idiosyncratic investment shock $F(\varepsilon) = 1 - (\varepsilon/\varepsilon_{\min})^{-\frac{1}{\eta}}$ and set $\varepsilon_{\min} = 1 - \eta$ so that the unconditional mean is 1. We set $\delta = 1.217\%$ so that the steady-state equity (or investment) return $R_k/q^k + (1 - \delta)$ is equal to 4% per annum. As is well known the equity premium is about 6% per annum in the data. Our steady-state target of 4% appears reasonable given that risk premium is absent in the deterministic steady state. We set $\eta = 0.358$ and $\mu = 0.197$ so that the real interest rate and the investment-to-GDP ratio in the steady state with a lower interest rate are equal to 1.9% per annum and 17.4%, respectively, as in the data.[15] The calibrated $\mu = 0.197$ is in line with those reported in Miao, Wang and Xu (2015) and Dong, Miao and Wang (2020).

B. Maximum Sustainable Deficit

Using the calibrated parameters given in Table 1, we can calculate the maximum sustainable deficit-to-output ratio $|s|/y$ as in Proposition 5. As shown in Figure 3, there are two steady states with primary deficits $s/y < 0$. The maximum sustainable deficit-to-output ratio $|s|/y$ is 0.45%, which is smaller than the number of 0.834% in Kaas (2016). When the economy is at its maximum sustainable deficit-to-output ratio, the annual net real interest rate is $4 * (R^r - 1) = 2\%$ and the debt-to-GDP ratio is $d/(4y) = 39.81\%$.

[15] We use the data of the nominal interest rates for the entire portfolio of the U.S. government bonds from Hall et al. (2018*a*) and the GDP deflator to obtain real interest rates.

As s/y increases from -0.45% to 0, the smaller steady-state interest rate declines until the investment cutoff decreases to ε_l. In the meantime, the larger steady-state interest rate increases to $1+g$ until the investment cutoff rises to ε_h. When s/y further increases from zero to a positive number, real interest rate R^r increases from $1+g$. Moreover, capital, output, and the debt-to-GDP ratio all increase. As $s/y \to +\infty$, $R^r \to (1+g)/\beta$ and $d/y \to +\infty$.

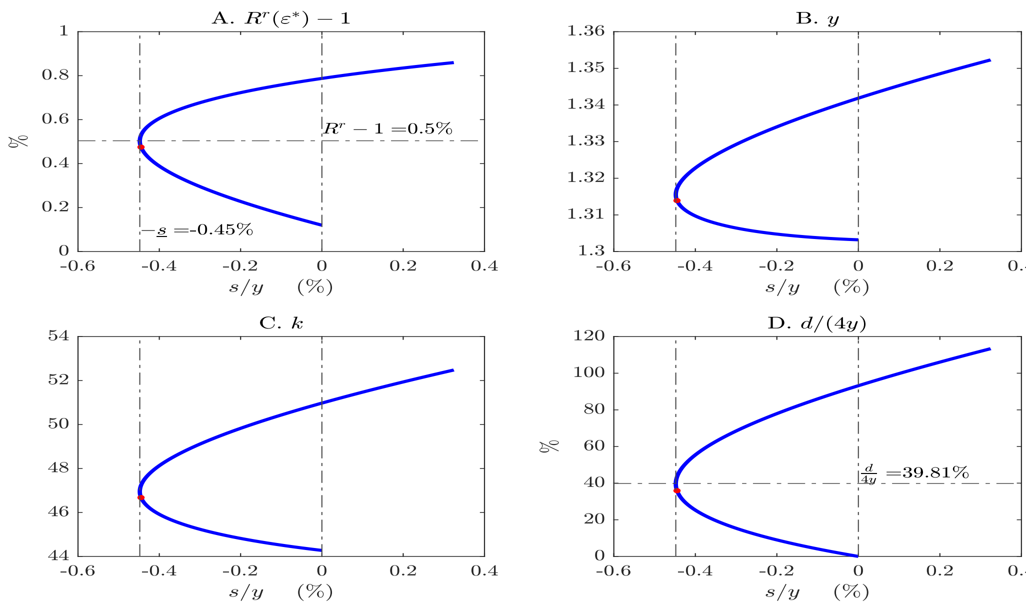

Figure 3. : Steady state values for various primary-surplus-to-output ratios s/y. The dot point in each curve shows the steady state of the model under the calibration in Table 1.

Figure 3 shows an interesting comparative statics property. As the long-run deficit-to-output ratio $|s|/y$ declines, the real interest rate, capital, output, and debt-to-GDP ratio in steady state L all decline. Intuitively, as deficit $|s|/y$ declines, the investment cutoff ε^* declines, the liquidity premium increases, and hence the real interest rate declines (see Figure 2). Then the government reduces debt issuance. But this hurts entrepreneurs because government bonds are their net worth used to finance investment spending. As a result, entrepreneurs accumulate less capital and the steady-state output declines. The opposite results obtain for steady state H as the primary deficit declines.

Notice that our model generated maximum deficit-to-output ratio is lower than those estimated in the literature. Using OLG models, Chalk (2000) finds that primary deficits up to 5.2% are sustainable, while Bullard and Russell (1999) calibrate a similar model with a primary deficit of 1.9%. Using an infinitely-lived agent model with financial frictions, Kaas (2016) finds that the maximum deficit-to-output ratio is 0.837%.

Chalk (2000) calibrates the real interest rate to a lower value of 1.2% per annum. If we follow our previous calibration strategy, but target his value of real interest rate with $R^r = 1 + 1.2\%/4$, we obtain $\delta = 1.135\%$, $\mu = 0.180$, and $\eta = 0.356$. The new calibration generated maximum sustainable deficit-to-output ratio is 0.53%. Given this ratio, the net real interest rate is $4 * (R^r - 1) = 1.84\%$ per annum and the debt-to-GDP ratio is $d/(4y) = 41.12\%$. Thus the maximum sustainable deficit-to-output ratio and debt-to-GDP ratio do not change much.

The difference in estimates is likely due to the structural differences between OLG and infinitely-lived agents models. One possible explanation is that OLG models with hump-shaped earnings profiles permit the government to roll over larger stocks of debt. For infinitely-lived agents models with credit constraints, such large deficits are not sustainable for plausible parameter values.

C. *Local Determinacy*

Given our calibrated parameter values in Table 1, there are two steady states for the detrended system as shown in Figure 3. In this subsection we study the stability of these steady states and local determinacy of equilibria around each of these steady states. Due to the complexity of our model, we are unable to derive analytical results. We thus use numerical methods. We first linearize the detrended system around a steady state and then study the determinacy and stability of the linearized system using Klein's (2000) method. The linearized system is presented in Online Appendix D.

We first consider a real version of our model by removing the pricing and monetary policy block and setting $\Pi_t = 1$ for all t. Following Kaas (2016), we set $\phi_d = 0$ in (24) or assume that T_t/Y_t is exogenously given. We summarize the detrended system by a system of 13 equations for 13 variables $\{R_t, R_{kt}, \lambda_t, \varepsilon_t^*, q_t^k, q_t^l, w_t, d_t, k_t, N_t, y_t, c_t, i_t\}$ given in Online Appendix B. The predetermined variables are $\{R_t, d_t, k_t\}$. The steady-state allocation remains the same as in the monetary model. Using the same calibration as in Table 1, we find that there is a unique equilibrium around steady state L and there is no bounded equilibrium around steady state H, a result similar to Kaas (2016).[16]

Now we consider our monetary model, which generates very different results. In this case we need to stabilize both inflation and public debt. The determinacy depends on the policy rules in (24) and (26). The critical parameters are the policy response coefficients (ϕ_d, ϕ_π). Figure 4 presents the determinacy region for the calibrated model under different policy mix (ϕ_d, ϕ_π). For ease of numerical computations, we consider only values of ϕ_d in $(-1, 1)$ and $\phi_\pi \geq 0$.

As shown in Figure 4, the vertical line $\phi_d^* = 1/\beta - 1$ and the horizontal line $\phi_\pi^* = 1$ partition the policy parameter space into four regions for the flexible-price model of Leeper (1991): (i) unique equilibrium for $\phi_d > 1/\beta - 1$ and $\phi_\pi > 1$ (active monetary policy and passive fiscal policy, regime M); (ii) unique equilibrium for $\phi_d < 1/\beta - 1$ and $\phi_\pi < 1$ (passive monetary policy and active fiscal, regime F); (iii) no bounded equilibrium for

[16]Formally, the number of stable eigenvalues is the same as (resp. smaller than) the number of predetermined variables for the linearized detrended equilibrium system around stead state L (resp. steady state H).

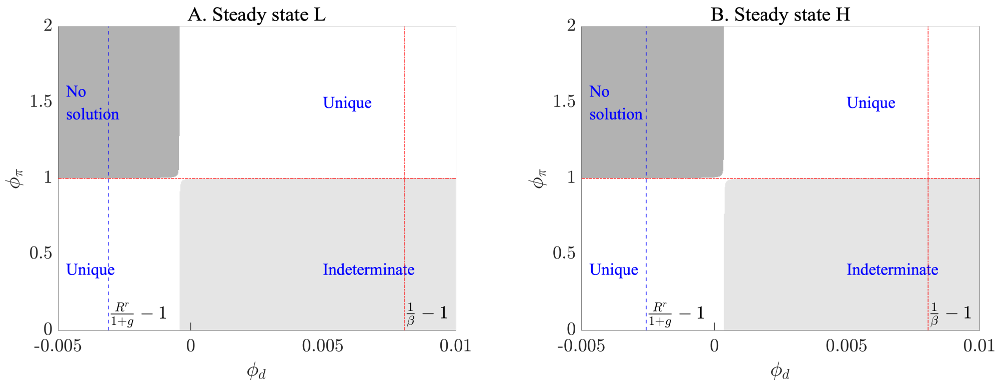

Figure 4. : Local determinacy regions for the policy mix parameters $\{\phi_d, \phi_\pi\}$ for $s/y = -0.445\%$.

$\phi_d < 1/\beta - 1$ and $\phi_\pi > 1$ (active monetary policy and active fiscal policy); and (iv) indeterminate equilibria for $\phi_d > 1/\beta - 1$ and $\phi_\pi < 1$ (passive monetary policy and passive fiscal policy). This result still holds for the standard DNK model without capital (e.g., Woodford (2003)).[17] It also holds for the standard DNK model with capital if the distortions in the economy are not too large (e.g., Lubik (2003) and Carlstrom and Fuerst (2005)).

By contrast, our model with financial frictions delivers different results. We focus our analysis on the determinacy around steady state L illustrated in Panel A of Figure 4, as the analysis for steady state H is similar. We find that there are three disjoint regions of the policy parameter space. The two regions that ensure a unique equilibrium in Leeper (1991) become one region in our model.[18] Unlike in Leeper (1991), the boundaries of this region are nonlinear. The Taylor principle threshold $\phi_\pi^* = 1$ is no longer the critical value to stabilize inflation and the steady-state net interest payment $R^r/(1+g) - 1$ is no longer the critical value to stabilize debt dynamics.

To understand the intuition, we use the linearized fiscal policy rule equation and the government budget constraint to derive

$$\widetilde{d}_t = \left(\frac{R^r}{1+g} - \phi_d\right)\widetilde{d}_{t-1} + \frac{R^r}{1+g}\frac{d}{y}\left(\widehat{R}_{t-1} - \widehat{\Pi}_t\right) + \frac{G_a}{y}\widehat{G}_{at} - z_{\tau,t}, \tag{47}$$

where a variable with a tilde denotes deviation from its steady state relative to output (e.g., $\widetilde{d}_t = (d_t - d)/y$) and a variable with a hat denotes log deviation from its steady state (e.g., $\widehat{\Pi}_t = \ln \Pi_t - \ln \Pi$). In the standard model without financial frictions (e.g., Leeper

[17] Woodford (2003, Proposition 4.11) considers a fiscal policy rule that reacts to the maturity value of real public debt $R_t d_t$, instead of d_t in (24). But the result is essentially the same.

[18] Canzoneri et al. (2011) and Cui (2016) obtain similar results in different models.

(1991)), the steady-state real interest expense is given by $R^r/(1+g)-1=1/\beta-1$ and the short-run real interest rate $\widehat{R}_{t-1}-\mathbb{E}_{t-1}\widehat{\Pi}_t$ is independent of public debt $\widetilde{d}_{t-1}$. Thus we obtain the standard critical value $1/\beta-1$. By contrast, the long-run real interest expense in our calibrated model with financial frictions is $R^r/(1+g)-1$, which is less than zero and lower than $1/\beta-1$. Importantly, the real interest rate $\widehat{R}_{t-1}-\mathbb{E}_{t-1}\widehat{\Pi}_t$ responds to public debt $\widetilde{d}_{t-1}$ due to the liquidity premium.[19] Thus the stability of debt cannot be determined by the coefficient of $\widetilde{d}_{t-1}$ in (47) alone. In particular, the real interest rate is positively related to the public debt value because an increase in public debt raises outside liquidity and reduces the liquidity premium, which reduces the demand for private bonds and thus raises the real interest rate.

Let us still apply Leeper's (1991) definition of passive/active policy thresholds $\phi_\pi^*=1$ and $\phi_d^*=R^r/(1+g)-1$.[20] We also keep his terminology of regime M and regime F. Then Figure 4 shows that both active and passive monetary policy can be combined with a passive fiscal policy to ensure price determinacy. There are two nonlinear disjoint boundaries for the determinacy region. Both boundaries are increasing curves. Unlike the standard DNK model, for a passive fiscal policy with $\phi_d>R^r/(1+g)-1$, public debt may not be stabilized even though the coefficient of $\widetilde{d}_{t-1}$ in (47) is less than 1, because the second interest rate term in (47) is positively related to $\widetilde{d}_{t-1}$. Thus debt value may explode unless it is revalued by surprise inflation. This can be achieved by a passive monetary policy with $\phi_\pi<1$.

An important feature of our model is that there may exist two steady states given the same long-run primary-deficit-to-output ratio (see Figures 2 and 3). While the determinacy property around steady state H is similar to that for the other steady state as illustrated in Panel B of Figure 4, the multiplicity of the steady states generates some interesting implications absent from the literature.

First, a fiscal and monetary policy mix is important not only for local determinacy of equilibria around a particular steady state, but also for selecting a particular steady state. We will show in Section III.E that the debt and tax target will play an important role.

Second, the determinacy region for the two steady states are different as illustrated in Figure 4. This means that, given the same policy response coefficients ϕ_d and ϕ_π, the local equilibrium is determinate around one steady state, but may be indeterminate around the other steady state. One particular example is that for an economy around steady state L, a policy mix with $\phi_d=0$ and $\phi_\pi=1.5$ implies that the model has a unique bounded solution and the economy is in regime M. However, for the same policy mix, if the economy is around steady state H, the model does not have any bounded solutions. In this case, the strong reaction of monetary policy to inflation ($\phi_\pi>1$) will increase the interest expense and lead to an explosive path for public debt. Another example is that around the steady state H, a policy mix with $\phi_d=0$ and $\phi_\pi=0.8$ can guarantee equilibrium determinacy. However, the same policy mix leads to indeterminacy around steady state L.

Third, Woodford (1995, 2001) discusses the interest-rate-peg policy which corresponds

[19]See Dominguez and Gomis-Porqueras (2019) for a similar discussion.

[20]More formally, monetary policy is active if $|\phi_\pi|>1$, and passive if $|\phi_\pi|<1$. Fiscal policy is active if $|R^r/(1+g)-\phi_d|>1$, and passive if $|R^r/(1+g)-\phi_d|<1$.

to $\phi_d = \phi_\pi = 0$ in our model. As shown in Panel B of Figure 4, these policy parameters determine a unique equilibrium around steady state H. However, they fall in the indeterminacy region for steady state L as shown in Panel A of Figure 4.[21] The intuition is as follows.

The fiscal rule $\phi_d = 0$ is an active policy in the standard DNK model, in which the steady-state interest rate R^* for log utility satisfies $R^* = (1+g)/\beta > (1+g)$. With positive interest expenses ($R^* > 1+g$) and without raising taxes for $\phi_d = 0$, the government must raise an explosive amount of debt eventually to cover accumulated interest expenses starting from any initially given government liabilities. To obtain a unique bounded equilibrium, debt value must be derived as a forward-looking solution and the initial inflation is adjusted to ensure the government budget constraint is satisfied (see equations (23) and (36)). This also suggests that the stabilization of public debt relies on the debt revaluation through surprise inflation, with the latter being possible if monetary policy is passive (e.g., $\phi_\pi = 0$). This is regime F discussed in Leeper (1991) and Woodford (1995, 2001).

By contrast, the fiscal rule $\phi_d = 0$ is a passive policy in our calibrated model because $R^r < (1+g)$. Importantly, the real interest rate $\widehat{R}_{t-1} - \mathbb{E}_{t-1}\widehat{\Pi}_t$ responds to debt value $\widetilde{d}_{t-1}$ due to the liquidity premium. We are unable to derive a closed-form solution to check determinacy. Our numerical results show that the fiscal rule $\phi_d = 0$ is passive enough to stabilize public debt by the fiscal authority itself for the local equilibrium around steady state L. When monetary policy is passive with $\phi_\pi = 0$, inflation is also stabilized given any initial level of inflation. Thus local equilibria around steady state L are indeterminate of degree one.

Our numerical results also show that the interest rate response to public debt is so strong that debt dynamics cannot be stabilized by the fiscal authority itself around steady state H for $\phi_d = 0$. We then obtain a unique saddle-path equilibrium around that steady state given $\phi_d = \phi_\pi = 0$ as in regime F of the standard DNK model discussed above.

D. Dynamic Responses

In this subsection we study the impulses responses of the calibrated model economy to a one percent shock to the lump-sum tax/transfer in (24), government spending in (25), and the nominal interest rate in (26), respectively.[22] We suppose that the economy initially stays at steady state L and study local dynamics of the detrended equilibrium system around this steady state after a shock. Focusing on steady state L is particularly interesting because it includes the debt rollover case $\phi_d = 0$ when $\phi_\pi > 1$, while this is not feasible around steady state H. For regime M, we set $\phi_d = 0$, $\phi_\pi = 1.5$, and for regime F, we set $\phi_d = R^r/(1+g) - 1 = -0.0031$, $\phi_\pi = 0.8$. These parameter values imply that taxes do not respond to public debt in regime M and taxes are cut by the amount of debt interest in response to an increase in public debt in regime F. These policy mixes can lead

[21] Bassetto and Cui (2018) find a similar result in different models.

[22] See Kim (2003) for impulse responses to various shocks in a standard DNK model without capital. We have conducted a similar analysis for a standard DNK model with capital. Such results are available upon request.

to equilibrium determinacy as shown in Figure 4. To introduce persistence, we set $\rho_G = 0.8$ and $\rho_m = 0.5$, but set $\rho_\tau = 0$.

We use numerical experiments to answer the following questions:[23] Can conventional contractionary monetary policy through open market sales of debt without raising future taxes (or surpluses) stabilize debt and inflation? What is the impact of a tax cut or an increase in government transfers/spending financed by debt, which is rolled over subsequently, on the macroeconomy? How does monetary policy coordinate with such fiscal policies to stabilize debt and inflation dynamics? How does the FTPL work in an economy with low interest rates?

Monetary Policy Shock. — We first consider the impact of a one percentage point increase in the nominal interest rate (a contractionary monetary policy shock) as shown in Figure 5. For regime M, we obtain the conventional impulse responses. In particular, the real interest rate rises, but consumption, investment, labor, output, and inflation all decline on impact. The evolution of the real value of public debt is backward-looking. The initial decline in inflation leads to a higher real value of debt liabilities. Without adjustments of taxes ($\phi_d = 0$), the government has to issue more public debt. The rising interest expense further raises future debt value. Given the low steady-state real interest rate $R^r < 1 + g$, debt can eventually pay for itself. However, debt rises for a few periods because the real interest rate rises with debt. It takes a long time for debt to revert back to its steady-state value as the long-run coefficient of $\widetilde{d}_{t-1}$ in (47) is $R^r/(1+g) = 0.9969$. This is in contrast to the conventional wisdom which calls for an increase in future taxes to finance debt because the conventional real interest rate is higher than the economic growth rate.

Figure 5 also shows the relationship among real interest $(\widehat{R}_t^r)$, real debt $(\widetilde{d}_t)$, liquidity premium $\left(\widehat{q}_t^l\right)$, and Tobin's Q $\left(\widehat{q}_t^k\right)$. In particular, both the liquidity premium and Tobin's Q decline as the nominal interest rate rises on impact.

For regime F, a positive interest-rate shock also generates a contractionary impact on quantities initially. But there are two major differences from regime M. First, consumption, investment, and output rise above the steady state for some periods before they revert. The intuition is that government liabilities rise given a positive shock to the nominal interest rate. In the next period, entrepreneurs make more investment and hire more workers due to the positive wealth effect of government bonds. Households raise consumption as their wages rise. The initial drop of consumption, investment, labor, and output is due to the initial rise of the real interest rate given the passive interest rate rule $\phi_\pi = 0.8$.

Second, inflation rises on impact, but the real value of public debt declines. According to the conventional interpretation based on the FTPL (e.g., Kim (2003)), the nominal government liabilities grow more rapidly relative to the present value of future surpluses when there is a positive shock to the nominal interest rate. In the next period, the real

[23] We use the perfect-foresight, deterministic simulations algorithm implemented in Dynare (Adjemian et al. (2011)). The algorithm adopts nonlinear methods that apply to models with large shocks or with the ZLB. The log-linear solutions for small shocks are quite close to the nonlinear solutions when the ZLB never binds.

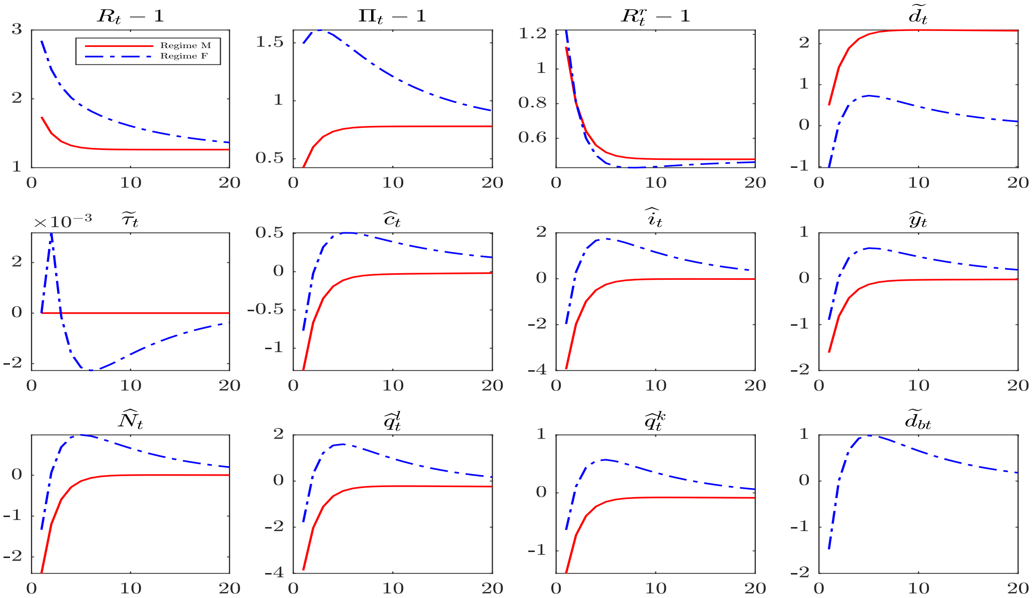

Figure 5. : Impulse response functions to a positive 1% innovation shock to the nominal interest rate starting from steady state L. All vertical axes represent percentage points.

value of government liabilities exceeds the present value at the given price level, which induces households and entrepreneurs to raise consumption and investment. This aggregate-demand increase pushes up inflation and output in the next period. Inflation also rises in the current period by the Phillips curve relation because prices are sticky.[24] This argument could fail because the present value discounted by the usual household SDF may explode when the real interest rate is lower than the economic growth rate.

By contrast, we decompose the real value of public debt into a fundamental component and a bubble component in regime F of our calibrated model with long-run fiscal deficits in Lemma 1. Both components are discounted by the household SDF and the implied discount rate is higher than the economic growth rate in the long run. The initial drop of consumption raises marginal utility Λ_t on impact and hence lowers the household SDF $\beta\Lambda_{t+1}/\Lambda_t$. Given fiscal deficits, the fundament value of public debt in (36) is negative and rises on impact. Given $\phi_d = R^r/(1+g) - 1 = -0.0031$ in (24), tax $\widetilde{\tau}_t$ does not change initially and then is negatively related to debt $\widetilde{d}_{t-1}$. The bubble component ($\widetilde{d}_{bt}$) of public debt in (36) declines due to the decline of the liquidity premium as shown in the last panel of Figure 5 and this effect dominates. Thus the initial price or inflation must rise to satisfy the government budget constraint as nominal debt is predetermined (see (23) and (36)). In regime F equation (36) is an equilibrium condition.

[24]In the flexible price model of Leeper (1991), the impact of the interest-rate shock is delayed without affecting the current inflation.

Government Spending Shock. — Next we study the impact of an initial increase in government spending by 1% from the steady state as shown in Figure 6. By the resource constraint, if consumption and investment do not respond, output would rise by about 0.1% given that the steady-state government spending to output ratio is about 11%.

Regime M gives the conventional story. An increase in government spending raises aggregate demand and inflation. Given the active interest-rate rule and sticky prices, both the nominal and real interest rates rise, thereby crowding out consumption and investment. Thus output rises by less than 0.1%. Moreover, the increase in government spending is financed by the public debt, which is rolled over without increasing taxes given $\phi_d = 0$. The real value of public debt evolves as a backward-looking variable and rises as interest rates rise. The debt can pay for itself as the long-run real interest rate is less than the economic growth rate, but the reversion to the steady state takes a long time. Thus the crowding-out effect on investment is long lasting.

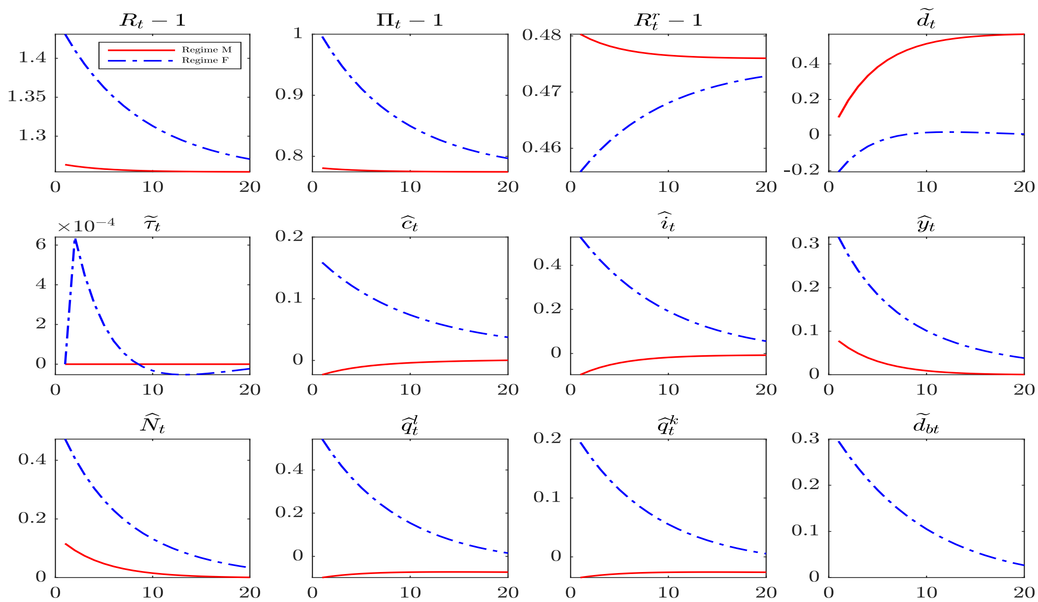

Figure 6. : Impulse response functions to a positive 1% government spending shock starting from steady state L. All vertical axes represent percentage points.

Regime F tells a different story. Households with non-Ricardian expectations do not think that an increase in government spending raises their tax burden. Quite the contrary, they think primary deficits rise even more as the government also cuts taxes given $\phi_d = R^r/(1+g) - 1 < 0$. Thus the fundamental value of public debt (which is negative in our calibrated model) declines. This effect dominates the rise of the bubble component in the last panel of Figure 6, causing the sum of the two present values on the right-hand side of (36) to fall. At the initial price level, the value of public debt held by entrepreneurs exceeds that sum, and this represents a positive wealth effect. Entrepreneurs increase investments and households increase consumption until the price level rises enough to eliminate the discrepancy. Given the passive interest-rate rule $\phi_\pi = 0.8$, the nominal interest rate rises on impact, but the real interest rate falls. This further stimulates consumption and investment. Firms also hire more labor. Thus the initial output response is 0.32%, much larger than 0.1% and the increase in inflation is about 10 times larger than in regime M. The impact fiscal multiplier ($\Delta y_0/\Delta G_{a,0}$) is 2.85 in regime F, while it is 0.70 in regime M. Thus our model does not need to rely on the ZLB to have a significant multiplier.

Transfer Shock. — Now we consider the impact of a 1% transfer shock as shown in Figure 7. This shock is effectively the same as the reduction of the initial lump-sum tax by 1% of output financed by debt. In the standard DNK model without financial frictions, this shock does not affect the real economy and inflation in regime M, because Ricardian equivalence holds and the determination of inflation is independent of the fiscal authority's behavior.

For our model with financial frictions, this shock has an impact on the real economy, but the quantitative effect is very small in regime M. Unlike in the standard DNK model, the real value of debt in our model does not increase one-to-one as the lump-sum transfer rises on impact because inflation responds to the transfer shock. The public debt can pay for itself given $\phi_d = 0$ as $R^r < 1 + g$. However, as the coefficient of debt in (47) is close to 1 as discussed earlier, it takes a long time for the debt to revert back to its steady-state value. Moreover, as households do not hold any bonds, they increase consumption and reduce labor supply when there is a lump-sum transfer or reduction in lump-sum taxes. But persistent debt crowds out entrepreneurs' investment, causing output to fall on impact.

By contrast, a positive transfer shock has a significant expansionary impact on the economy in regime F. According to the standard FTPL, the real value of public debt declines in response to the shock because its present value of future surpluses falls, holding the discount rate (real interest rate) constant. Then the initial price level or inflation must increase to balance the government budget. Given the passive interest-rate rule $\phi_\pi = 0.8$, the nominal interest rate increases less than the rise of inflation so that the real interest rate declines on impact, leading to an economic expansion. Moreover, the positive wealth effect of public debt on entrepreneurs stimulates investment significantly. As discussed earlier, this argument could fail in the standard FTPL when the real interest rate is less than the economic growth rate and when the government runs persistent deficits as the present value discounted by the real interest rate would explode.

In our model public debt contains a bubble component as shown in Lemma 1. In response to a positive transfer shock, the (negative) fundamental value of public debt falls. But the bubble component rises due to the increase in the liquidity premium so that the real value of public debt D_t rises on impact. This increase is lower than the increase of the fiscal deficit (or the decline of surplus S_t) so that the right-hand side of (23) declines on impact. To satisfy (23) or (36), the initial price level or inflation rises by about 2.5 percentage

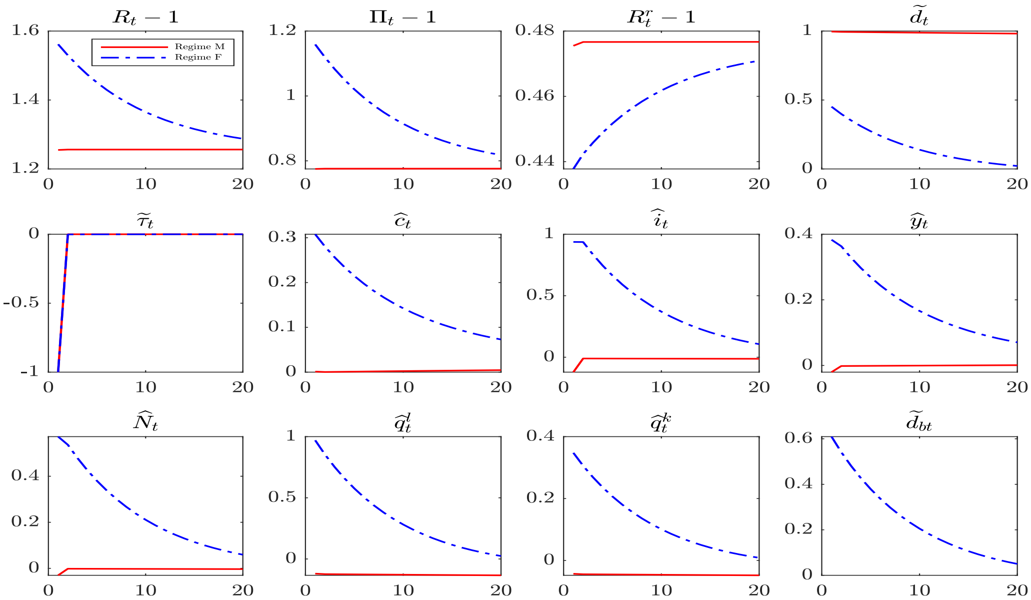

Figure 7. : Impulse response functions to a positive 1% fiscal transfer shock to households starting from steady state L. All vertical axes represent percentage points.

points. Given the fiscal policy parameter $\phi_d = R^r/(1+g) - 1 = -0.0031$, the government also cuts lump-sum taxes from the second period on. But the real debt burden is reduced through persistent inflation. Then the rest of the usual FTPL goes through as discussed earlier.

The above policy experiments merit further discussions. First, the above positive transfer shock essentially increases primary deficits because our calibrated economy features deficits in the steady state. Our results suggest that temporarily increasing deficits can lead to a short-run expansion and persistent inflation in regime F. Second, because entrepreneurs make investment in our model, making transfers to entrepreneurs instead of households can relax their credit constraints, and hence it may have a larger stimulative effect on investment. In additional results available upon request, we find that it is indeed the case but the quantitative effect is similar to that in Figure 7, when each entrepreneur receives equal transfer. If we allow taxes to respond to debt in regime M, then debt can revert to the steady state much faster and hence its crowding-out effect will be weaker. The positive wealth effect can dominate the crowding-out effect, causing the initial investment to rise in regime M. Third, we find that although debt-financed transfer and rolling over debt can have zero fiscal cost given low interest rates, its welfare effect on households is very small (close to zero). This is due to the large crowding-out effect in regime M and the large increase of labor in regime F.

E. Fiscal Target as an Equilibrium Selection Device

As argued by Beck-Friis and Willems (2017) and Billi and Walsh (2021), raising the debt target can improve welfare in regime F in a standard DNK model because of the positive wealth effect of public bonds. In this subsection we show that the debt target can be used as an equilibrium selection device in our model with multiple equilibria.

To illustrate this point, we conduct two numerical experiments. First, under our baseline calibration in Table 1, there are two steady states with $R^r < 1+g$. The debt-to-GDP ratio is equal to 35.9% and 43.4% for steady states L and H, respectively. Suppose that the economy is initially in steady state L. In period 1, the government announces that the debt/GDP target is permanently set to the level 43.4% in steady state H. Then the steady-state tax/output ratio τ/y in (24) must change accordingly. Can the economy transition from steady state L to steady state H? The answer depends on the fiscal-monetary policy regime. As an example, we set $\phi_\pi = 1.5$ and $\phi_d = 0.2$. We can verify that this policy mix represents regime M and generates a unique equilibrium around both steady states.

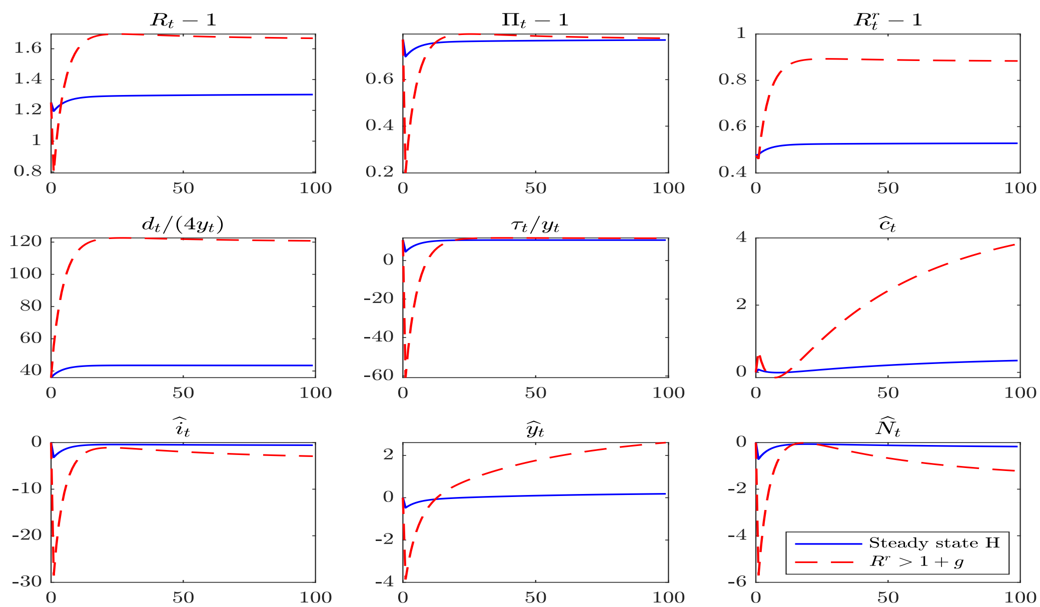

Figure 8. : Transitional dynamics with different fiscal targets under regime M. We set $\phi_\pi = 1.5$ and $\phi_d = 0.2$ such that the economy is always in the monetary regime. The hatted variables denote the percentage deviations from the original steady state L. All vertical axes are in percentage points.

Figure 8 presents the transition dynamics. We find that the government cuts taxes in period 1 under the fiscal rule (24) with $\phi_d = 0.2$ and the higher debt target. The household raises consumption and reduces labor in period 1 due to the positive wealth effect. The government gradually issues more debt over time to achieve the higher debt target. Higher debt crowds out investment and reduces aggregate demand in the short run. Thus inflation and output decline in the short run. Under the active monetary policy with $\phi_\pi = 1.5$, the nominal interest rate decreases on impact so that the real interest rate also decreases and then both rates gradually rise to higher levels.

Because government bonds provide liquidity and are net worth in our model, higher debt eventually stimulates consumption and aggregate demand. But aggregate investment reaches a lower level than that in the initial steady state because fewer efficient entrepreneurs make investment, albeit each investing entrepreneur makes more investment. As the average investment efficiency is higher, the economy eventually reaches steady state H with higher output and higher capital.

Second, we study an experiment in which the economy can transit from steady state L with $R^r < 1+g$ to the steady state with $R^r > 1+g$. Proposition 4 shows that the economy has a unique steady state with $R^r > 1+g$ if the economy has a long-run fiscal surplus. As an example, we set the long-run surplus-to-output ratio as 4.45%. The implied steady-state debt/GDP ratio is given by $d/(4y) = 120\%$. We still choose the same policy mix $\phi_\pi = 1.5$ and $\phi_d = 0.2$, which also represents regime M and generates a unique equilibrium around the $R^r > 1+g$ steady state. Figure 8 shows that the transition dynamics are similar to those in the first experiment. The main differences are that the $R^r > 1+g$ steady state features higher consumption, capital, and output levels, and that it takes a longer time to converge. In order to reach steady state H, only 26 periods (or 6.5 years) are required, while reaching the steady state with $R^r > 1+g$ takes 143 periods (or 35.75 years) as the economy needs to accumulate more capital.

IV. Policy Interactions in a Liquidity Trap

In this section we study the impact of monetary and fiscal policy interactions when negative demand shocks cause the economy to enter a liquidity trap. We suppose that negative demand shocks originate from adverse financial shocks due to a credit crunch. As shown in Buera and Moll (2015), Cui (2016), and Buera and Nicolini (2020), a credit crunch in the form of tightening of credit/collateral constraints can cause the ZLB to bind because reducing productive entrepreneurs' borrowing capacity will reduce the supply of private bonds and hence interest rates.

To introduce a ZLB, we modify the interest rate rule (26) as

$$R_t = \max\left\{1,\ R\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi}\right\}. \tag{48}$$

We also allow the parameter μ in the credit constraints (9) to be time varying, denoted by μ_t. For simplicity we consider a perfect foresight equilibrium around steady state L following a deterministic negative shock to μ_t. In particular, let μ_t decrease by 50% for $t = 0, 1, ..., 7$ from the baseline value in Table 1 and then return to the baseline value for

$t \geq 8$. We shut down all other shocks in the model.

Figure 9 presents the dynamic responses for the following four policy mixes:

- Policy 1, Regime M: $\phi_d = 0.2$, $\phi_\pi = 1.5$.
- Policy 2, Regime M: $\phi_d = 0$, $\phi_\pi = 1.5$, debt rollover.
- Policy 3, Regime F: $\phi_d = R^r/(1+g) - 1 = -0.0031$, $\phi_\pi = 0.8$.
- Policy 4, Regime F: $\phi_d = -0.2$, $\phi_\pi = 0.8$.

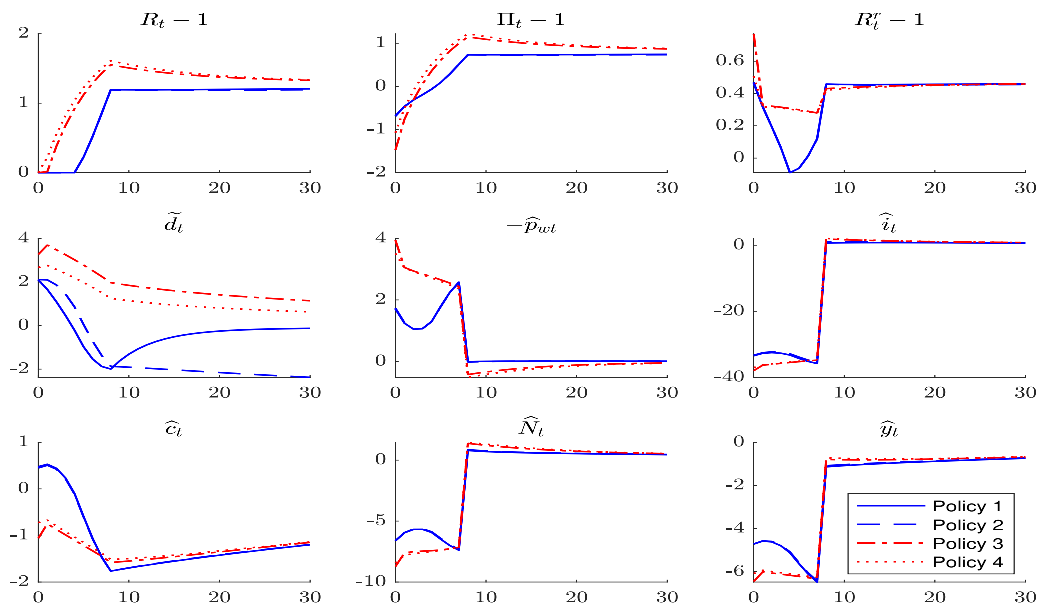

Figure 9. : Dynamic responses to a credit crunch for 4 policy mixes. All vertical axes represent percentage points.

As shown in Section III.C, the first two policy mixes represent regime M and the last two represent regime F. Each of them ensures a unique equilibrium around steady state L. Figure 9 shows that the ZLB binds for $t = 0, 1, 2, 3, 4$ under policy 1. On impact, inflation, output, investment, and labor all decline as aggregate demand falls, but consumption and public debt rise. As is well known, demand shocks cannot generate comovement among consumption, investment, and output in a standard real business cycle model. Basu and Bundick (2017) argue that the countercyclical price markup channel in a DNK model can help resolve this issue. In our model the price markup $1/p_{wt}$ rises on impact in response to the negative demand shock, causing the labor demand to fall. Thus output also falls, but this effect is not strong enough to cause consumption to decline on impact as investment declines too much under policy 1.[25]

The initial deflation raises the government real debt liabilities so that the government must raise new debt to fulfill its budget. Given $\phi_d = 0.2$ under policy 1, the government raises taxes to pay off its debt on impact. As the real interest rate is lower than the economic growth rate and declines in the short run, the real debt value falls below its steady state level before it eventually rises back to that level.

Under policy 2, there is no fiscal policy response and the government simply rolls over debt. Compared to policy 1, policy 2 implies that the real debt value stays at lower levels for a longer time. The reason is that the government cuts taxes when the real debt value falls below its steady state level under policy 1 and thus it must issue relatively more debt than under policy 2. Because public debt provides liquidity services and crowds in investment, we find that policy 1 dominates policy 2 in terms of welfare by computing household life-time utility, though the dynamic responses of aggregate quantities under policies 1 and 2 are quite similar as shown in Figure 9. Under our calibration with small financial frictions, the deviation from Ricardian equivalence is small in regime M and thus the real variables remain similar for different specifications of the fiscal rule, even with $\phi_d < 0$.

We now consider policies 3 and 4 under regime F. For both policies, there is deflation on impact, but inflation gradually rises above the target level before it returns to that level. Because the government actually cuts taxes when debt rises, it must rely on inflation to decrease the real value of public debt such that the government budget constraints can be satisfied. Because of the weak responses of the nominal interest rate ($\phi_\pi = 0.8$), the ZLB binds for 1 and 2 periods under policies 3 and 4, respectively, less frequently than under polices 3 and 4. The real value of public debt stays above its steady-state level before it returns to that level.

Consumption, investment, labor, and output all decline on impact, in contrast to polices 1 and 2 in regime M. This is due to the larger increase in the price markup under policies 3 and 4 in regime F. The short-run negative impact on the economy is larger under policies 3 and 4. But in the medium and long run, the economy recovers faster from the recession. The main reason is that higher inflation in the future stimulates aggregate demand.

We find that policies 3 and 4 in regime F dominates policies 1 and 2 in regime M in terms of welfare.[26] In regime F, future higher inflation helps shorten the ZLB episode and thus avoids the disruptive deflation spirals that can arise in regime M. We also find that policy 4 with $\phi_d = -0.2$ dominates policy 3 with $\phi_d = -0.0031$ in terms of welfare because the former policy can generate even higher future inflation. Intuitively, with a smaller ϕ_d, the government cuts taxes more heavily in response to the initial increase in public debt. This generates a larger positive wealth effect, leading to a larger increase in

[25] We find that a smaller ϕ_π causes the price markup to rise more and thus consumption is more likely to decline on impact.

[26] Billi and Walsh (2021) find a similar result in a standard DNK model with an occasionally binding ZLB, in which interest rates are higher than the economic growth rate. Away from the ZLB, Schmitt-Grohé and Uribe (2007) show that regime M dominates regime F in a standard DNK model.

future inflation. When we search policy parameters $\phi_d \in [-0.2, 0.2]$ and $\phi_\pi \in [0, 3]$ to ensure a unique equilibrium to maximize household utility (1), we find that the regime F policy mix at the corner $\phi_d = -0.2$ and $\phi_\pi = 0$ is optimal. To see why $\phi_\pi = 0$ dominates $\phi_\pi > 0$, consider the impact of a negative demand shock. For $\phi_\pi > 0$, the nominal interest rate must drop and thus the government issues less debt to satisfy its budget constraints as interest expenses decline, compared to the case of $\phi_\pi = 0$. As a result of the reduced supply of public debt, its positive wealth and liquidity effects become weaker for $\phi_\pi > 0$.

So far, we have focused on the equilibrium around steady state L with long-run fiscal deficits. As shown in Section II, the economy has another steady state with a higher real interest rate. By contrast, the economy has a unique steady state in which the real interest rate is higher than the economic growth rate if there is a long-run fiscal surplus. We have conducted a similar analysis for the unique equilibrium around each of these two steady states (see Online Appendix E). Unlike our previous results, we find that the debt rollover policy $\phi_d = 0$ is sustainable only in regime F, i.e., we must have a passive monetary policy with $\phi_\pi < 1$. But we still find that regime F dominates regime M in terms of welfare in response to adverse financial shocks. Moreover, the debt rollover policy is not optimal.

V. Conclusion

We have provided a DNK model with financial frictions to study the interactions of monetary and fiscal polices in a world with low interest rates and high public debt. The main challenge for interest rates lower than the economic growth rate is how to value public debt. Our key insight is that the public debt value contains a bubble component generated by the liquidity service. Once taking this component into account, we can modify the usual FTPL and apply the standard tool to analyze the interactions of fiscal and monetary polices.

We confirm a result in Blanchard (2019*a*) that public debt may have no fiscal cost in a world with low interest rates, i.e., debt rollover can be feasible. Such a fiscal policy can be combined with an active monetary policy to stabilize debt and inflation if the steady-state interest rate is sufficiently low. By contrast, the debt rollover policy must be combined with a passive monetary policy when interest rates are higher than the economic growth rate. In both cases, this fiscal policy is not optimal in terms of welfare, especially when the economy enters a liquidity trap.

We also find that the debt-financed tax cuts/transfers in regime M have a very small stimulative effect due to the large crowding-out effect of persistent debt. But this fiscal policy in regime F has a large stimulative effect and generates high and persistent inflation.

Our paper has focused on positive policy questions by assuming lump-sum taxes. It would be interesting to study some normative questions by assuming distortionary taxes: What are optimal monetary and fiscal policies in a world with low interest rates? What is the welfare cost of a monetary and fiscal policy given persistent primary deficits and low interest rates? We leave these questions for future research.

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Fiscal and Monetary Policy Interactions in a Model with Low Interest Rates

Online Appendix

By Jianjun Miao and Dongling Su

Appendix A. Proofs

Proof of Proposition 1. — The entrepreneur's objective is to solve the following dynamic programming problem:

$$V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) = \max_{\{I_{jt}, D_{jt}, B_{jt}\}} C_{jt} + \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}\left(K_{jt}, B_{jt}, D_{jt}, \varepsilon_{jt+1}\right), \tag{A1}$$

subject to

$$K_{jt} = (1-\delta) K_{jt-1} + \varepsilon_{jt} I_{jt}, \tag{A2}$$

$$B_{jt} \geq -\mu K_{jt-1}, \tag{A3}$$

$$C_{jt} + I_{jt} + B_{jt} + D_{jt} = R_{kt} K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}, \tag{A4}$$

$$C_{jt} \geq 0. \tag{A5}$$

We conjecture that the value function takes the following form

$$V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) = \phi_t^k\left(\varepsilon_{jt}\right) K_{jt-1} + \phi_t^b\left(\varepsilon_{jt}\right) B_{jt-1} + \phi_t^d\left(\varepsilon_{jt}\right) D_{jt-1}, \tag{A6}$$

where $\phi_t^i(\varepsilon_{jt})$, $i \in \{k, b, d\}$, satisfy

$$q_t^k = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^k(\varepsilon) dF(\varepsilon), \tag{A7}$$

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^b(\varepsilon) dF(\varepsilon), \tag{A8}$$

$$1 = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi_{t+1}^d(\varepsilon) dF(\varepsilon). \tag{A9}$$

Substituting (A2), (A4), and the above conjecture into the Bellman equation (A1), we have

$$
\begin{aligned}
& V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) \\
& = \max_{\{I_{jt}, D_{jt}, B_{jt}\}} \left(R_{kt} + (1-\delta)\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi^k_{t+1}(\varepsilon) dF(\varepsilon)\right) K_{jt-1} \\
& \quad + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1} + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi^k_{t+1}(\varepsilon) dF(\varepsilon) \varepsilon_{jt} - 1\right] I_{jt} \\
& \quad + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi^b_{t+1}(\varepsilon) dF(\varepsilon) - 1\right] B_{jt} + \left[\beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \int \phi^d_{t+1}(\varepsilon) dF(\varepsilon) - 1\right] D_{jt}.
\end{aligned} \tag{A10}
$$

Optimal choices of B_{jt} and D_{jt} imply that (A8) and (A9) must hold in equilibrium. Otherwise, all entrepreneurs will either save or borrow at the same time, contradicting the market-clearing conditions for bonds.

Since $I_{jt} \geq 0$ and $C_{jt} \geq 0$, it follows that $I_{jt} = 0$ if $\varepsilon_{jt} < 1/q^k_t \equiv \varepsilon^*_t$; but the firm makes as much investment as possible so that $C_{jt} = 0$ if $\varepsilon_{jt} > \varepsilon^*_t$. It follows from (A4) that when $\varepsilon_{jt} > \varepsilon^*_t$, we have

$$
I_{jt} = -B_{jt} - D_{jt} + R_{kt} K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}, \tag{A11}
$$

$$
D_{jt} = 0, \quad B_{jt} = -\mu K_{jt-1}. \tag{A12}
$$

Consider first the case where $\varepsilon_{jt} < \varepsilon^*_t$ and $I_{jt} = 0$. The entrepreneurs are indifferent between borrowing and saving. Substituting the decision rules into (A10) and reorganizing yield

$$
\begin{aligned}
& V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right) \\
& = \max_{\{I_{jt}, D_{jt}, B_{jt}\}} \left(R_{kt} + (1-\delta) q^k_t\right) K_{jt-1} + \frac{R_{t-1}}{\Pi_t} B_{jt-1} + \frac{R_{t-1}}{\Pi_t} D_{jt-1}.
\end{aligned}
$$

Notice that (A8) and (A9) ensure that the indeterminacy of B_{jt} and D_{jt} does not affect the value function.

Matching the coefficients, we have

$$
\begin{aligned}
\phi^k_t(\varepsilon_{jt}) &= R_{kt} + (1-\delta) q^k_t, \\
\phi^b_t(\varepsilon_{jt}) = \phi^d_t(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}.
\end{aligned}
$$

Next we consider the case where $\varepsilon_{jt} > \varepsilon^*_t$. Substituting (A11) and (A12) into (A10) and reorganizing yield

$$\begin{aligned}
&V_t\left(K_{jt-1}, B_{jt-1}, D_{jt-1}, \varepsilon_{jt}\right)\\
=&\max_{\{I_{jt}, D_{jt}, B_{jt}\}}\left(R_{kt}+(1-\delta)q_t^k+R_{kt}\left(q_t^k\varepsilon_{jt}-1\right)-\mu\left(1-q_t^k\varepsilon_{jt}\right)\right)K_{jt-1}\\
&+\frac{R_{t-1}}{\Pi_t}\left(q_t^k\varepsilon_{jt}\right)B_{jt-1}+\frac{R_{t-1}}{\Pi_t}\left(q_t^k\varepsilon_{jt}\right)D_{jt-1}.
\end{aligned}$$

Matching the coefficients yields

$$\begin{aligned}
\phi_t^k(\varepsilon_{jt}) &= R_{kt}\left(1+\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1\right)\right)+(1-\delta)q_t^k+\mu\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1\right),\\
\phi_t^b(\varepsilon_{jt}) = \phi_t^d(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}\left(q_t^k\varepsilon_{jt}\right)=\frac{R_{t-1}}{\Pi_t}\left(1+\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1\right)\right).
\end{aligned}$$

Combining the two cases above, we have

$$\begin{aligned}
\phi_t^k(\varepsilon_{jt}) &= R_{kt}\left(1+\max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1,0\right)\right)+(1-\delta)q_t^k+\mu\max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1,0\right),\\
\phi_t^b(\varepsilon_{jt}) = \phi_t^d(\varepsilon_{jt}) &= \frac{R_{t-1}}{\Pi_t}\left(1+\max\left(\frac{\varepsilon_{jt}}{\varepsilon_t^*}-1,0\right)\right),
\end{aligned}$$

for $\varepsilon_{jt}\in[\varepsilon_{\min},\varepsilon_{\max}]$. Substituting these two equations into (A7), (A8) and (A9), we obtain (13) and (14).

Finally, for the entrepreneur's objective to be finite, the value function must satisfy the following condition by the Bellman equation (A1):

$$\lim_{i\to\infty}\mathbb{E}_t\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}V_{t+i}\left(K_{j,t+i-1}, B_{j,t+i-1}, D_{j,t+i-1}, \varepsilon_{j,t+i}\right)=0.$$

Using equations (A6)-(A9) we can derive the transversality condition (16). Q.E.D.

PROOF OF LEMMA 1. — To simplify notations, we define

$$M_{t+1}=\frac{\beta\Lambda_{t+1}}{\Lambda_t},\ M_{t+1}^l=\frac{\beta\Lambda_{t+1}}{\Lambda_t}\left(1+q_{t+1}^l\right),\ x_t=\frac{D_{t-1}R_{t-1}}{\Pi_t}. \tag{A13}$$

Then we can rewrite (34) as

$$x_t=S_t+\mathbb{E}_tM_{t+1}x_{t+1}+\mathbb{E}_t\left(M_{t+1}^l-M_{t+1}\right)x_{t+1}.$$

Leading the above equation by one period and multiplying by M_{t+1}, we obtain

$$M_{t+1}x_{t+1} = M_{t+1}S_{t+1} + \mathbb{E}_{t+1}M_{t+1}M_{t+2}x_{t+2} + \mathbb{E}_{t+1}M_{t+1}\left(M^l_{t+2} - M_{t+2}\right)x_{t+2}.$$

Following similar procedures recursively until period $t+T$, we have

$$\begin{aligned} M_{t+1}M_{t+2}...M_{t+T}x_{t+T} &= M_{t+1}M_{t+2}...M_{t+T}S_{t+T} + \mathbb{E}_{t+T}M_{t+1}M_{t+2}M_{t+T+1}x_{t+T+1} \\ &\quad + \mathbb{E}_{t+T}M_{t+1}M_{t+2}...M_{t+T+1}\left(M^l_{t+T+1} - M_{t+T+1}\right)x_{t+T+1}. \end{aligned}$$

Taking conditional expectations $\mathbb{E}_t$ on the two sides of above system of $T+1$ equations and using (A13), we obtain

(A14)

$$\frac{D_{t-1}R_{t-1}}{\Pi_t} = \mathbb{E}_t\sum_{i=0}^{T}\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}S_{t+i} + \mathbb{E}_t\sum_{i=0}^{T}\frac{\beta^{i+1}\Lambda_{t+i+1}}{\Lambda_t}q^l_{t+i+1}\frac{D_{t+i}R_{t+i}}{\Pi_{t+i+1}} + \mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{T+1}}.$$

Summing over j in (16) and using the market-clearing conditions, we have

$$\lim_{i\to\infty}\mathbb{E}_t\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}\left(q^k_{t+i}K_{t+i} + D_{t+i}\right) = 0.$$

Since $K_{t+i} > 0$ and $q^k_{t+i} > 0$, we have

$$\lim_{i\to\infty}\mathbb{E}_t\frac{\beta^i\Lambda_{t+i}}{\Lambda_t}D_{t+i} = 0. \tag{A15}$$

Since $q^l_{t+1+T} \geq 0$, it follows from (14) that

$$\begin{aligned} 0 \leq \mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} &\leq \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}\beta\frac{\Lambda_{t+1+T}}{\Lambda_{t+T}}(1+q^l_{t+1+T})\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} \\ &= \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}\mathbb{E}_{t+T}\frac{\beta\Lambda_{t+1+T}}{\Lambda_{t+T}}(1+q^l_{t+1+T})\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} = \mathbb{E}_t\frac{\beta^T\Lambda_{t+T}}{\Lambda_t}D_{t+T}. \end{aligned}$$

Thus,

$$\lim_{T\to\infty}\mathbb{E}_t\frac{\beta^{T+1}\Lambda_{t+1+T}}{\Lambda_t}\frac{D_{t+T}R_{t+T}}{\Pi_{t+T+1}} = 0.$$

Taking limit in (A14) as $T\to\infty$ gives (36). Q.E.D.

PROOF OF LEMMA 2. — Taking derivative of $R_k(\varepsilon^*)$ in (39) and reorganizing yields

$$\frac{\partial R_k(\varepsilon^*)}{\partial \varepsilon^*} = \frac{\mu \int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon^*)}{\left[\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \max(\varepsilon, \varepsilon^*)\, dF(\varepsilon)\right]^2}. \tag{A16}$$

The numerator in (A16) is strictly decreasing in ε^*, with the maximum and the minimum being $\mu\mathbb{E}[\varepsilon] \geq 0$ and $-(\beta^{-1}(1+g) - 1 + \delta) < 0$, respectively. Hence, by the intermediate value theorem, there exists a unique threshold $\varepsilon_k \in [\varepsilon_{\min}, \varepsilon_{\max}]$ such that

$$\mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon_k) = 0.$$

And it follows that $\partial R_k(\varepsilon^*)/\partial \varepsilon^* > 0$ if $\varepsilon^* < \varepsilon_k$; $\partial R_k(\varepsilon^*)/\partial \varepsilon^* \leq 0$ if $\varepsilon^* \geq \varepsilon_k$. Moreover, we have $\varepsilon_k = \varepsilon_k(\mu)$ strictly increasing and $\lim_{\mu \to 0} \varepsilon_k = \varepsilon_{\min}$. Q.E.D.

PROOF OF LEMMA 3. — By Lemma 2, on $[\varepsilon_k, \varepsilon_{\max}]$, $R_k(\varepsilon^*)$ is decreasing and thus $\Phi(\varepsilon^*)$ is increasing. By (39), we compute

$$R_k(\varepsilon_k) = \frac{(1+g)/\beta - 1 + \delta - \mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) + \mu\varepsilon_k(1 - F(\varepsilon_k))}{\varepsilon_k F(\varepsilon_k) + \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}. \tag{A17}$$

By Lemma 1, we have

$$\frac{\partial R_k(\varepsilon^*)}{\partial \varepsilon^*}\Big|_{\varepsilon_k} = 0.$$

Thus,

$$\mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\beta^{-1}(1+g) - 1 + \delta)F(\varepsilon_k) = 0. \tag{A18}$$

Using this equation, we can eliminate $F(\varepsilon_k)$ in (A17) to obtain

$$R_k(\varepsilon_k) = \frac{(1+g)/\beta - 1 + \delta - \mu \int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}.$$

Substituting this expression into (40) yields

$$\Phi(\varepsilon_k) = -\frac{(\beta^{-1} - 1)(1+g)}{\int_{\varepsilon_k}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)} < 0.$$

Since $\Phi(\varepsilon_{\max}) = +\infty$ and $\Phi(\varepsilon_k) < 0$ and Φ is increasing on $[\varepsilon_k, \varepsilon_{\max}]$, it follows from the intermediate value theorem that there exists a unique value $\varepsilon_l \in (\varepsilon_k, \varepsilon_{\max})$ such that $\Phi(\varepsilon_l) = 0$.

By (39), we have

$$R_k(\varepsilon_{\min}) = \frac{(1+g)/\beta - 1 + \delta - \mu(\mathbb{E}[\varepsilon] - \varepsilon_{\min})}{\mathbb{E}[\varepsilon]}.$$

Substituting this expression into (40) yields

$$\Phi(\varepsilon_{\min}) = -\frac{(\beta^{-1}-1)(1+g) + \mu\varepsilon_{\min}}{\mathbb{E}\varepsilon} < 0.$$

When $\mu = 0$, we have

$$\Phi(\varepsilon_{\min}) = -\frac{(\beta^{-1}-1)(1+g)}{\mathbb{E}\varepsilon} < 0.$$

By (A18), ε_k is an implicit continuous function of μ and $\varepsilon_k \to \varepsilon_{\min}$ as $\mu \to 0$. By continuity, for sufficiently small $\mu \geq 0$, we have $\Phi(\varepsilon^*) < 0$ for $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_k]$. Q.E.D.

Proof of Proposition 2. — By the assumption and Lemma 3, the investment cutoff ε^* in any steady state must satisfy $\varepsilon^* \geq \varepsilon_k$. Since $\Phi(\varepsilon_l) = 0$, by (40) and setting $\varepsilon^* = \varepsilon_l$, we have $d = 0$. Thus (41) or (43) is satisfied for $s = 0$. We deduce that $\varepsilon^* = \varepsilon_l$ is the steady-state cutoff for $s = 0$. This is the only steady state with $d = 0$ because $\Phi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$ by Lemma 2.

Suppose that there is another steady state with $d > 0$ if $R^r(\varepsilon_l) > 1+g$. Then (41) implies that $R^r(\varepsilon^*) = 1+g$ for $s = 0$. Since $R^r(\varepsilon^*)$ increases with ε^* and since $R^r(\varepsilon_l) > 1+g$, we must have the steady state cutoff $\varepsilon^* < \varepsilon_l$. Since $R_k(\varepsilon^*)$ decreases with ε^* on $(\varepsilon_k, \varepsilon_l)$, it follows (40) that Φ increases with ε^* on $(\varepsilon_k, \varepsilon_l)$. Thus we have $\Phi(\varepsilon^*) < \Phi(\varepsilon_l) = 0$ for $\varepsilon^* \in (\varepsilon_k, \varepsilon_l)$, contradicting equation (40) as $d > 0$ and $R^r > 0$.

If $R^r(\varepsilon_l) < 1+g$, we show that there is another steady state with $d > 0$. It follows from (41) we must have $R^r = 1+g$. Since $R^r(\varepsilon^*)$ is a continuous and increasing function and since $R^r(\varepsilon_l) < 1+g$ and $R^r(\varepsilon_{\max}) = (1+g)/\beta > 1+g$, by the intermediate value theorem there is a unique solution $\varepsilon^* = \varepsilon_h \in (\varepsilon_l, \varepsilon_{\max})$ such that $R^r(\varepsilon^*) = 1+g$. We then have $R^r = R^r(\varepsilon_h) = 1+g$ in the steady state. It follows from (40) that $R^r d/k = \Phi(\varepsilon_h)$. Q.E.D.

Proof of Proposition 3:. — Recall that ε_l satisfies $\Phi(\varepsilon_l^*) = 0$. Total differentiating this equation and reorganizing yield

$$\frac{\mathrm{d}\varepsilon_l}{\mathrm{d}\mu} = -\frac{\left(1 + \frac{\partial R_k(\varepsilon_l)}{\partial \mu}\right)\int_{\varepsilon_l}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)}{\frac{\partial R_k(\varepsilon_l)}{\partial \varepsilon_l}\int_{\varepsilon_l}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon) - (\mu + R_k(\varepsilon_l))\varepsilon_l F'(\varepsilon_l)}.$$

By (39), we have $1 + \partial R_k(\varepsilon_l)/\partial\mu > 0$ and that $\partial R_k(\varepsilon_l)/\partial\varepsilon_l < 0$. Thus we have $\mathrm{d}\varepsilon_l/\mathrm{d}\mu > 0$. By (38), $R^r(\varepsilon^*)$ increases with ε^*. It follows that both ε_l and $R^r(\varepsilon_l)$ increase with μ.

Q.E.D.

Proof of Proposition 4:. — By Lemma 2, for a sufficiently small $\mu \geq 0$, we only need to consider steady-state the investment cutoffs in $[\varepsilon_k, \varepsilon_{\max}]$. It follows from Lemma 1 that $R_k(\varepsilon^*)$ is a decreasing function of $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$. Thus $\Phi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_k, \varepsilon_{\max}]$ by (40). We also know that $R^r(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_{\min}, \varepsilon_{\max}]$. By (43) we have

$$\Psi(\varepsilon^*) = \left[1 - \frac{1+g}{R^r(\varepsilon^*)}\right] \frac{\alpha p_w}{R_k(\varepsilon^*)} \Phi(\varepsilon^*). \tag{A19}$$

Thus $\Psi(\varepsilon^*)$ is a product of three increasing functions on $[\varepsilon_k, \varepsilon_{\max}]$. Since $\Phi(\varepsilon_l) = 0$ and $\Phi(\varepsilon^*) < \Phi(\varepsilon_l) = 0$ for $\varepsilon^* \in [\varepsilon_k, \varepsilon_l]$, we will focus on the region $[\varepsilon_l, \varepsilon_{\max}]$ as equation (40) must hold with $R^r d \geq 0$. On this region $\Phi(\varepsilon^*) \geq 0$.

Suppose that $R^r(\varepsilon_l) > 1 + g$. Then we have

$$1 - \frac{1+g}{R^r(\varepsilon^*)} > 1 - \frac{1+g}{R^r(\varepsilon_l)} > 0$$

for $\varepsilon^* > \varepsilon_l > \varepsilon_k$. Since $\Phi(\varepsilon_l) = 0$, we have $\Phi(\varepsilon^*) > 0$ for $\varepsilon^* > \varepsilon_l$. Thus, as a product of three positive increasing functions on $[\varepsilon_l, \varepsilon_{\max}]$, $\Psi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_l, \varepsilon_{\max}]$. Since $\Psi(\varepsilon_l) = 0$ and $\Psi(\varepsilon_{\max}) = +\infty$, it follows from the intermediate value theorem that there exists a unique solution $\varepsilon_p \in (\varepsilon_l, \varepsilon_{\max})$ to equation (43). Then $R^r(\varepsilon_p) > R^r(\varepsilon_l) > 1 + g$.

Suppose that $R^r(\varepsilon_l) < 1 + g$. Then Proposition 2 shows that there exists $\varepsilon_h \in (\varepsilon_l, \varepsilon_{\max})$ such that $R^r(\varepsilon_h) = 1 + g$ and $\Psi(\varepsilon_h) = 0$. Thus $R^r(\varepsilon^*) > 1 + g$ for $\varepsilon^* \in [\varepsilon_h, \varepsilon_{\max}]$ by the monotonicity of $R^r(\varepsilon^*)$. It follows that $\Psi(\varepsilon^*)$ increases with $\varepsilon^* \in [\varepsilon_h, \varepsilon_{\max}]$ because $\Psi(\varepsilon^*)$ is a product of three positive increasing functions on $[\varepsilon_h, \varepsilon_{\max}]$. The intermediate value theorem implies that there exists a unique cutoff $\varepsilon_p \in (\varepsilon_h, \varepsilon_{\max})$ such that $\Psi(\varepsilon_p) = s/y > 0$. Then we have $R^r(\varepsilon_p) > R^r(\varepsilon_h) = 1 + g$.

For $\varepsilon^* \in (\varepsilon_l, \varepsilon_h)$, we have $R^r(\varepsilon^*) < R^r(\varepsilon_h) = 1 + g$ and thus $\Psi(\varepsilon^*) < 0$. There cannot exist a steady state with $s/y > 0$ by (43). Q.E.D.

Proof of Proposition 5:. — As in the proof of Proposition 4, for a sufficiently small $\mu \geq 0$, we only need to consider the region $[\varepsilon_l, \varepsilon_{\max}]$ for the steady state investment cutoff. By assumption, $R^r(\varepsilon_l) < 1 + g$. By the proof of Proposition 4, $\Psi(\varepsilon^*)$ is positive and increases with $\varepsilon^* \in (\varepsilon_h, \varepsilon_{\max}]$. But $\Psi(\varepsilon^*)$ is negative for $\varepsilon^* \in (\varepsilon_l, \varepsilon_h)$. Moreover, $\Psi(\varepsilon_h) = \Psi(\varepsilon_l) = 0$. Let $\underline{s}$ be defined as in the proposition. By the intermediate value theorem, for any $s/y \in (-\underline{s}, 0)$, there exist at least two steady-state cutoffs ε_l^* and ε_h^* with $\varepsilon_l < \varepsilon_l^* < \varepsilon_h^* < \varepsilon_h$ such that (43) holds. Q.E.D.

Appendix B. Detrended Equilibrium System

The model exhibits long-run growth. To find a steady state and to study the dynamics around a steady state, we need to detrend the model around a long-run growth path. We consider transformations of $x_t = X_t/A_t$ for any variable $X_t \in \{K_t,\ D_t,\ S_t,\ Y_t,\ W_t,\ C_t,\ I_t\}$. For the marginal utility, we denote $\lambda_t = A_t\Lambda_t$. Then the detrended system can be summarized by the following 20 equations in 20 variables $\{R_{kt}, k_t,\ R_t,\ q_t^k,\ q_t^l,\ \varepsilon_t^*,\ d_t,\ \tau_t,\ \Pi_t,\ p_t^*,\ \Gamma_t^a,\ \Gamma_t^b,\ \Delta_t,\ w_t,\ \lambda_t,\ p_{wt},\ N_t,\ y_t,\ c_t,\ i_t\}$, where $\{R_{-1},\ \Delta_{-1},\ d_{-1},\ k_{-1}\}$ and $\{z_{mt}, z_{\tau,t}, G_{at}\}$ are given exogenously:

1) The capital return,

$$R_{kt} = \alpha\left(1+g\right)^{1-\alpha} p_{wt} k_{t-1}^{\alpha-1} N_t^{1-\alpha}. \tag{B1}$$

2) Evolution of capital,

$$(1+g)k_t = (1-\delta)k_{t-1} + \left(\left(\mu + R_{kt}\right)k_{t-1} + \frac{R_{t-1}}{\Pi_t} d_{t-1}\right)\int_{\epsilon_t^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon). \tag{B2}$$

3) The nominal interest rate,

$$1 = \frac{\beta}{1+g}\mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t}\frac{R_t}{\Pi_{t+1}}\left(1 + q_{t+1}^l\right). \tag{B3}$$

4) Tobin's Q,

(B4)

$$q_t^k = \frac{\beta}{1+g}\mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} R_{kt+1}\left(1+q_{t+1}^l\right) + \frac{\beta}{1+g}\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^k(1-\delta) + \frac{\beta\mu}{1+g}\mathbb{E}_t\frac{\lambda_{t+1}}{\lambda_t} q_{t+1}^l.$$

5) Liquidity premium,

$$q_t^l = \int_{\varepsilon_t^*}^{\varepsilon_{\max}} \left(q_t^k\varepsilon - 1\right) dF(\varepsilon). \tag{B5}$$

6) Investment cutoff,

$$\varepsilon_t^* = 1/q_t^k. \tag{B6}$$

7) Government budget constraint,

$$\frac{R_{t-1}}{\Pi_t}\frac{d_{t-1}}{1+g} = \tau_t - G_{at} + d_t. \tag{B7}$$

8) Fiscal policy rule,

$$(\tau_t - \tau)/y = \phi_d(d_{t-1} - d)/y + z_{\tau,t}. \tag{B8}$$

9) Monetary policy rule,

$$R_t = R\left(\frac{\Pi_t}{\Pi}\right)^{\phi_\pi} \exp(z_{mt}). \tag{B9}$$

10) Pricing rule,

$$p_t^* = \frac{\sigma}{\sigma - 1}\frac{\Gamma_t^a}{\Gamma_t^b}. \tag{B10}$$

11) Numerator in the pricing rule,

$$\Gamma_t^a = \lambda_t p_{wt} y_t + \beta\xi\mathbb{E}_t\left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma}\Gamma_{t+1}^a. \tag{B11}$$

12) Denominator in the pricing rule,

$$\Gamma_t^b = \lambda_t y_t + \beta\xi\mathbb{E}_t\left(\frac{\Pi_{t+1}}{\Pi}\right)^{\sigma-1}\Gamma_{t+1}^b. \tag{B12}$$

13) Evolution of inflation,

$$1 = \left[\xi\left(\frac{\Pi}{\Pi_t}\right)^{1-\sigma} + (1-\xi)\,p_t^{*1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{B13}$$

14) Price dispersion,

$$\Delta_t = (1-\xi)p_t^{*-\sigma} + \xi\left(\frac{\Pi}{\Pi_t}\right)^{-\sigma}\Delta_{t-1}. \tag{B14}$$

15) Labor demand,

$$w_t = (1-\alpha)(1+g)^{-\alpha} p_{wt} k_{t-1}^{\alpha}(N_t)^{-\alpha}. \tag{B15}$$

16) Labor supply,

$$w_t = \frac{\psi}{\lambda_t}. \tag{B16}$$

17) Marginal utility,

$$\lambda_t = 1/c_t. \tag{B17}$$

18) Aggregate output,

$$y_t = \Delta_t^{-1} (1+g)^{-\alpha} k_{t-1}^{\alpha} (N_t)^{1-\alpha}. \tag{B18}$$

19) Aggregate investment,

$$(1+g) i_t = \left((\mu + R_{kt}) k_{t-1} + \frac{R_{t-1}}{\Pi_t} d_{t-1} \right) (1 - F(\varepsilon_t^*)). \tag{B19}$$

20) Resource constraint,

$$c_t + i_t + G_{at} = y_t. \tag{B20}$$

For the real version of our model, we set $p_{wt} = 1 - 1/\sigma$, $\Pi_t = \Delta_t = 1$, and $R_t = R_t^r$ in the above system and the detrended equilibrium system consists of 13 equations (B1)-(B7), and (B15)-(B20) in 13 variables $\{R_t,\, R_{kt},\, \lambda_t,\, \varepsilon_t^*,\, q_t^k,\, q_t^l,\, w_t,\, d_t,\, k_t,\, N_t,\, y_t,\, c_t,\, i_t\}$.

Appendix C. Steady-State System

We study the nonstochastic steady state of the detrended system with s/y and Π being exogenously given. Define real interest rate as $R^r = R/\Pi$. Let variables without time subscripts denote their steady state values. By the steady-state version of (B13), we obtain $p^* = 1$. It then follows from (B14) that $\Delta = 1$. Combining (B10), (B11), and (B12), we have $p_w = 1 - 1/\sigma$, $\Gamma^a = p_w \Gamma^b = p_w \lambda y/(1 - \beta\xi)$. With w and λ being eliminated by using (B16) and (B17), and noting that $z_\tau = z_m = 0$, we obtain a steady-state system of 11 equations in 11 variables $\{R^r,\, R_k,\, \varepsilon^*,\, q^k,\, d,\, k,\, N,\, y,\, c,\, i, q^l\}$:

1) The capital return,

$$R_k = (1 - 1/\sigma)\, \alpha\, (1+g)^{1-\alpha} k^{\alpha-1} N^{1-\alpha}. \tag{C1}$$

2) Evolution of capital,

$$(g+\delta)k = \left(\left(\mu + R_k\right)k + R^r d\right)\int_{\epsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon). \tag{C2}$$

3) Nominal interest rate,

$$1 = \frac{\beta}{1+g} R^r \left(1+q^l\right). \tag{C3}$$

4) Tobin's Q,

$$q^k = \frac{\beta}{1+g} R_k \left(1+q^l\right) + \frac{\beta}{1+g} q^k (1-\delta) + \frac{\beta}{1+g} \mu q^l. \tag{C4}$$

5) Liquidity premium,

$$q^l = \int_{\varepsilon^*}^{\varepsilon_{\max}} \left(q^k \varepsilon - 1\right) dF(\varepsilon). \tag{C5}$$

6) Investment cutoff,

$$\varepsilon^* = 1/q^k. \tag{C6}$$

7) Government budget constraint,

$$\left(\frac{R^r}{1+g} - 1\right)\frac{d}{y} = \frac{s}{y}. \tag{C7}$$

8) Labor demand,

$$\psi c = \left(1 - 1/\sigma\right)\left(1-\alpha\right)\left(1+g\right)^{-\alpha} k^{\alpha} N^{-\alpha}. \tag{C8}$$

9) Aggregate output,

$$y = \left(1+g\right)^{-\alpha} k^{\alpha} N^{1-\alpha}. \tag{C9}$$

10) Aggregate investment,

$$(1+g)i = \left[\left(\mu + R_k\right)k + R^r d\right]\left(1 - F(\varepsilon^*)\right). \tag{C10}$$

11) Resource constraint,

$$c + i + G_a = y. \tag{C11}$$

As discussed in Section II, the investment cutoff ε^* can be solved for by combining (C3), (C4), (C5), (C6), and (C7). Given the inflation target Π, we obtain the nominal interest rate $R = R^r(\varepsilon^*)\Pi$. By (C6), $q^k = 1/\varepsilon^*$. By (C5), we derive q^l. With $R^r = R^r(\varepsilon^*)$, $R_k = R_k(\varepsilon^*)$ and $R^r d/k = \Phi(\varepsilon^*)$, we can determine y/k from $R_k = (1-1/\sigma)(1+g)\alpha y/k$ and $d/k = \Phi(\varepsilon^*)/R^r(\varepsilon^*)$. Noticing that equation (C10) pins down the value of i/k, we can derive $i/y = (i/k)/(y/k)$. Using the resource constraint and the exogenously given G_a/y by calibration, we obtain $c/y = 1 - G_a/y - i/y$. Dividing (C8) over (C9) and reorganizing yield the steady-state value of labor:

$$N = (1-1/\sigma)\frac{1-\alpha}{\psi}/(\frac{c}{y}).$$

Then by noting that $R_k = R_k(\varepsilon^*) = (1-1/\sigma)\alpha(1+g)^{1-\alpha}k^{\alpha-1}N^{1-\alpha}$, we can solve for k. Combining with the ratios given above, we can then determine y, d, i, c, w, and s. Finally, we have $\Gamma_a = (1-1/\sigma)\Gamma_b = (1-1/\sigma)(y/c)/(1-\beta\xi)$.

Appendix D. Linearized System

Let $\widehat{x}_t = (x_t - x)/x$ denote the log deviation from steady state for any variable x_t except for the surplus s_t and public debt d_t. For these two variables we consider level deviation relative to output, $\widetilde{d}_t = (d_t - d)/y$ and $\widetilde{\tau}_t = (\tau_t - \tau)/y$, instead of log deviation, because d may be zero and τ may be negative.

By standard linearization of the DNK model, we know the deviation of price dispersion $\widehat{\Delta}_t$ is of second-order. Thus we ignore the law of motion for the price dispersion. Moreover, the supply block can be summarized by the New-Keynesian Phillips curve. Hence, we can further eliminate $\widehat{p}_t^*$, $\widehat{\Gamma}_t^a$, and $\widehat{\Gamma}_t^b$. Then the linearized model can be summarized by a system of 16 equations in 16 variables, $\widehat{R}_{kt}$, $\widehat{k}_t$, $\widehat{R}_t$, $\widehat{q}_t^k$, $\widehat{q}_t^l$, $\widehat{\varepsilon}_t^*$, $\widetilde{d}_t$, $\widetilde{\tau}_t$, $\widehat{\Pi}_t$, $\widehat{p}_{wt}$, $\widehat{w}_t$, $\widehat{\lambda}_t$, $\widehat{N}_t$, $\widehat{y}_t$, $\widehat{c}_t$, and $\widehat{i}_t$, where $\widehat{R}_{-1}$, $\widetilde{d}_{-1}$, and $\widehat{k}_{-1}$ are predetermined, and $z_{\tau,t}$, z_{mt}, and $\widehat{G}_{at}$ are exogenous AR(1) processes:

1) The capital return,

$$\widehat{R}_{kt} = \widehat{p}_{wt} + (\alpha - 1)\widehat{k}_{t-1} + (1-\alpha)\widehat{N}_t. \tag{D1}$$

2) Evolution of capital,

(D2)
$$\begin{aligned}(1+g)\widehat{k}_t =&(1-\delta)\widehat{k}_{t-1} - \left(\mu + R_k + \frac{R^r d}{k}\right)\varepsilon^{*2} f(\varepsilon^*)\widehat{\varepsilon}^*_t \\ &+ \int_{\varepsilon^*}^{\varepsilon_{\max}} \varepsilon dF(\varepsilon)\left((\mu+R_k)\widehat{k}_{t-1} + R_k\widehat{R}_{kt} + \frac{R^r d}{k}\left(\widehat{R}_{t-1} - \widehat{\Pi}_t\right) + \frac{R^r y}{k}\widetilde{d}_{t-1}\right).\end{aligned}$$

3) Nominal interest rate,

(D3)
$$\widehat{R}_t - \mathbb{E}_t\widehat{\Pi}_{t+1} = \mathbb{E}_t\left(\widehat{\lambda}_t - \widehat{\lambda}_{t+1}\right) - \frac{q^l}{1+q^l}\mathbb{E}_t\widehat{q}^l_{t+1}.$$

4) Tobin's Q,

(D4)
$$\begin{aligned}\widehat{q}^k_t =&\mathbb{E}_t\left(\widehat{\lambda}_{t+1} - \widehat{\lambda}_t\right) + \frac{\beta}{1+g}\frac{R_k(1+q^l)}{q^k}\mathbb{E}_t\widehat{R}_{kt+1} \\ &+ \frac{\beta}{1+g}\frac{(\mu+R_k)\,q^l}{q^k}\mathbb{E}_t\widehat{q}^l_{t+1} + \frac{\beta}{1+g}(1-\delta)\mathbb{E}_t\widehat{q}^k_{t+1}.\end{aligned}$$

5) Liquidity premium,

(D5)
$$\widehat{q}^l_t = -\frac{\int_{\varepsilon^*}^{\varepsilon_{\max}}\varepsilon dF}{q^l\varepsilon^*}\widehat{\varepsilon}^*_t.$$

6) Investment cutoff,

(D6)
$$\widehat{\varepsilon}^*_t = -\widehat{q}^k_t.$$

7) Government budget constraint,

(D7)
$$\widetilde{\tau}_t + \widetilde{d}_t = \frac{G_a}{y}\widehat{G}_{at} + \frac{R^r}{1+g}\widetilde{d}_{t-1} + \frac{R^r}{1+g}\frac{d}{y}\left(\widehat{R}_{t-1} - \widehat{\Pi}_t\right).$$

8) Fiscal policy rule,

(D8)
$$\widetilde{\tau}_t = \phi_d\widetilde{d}_{t-1} + z_{\tau,t}.$$

9) Monetary policy rule,

(D9)
$$\widehat{R}_t = \phi_\pi \widehat{\Pi}_t + z_{mt}.$$

10) New-Keynesian Phillips curve,

(D10)
$$\widehat{\Pi}_t = \beta \mathbb{E}_t \widehat{\Pi}_{t+1} + \kappa \widehat{p}_{wt},$$

where $\kappa = (1-\xi)(1-\beta\xi)/\xi$.

11) Labor demand,

(D11)
$$\widehat{w}_t = \widehat{p}_{wt} + \alpha \widehat{k}_{t-1} - \alpha \widehat{N}_t.$$

12) Labor supply,

(D12)
$$\widehat{w}_t = -\widehat{\lambda}_t.$$

13) Marginal utility,

(D13)
$$\widehat{\lambda}_t = -\widehat{c}_t.$$

14) Aggregate output,

(D14)
$$\widehat{y}_t = \alpha \widehat{k}_{t-1} + (1-\alpha)\widehat{N}_t.$$

15) Aggregate investment,

(D15)
$$\begin{aligned}(1+g)\frac{i}{k}\widehat{i}_t = [1-F(\varepsilon^*)]\left[(\mu + R_k)\widehat{k}_{t-1} + R_k \widehat{R}_{kt} + \frac{R^r d}{k}\widehat{R}_{t-1} - \frac{R^r d}{k}\widehat{\Pi}_t + \frac{R^r y}{k}\widetilde{d}_{t-1}\right] \\ - \left(\mu + R_k + \frac{R^r d}{k}\right) f(\varepsilon^*)\varepsilon^* \widehat{\varepsilon}^*_t.\end{aligned}$$

16) Resource constraint,

(D16)
$$\frac{c}{y}\widehat{c}_t + \frac{i}{y}\widehat{i}_t + \frac{G_a}{y}\widehat{G}_{at} = \widehat{y}_t.$$

Appendix E. Additional Results

In this appendix we present some additional results not reported in the main text. First, Figure E1 shows the determinacy region for the steady state in which the interest rate is higher than the economic growth rate. We set the long-run $s/y = 4.45\%$ and fix other parameter values as in Table 1. The implied debt to GDP ratio is 120%.

Next we study welfare for different policy parameter mixes $\phi_d \in [-0.2, 0.2]$ and $\phi_\pi \in [0, 3]$ given adverse financial shocks as in Section IV. We consider parameter values in the set such that the model admits a unique equilibrium. Figures E2, E3, and E4 present the welfare losses in terms of the consumption equivalent relative to the steady state without the financial shock for the equilibria around the three steady states, respectively. We find that the welfare loss is the smallest when $\phi_d = -0.2$ and $\phi_\pi = 0$ in regime F.

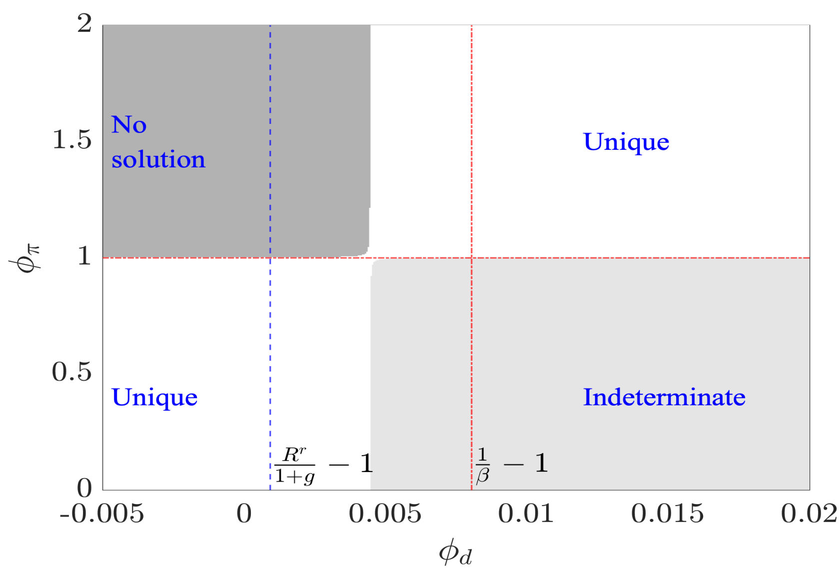

Figure E1. : Determinacy region for the steady state with $R^r > 1 + g$.

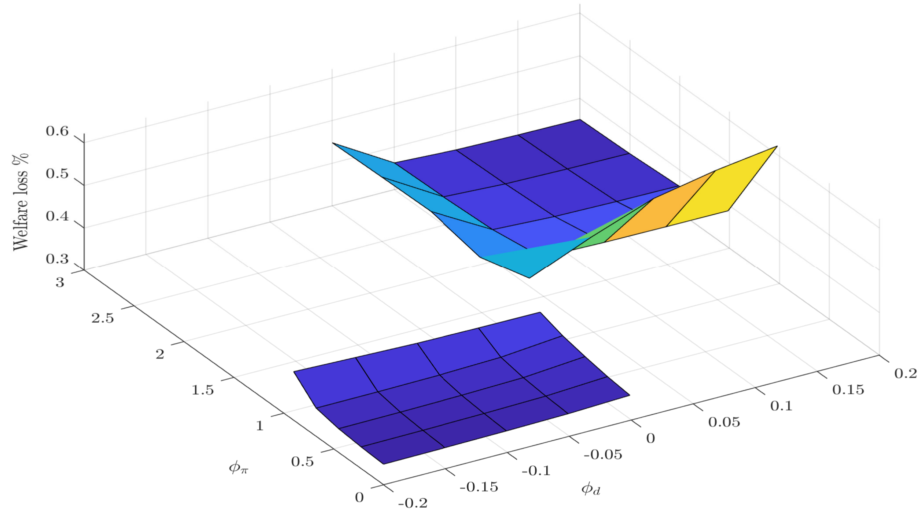

Figure E2. : Welfare loss in response to financial shocks under different policy mixes around steady state L.

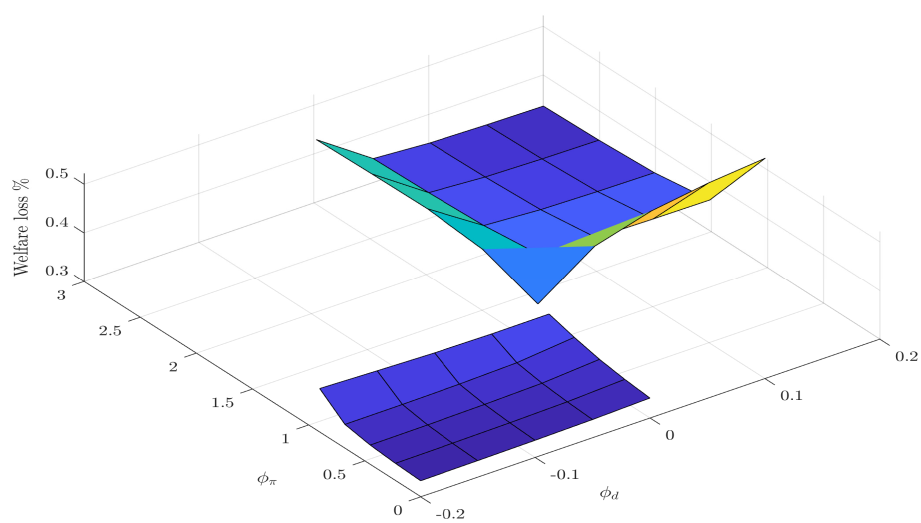

Figure E3. : Welfare loss in response to financial shocks under different policy mixes around steady state H.

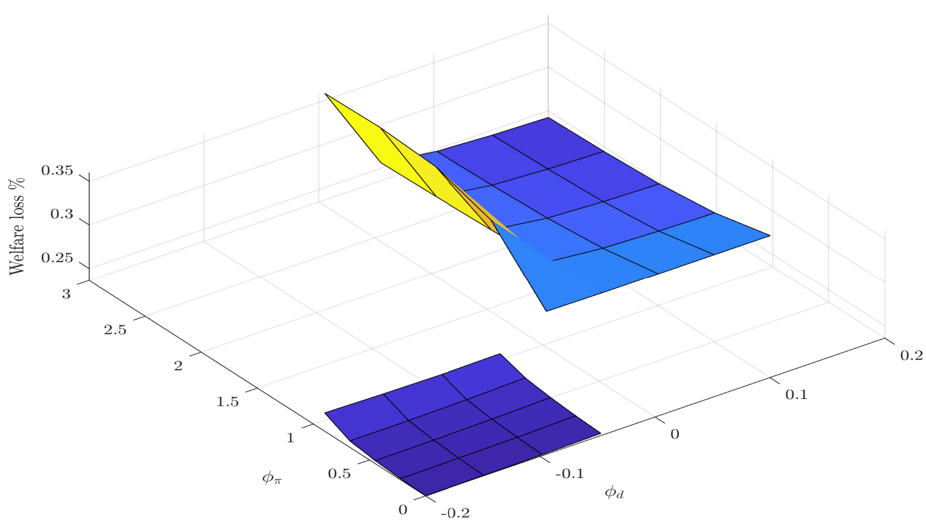

Figure E4. : Welfare loss in response to financial shocks under different policy mixes around the steady state with $R^r > 1 + g$.

Appendix F. The Standard New Keynesian Block

Retailers are monopolistically competitive. Their role is to introduce nominal price rigidities. In each period t they buy intermediate goods from entrepreneurs at the real price p_{wt} and sell good j at the nominal price P_{jt}. Intermediate goods are transformed into final goods according to the CES aggregator

$$Y_t = \left[\int_0^1 Y_{jt}{}^{\frac{\sigma-1}{\sigma}} dj \right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma > 1.$$

Thus retailers face demand given by

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\sigma} Y_t, \tag{F1}$$

where the price index is given by

$$P_t \equiv \left[\int_0^1 P_{jt}{}^{1-\sigma} dj \right]^{\frac{1}{1-\sigma}}. \tag{F2}$$

Aggregating equation (F1) yields aggregate output equation (31).

To introduce price stickiness, we assume that each retailer is free to change its price in any period only with probability $1-\xi$, following Calvo (1983). To introduce trend inflation, we follow Erceg, Henderson and Levin (2000) and assume that whenever the retailer is not allowed to reset its price, its price is automatically increased at the steady-state inflation rate. The retailer selling good j chooses the nominal price P_{jt}^* in period t to maximize the discounted present value of real profits

$$\max_{P_t^*} \sum_{k=0}^{\infty} \xi^k \mathbb{E}_t \left[\frac{\beta^k \Lambda_{t+k}}{\Lambda_t} \left(\frac{\Pi^k P_{jt}^*}{P_{t+k}} - p_{w,t+k} \right) Y_{jt+k}^* \right],$$

subject to the demand curve

$$Y_{jt+k}^* = \left(\frac{\Pi^k P_{jt}^*}{P_{t+k}} \right)^{-\sigma} Y_{t+k}, \ k \geq 0,$$

where Π denotes the steady-state inflation target. We use the household intertemporal marginal rate of substitution as the stochastic discount factor because retailers must hand in all profits to households who are the shareholders.

The first-order condition gives the pricing rule

$$P_{jt}^* = P_t^* \equiv \frac{\sigma}{\sigma - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} p_{w,t+k} P_{t+k}^{\sigma} Y_{t+k} (\Pi^k)^{-\sigma}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\xi)^k \Lambda_{t+k} P_{t+k}^{\sigma-1} (\Pi^k)^{1-\sigma} Y_{t+k}}$$

for all j. Let $p_t^* = P_t^*/P_t$. We can then write the pricing rule in a recursive form as follows

$$p_t^* = \frac{\sigma}{\sigma - 1} \frac{\Gamma_t^a}{\Gamma_t^b},$$

where

$$\begin{aligned} \Gamma_t^a &= \Lambda_t p_{wt} Y_t + \beta\xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma} \Gamma_{t+1}^a, \\ \Gamma_t^b &= \Lambda_t Y_t + \beta\xi \mathbb{E}_t \left(\frac{\Pi_{t+1}}{\Pi} \right)^{\sigma-1} \Gamma_{t+1}^b. \end{aligned}$$

It follows from (F2) and Calvo price setting that

$$1 = \left[\xi \left(\frac{\Pi}{\Pi_t} \right)^{1-\sigma} + (1-\xi)\, p_t^{*1-\sigma} \right]^{\frac{1}{1-\sigma}}.$$