Macro-Financial Volatility under Dispersed Information[*]

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Abstract

We provide a production-based asset pricing model under dispersed information. In the log-linearized equilibrium system, aggregate output and equity prices depend on the higher-order beliefs about average forecasts of aggregate demand and individual stochastic discount factors, respectively. We prove that the presence of dispersed information reduces aggregate output volatility under very general information structures, provided agents are informationally small. On the other hand, equity volatility can be arbitrarily high as the volatility of the idiosyncratic shock approaches infinity. We show our analytical results using the frequency-domain techniques. As a methodological contribution, we illustrate how a two-step spectral factorization method can be used to obtain closed-form solutions if the signal extraction problem involves more shocks than signals, applicable to a large class of models with information frictions.

Keywords: Dispersed Information, Frequency Domain Analysis, Higher-order Beliefs, Asset Pricing, Business Cycle

JEL Classifications: C63, E32, E44,G12.

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1 Introduction

Finance is littered with puzzles; one prominent and persistent puzzle is the observation by Shiller (1981) that aggregate stock prices are too volatile relative to the expected present value of dividends. Several studies have identified resolutions to the puzzle, including nonstandard preferences and nonstationary dividend processes, that argue the issue is with the inputs to the expectations operator for future dividends.[1] An alternative approach is the dispersed information environment of Kasa, Walker, and Whiteman (2014). In their model the issue is that the wrong expectations operator is used – the relevant expected value is taken using the average expectations operator, which in general does not satisfy a law of iterated expectations.

Another aspect of the equity volatility puzzle is that macroeconomic quantities – aggregate output, consumption, and dividends – are too smooth relative to equity prices. In actual economies all these quantities are endogenous and respond to the same shocks that drive equity price movements. The goal of our paper is to understand whether a simple production-based asset pricing model is able to deliver both smooth macroeconomic dynamics and highly volatile equity prices, without introducing complex exogenous shocks or nonstandard preferences. We provide a positive answer to this question by developing a model of a dispersed-information island economy along the lines of Lorenzoni (2009) and Angeletos and La'O (2010, 2013), extended to include a centralized stock market.

Maintaining dynamic and persistent information frictions is crucial for our results. Such frictions often lead to the technical problem of "forecasting the forecasts of others" (Townsend (1983)). That is, the state space for the model solution contains an infinite number of higher-order expectations so that the time-domain methods become largely intractable. Therefore we use the frequency-domain methods to circumvent this obstacle after we log-linearize the equilibrium system. The frequency domain methods have a long history in economics, and are particularly useful for dealing with incomplete information models.[2] Using tools from harmonic and complex analysis, we obtain analytical solutions for the log-linearized equilibrium system and derive the volatility measures in closed-form. These tools also allow us to prove equilibrium existence and uniqueness in a transparent manner.

The first main result of our paper is that higher-order expectations under dispersed information always reduce the volatility of business cycle fluctuations in the real economy. We establish this result by showing that the volatility of aggregate output (or consumption) under full information gives an upper bound for that under dispersed information. The key assumption for this result

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[1]See Campbell (1999) and Cochrane (2001) for surveys.

[2]Futia (1981), Hansen and Sargent (1980, 1981a, 1981b), Whiteman (1983, 1985), and Taub (1989) are among the first generation of research that employ these methods. Recent developments include Kasa (2000), Walker (2007), Bernhardt, Seiler, and Taub (2010), Makarov and Rytchkov (2012), Kasa, Walker, and Whiteman (2014), Rondina and Walker (2015), Tan and Walker (2015), and Huo and Takayama (2015).

is that agents in the economy are *informationally small* in the sense that there is a continuum of agents with private information and the idiosyncratic shock component of the private information washes out in the aggregate. Since aggregate output depends on the average forecast of aggregate demand, there is no need for any agent to forecast the behavior of any other particular agent's action to predict aggregate output. The slow learning effect brought in by signal extraction leads to dampened fluctuations, while the speculative effect of the forecasting the forecasts of others in models with finitely many agents completely vanishes due to the law of large numbers.

Next we consider equity volatility under dispersed information. To illustrate our key mechanism transparently, we adopt a univariate signal structure with aggregate and island-specific idiosyncratic total factor productivity (TFP) shocks. Our second main result demonstrates that when the volatility of idiosyncratic shocks approaches infinity, an endogenous unit root appears in the stochastic processes of investors' shareholdings and the aggregate equity price. Thus the equity price becomes infinitely volatile when the volatility of idiosyncratic shocks approaches infinity. This is the most important result of our paper and may seem surprising because aggregate equity prices only respond to aggregate shocks but not idiosyncratic shocks. The key is that the response coefficient endogenously varies with the idiosyncratic shock volatility – as idiosyncratic volatility rises, a feedback loop emerges and raises the sensitivity of equity prices to aggregate shocks. Our theoretical result has an appealing quantitative implication in that we can choose a relatively low volatility of aggregate shocks to match the low volatility of aggregate consumption and choose a relatively high volatility of idiosyncratic shocks to match the high volatility of equity prices as in the data.

The mechanism for equity volatility is through the confusion effect on the average forecast of the individual stochastic discount factors (SDFs) when agents are unable to determine whether a given change in the information signal is due to aggregate or idiosyncratic shocks. An agent's SDF is equal to the intertemporal marginal rate of substitution. Optimality of consumption and portfolio choices implies that equity prices must satisfy Euler equations for agents on all islands with their individual SDFs being used. Due to dispersed information, the average of the expected SDFs is not equal to the expected average SDFs. Thus the variation in the distribution of individual consumption matters to equity prices. Since individual shareholdings and labor supply affect individual consumption and SDFs, their responses to idiosyncratic shocks affect equity volatility.[3] As a result the effect on the equity price is different from the effect on aggregate output, which depends on the average forecast of aggregate demand instead of individual behavior.

Given the fixed supply of equity shares, equilibrium asset trading responds to idiosyncratic shocks only: an aggregate shock to buy for all agents (properly interpreted) would simply lead to a rise in equity prices, whereas an idiosyncratic shock would generate a transfer of assets be-

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[3]This intuition is similar to that in the incomplete markets models of Mankiw (1986) and Constantinides and Duffie (1996).

tween individuals (those with good idiosyncratic shocks buy from those with bad ones). When these idiosyncratic shocks are corrupted by aggregate shocks that cannot be filtered out, confusion leads to idiosyncratic volatility bleeding into equity prices because individual errors are correlated across individuals. Technically speaking, the cross-sectional integral of expectations of idiosyncratic variables does not obey the law of large numbers under dispersed information. Hence responses of individual shareholdings to idiosyncratic shocks are transmitted into aggregate equity prices via the correlated signals. We show that the confusion effect by itself has a limited impact in a two-period example. By contrast, in our dynamic model, equity prices depend on higher-order beliefs about the average forecast of future individual shareholdings, which in turn depend on the individual forecast of future equity prices due to an intertemporal hedging incentive. This dynamic interaction causes equity prices and shareholdings to be highly persistent and the persistence varies with the idiosyncratic shock volatility.

If the idiosyncratic shock volatility is very large, each agent mistakenly believes that any change in the TFP is driven almost entirely by an idiosyncratic shock and hence he acts as if the source of the shock is known, leading to trading behavior that resembles the choices under full information. We show that individual consumption and shareholdings follow a random walk under full information as in Graham and Wright (2010). Unlike the case under full information in which the permanent shifts cancel out in the cross-sectional aggregation, correlated estimation errors under dispersed information cause the permanent shifts in shareholdings to be transmitted into permanent shifts in equity prices in the limit when the idiosyncratic shock volatility approaches infinity. We decompose the equity price into a present-value component under a constant SDF (i.e, the infinite sum of higher-order expectations about future aggregate dividends) and a component of the infinite sum of higher-order expectations about individual SDFs. We find that it is the second component that generates the unit root and hence a large equity volatility.

We establish our preceding results by assuming that agents use exogenous signals only to perform forecasting in the baseline model. We extend our analysis to multivariate cases, in which agents forecast using information learned from endogenous variables. Following Taub (1989) and Rondina and Walker (2015), we suppose that agents also receive a noisy price signal for forecasting; the noise prevents the revelation of the aggregate shock information. We prove that our key insights and main results hold in this more general case.

In addition to the contributions above, we implement a two-step spectral factorization method in Rozanov (1967) to find the Wold representation for the non-square signal system. Once we have obtained the Wold representation, we can apply the Wiener-Hopf prediction formula to compute conditional expectations. The rest of our solution method follows the classical approach to solving linear rational expectations models (e.g., Whiteman (1983), Kasa, Walker, and Whiteman (2014), Rondina and Walker (2015), and Tan and Walker (2015)). Our procedure extends the existing

literature on models with private information to non-square environments with more underlying shocks than signals, and we allow signals to follow arbitrary ARMA(p,q) processes. The restriction that the numbers of signals and shocks are the same is quite limited in application. Given this restriction, equilibrium will be fully revealing unless there is non-invertibility from signals to shocks. If there are more shocks than signals, agents can never fully learn the true state of the economy and prediction errors are typically persistent.

Huo and Takayama (2015) develop a state-space spectral decomposition approach to deal with the non-square signal system. They first find a state-space representation of the signal process and then use the resulting innovation representation and factorization identity matrix to find the Wold representation. Their approach is computationally convenient since it can be solved using fast Ricatti equation methods and the Kalman filter. One drawback is that it is difficult to find an analytical solution because the Ricatti equation typically does not admit an analytical solution for high-dimensional systems. By contrast, our approach is constructive and can deliver analytical solutions in a much wider range of models.

Our paper is related to two strands of the literature. The first strand is on asset pricing under dispersed information (e.g., Bacchetta and van Wincoop (2008), Kasa, Walker, and Whiteman (2014), and Rondina and Walker (2015)).[4] Bacchetta and van Wincoop (2008) argue that the equity price volatility is reduced under dispersed information and higher-order expectations. Using the frequency domain methods, Kasa, Walker, and Whiteman (2014) show that the equity price volatility can be excessively high given two-types of agents. Their intuition is in a similar spirit of Harrison and Kreps (1978) and Scheinkman and Xiong (2003), where higher-order beliefs lead to speculative bubbles. Our paper differs from this literature in three important ways. First, in our environment with a continuum of informationally small agents, higher-order beliefs do not lead to high volatilities per se. In fact, they dampen aggregate output volatility rather than amplifying it. It is the higher-order beliefs about the average forecast of individual SDFs that generates massive fluctuations in the financial market. Second, all these papers study endowment economies in which consumption and dividends are exogenously given. They cannot address the issue of why macroeconomic quantities are too smooth relative to equity prices. Finally, many papers in this literature assume constant exogenous SDFs, whereas our SDFs are endogenous, heterogeneous, and time-varying; as noted already, this feature is key for our result.

Our paper is also related to the large literature that incorporates dispersed information into macroeconomics.[5] Important recent papers include Lorenzoni (2009), Angeletos and La'o (2010, 2013), and Benhabib, Wang, and Wen (2015). Angeletos and La'o (2013) point out that dispersed

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[4]Our paper is also related to the literature on noisy rational expectations models, which is too large for us to cite all related papers. Important contributions include Grossman and Stiglitz (1980), Hellwig (1980), Kyle (1985), Wang (1994), Bernhardt and Miao (2004), Bernhardt, Seiler, and Taub (2010), Albagli, Hellwig and Tsyvinski (2015), and Albuquerque and Miao (2015).

[5]See Angeletos and Lian (2016) for a survey.

information on its own right may dampen output volatility as our model shows. Our contribution is to formally prove this result under general assumptions in the frequency domain. To generate a large volatility, they introduce an aggregate sentiment shock, which affects higher-order expectations. Benhabib, Wang, and Wen (2015) show that endogenous information can arise when each firm observes an endogenous private signal about its demand, which in turn depends on the behavior of other firms. This literature typically focuses on business cycle dynamics instead of asset price volatilities.

Three recent papers consider both business cycles and asset prices. Benhabib, Liu, and Wang (2016) build an overlapping-generations model to show that exuberant financial market sentiments of high output and high demand for capital increase the price of capital, which signals strong fundamentals of the economy to the real side and consequently leads to an actual boom in real output and employment. Their model can also generate asymmetric nonlinear asset prices. Hassan and Mertens (2014, 2016) introduce dispersed information into dynamic stochastic general equilibrium models with physical capital. They use an approximation method in the time domain to solve their models numerically. Hassan and Mertens (2014) introduce noise traders to prevent equilibrium from fully revealing, while Hassan and Mertens (2016) replace noise traders with near rational traders who make small correlated errors. They show numerically that an exogenous aggregate shock to the conditional expectation can generate sizable variations in the equity market. Correlated mistakes in our model arise endogenously from the signal extraction problem due to higher-order expectations rather than extra exogenous shocks. Moreover, our novel analytical solutions and limiting results are transparently derived in the frequency domain.

2 Basic Intuition

We use a simple two-period model of an endowment economy to illustrate the basic intuition behind our analysis. Suppose that there is a continuum of agents indexed by $i \in I = [0, 1]$ who trade a single stock with a unit supply in period 1. The stock pays random dividends D in period 2. Each agent i is endowed with one unit of the stock and random labor income L_i in period 1. He derives utility from consumption C_{i1} and C_{i2} in the two periods according to the function

$$\mathbb{E}_i \left[\frac{C_{i1}^{1-\gamma}}{1-\gamma} + \beta \frac{C_{i2}^{1-\gamma}}{1-\gamma} \right],$$

where $\mathbb{E}_i$ denotes the conditional expectation operator, $\beta \in (0, 1)$ is the subjective discount factor, and γ is the risk aversion parameter. His budget constraints are given by

$$C_{i1} + QS_i = Q + L_i, \quad C_{i2} = DS_i,$$

where Q and S_i denote the stock price and shareholdings, respectively.

Suppose that dividends and labor income satisfy

$$\log D = \log \bar{D} + x_d \varepsilon_a, \ \log L_i = \log \bar{L} + x_l \varepsilon_i,$$

where $\bar{D}$, x_d, $\bar{L}$, and x_l are exogenous constants, and ε_a and ε_i are independent random normal variables with means zero and variances σ_a^2 and σ_i^2. Suppose that the labor income shock is purely idiosyncratic in that $\int_I \epsilon_i di = 0$.

At the beginning of period 1, each agent i receives a signal $X_i = \epsilon_a + \epsilon_i$, but does not observe ϵ_a and ϵ_i separately. All agents do not communicate their signals with each other. To prevent information revelation, we assume that agents do not use the price information to perform forecasting even though they observe the stock price. Based on his own information signal X_i, each agent i solves his utility maximization problem. We obtain the Euler equation

$$Q = \mathbb{E}_i \left[\beta \frac{C_{i1}^\gamma}{C_{i2}^\gamma} D \right].$$

In equilibrium $\int_I S_i di = 1$. It is straightforward to show that the deterministic equilibrium when $\epsilon_a = \epsilon_i = 0$ is given by

$$\bar{S}_i = 1, \ \bar{C}_{i1} = \bar{L}, \ \bar{C}_{i2} = \bar{D}, \ \bar{Q} = \beta \left(\bar{L}/\bar{D} \right)^\gamma \bar{D}.$$

Now we log-linearize the stochastic equilibrium around the deterministic equilibrium and use a lower case variable to denote its log deviation from its deterministic equilibrium value. We then obtain the log-linearized Euler equation

$$q = \mathbb{E}_i[\gamma(c_{i1} - c_{i2}) + d]. \tag{1}$$

Next we substitute the log-linearized budget constraints into this Euler equation to get the log-linearized trading strategy

$$s_i = \frac{\mathbb{E}_i[(1-\gamma)d] - q}{\gamma(1 + \bar{Q}/\bar{L})} + \frac{\mathbb{E}_i[l_i]}{1 + \bar{Q}/\bar{L}}. \tag{2}$$

This expression is akin to Merton's (1969) result: the trading strategy consists of a mean-variance efficient component and a hedging component against idiosyncratic labor income.

Substituting the log-linearized budget constraints into (1) and aggregating the resulting equation over $i \in [0,1]$ using the log-linearized market-clearing condition $\int_I s_i di = 0$, we can derive

$$q = (1-\gamma)\bar{\mathbb{E}}_i[d] + \gamma\bar{\mathbb{E}}_i[l_i], \tag{3}$$

where $\bar{\mathbb{E}}_i [\cdot] \equiv \int \mathbb{E}_i [\cdot] di$. By the Gaussian projection theorem,

$$\mathbb{E}_i \epsilon_i = \frac{\sigma_i^2}{\sigma_a^2 + \sigma_i^2} X_i \equiv \tau_i X_i, \ \mathbb{E}_i \epsilon_a = \frac{\sigma_a^2}{\sigma_a^2 + \sigma_i^2} X_i \equiv \tau_a X_i.$$

We can then show that

$$\mathbb{E}_i[d] = x_d \tau_a (\epsilon_a + \epsilon_i), \ \ \bar{\mathbb{E}}_i[d] = x_d \tau_a \epsilon_a. \tag{4}$$

This implies that, when agents are informationally small, the market average forecast of aggregate fundamentals becomes smoother than the individual forecast in that $Var\left(\bar{\mathbb{E}}_i[d]\right) < Var\left(\mathbb{E}_i[d]\right)$. It is also easy to check that $Var\left(\mathbb{E}_i[d]\right) < Var\left(d\right)$. We will show that this dampening result applies to our general dynamic model when aggregate fundamentals are endogenous (see Lemma 2). An immediate implication is that dispersed information does not help generate a large equity volatility when $\gamma = 0$. In this case equity volatility decreases with σ_i and approaches zero when $\sigma_i \to \infty$, because $\lim_{\sigma_i \to \infty} \tau_a = 0$. Thus we need risk aversion $\gamma > 0$ and hence volatile SDFs.

Consider the second term on the right side of equation (3), which comes from the average forecast of individual SDFs. If agents can communicate with each other so that information is homogenous, this term will vanish $\bar{\mathbb{E}}_i[l_i] = \mathbb{E}_i[\int_I l_i di] = 0$. Under dispersed information without communication, we have

$$\mathbb{E}_i[l_i] = x_l \tau_i (\epsilon_a + \epsilon_i), \ \ \bar{\mathbb{E}}_i[l_i] = x_l \tau_i \epsilon_a. \tag{5}$$

Thus the second term in (3) contributes additional aggregate volatility in the equity price. Using (4) and (5), we obtain the equilibrium price and shareholdings

$$q = \left[(1 - \gamma) x_d \tau_a + \gamma x_l \tau_i\right] \epsilon_a, \ \ s_i = \frac{x_l \tau_i}{1 + \bar{Q}/\bar{L}} \epsilon_i.$$

This result implies that equilibrium stock prices only respond to aggregate shocks, while equilibrium shareholdings only respond to idiosyncratic shocks. When agents are unable to distinguish between the aggregate and idiosyncratic shocks, they make correlated mistakes in forecasts of idiosyncratic labor income due to the aggregate shock ϵ_a. This correlated mistake does not average out and therefore is transmitted into the aggregate price, as shown in equations (3) and (5). However, this additional volatility only has a limited effect since $\tau_i \in (0, 1)$. Even if idiosyncratic shocks are arbitrarily volatile, aggregation cancels them out and τ_i approaches the upper bound of one. Unless we assume a very high value for x_l, the quantitative effect on equity prices will be small.

In the next section we extend this simple example to an infinite-horizon setup. We will endogenize labor income and dividends by introducing the production side of the economy so that x_d and x_l are endogenous. In the infinite-horizon model the trading strategy will include intertemporal hedging demand against future investment opportunities so that shareholdings today depend on the forecasts of future shareholdings, labor income, and equity prices. On the other hand, equity prices are forward-looking and depend on the higher-order beliefs about the average forecasts of future individual shareholdings. Interpreted through the lens of the two-period model, this dynamic interaction makes shareholdings and equity prices highly persistent and generates a positive connection between σ_i and x_l that causes equity volatility to increase without bound as $\sigma_i \to \infty$.

3 Model

We consider a variation of the classical dispersed-information (real) business cycle models of Lorenzoni (2009) and Angeletos and La'O (2010, 2013). The economy consists of a continuum of islands with a Lebesgue measure of one. Information is dispersed across islands but identical within each island. There is a representative household and a representative firm on each island. Each firm is monopolistically competitive and produces a specialized good using labor input only, while consumers have Dixit-Stiglitz preferences over varieties. Labor is immobile across islands, but consumption goods of all varieties are freely mobile. The equity market is operated through a mutual fund which owns the firms and issues equity shares to households. The stock price therefore reflects the average valuation of firms in the economy. Normalize the aggregate stock supply to one.

3.1 Information Structure

Time is discrete and indexed by $t = 0, 1, ...$ Uncertainty is generated by the TFP shock A_{it} in each island $i \in [0, 1]$, which satisfies

$$A_{it} = A_t \exp(\epsilon_{it}), \tag{6}$$

where A_t represents the aggregate component that affects all firms in all islands and ϵ_{it} represents the idiosyncratic component that is independent of A_t and affects the firm in island i only. All agents observe A_{it} at time t, but cannot distinguish between the aggregate and idiosyncratic components. Let

$$\log A_t = \rho_a \log A_{t-1} + \epsilon_{at}, \tag{7}$$

where ϵ_{at} and ϵ_{it} are identically and independently distributed over time and drawn from the normal distributions with means zero and variances σ_a^2 and σ_i^2, respectively. Moreover, assume that the law of large number (LLN) holds for ϵ_{it} so that

$$\int_I \epsilon_{it} di = 0. \tag{8}$$

Suppose that the representative household in each island i observes the current wage in its island and the current prices of specialized goods in all islands when choosing consumption varieties and labor supply. This household uses only the history of exogenous signals $\{A_{it-n}\}_{n=0}^{\infty}$ to forecast future equity prices and dividends when making shareholding decisions. The representative firm in each island i observes the current wage in its island and makes production and employment decisions before the current output price is realized. As with the household, the firm uses only the history of exogenous signals $\{A_{it-n}\}_{n=0}^{\infty}$ to form static expectations about output demand. Admittedly, our assumption on the information structure is restrictive in the sense that we rule out endogenous signals such as equity prices and aggregate dividends. But it is technically convenient as it prevents equilibrium from being fully revealing and mitigates the algebraic burden without

changing our main results.[6] In Section 7 we extend our model to study the case with signals generated by endogenous variables.

3.2 Households

A representative household on each island $i \in I \equiv [0,1]$ derives utility from the composite good consumption $\{C_{it}\}$ and labor supply $\{N_{it}\}$ according to the utility function of Greenwood, Huffman, and Hercowitz (1988):

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \log\left(C_{it} - \frac{N_{it}^{1+\phi}}{1+\phi}\right)\right],$$

where $\beta \in (0,1)$, $\phi > 0$,

$$C_{it} = \left[\int_I C_{it}(j)^{\frac{\varsigma-1}{\varsigma}} \, dj\right]^{\frac{\varsigma}{\varsigma-1}},$$

and $C_{it}(j)$ denotes the consumption of good j demanded by the household on island i. Here $\varsigma > 1$ denotes the inter-island elasticity of substitution that determines the degree of strategic complementarity.

The household faces the following intertemporal budget constraint

$$\int_I C_{it}(j) P_t(j) \, dj + Q_t S_{it+1}^h = S_{it}^h (Q_t + D_t) + W_{it} N_{it}, \tag{9}$$

where $P_t(j)$, Q_t, S_{it}^h, D_t, and W_{it} represent the price of good j, the stock price, share holdings, aggregate dividends, and the wage rate in island i, respectively.

First-order conditions give

$$W_{it} = N_{it}^\phi, \tag{10}$$

$$C_{it}(j) = \left[\frac{P_t(j)}{P_t}\right]^{-\varsigma} C_{it}, \tag{11}$$

$$\mathbb{E}_{it}\left[M_{it+1}(Q_{t+1} + D_{t+1})\right] = Q_t, \tag{12}$$

where the SDF M_{it+1} is given by

$$M_{it+1} = \frac{\beta\left(C_{it} - N_{it}^{1+\phi}/(1+\phi)\right)}{C_{it+1} - N_{it+1}^{1+\phi}/(1+\phi)}, \tag{13}$$

and

$$P_t \equiv \left[\int_I P_t(j)^{1-\varsigma} \, di\right]^{\frac{1}{1-\varsigma}}$$

is the price index of the composite good satisfying

$$\int_I C_{it}(j) P_t(j) \, dj = P_t C_{it}.$$

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[6]See Angeletos and La'O (2013) and Huo and Takayama (2015) for similar assumptions.

Here $\mathbb{E}_{it}$ denotes the conditional expectation given the infinite history of signals $\{A_{it-n}\}_{n=0}^{\infty}$. Our adopted utility function implies that the labor supply in (10) is independent of C_{it} and hence simplifies our analysis, but it is not crucial for our main results.

We normalize the composite goods price P_t to one so that the budget constraint (9) becomes

$$C_{it} + Q_t S_{it+1}^h = S_{it}^h \left(Q_t + D_t \right) + W_{it} N_{it}. \tag{14}$$

Aggregating (11) over $i \in I$ yields the total demand for good $j \in [0,1]$,

$$Y_{jt} = \int_I C_{it}(j) \, di = \left[P_t(j) \right]^{-\varsigma} Y_t, \tag{15}$$

where Y_t denotes aggregate consumption

$$Y_t = \int_I C_{it} di \equiv C_t. \tag{16}$$

3.3 Firms

The representative firm in island $i \in [0,1]$ has a production function

$$Y_{it} = A_{it} N_{it}^{\alpha}, \quad \alpha \in (0,1), \tag{17}$$

where A_{it} satisfies (6). The firm solves the static profit maximization problem

$$\pi_{it} = \max_{N_{it}} \ \mathbb{E}_{it} \left[P_t(i) \right] Y_{it} - W_{it} N_{it}$$

subject to the demand schedule in (15) for $j = i$. Since the production and labor demand choice is made before observing the output price, the firm needs to form static conditional expectation about the price $P_t(i)$ given the infinite history of signals $\{A_{it-n}\}_{n=0}^{\infty}$. Since Y_{it} and N_{it} are observable choice variables, the firm essentially forms conditional expectations about the aggregate demand Y_t. Simple algebra yields the labor demand condition

$$\alpha \left(1 - \frac{1}{\varsigma} \right) \frac{Y_{it}^{\left(1 - \frac{1}{\varsigma} \right)} \mathbb{E}_{it} \left[Y_t^{\frac{1}{\varsigma}} \right]}{N_{it}} = W_{it}. \tag{18}$$

3.4 Equilibrium Characterization in the Time Domain

There is one aggregate mutual fund that issues equity shares and collects dividends from individual islands. The aggregate dividend satisfies $D_t = \int_I \pi_{it} di$ and aggregate output satisfies $Y_t = \int_I Y_{it} di$. The mutual fund distributes the dividend to shareholders. The market-clearing condition for the stock is given by

$$\int_I S_{it}^h \, di = 1. \tag{19}$$

A competitive equilibrium with dispersed information is characterized by a system of 9 equations (10), (11), (12), (14), (15), (16) (17), (18), and (19) for 9 variables W_{it}, N_{it}, S_{it}^h, C_{it}, $C_{it}(j)$, $Y_t(j)$, $P_t(j)$, Q_t, and Y_t, where D_t satisfies

$$\int_I W_{it} N_{it} di + D_t = Y_t. \tag{20}$$

This equation follows from aggregating (14) using (16) and (19).

Since the equilibrium system is nonlinear and does not admit an explicit solution, we derive a log-linearized solution (see Appendix A). We use a lower case variable to denote the log deviation from the non-stochastic steady state. We impose the following assumption on the parameters so that there exists a unique deterministic steady-state equilibrium.

Assumption 1 *The parameter values satisfy $\alpha, \beta \in (0, 1)$, $\phi > 0$, $\varsigma > 1$.*

We first use (10), (18), and (17) to eliminate W_{it} and N_{it} to derive

$$y_{it} = \frac{1}{\xi} a_{it} + \theta \mathbb{E}_{it}[y_t], \tag{21}$$

and

$$y_{it} = a_{it} + \alpha n_{it}, \tag{22}$$

where we define

$$\xi \equiv \frac{1 + \phi - \alpha(1 - 1/\varsigma)}{1 + \phi} > 0, \ \theta \equiv \frac{\alpha}{\alpha + (1 - \alpha + \phi)\varsigma} \in (0, 1).$$

The parameter θ describes the degree of strategic complementarity (see Angeletos and La'O (2013) and Huo and Takayama (2015)). Aggregating (21) over the continuum, we have

$$y_t = \frac{1}{\xi} \int_I a_{it} di + \theta \overline{\mathbb{E}}_{it}[y_t], \tag{23}$$

where the average expectation operator is defined as

$$\overline{\mathbb{E}}_{it}[\cdot] \equiv \int_I \mathbb{E}_{it}[\cdot] di.$$

Log-linearizing (14) and (12) yields

$$\alpha_1 s_{it+1}^h = \alpha_2 s_{it}^h + \mathbb{E}_{it}\left[\alpha_3 s_{it+2}^h + \Delta b_{it+1}\right] + \mathbb{E}_{it}[\beta q_{t+1} + (1 - \beta) d_{t+1}] - q_t, \tag{24}$$

where $\Delta b_{it+1} \equiv b_{it} - b_{it+1}$ and

$$b_{it} = \alpha_4 d_t + \alpha_5 n_{it}. \tag{25}$$

Unlike the two-period model, agent i has an intertemporal hedging incentive so that his shareholdings depend on his forecasts of his future shareholdings, labor income, and equity prices. Using (10) and (20) we obtain

$$\alpha_6 d_t + \alpha_7 n_t = y_t, \tag{26}$$

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where $n_t = \int_I n_{it} di$. Here the coefficients $\alpha_1, \alpha_2, ..., \alpha_7$ are defined in Appendix A. Define the parameter $\lambda_s \equiv \alpha_2/\alpha_1$. In Appendix A we show that $\lambda_s \in (1/2, 1)$ and $\alpha_1 = \alpha_2 + \alpha_3$. These two properties are important for our results and also hold for general utility functions.

Aggregating (24) and using (19), we show that equity prices satisfy

$$q_t = \overline{\mathbb{E}}_{it} \left[\alpha_3 s^h_{it+2} + \Delta b_{it+1} \right] + \overline{\mathbb{E}}_{it} \left[\beta q_{t+1} + (1 - \beta) d_{t+1} \right]. \tag{27}$$

The first term is the average forecast of the individual SDFs, which depend on shareholdings and labor income. Iterating (27) forward, we find that the equity price is determined by an infinite number of forward-looking higher-order expectations about aggregate dividends and individual shareholdings and labor income.

In summary, we characterize the log-linearized equilibrium by a system of 6 equations (21), (22), (23), (24), (26), and (27) for 6 variables $y_{it}, n_{it}, y_t, s^h_{it}, d_t$, and q_t. We are looking for causal covariance stationary equilibrium processes.

3.5 Full Information Benchmark

Before we solve for the equilibrium under dispersed information, we present the equilibrium under full information. In this case all agents have rational expectations using all available information and hence equations (23) and (27) become

$$y_t = \frac{1}{\xi} a_t + \theta \mathbb{E}_t [y_t], \tag{28}$$

$$q_t = \mathbb{E}_t [\Delta b_{t+1}] + \mathbb{E}_t [\beta q_{t+1} + (1 - \beta) d_{t+1}], \tag{29}$$

where $b_t = \alpha_4 d_t + \alpha_5 n_t$. It follows that

$$y_t^{FI} = \frac{1}{(1 - \theta) \xi} a_t, \tag{30}$$

where a variable with a superscript "FI" denotes its full information value. We then use (21) and (26) to derive

$$n_t^{FI} = \frac{1 - (1 - \theta) \xi}{\alpha (1 - \theta) \xi} a_t, \quad d_t^{FI} = \frac{\alpha - \alpha_7 [1 - (1 - \theta) \xi]}{\alpha \alpha_6 (1 - \theta) \xi} a_t.$$

Applying the method of undetermined coefficients to (29) yields $q_t = y_t = c_t$. Thus the model under full information cannot simultaneously generate smooth output and highly volatile equity prices.

A subtle but important observation in the full information case is that the processes of individual consumption and shareholdings contain a unit root. Applying the method of undetermined coefficients to (24) under full information yields

$$s^{h,FI}_{it+1} = s^{h,FI}_{it} + \chi_s \epsilon_{it}, \quad \chi_s \equiv \frac{\alpha_5 (1/\xi - 1)}{\alpha (\alpha_1 - \alpha_2)}.$$

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This in turn implies that individual consumption possesses contain a random walk component using the log-linearized budget constraint:

$$c_{it}^{FI} = c_{it-1}^{FI} + y_t^{FI} - y_{t-1}^{FI} + \chi_c \epsilon_{it} + \left(\frac{D}{C} \chi_s - \chi_c \right) \epsilon_{it-1},$$

where $\chi_c \equiv \frac{WN}{C}(1 + \phi)\frac{1/\xi - 1}{\alpha} - \chi_s \frac{Q}{C}$, and W, N, Q, D, and C are steady state values given in Appendix A. This result is similar to that in Graham and Wright (2010), while the LLN condition (8) and the full-information assumption ensure that such permanent shifts in idiosyncratic consumption and shareholdings cancel out in the aggregate. In particular,

$$\int_I \mathbb{E}_t \left[\alpha_3 s_{it+2}^h \right] di = \mathbb{E}_t \int_I \left[\alpha_3 s_{it+2}^h \right] di = 0.$$

Under dispersed information, however, such interchange of integration operators is invalid because agents have different information sets, and the interconnection between shareholding choices and the equity price leads to our key results for the financial market.

4   Computing Expectations in the Frequency Domain

To solve the log-linearized equilibrium system under dispersed information, we need to deal with the problem of forecasting the forecast of others as revealed by equations (23) and (27). We use equation (23) to illustrate this issue. Iterating (23) yields

$$y_t = \frac{1}{\xi} \sum_{k=0}^{\infty} \theta^k \ \overline{\mathbb{E}}_{it}^{(k)} \left[\int_I a_{it} di \right] + \lim_{k \to \infty} \theta^k \overline{\mathbb{E}}_{it}^{(k)} \left[y_t \right],$$

where the k-order average expectation is the repeated integral

$$\overline{\mathbb{E}}_{it}^{(k)} [\cdot] = \underbrace{\int_I \mathbb{E}_{it} \int_I \mathbb{E}_{it} \cdots \int_I \mathbb{E}_{it} [\cdot] \underbrace{di \cdots di}_{k}}_{k}.$$

Under homogeneous information, higher-order expectations collapse to first-order expectations. Under dispersed information, aggregate output depends on an infinite number of higher-order expectations. Solving these higher-order expectations in the time domain is challenging. Therefore we adopt the frequency domain approach.

We present this approach and our extension in a general framework. In the model of Section 3 the information used for forecasting is the history of one dimensional signals $\{A_{it-n}\}_{n=0}^{\infty}$. Now suppose that the signal is an ℓ-dimensional variable X_t, defined in terms of infinite-order moving average processes.[7] Let $\mathbb{C}$ denote the complex plane, $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$.

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[7]We can extend the definition to contain information about future innovations (e.g. Bachetta and Wincoop, 2008).

Definition 1 *(signal representation) The $\ell-$dimensional real-valued signal process $\{X_t\}$ is linearly regular and admits representation*

$$\underset{\ell \times 1}{X_t} = \underset{\ell \times k}{H\left(L\right)} \underset{k \times 1}{\eta_t}, \quad \ell \leq k,$$

where L denotes the lag operator, $\{\eta_t\}$ represents structural Gaussian innovations with mean zero and covariance matrix Σ_η, and $H\left(z\right)$ is an $\ell \times k$ matrix analytic function defined on the open unit disk $\mathbb{D}$ in the matrix-valued Hardy space $\mathbf{H}^2\left(\mathbb{D}\right)$.[8]

We call $H\left(\cdot\right)$ the signal matrix or the transfer function as in the mathematics literature. In our model of Section 3, $\ell = 1$, $k = 2$, $X_t = a_{it}$,

$$\eta_t = \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \end{bmatrix}, \quad \text{and } H\left(L\right) = \begin{bmatrix} \frac{1}{1-\rho_a L} & 1 \end{bmatrix}, \tag{31}$$

for agents on island i. A key step in the solution procedure involves finding the conditional expectations in the signal extraction problem. In classical forecasting problems, conditional expectations are computed using the Wiener-Kolmogorov prediction formula, or the Hansen-Sargent formula for geometrically discounted processes. The corresponding frequency domain space for $H\left(\cdot\right)$ is $\mathbf{H}^2\left(\sqrt{\beta}\right)$, which consists of all functions that are analytic in the domain $|z| < \sqrt{\beta} < 1$ and square integrable on the boundary. This restriction allows some non-stationary processes provided they do not explode too quickly. In our model with dispersed information, the computation of conditional expectations is more involved in the sense that our problem involves non-forward-looking projections as in (23). Moreover, the model's endogenous processes are not necessarily geometrically discounting. Therefore we restrict our attention to the Hardy space $\mathbf{H}^2\left(\mathbb{D}\right)$ which ensures stationarity and resort to the well-known Wiener-Hopf prediction theory.

To simplify the signal extraction problem, it is useful to assume a maximal rank condition for the signal process so that no redundant information is contained in X_t.

Assumption 2 *The $\ell-$dimensional signal process X_t has maximal rank, i.e. the rank of its associated spectral density $f_x\left(\omega\right)$ equals its dimension:*

$$rank\left(f_x\left(\omega\right)\right) = \ell$$

for almost all $\omega \in [-\pi, \pi]$.

An important methodological contribution of our paper is that we study a non-square signal representation in that $\ell < k$. The existing literature focuses on the case of square signal representations with $\ell = k$ (e.g., Kasa, Walker, and Whiteman (2014), and Rondina and Walker (2015)). To use the Wiener-Hopf prediction formula, we need the Wold fundamental representation for the signal process. For the case of non-square signal representation, finding the Wold representation is non-trivial. We use spectral factorization techniques to solve this problem.

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[8]See online Appendix G for the definition of the Hardy space.

4.1 A Two-Step Spectral Factorization Method

Our goal is to find a Wold representation for $\{X_t\}$. Formally, we are looking for an analytic matrix function $\Gamma(\cdot)$ in the Hardy space $\mathbf{H}^2(\mathbb{D})$ such that

$$\underset{\ell \times 1}{X_t} = \underset{\ell \times \ell}{\Gamma(L)} \underset{\ell \times 1}{v_t}, \quad f_x(\omega) = \Gamma\left(e^{-i\omega}\right) \Gamma^*\left(e^{-i\omega}\right), \quad \omega \in [-\pi, \pi], \tag{32}$$

where asterisk denotes the conjugate transpose, $\{v_t\}$ is some mutually uncorrelated Wold (fundamental) innovation process with mean zero and an identity covariance matrix, f_x is the spectral density, and $\Gamma(\cdot)$ is an outer analytic function.[9] In the mathematics literature $\Gamma(\cdot)$ is also called the outer spectral factor.

For the square signal case with $\ell = k$, we can directly apply the Beurling-Blaschke factorization method to derive the Wold representation as in Kasa, Walker, and Whiteman (2014) and Rondina and Walker (2015). However, this method does not apply to the non-square case with $\ell < k$. We propose a two-step spectral factorization procedure. In step 1 we apply the covolution theorem to find the spectral density $f_x(\omega)$ of the signal process $\{X_t\}$. Then we use the Rozanov (1967) theorem to find a lower triangular decomposition of $f_x(\omega)$. In step 2 we apply the Beurling-Blaschke factorization method to the lower triangular matrix.

We start with the following result.

Lemma 1 *Suppose that X_t is the vector of signals defined in Definition 1 and Assumption 2 holds. Moreover, the transfer function $H(z)$ is a non-square rational matrix function with dimension $k > \ell$. Then the spectral density $f_x(\omega)$ is an $\ell \times \ell$ rational matrix function defined on $[-\pi, \pi]$ and*

$$f_x(\omega) = H\left(e^{-i\omega}\right) \Sigma_\eta H^*\left(e^{-i\omega}\right) = H(z) \Sigma_\eta H^T\left(\frac{1}{z}\right), \quad z = e^{-i\omega},$$

where the superscript T denotes the transpose of a matrix. Furthermore, $f_x(\omega)$ is a Hermitian normal matrix that is non-negative definite for almost all $\omega \in [-\pi, \pi]$. If we extend the definition of z to the entire complex plane $\mathbb{C}$, then the autocovariance generating function is given by $S_x(z) = H(z) \Sigma_\eta H^T(1/z)$, but without the Hermitian non-negativeness property for general $z \in \mathbb{C}$.

Lemma 1 allows us to transform the non-square signal transfer matrix function into the square spectral density matrix $f_x(\omega)$. Based on this lemma, the first step of the spectral factorization method is to decompose $f_x(\omega)$ into triangular matrix functions using Rozanov's (1967) analytical method.

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[9]Note that the Wold fundamental innovations can have non-diagonal, non-normalized covariance matrices. Using the unitary eigen-decomposition of the covariance matrix, we can obtain the orthonormal Wold representations with an identity covariance matrix.

Proposition 1 *Given an $\ell \times \ell$ rational spectral density matrix $f_x(\omega)$ with full rank almost everywhere, there exists an $\ell \times \ell$ lower triangular matrix function $\widetilde{\Gamma}(e^{-i\omega})$ whose elements are rational functions such that*

$$f_x(\omega) = \widetilde{\Gamma}(e^{-i\omega})\widetilde{\Gamma}^*(e^{-i\omega}).$$

All elements of the matrix function

$$
\widetilde{\Gamma}(z) = \begin{bmatrix}
\widetilde{\Gamma}_{11}(z) & 0 & ... & 0 \\
\widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & ... & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & ... & \widetilde{\Gamma}_{\ell\ell}(z)
\end{bmatrix}
$$

are analytic in the closed unit disk $\mathbb{T} \cup \mathbb{D}$ and hence in the $\mathbf{H}^2(\mathbb{D})$ space. Moreover, $\widetilde{\Gamma}(e^{-i\omega})$ has full rank in $\mathbb{D}$ except for at most a finite number of points.

The proof is constructive and relies on the reciprocal symmetry property of the univariate rational polynomial roots. We employ a LDU decomposition as the key intermediate step. Although the matrix operations involved are lengthy, the method is almost entirely analytical so that the resulting decomposition has a closed-form expression. Note that the matrix decomposition in Proposition 1 is similar to a Cholesky decomposition except that the diagonal elements may not be real and positive. Importantly, we need all elements in the triangular matrix $\widetilde{\Gamma}(z)$ to be in the Hardy space.

The resulting analytic matrix $\widetilde{\Gamma}(z)$ is not the Wold outer spectral factor as its determinant vanishes at finitely many points inside the unit disk. Without loss of generality, let $\{z_1, z_2, ...z_n\}$ be the finite set of distinct points such that $\det\left(\widetilde{\Gamma}(z_j)\right) = 0$, $|z_j| < 1$, $j \in \{1, 2, ...n\}$. Let $\overline{z}_j$ denote the conjugate of z_j. We assume that all zeros are of order 1 (this property is generic).

The second step of our spectral factorization method employs a multivariate version of the Beurling-Blaschke inner-outer factorization theorem to remove any zeros inside the unit disk.

Proposition 2 *The Wold outer spectral factor $\Gamma(z)$ is given by the inner-outer factorization for Hardy space functions*

$$\Gamma(z) = \widetilde{\Gamma}(z)\prod_{j=1}^{n} V_j^{-1}B_j(z),$$

where the $\ell \times \ell$ Blaschke matrices $B_j(z)$ are (inverse) inner matrix functions of the form

$$
B_j(z) = \begin{bmatrix}
1 & 0 & ... & 0 \\
0 & 1 & ... & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & ..... & \frac{1-\overline{z}_j z}{z-z_j}
\end{bmatrix},
$$

and the constant unitary matrix V_j is given by the singular value decomposition of $\widetilde{\Gamma}(z)$ evaluated at the zeros

$$\widetilde{\Gamma}(z_j) = U_j D V_j,$$

where D is a diagonal matrix containing the singular values.

The constant unitary matrices V_j remove the unwelcome poles brought in by the Blaschke factors. There are different ways of computing these matrices, and we use the eigen-decomposition method. In particular, the orthonormal column vectors of V_j can be directly picked from normalized linear independent eigenvectors of the Hermitian matrix $G_j(z_j) = \widetilde{\Gamma}^*(z_j)\widetilde{\Gamma}(z_j)$, which are automatically pairwise-orthogonal for distinct eigenvalues. For more complicated systems, the eigenvectors can be found easily using symbolic toolboxes in Matlab or Mathematica. In the online appendix, we provide a working example of a 2×3 signal system.

As an alternative to our approach, the spectral factorization can also be obtained using a state-space approach. Bart *et al.* (2007) and Lindquist and Picci (2015) contain extensive treatments of the state-space spectral factorization methods. Taub (2010) and Huo and Takayama (2014) apply this method to economics. In particular, any finite dimensional linear stochastic system with rational transfer matrix functions admits a state-space representation. Finding the outer spectral factor then involves solving an algebraic Riccati equation that links to the state space system. The Riccati equation approach has numerical advantages with fast recursive iteration algorithms, but closed-form expressions are rarely available. Moreover, the state-space representation is not unique, so any such representation will be arbitrary. Our analytical spectral factorization method is constructive and has the advantage of admitting a closed-form solution for many problems, especially for low-dimensional systems where explicit formulas exist for the roots of polynomials.

4.2 Wiener-Hopf Prediction Formula

Using the Wold representation for the signal process, we can compute the conditional expectations given the history of signals. Since in our model agents need to perform optimal linear filtering to estimate unobserved shocks, we use the Wiener-Hopf prediction formula, a generalization of the Wiener-Kolmogorov forecasting formula.

Consider any random vector Θ_t satisfying $\Theta_t = G(L)\eta_t$, where $G(z)$ is a matrix analytic function in some matrix-valued Hardy space, we wish to compute the conditional expectation $\mathbb{E}[L^m\Theta_t| \{X_{t-n}\}_{n=0}^\infty]$ given the history of signals $\{X_{t-n}\}_{n=0}^\infty$, where m is any integer. The Wiener-Hopf prediction formula gives

$$\mathbb{E}[L^m\Theta_t| \{X_{t-n}\}_{n=0}^\infty] = \Xi(L)X_t, \tag{33}$$

where the analytic matrix function $\Xi(z)$ is given by

$$\Xi(z) = \left[z^m S_{\Theta x}(z) \Gamma^{-1}\left(\frac{1}{z}\right)^T \right]_+ \Gamma^{-1}(z). \tag{34}$$

Here $\Gamma(z)$ is the Wold outer spectral factor derived in the previous subsection and $S_{\Theta x}(z) = G(z)\Sigma_\eta H^T(1/z)$ is the covariance generating function. The annihilation operator $[\cdot]_+$ is linear and is used to remove the principal part of the Laurent series expansion of the analytic functions around a common region of convergence.[10] This formula reduces to the Wiener-Kolmogorov formula when $\Theta_t = X_t$ so that $\Xi(z) = [z^m \Gamma(z)]_+ \Gamma^{-1}(z)$. If the forecasted objects follow geometrically discounted processes, the formula reduces to the Hansen-Sargent optimal prediction formula.

So far we have assumed that the signal system admits known rational transfer functions with ARMA(p,q) processes. The signal extraction problem becomes more involved if we introduce endogenous information when agents learn from endogenous variables. In this case $H(L)$ contains unknown endogenous coefficients (see Section 7). The prediction formula (33) implies that the orthogonal projection is in general a non-linear operator in the functional space of analytic functions. Due to the non-linearity in projection, the equilibrium fixed point may not be represented by rational analytic functions. In other words, model equilibria may not admit finite-dimensional Markovian dynamics. This result is akin to previous findings by Makarov and Rytchkov (2011), but is independent of the forecasting the forecast of others problem. In this case one may use the functional space techniques to determine equilibrium existence and uniqueness and resort to numerical methods for equilibrium computation.[11]

5 Business Cycle Volatility

In this section we show that the output volatility under dispersed information is lower than that under full information. This result may seem counterintuitive because speculation due to dispersed information might be expected to generate high volatility. We will show that our result is quite general and can be established without explicitly solving the model.

Conjecture that the solution for output in island i takes the following form

$$y_{it} = M_y(L)\epsilon_{at} + M_y^i(L)\epsilon_{it}, \tag{35}$$

where $M_y(z)$ and $M_y^i(z)$ are some analytic functions in $\mathbf{H}^2(\mathbb{D})$. As a convention we use $M_x(\cdot)$ to denote the frequency domain decision rule of any endogenous variable x with respect to the

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[10]See Kailath, Sayed, and Hassibi (2000) for a textbook proof of the Wiener-Hopf prediction formula. Hansen and Sargent (1980) provide a practical method of computing the annihilation operator using elementary complex analysis.

[11]The previous literature circumvents this non-linearity problem by designing a square signal system with designated zeros structure such that the Blaschke matrices and the Wold fundamental matrices are independent of the functional form of the endogenous information. In that case the orthogonal projection is linear in the signal and the model has a finite-state representation.

aggregate shock ϵ_{at}, and use $M_x^i(\cdot)$ to denote the decision rule with respect to the idiosyncratic shocks ϵ_{it}. Then aggregate output satisfies

$$y_t = \int_I y_{it} di = M_y(L)\epsilon_{at}. \tag{36}$$

We first present a lemma summarizing the behavior of higher-order expectations in our model, which is central for determining business cycle volatility when information is dispersed.

Lemma 2 *We have*

$$Var\left(\overline{\mathbb{E}}_{it}[y_t]\right) < Var\left(\mathbb{E}_{it}[y_t]\right) \le Var(y_t).$$

The second inequality is merely the orthogonality condition associated with the conditional expectation. The nontrivial part is the first inequality, whose proof is outlined here. By the Wiener-Hopf prediction formula,

$$\mathbb{E}_{it}[y_t] = a_y(L)a_{it} = a_y(L)(a_t + \epsilon_{it}),$$

where $a_y(z)$ can be computed using (34). By the LLN (8),

$$\overline{\mathbb{E}}_{it}[y_t] = a_y(L)\left(a_t + \int_I \epsilon_{it} di\right) = a_y(L)a_t.$$

By the Parseval theorem,

$$Var\left(\overline{\mathbb{E}}_{it}[y_t]\right) = \|a_y(z)\|_{\mathbf{H}^2}^2 \sigma_a^2 < \|a_y(z)\|_{\mathbf{H}^2}^2 \left(\sigma_a^2 + \sigma_i^2\right) = Var\left(\mathbb{E}_{it}[y_t]\right), \tag{37}$$

where $\|\cdot\|_{\mathbf{H}^2}$ denotes the norm in the Hardy space. The intuition is that variance of the average expectations about aggregate output is smaller than the variance of individual expectations about aggregate output, when individual agents' effect on the aggregate equilibrium is infinitesimal so that the LLN can be applied. This feature is in sharp contrast with models that assume finitely-many uninformed agents, such as Kasa, Walker, and Whiteman (2014).

Using the preceding lemma, we show that the business cycle volatility is dampened under dispersed information relative to a full-information environment.

Theorem 1 *If idiosyncratic shocks satisfy the LLN (8), then the variance of output under dispersed information is bounded above by the variance under full information*

$$Var\left(y_t^{FI}\right) > Var(y_t).$$

The proof is a simple application of the triangle inequality in Hilbert spaces. Note that this theorem is applicable to general information structures, exogenous or endogenous, univariate or multivariate. Adding confidence, noise, or higher-order sentiment shocks would also not change the

results. This is because the critical step in the proof is Lemma 2, which applies to general high-dimensional non-square signals (see Section 7). Bacchetta and van Wincoop (2008) and Angeletos and La'O (2013) infer similar results based on the variance bound in the time domain, but our theorem is the first formal statement of the result and its proof uses the frequency domain methods.

Contrary to the common intuition, here the presence of higher-order beliefs and the forecasting the forecasts of others problem dampens business cycle fluctuations. To understand the economic rationale behind this result, we can decompose the effect of dispersed information and higher-order expectations into two mechanisms. The first mechanism is associated with slow learning of the unobserved states. Slow learning creates inertia in endogenous variables, and more importantly in the higher-order average expectations of endogenous and exogenous variables, which leads to low volatility. The second mechanism is associated with the forecasting the forecasts of others. Agents have a speculative motive if other agents overreact to news. This mechanism is strong for informationally-influential participants in models with finitely many agents (Kasa, Walker, Whiteman (2014)). It is also at work in the heterogeneous prior setup (Scheinkman and Xiong (2003)). When each agent is informationally negligible as in our model, the second mechanism completely vanishes since there is no need to forecast any particular agent's forecast. What matters is the forecast of the average. Thus the first mechanism dominates and leads to the volatility bounds we deliver above.

6 Equity Price Volatility

We now turn to the financial side of the model. The main result of this section is that equity volatility will converge to infinity as the variance of the idiosyncratic TFP shock converges to infinity. In contrast to the previous section, we need to derive an explicit model solution to establish this result. We will also prove the existence and uniqueness of equilibrium by extensively using the methods developed in Section 4.

6.1 Equilibrium Solution

Conjecture that the aggregate equity price q_t can only depend on aggregate innovations,

$$q_t = M_q(L)\,\epsilon_{at}, \tag{38}$$

and individual shareholdings can only depend on idiosyncratic innovations,

$$s_{it+1}^h = M_s^i(L)\,\epsilon_{it}, \tag{39}$$

where $M_q(z), M_s^i(z) \in \mathbf{H}^2(\mathbb{D})$. The intuition for these conjectures is as follows. Suppose that q_t depends on the idiosyncratic innovations; since the asset-pricing equation (27) implies that idiosyncratic innovations will be averaged out by the LLN, we have a contradiction. Similarly, suppose

that s_{it+1}^h depends on aggregate innovations; then aggregate shareholdings will be stochastic, contradicting the assumption of a constant stock supply. The following result delivers the link between equity prices and individual shareholdings.

Proposition 3 *In equilibrium we have*

$$M_q(z) = \alpha_1 \frac{1 - \lambda_s z}{1 - \rho_a z} M_s^i(z). \tag{40}$$

This proposition shows how the equity price is endogenously determined by the trading behavior of individual investors under dispersed information. In particular, it indicates the connection between the trading volume and the equity price volatility. If investors frequently adjust their shareholding positions, the equity price will become more volatile. However, this relation vanishes under full-information, as the cross-sectional aggregation results in an equilibrium equity price that is close to the representative agent case.

Theorem 2 *There is a unique equilibrium under dispersed information in which individual output, aggregate output, equity prices, and shareholdings admit rational function representations in the frequency domain given in equations (35), (36) (38), and (39).*

In Appendix C we provide an explicit solution to the equilibrium. In particular we prove that

$$M_q(z) = \frac{(1 - \lambda_s z) N_q(z)}{\sigma_m \rho_a P_q(z)}, \tag{41}$$

where σ_m is some constant, $P_q(z) = (1 - m_1 z)(1 - m_2 z)(1 - m_3 z)(1 - m_4 z)$ is a polynomial function satisfying $|m_1| \geq |m_2| > 1 > |m_3| \geq |m_4|$, and $N_q(z)$ is a real rational analytic function in the closed unit disk. Despite the presence of the infinite number of higher-order expectations formed by agents, the ARMA(p,q) representation of (41) allows us to compute the equity price volatility in closed-form via the integral method using the Parseval theorem. More importantly, the explicit expression also highlights some crucial analytical properties of the equity price fluctuations under dispersed information. We are particularly interested in the limit property as $\sigma_i \to \infty$.

6.2 Limiting Result

In Appendix C we prove the following result.

Theorem 3 *For any finite $\sigma_a \in (0, \infty)$, we have*

$$\lim_{\sigma_i \to \infty} Var(q_t) = \sigma_a^2 \lim_{\sigma_i \to \infty} \|M_q(z)\|_{\mathbf{H}^2}^2 \to \infty.$$

This theorem is somewhat surprising. Although idiosyncratic shocks have no effect on the movement of the equity price, the equity price becomes arbitrarily volatile as the volatility of the

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idiosyncratic shock approaches infinity given any finite aggregate TFP volatility. This result is independent of the values of ρ_a and σ_a. Therefore, our model is able to generate a highly volatile equity price, while the aggregate shock variance σ_a^2 can be chosen to match the low volatility of aggregate consumption.

To illustrate the quantitative implication of this result, we calibrate our model at quarterly frequency. We set $\beta = 0.99$, $\alpha = 0.67$, $\varsigma = 9$ (implying a steady-state markup of 12.5%), $\phi = 1$ (implying a unitary Frisch elasticity of labor supply), $\sigma_a = 0.7\%$, and $\rho_a = 0.9$. Panel A of Figure 1 plots the four roots in modulus of $P_q(z)$ against σ_i. The panel shows that the two roots inside the unit circle are real numbers for all $\sigma_i > 0$, but the two roots outside the unit circle are complex conjugates for small σ_i. As σ_i approaches infinity, the smaller root $(1/m_3)$ outside the unit circle converges to a unit root. Panels B and C of Figure 1 plot equity volatility and output volatility against σ_i. We already showed that equity volatility and output volatility are the same and independent of σ_i under full information. But equity volatility rises with σ_i and approaches infinity as $\sigma_i \to \infty$ under dispersed information, while output volatility decreases with σ_i and approaches to a finite number. Thus our model under dispersed information can match the low output volatility and the high equity volatility observed in the data by choosing a suitable value for σ_i. For example, output volatility and equity volatility are equal to 2.3% and 13% when $\sigma_i = 10\%$.
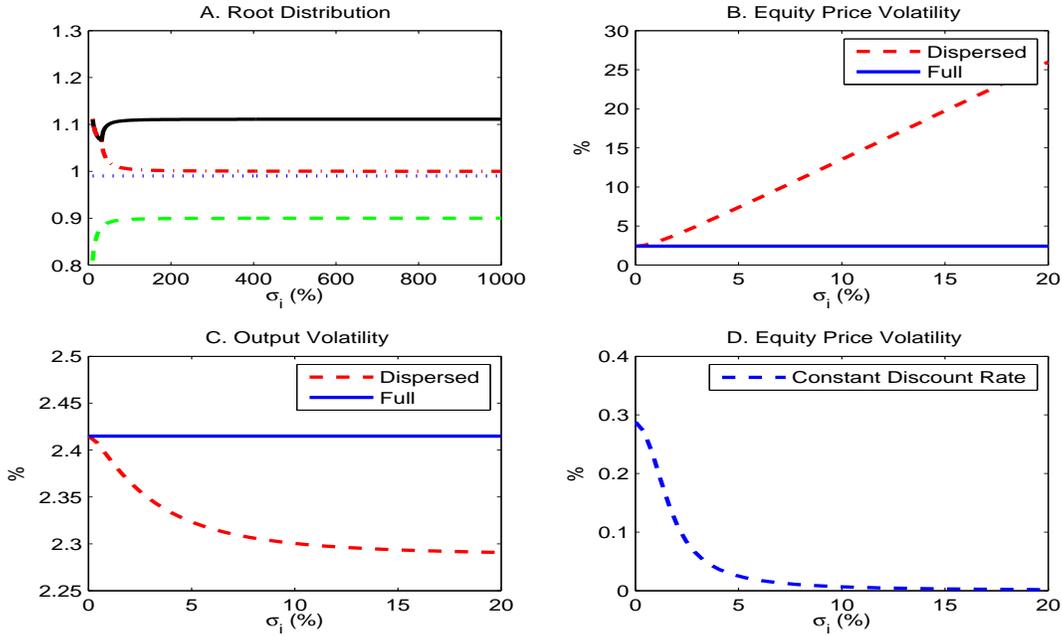
Figure 1: This figure plots the root distribution, equity volatility, and output volatility as functions of σ_i.

To understand the economic mechanism generating the high equity price volatility, we rewrite (27) as

$$q_t = \int_I \mathbb{E}_{it} \left[\beta q_{t+1} + (1 - \beta) d_{t+1} \right] di + \int_I \mathbb{E}_{it} m_{it+1} di. \tag{42}$$

where we can show that

$$\int_I \mathbb{E}_{it} \left[m_{it+1} \right] di = \int_I \mathbb{E}_{it} \left[\alpha_3 s_{it+2}^h + \Delta b_{it+1} \right] di,$$

$b_{it} = \alpha_4 d_t + \alpha_5 n_{it}$, and $\Delta b_{it+1} = b_{it} - b_{it+1}$. Iterating forward gives

$$
\begin{aligned}
q_t &= \left(\bar{\mathbb{E}}_{it} \left[m_{it+1} \right] + \beta \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \left[m_{it+2} \right] + \beta^2 \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \bar{\mathbb{E}}_{it+2} \left[m_{it+3} \right] + ... \right) \\
&\quad + (1 - \beta) \left(\bar{\mathbb{E}}_{it} \left[d_{t+1} \right] + \beta \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \left[d_{t+2} \right] + \beta^2 \bar{\mathbb{E}}_{it} \bar{\mathbb{E}}_{it+1} \bar{\mathbb{E}}_{it+2} \left[d_{t+3} \right] + ... \right).
\end{aligned}
$$

Thus the equity price consists of a present-value component under a constant SDF (i.e, the infinite sum of higher-order expectations about future aggregate dividends) and a component of the infinite sum of higher-order expectations about individual SDFs. Panel D of Figure 1 shows the volatility of the present-value component and indicates that it decreases with σ_i and approaches zero. Thus higher-order expectations about aggregate dividends and the failure of law of iterate expectations do not lead to excess volatility per se. The intuition is similar to that described in Sections 2 and 5. We now turn to the component of higher-order expectations about individual SDFs, which depend on the average forecast of individual shareholdings and labor income.

Unlike in the case of full information studied in Section 3.5, the average forecast of individual shareholdings is not equal to the forecast of the average shareholdings,

$$\int_I \mathbb{E}_{it} \left[s_{it+2}^h \right] di \neq \mathbb{E}_{it} \int_I \left[s_{it+2}^h \right] di = 0, \quad \forall i \in [0, 1].$$

Thus individual shareholding choices affect aggregate equity prices.

Since the equity shares are in constant supply, investors respond to idiosyncratic TFP shocks instead of aggregate TFP shocks. Due to incomplete information, they interpret the innovation in the signal of the (composite) TFP as a change in their private opportunities. This confusion induces investors to adjust their shareholdings accordingly. When the volatility of the idiosyncratic TFP shock is large relative to the volatility of the aggregate TFP shock, the volatility of individual shareholdings is also large. By Proposition 3, the individual responses to the idiosyncratic TFP shock are transmitted to the aggregate stock prices, causing a large equity volatility. By Lemma 8, an endogenous unit root arises when $\sigma_i \to \infty$. This channel accounts for the autoregressive component in the equity price process. On the other hand, since b_{it} contains an idiosyncratic component n_{it}, the volatility of $\int_0^1 \mathbb{E}_{it} \left[\Delta b_{it+1} \right] di$ does not converge to zero as $\sigma_i \to \infty$. In other words, this component ensures a positive lower bound for the moving average component in the equity price process. Combining the preceding autoregressive and moving average components

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delivers the result in Theorem 3. We should emphasize that there is a key assumption for our result. That is, the elasticity of labor supply $1/\phi > 0$ must be finite so that $\alpha_5 \neq 0$. If $\phi = 0$, then $\alpha_5 = 0$ by appendix A. In this case, the moving average component approaches zero as $\sigma_i \to \infty$, so that the limiting result in Theorem 3 is not valid.

7   Extension with Endogenous Information

In this section we will show that the striking results we obtained previously do not hinge on the simplified assumption of univariate exogenous signal structure by extending our analysis to the case of multivariate signals with endogenous information. A natural endogenous information signal is the equity price. One key issue of incorporating the additional information contained in the equity price is that the equity price will reveal the aggregate shock information so that the representative agent in each island will not face the signal extraction problem. One way to prevent equilibrium from revealing such information is to impose assumptions such that the signal matrix is non-invertible as in Rondina and Walker (2015). The other way is to introduce an additional shock so that the number of signals remains smaller than the number of shocks. We will adopt the second approach and follow Taub (1989) and Rondina and Walker (2015) by introducing an aggregate noise shock with a particular structure that allows us to still solve the model analytically.

Following Lorenzoni (2009) and Rondina and Walker (2015), we suppose that the agent in each island i observes noisy equity prices so that his signal system is given by

$$X_{it} = \begin{bmatrix} \epsilon_{at} + \epsilon_{it} \\ q_t + u_t \end{bmatrix},$$

where the first signal is the sum of aggregate and idiosyncratic TFP innovations and the second is a noisy signal of the equity price. Here u_t represents an additional aggrgegate shock. In the literature on rational expectations equilibria with private information, there are various ways of engineering additional noises that serve the same purpose of preventing full information revelation (e.g. introducing additional structural shocks, noise traders, or random matching processes). Our formulation ensures a tractable spectral factorization that can be implemented using the analytical method we derived earlier.

We assume that the aggregate TFP shock a_t and the noise shock u_t have the general representation

$$a_t = a\,(L)\,\epsilon_{at},\ \ u_t = u(L)\epsilon_{ut},$$

where $a(z)$ and $u(z)$ are (rational) analytic functions in $\mathbf{H}^2(\mathbb{D})$ space and $\{\epsilon_{ut}\}$ is an IID Gaussian process with mean zero and variance σ_u^2. The AR(1) process in (7) is a special case. As discussed in Section 3.1, we also assume that there are other endogenous information that are observable to agents but not used in the signal extraction problem.

To solve the model, we first define the conditional expectations that form agents' forecasts of discounted future returns in the stock market

$$\chi_{it} \equiv \mathbb{E}_{it}\left[\alpha_3 s^h_{it+2} + \Delta b_{it}\right] + \mathbb{E}_{it}\left[\beta q_{t+1} + (1-\beta) d_{t+1}\right].$$

Using the Wiener-Hopf prediction formula, we can write

$$\chi_{it} = \pi(L)X_{it} = \pi_1(L)\left(\epsilon_{at} + \epsilon_{it}\right) + \pi_2(L)\left(q_t + u_t\right), \tag{43}$$

where $\pi_1(z)$ and $\pi_2(z)$ are endogenous rational functions to be determined. Since $q_t = \int_I \chi_{it} di$, we have

$$q_t = \frac{\pi_1(L)}{1 - \pi_2(L)}\epsilon_{at} + \frac{\pi_2(L)}{1 - \pi_2(L)}u(L)\epsilon_{ut}. \tag{44}$$

We can then write the signal representation as

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} 1 & 1 & 0 \\ \frac{\pi_1(L)}{1-\pi_2(L)} & 0 & \frac{u(L)}{1-\pi_2(L)} \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix}. \tag{45}$$

We apply our method in Section 4 to derive the Wold representation for the preceding non-square signal system. To simplify the computation of spectral factorization, we will impose an assumption on $u(z)$ such that $u(z) = \pi_1(z)$, originally suggested by Taub (1989) and applied by Rondina and Walker (2015). We can then establish an equilibrium existence and uniqueness result formally stated in online Appendix D.

Do our two main results, Theorems 1 and 3, also hold in the model of this section? Consider Theorem 1 first, whose proof relies critically on Lemma 2. To see this lemma also holds in the extended model, we apply the Wiener-Hopf prediction formula and (45) to derive that

$$\mathbb{E}_{it}[y_t] = M_y(L)\eta_{it} = M^a_y(L)\epsilon_{at} + M^i_y(L)\epsilon_{it} + M^u_y(L)\epsilon_{ut},$$

where $M_y(z)$, $M^a_y(z)$, $M^i_y(z)$, and $M^u_y(z)$ are some functions to be determined in Appendix D. As long as the LLN (8) holds, we can apply the previous argument as in (37) to show that $Var\left(\overline{\mathbb{E}}_{it}[y_t]\right) < Var\left(\mathbb{E}_{it}[y_t]\right)$. Thus Lemma 2 and Theorem 1 still hold in our extended model in this section.

Next consider Theorem 3. Since we have introduced an additional aggregate noise shock u_t into the model, we decompose the equity price in (44) as $q_t = q^f_t + q^n_t$, where

$$q^f_t \equiv \frac{\pi_1(L)}{1 - \pi_2(L)}\epsilon^a_t \text{ and } q^n_t = \frac{\pi_2(L)}{1 - \pi_2(L)}u(L)\epsilon_{ut}$$

represent the components driven by the fundamental TFP shock and by the aggregate price noise, respectively. In online Appendix D we formally prove that

$$\lim_{\sigma_i \to \infty} \|\pi_1(z)\|_{\mathbf{H}^2} = \infty \text{ and } \lim_{\sigma_i \to \infty} Var(q^f_t) = \sigma^2_a \lim_{\sigma_i \to \infty} \left\|\frac{\pi_1(z)}{1 - \pi_2(z)}\right\|^2_{\mathbf{H}^2} = \infty.$$

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This result says that the volatility of the fundamental-driven equity price component explodes to infinity as $\sigma_i \to \infty$. But due to the assumption of $u(z) = \pi_1(z)$, the volatility of the noise component of the equity price also approaches infinity. To isolate the effect of the exogenous price noise, we assume that $u(L) = \frac{\pi_1(L)}{\sigma_i}$. This normalization ensures that the price noise volatility remains positive and finite in the limit as $\sigma_i \to \infty$. In this case the previous results still hold except that we replace σ_u with $\frac{\sigma_u}{\sigma_i}$. Then we can decompose the equity price volatility using the Parseval theorem as

$$Var(q_t) = \left\| \frac{\pi_1(z)}{1 - \pi_2(z)} \right\|_{\mathbf{H}^2}^2 \sigma_a^2 + \left\| \frac{\pi_2(z)\pi_1(z)}{1 - \pi_2(z)} \right\|_{\mathbf{H}^2}^2 \frac{\sigma_u^2}{\sigma_i^2}.$$

While the fundamental-driven equity price volatility explodes as $\sigma_i \to \infty$, the volatility of the noise-driven component approaches to a finite limit. Thus the arbitrarily large equity price volatility in our model as $\sigma_i \to \infty$ is not driven by the exogenous aggregate price noise.

The intuition for the model in this section is similar to that for the model in Section 3. The equity price depends on higher-order expectations about the average forecasts of individual SDFs and hence individual shareholdings. As long as the equity price does not reveal aggregate shocks so that each agent faces the signal extraction problem, the higher expectations about shareholdings cannot be averaged out. Thus the persistent movement of shareholdings is transmitted into the persistent movement of equity prices due to the confusion mechanism. As $\sigma_i \to \infty$, the shareholding process contains a unit root, which is transmitted into the unit root in the equity price process, so that equity volatility approaches infinity.

8 Conclusion

We have developed a model of a production economy with dispersed information that features smooth aggregate consumption dynamics and highly volatile equity prices. The key elements of our model are not assumptions on nonstandard preferences, bubbles, or sentiments, but the introduction of dispersed information and the endogeneity of SDFs that are time-varying and heterogeneous across population. The key for our model result is due to the different impact of the higher-order beliefs about the average forecasts of aggregate demand and the individual SDFs, together with the dynamic interaction between shareholdings and equity prices. As a technical contribution, our two-step spectral factorization methodology can be applied to many other contexts that involves solving signal extraction problems with non-square systems.

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Appendix

A   Steady State and Log-linearized System

We can easily show that the (symmetric) deterministic steady state is given by

$$
\begin{aligned}
N_i &= N = \left(\alpha\left(1-\frac{1}{\varsigma}\right)\right)^{\frac{1}{\phi-\alpha+1}}, \\
Y_i &= C_i = C = Y = \left(\alpha\left(1-\frac{1}{\varsigma}\right)\right)^{\frac{\alpha}{\phi-\alpha+1}}, \\
D &= \left(1-\left(1-\frac{1}{\varsigma}\right)\alpha\right)\left(\alpha\left(1-\frac{1}{\varsigma}\right)\right)^{\frac{\alpha}{\phi-\alpha+1}}, \\
Q &= \frac{\beta}{1-\beta}D, \ S_i^h = 1, \\
W_i &= W = \left(1-\frac{1}{\varsigma}\right)\alpha N^{\alpha-1}.
\end{aligned}
$$

Given Assumption 1, all equilibrium variables are positive and

$$
C - \frac{N^{1+\phi}}{1+\phi} > 0.
$$

Thus a unique deterministic steady-state equilibrium exists.

Log-linearizing equation (10) yields $w_{it} = \phi n_{it}$. Using this equation and log-linearizing the budget constraint, we obtain

$$
\begin{aligned}
Cc_{it} + Qs_{it+1}^h &= (Q+D)\,s_{it}^h + Dd_t + WN\,(n_{it}+w_{it}) \\
&= (Q+D)\,s_{it}^h + Dd_t + WN\,(1+\phi)\,n_{it}. \tag{A.1}
\end{aligned}
$$

Log-linearizing the Euler equation (12) yields

$$
q_t = \mathbb{E}_{it}\left[\beta q_{t+1} + (1-\beta)\,d_{t+1}\right] + \mathbb{E}_{it}m_{it+1}. \tag{A.2}
$$

Log-linearizing the SDF yields

$$
\left(C - \frac{N^{1+\phi}}{1+\phi}\right)m_{it+1} = C\,(c_{it}-c_{it+1}) - N^{\phi+1}\,(n_{it}-n_{it+1}).
$$

Substituting c_{it} from the budget constraint (A.1) into the preceding equation yields

$$
\begin{aligned}
\left(C - \frac{N^{1+\phi}}{1+\phi}\right)m_{it+1} &= (Q+D)\,s_{it}^h + Dd_t + WN\,(1+\phi)\,n_{it} - Qs_{it+1}^h \\
&\quad - (Q+D)\,s_{it+1}^h - Dd_{t+1} - WN\,(1+\phi)\,n_{it+1} + Qs_{it+2}^h \\
&\quad - N^{\phi+1}\,(n_{it}-n_{it+1}) \\
&= (Q+D)\,s_{it}^h - (2Q+D)\,s_{it+1}^h + Qs_{it+2}^h \\
&\quad + \left[WN\,(1+\phi) - N^{\phi+1}\right](n_{it}-n_{it+1}) + D\,(d_t - d_{t+1}).
\end{aligned}
$$

Plugging this equation into (A.2) yields

$$\frac{2Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} s^h_{it+1} = \frac{Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} s^h_{it} + \mathbb{E}_{it} \left[\frac{Q}{C - \frac{N^{1+\phi}}{1+\phi}} s^h_{it+2} + \Delta b_{it+1} \right] + \mathbb{E}_{it} \left[\beta q_{t+1} + (1 - \beta) d_{t+1} \right] - q_t$$

where $\Delta b_{it+1} = b_{it} - b_{it+1}$ and

$$b_{it} = \frac{D}{C - \frac{N^{1+\phi}}{1+\phi}} d_t + \frac{\left[WN(1 + \phi) - N^{\phi+1} \right]}{C - \frac{N^{1+\phi}}{1+\phi}} n_{it}.$$

We define

$$\alpha_1 = \frac{2Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \quad \alpha_2 = \frac{Q + D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0,$$

$$\alpha_3 = \frac{Q}{C - \frac{N^{1+\phi}}{1+\phi}} > 0, \quad \alpha_4 = \frac{D}{C - \frac{N^{1+\phi}}{1+\phi}} > 0,$$

$$\alpha_5 = \frac{(1 + \phi) WN - N^{\phi+1}}{C - \frac{N^{1+\phi}}{1+\phi}}.$$

Log-linearizing (17) yields

$$y_{it} = a_{it} + \alpha n_{it} \implies n_{it} = \frac{1}{\alpha} (y_{it} - a_{it}).$$

Aggregating leads to $n_t = \frac{1}{\alpha} (y_t - a_t)$. Log-linearizing (20) yields $y_t = \alpha_6 d_t + \alpha_7 n_t$, where

$$\alpha_6 = \frac{D}{Y}, \quad \alpha_7 = \frac{(1 + \phi) WN}{Y}.$$

We can show that $\alpha_1 = \alpha_2 + \alpha_3$ and

$$\lambda_s \equiv \frac{\alpha_2}{\alpha_1} = \frac{Q + D}{2Q + D} \in \left(\frac{1}{2}, 1 \right).$$

We have shown that this result also holds for general utility functions. The analysis is available upon request.

B   Proofs of Results in Section 5

Proof of Lemma 2:   By the Weiner-Hopf prediction formula,

$$\mathbb{E}_{it} [y_t] = a_y(L) a_{it} = a_y(L) (a_t + \epsilon_{it}),$$

where $a_y(z)$ can be computed using (34). Thus

$$\overline{\mathbb{E}}_{it} [y_t] = a_y(L) \left(a_t + \int_I \epsilon_{it} di \right) = a_y(L) a_t.$$

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By the Parseval theorem,

$$Var\left(\overline{\mathbb{E}}_{it}\left[y_t\right]\right) = \|a_y\left(z\right)\|_{\mathbf{H}^2}^2 \sigma_a^2 < \|a_y\left(z\right)\|_{\mathbf{H}^2}^2 \left(\sigma_a^2 + \sigma_i^2\right) = Var\left(\mathbb{E}_{it}\left[y_t\right]\right).$$

We can write $\mathbb{E}_{it}\left[y_t\right] + e_t = y_t$, where e_t is uncorrelated with $\mathbb{E}_{it}\left[y_t\right]$. Thus

$$Var\left(y_t\right) \geq Var\left(\mathbb{E}_{it}\left[y_t\right]\right).$$

Combining the two inequalities above gives us the desired result.   Q.E.D.

Proof of Theorem 1:   By equation (23),

$$Var\left(y_t\right) = Var\left(\frac{a_t}{\xi} + \theta\overline{\mathbb{E}}_{it}\left[y_t\right]\right).$$

Using the triangular inequality and Lemma 2, we have

$$\sqrt{Var\left(y_t\right)} \leq \sqrt{Var\left(a_t/\xi\right)} + \theta\sqrt{Var\left(\overline{\mathbb{E}}_{it}\left[y_t\right]\right)} < \frac{\sigma_a}{\xi} + \theta\sqrt{Var\left(y_t\right)}.$$

Thus

$$\sqrt{Var\left(y_t\right)} < \frac{\sigma_a}{\left(1 - \theta\right)\xi}.$$

Using (30), we obtain the desired result.   Q.E.D.

C   Proofs of Results in Section 6

We first use Propositions 1 and 2 to derive the Wold representation for the signal a_{it} (see online Appendix F),

$$a_{it} = \Gamma(L)v_{it} = \sigma_w\frac{(1 - \lambda_w L)}{(1 - \rho_a L)}v_{it}, \tag{C.1}$$

where v_{it} denotes the one dimensional Gaussian Wold innovation with zero mean and unit variance, the moving average parameter λ_w is given by

$$\lambda_w = \frac{1}{2\rho_a}\left[(1 + \tau + \rho_a^2) - \sqrt{\tau^2 + 2\tau + 2\tau\rho_a^2 + 1 - 2\rho_a^2 + \rho_a^4}\right], \tag{C.2}$$

and the variance σ_w^2 is given by

$$\sigma_w^2 = \frac{\rho_a\sigma_i^2}{\lambda_w}. \tag{C.3}$$

Here $\tau \equiv \sigma_a^2/\sigma_i^2 \in (0, \infty)$ denotes the relative volatility of the aggregate shock to the idiosyncratic shock.

Next, substituting equations (35) and (36) into equation (21) and matching coefficients, we obtain

$$M_y\left(z\right) = \frac{1}{\left(1 - \rho_a z\right)\xi} + \frac{\theta a_y\left(z\right)}{\left(1 - \rho_a z\right)}, \tag{C.4}$$

$$M_y^i\left(z\right) = \frac{1}{\xi} + \theta a_y\left(z\right), \tag{C.5}$$

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where $a_y(z)$ can be derived from the Wiener-Hopf prediction formula (34) (see online Appendix F):

$$a_y(z) = \frac{\sigma_a^2}{\sigma_w^2} \left[z M_y(z) - \lambda_w M_y(\lambda_w) \right] \frac{1 - \rho_a z}{(1 - \lambda_w z)(z - \lambda_w)}. \tag{C.6}$$

Substituting (C.6) into (C.4) yields

$$M_y(z) = \frac{(1 - \lambda_w z)(z - \lambda_w) - (1 - \rho_a z) \lambda_w \xi \theta \sigma_a^2 \sigma_w^{-2} M_y(\lambda_w)}{\xi \left[(1 - \lambda_w z)(z - \lambda_w) - \theta \sigma_a^2 \sigma_w^{-2} z \right] (1 - \rho_a z)}. \tag{C.7}$$

Note that this is a fixed point equation in the sense that the endogenous variable $M_y(\lambda_w)$ appears on the right-side of the equation above. We use the standard pole-removing procedure to pin down this variable. Specifically, we can show that the quadratic equation

$$(1 - \lambda_w z)(z - \lambda_w) - \theta \frac{\sigma_a^2}{\sigma_w^2} z = 0 \tag{C.8}$$

has two real reciprocal roots. Let ϑ denote the root inside the unit circle. Then we have the following result:

Proposition 4 *Equation (C.8) has two real roots ϑ and $1/\vartheta$ with $|\vartheta| < 1$. There is a unique solution to equation (C.7) given by*

$$M_y(z) = \frac{(\kappa - z)\vartheta}{\xi(1 - \vartheta z)(1 - \rho_a z)},$$

where

$$\kappa \equiv \frac{\lambda_w^2 (1 - \rho_a \vartheta) + (1 - \lambda_w \vartheta)(1 - \rho_a \lambda_w)}{\lambda_w (1 - \rho_a \vartheta)}.$$

Moreover,

$$M_y^i(z) = M_y(z)(1 - \rho_a z) = \frac{(\kappa - z)\vartheta}{\xi(1 - \vartheta z)}.$$

Proof of Proposition 4: Consider the real-coefficients, complex polynomial equation given by (C.8),

$$(1 - \lambda_w z)(z - \lambda_w) - \theta \frac{\sigma_a^2}{\sigma_w^2} z = -\lambda_w z^2 + \left(1 + \lambda_w^2 - \theta \frac{\sigma_a^2}{\sigma_w^2} \right) z - \lambda_w = 0.$$

Since the coefficients are symmetric, it follows immediately that the two roots satisfy

$$z^- z^+ = 1 \text{ and } z^- + z^+ = \frac{1 + \lambda_w^2 - \theta \sigma_a^2 / \sigma_w^2}{\lambda_w}.$$

If the two roots are complex, then by the complex conjugate theorem they are located on the unit cycle. We will show that both roots are real and reciprocal. The quadratic equation has real roots if and only if

$$\left(\frac{1 + \lambda_w^2 - \theta \sigma_a^2 / \sigma_w^2}{\lambda_w} \right)^2 - 4 \geq 0.$$

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It suffices to show that $1 + \lambda_w^2 - \theta\sigma_a^2/\sigma_w^2 \geq 2\lambda_w > 0$, which is equivalent to the condition

$$(1 - \lambda_w)^2 \geq \theta\frac{\sigma_a^2}{\sigma_w^2} = \theta\frac{\tau\lambda_w}{\rho_a},$$

where the last equality follows from the definition of $\sigma_w^2 = \frac{\rho_a\sigma_i^2}{\lambda_w}$ and $\tau = \sigma_a^2/\sigma_i^2$. Since $\theta \in (0,1)$ is the strategic complementarity parameter, it suffices to show that

$$(1 - \lambda_w)^2 \geq \frac{\tau\lambda_w}{\rho_a},$$

or

$$\lambda_w^2 - \left(2 + \frac{\tau}{\rho_a}\right)\lambda_w + 1 \geq 0. \tag{C.9}$$

Note that λ_w is itself a function of ρ_a and τ. If for any parameter choice ρ_a and τ, $\lambda_w(\rho_a, \tau)$ is located to the left of the smaller root of the quadratic equation

$$x^2 - \left(2 + \frac{\tau}{\rho_a}\right)x + 1 = 0,$$

then (C.9) holds. That is, we need to show that

$$\lambda_w \leq \frac{\tau + 2\rho_a - \sqrt{\tau^2 + 4\rho_a\tau}}{2\rho_a}, \tag{C.10}$$

for all τ and ρ_a. Without loss of generality, fix any $\rho_a \in (0,1)$ and define

$$f(x) = \lambda_w - \frac{1}{2\rho_a}\left(x + 2\rho_a - \sqrt{x^2 + 4\rho_a x}\right)$$
$$= \frac{1}{2\rho_a}\left[\left(1 + x + \rho_a^2 - \sqrt{(1 + x - \rho_a^2)^2 + 4\rho_a^2 x}\right) - \left(x + 2\rho_a - \sqrt{x^2 + 4\rho_a x}\right)\right].$$

where we have substituted the expression for λ_w with τ replaced by x. We wish to show that $f(x) \leq 0$ for all $x > 0$. We can show that $\lim_{x \to 0} f(x) = \rho_a - 1 < 0$ and $\lim_{x \to \infty} f(x) = 0$.

The derivative of $f(x)$ is

$$f'(x) = \frac{1}{2\rho_a}\left[\frac{x + 2\rho_a}{\sqrt{x^2 + 4\rho_a x}} - \frac{1 + x + \rho_a^2}{\sqrt{(1 + x - \rho_a^2)^2 + 4\rho_a^2 x}}\right].$$

We wish to show that $f'(x) > 0$ for all $x > 0$. This is equivalent to

$$\frac{x + 2\rho_a}{\sqrt{x^2 + 4\rho_a x}} > \frac{1 + x + \rho_a^2}{\sqrt{(1 + x - \rho_a^2)^2 + 4\rho_a^2 x}}$$
$$\iff \frac{x^2 + 2(1 + \rho_a^2)x + (1 - \rho_a^2)^2}{x^2 + 4\rho_a x} > \frac{(1 + x + \rho_a^2)^2}{(x + 2\rho_a)^2}. \tag{C.11}$$

Lemma 3 *If $\frac{a}{b} > 1$, with $a > 0$, $b > 0$, and if $0 < c < b$, then*

$$\frac{a}{b} < \frac{a - c}{b - c}.$$

Proof: It follows from simple algebra. □

We can easily check

$$\frac{(1 + x + \rho_a^2)^2}{(x + 2\rho_a)^2} > 1.$$

Applying the preceding lemma, we obtain

$$
\begin{aligned}
\frac{(1 + x + \rho_a^2)^2}{(x + 2\rho_a)^2} &< \frac{(1 + x + \rho_a^2)^2 - 4\rho_a^2}{(x + 2\rho_a)^2 - 4\rho_a^2} \\
&= \frac{1 + x^2 + 2x + \rho_a^4 + 2\rho_a^2 + 2\rho_a^2 x - 4\rho_a^2}{x^2 + 4\rho_a^2 + 4\rho_a x - 4\rho_a^2} \\
&= \frac{x^2 + 2(1 + \rho_a^2)x + (1 - \rho_a^2)^2}{x^2 + 4\rho_a x},
\end{aligned}
$$

which is inequality (C.11). So $f'(x) > 0$ for all $x \in (0, \infty)$.

In summary, we have shown that $f(x)$ is continuous, monotonically increasing on $x \in (0, \infty)$, and $\lim_{x \to 0} f(x) < 0$, $\lim_{x \to \infty} f(x) = 0$. Elementary calculus implies that $f(x) \leq 0$ for all $x \in (0, \infty)$. Thus we have established (C.10) and all roots of equation (C.8) are real and reciprocal. We let $\vartheta = z^-$ denote the root inside the unit circle. In fact, one can readily verify, using the quadratic function property and the Rouché's theorem, that $\lambda_w \leq \vartheta \leq \rho_a$. Since these inequalities are inessential for our results, we omit the proof here.

Now consider the expression for the aggregate output y_t in equation (C.7),

$$M_y(z) = \frac{(1 - \lambda_w z)(z - \lambda_w) - (1 - \rho_a z)\lambda_w \xi \theta \sigma_a^2 \sigma_w^{-2} M_y(\lambda_w)}{\xi\left[(1 - \lambda_w z)(z - \lambda_w) - \theta \sigma_a^2 \sigma_w^{-2} z\right](1 - \rho_a z)}. \tag{C.12}$$

Using the results we just proved, we can rewrite it as

$$M_y(z) = \frac{(1 - \lambda_w z)(z - \lambda_w) - (1 - \rho_a z)\lambda_w \xi \theta \sigma_a^2 \sigma_w^{-2} M_y(\lambda_w)}{-\xi \lambda_w (1 - \vartheta z)(1 - \frac{1}{\vartheta}z)(1 - \rho_a z)}. \tag{C.13}$$

Using the standard pole-removing procedure, we set $M_y(\lambda_w)$ to remove the pole at $z = \vartheta < 1$ in the denominator, ensuring the analyticity of $M_y(z)$ inside the unit disc. Observe that the numerator is again a quadratic system

$$(1 - \lambda_w z)(z - \lambda_w) - (1 - \rho_a z)\lambda_w \xi \theta \sigma_a^2 \sigma_w^{-2} M_y(\lambda_w) = -\lambda_w\left[(r_1 - z)(r_2 - z)\right],$$

where r_1 and r_2 are the quadratic roots. It is easy to see that

$$r_1 + r_2 = \frac{1 + \lambda_w^2 + \rho_a \lambda_w \xi \theta \sigma_a^2 \sigma_w^{-2} M_y(\lambda_w)}{\lambda_w}. \tag{C.14}$$

Without loss of generality, set $r_1 = \vartheta$. Then we have

$$M_y(z) = \frac{-\lambda_w\left[(\vartheta - z)(r_2 - z)\right]}{-\xi \lambda_w (1 - \vartheta z)(1 - \frac{1}{\vartheta}z)(1 - \rho_a z)} = \frac{\vartheta(r_2 - z)}{\xi(1 - \vartheta z)(1 - \rho_a z)}. \tag{C.15}$$

Evaluating the numerator of (C.13) at $z = \vartheta$, $M_y(\lambda_w)$ has to satisfy the zero restriction,

$$(1 - \lambda_w\vartheta)(\vartheta - \lambda_w) - (1 - \rho_a\vartheta)\lambda_w\xi\theta\sigma_a^2\sigma_w^{-2}M_y(\lambda_w) = 0,$$

which implies

$$M_y(\lambda_w) = \frac{(1 - \lambda_w\vartheta)(\vartheta - \lambda)}{(1 - \rho_a\vartheta)\lambda_w\xi\theta\sigma_a^2\sigma_w^{-2}}. \tag{C.16}$$

Now substituting (C.16) into (C.14) and using the fact that $r_1 = \vartheta$, we can derive

$$r_2 = \frac{\lambda_w^2(1 - \rho_a\vartheta) + (1 - \lambda_w\vartheta)(1 - \rho_a\lambda_w)}{\lambda_w(1 - \rho_a\vartheta)}.$$

Define $\kappa = r_2$. We use (C.15) to obtain the expression for $M_y(z)$ in the proposition. The expression for $M_y^i(z)$ in the proposition follows from (C.4) and (C.5).   Q.E.D.

Proof of Proposition 3:   Consider equations (24) and (27),

$$\alpha_1 s_{it+1}^h = \alpha_2 s_{it}^h + \mathbb{E}_{it}\left[\alpha_3 s_{it+2}^h + \Delta b_{it+1}\right] + \mathbb{E}_{it}\left[\beta q_{t+1} + (1 - \beta)d_{t+1}\right] - q_t \tag{C.17}$$

$$q_t = \int_I \mathbb{E}_{it}\left[\alpha_3 s_{it+2} + \Delta b_{it+1}\right]di + \int_I \mathbb{E}_{it}\left[\beta q_{t+1} + (1 - \beta)d_{t+1}\right]di, \tag{C.18}$$

where $\Delta b_{it+1} = b_{it} - b_{it+1}$ and $b_{it} = \alpha_4 d_t + \alpha_5 n_{it}$. Using the structural equation $y_{it} = (a_t + \epsilon_{it}) + \alpha n_{it}$ and equations (C.4) and (C.5), it is easy to derive

$$\begin{aligned}
n_{it} &= \frac{1}{\alpha}\left\{\left[M_y(L) - \frac{1}{1 - \rho_a L}\right]\epsilon_{at} + \left[M_y^i(L) - 1\right]\epsilon_{it}\right\} \\
&= \frac{1}{\alpha}\left\{\frac{1}{1 - \rho_a L}\left[M_y^i(L) - 1\right]\epsilon_t^a + \left[M_y^i(L) - 1\right]\epsilon_{it}^z\right\} \\
&\equiv M_n(L)\epsilon_{at} + M_n^i(L)\epsilon_{it}.
\end{aligned}$$

It follows that

$$M_n^i(z) = (1 - \rho_a z)M_n(z). \tag{C.19}$$

Define

$$q_{it}^e = \mathbb{E}_{it}\left[\alpha_3 s_{it+2}^h + \Delta b_{it+1}\right] + \mathbb{E}_{it}\left[\beta q_{t+1} + (1 - \beta)d_{t+1}\right].$$

Then

$$\begin{aligned}
q_{it}^e &= \mathbb{E}_{it}\left[\alpha_3 s_{it+2}^h + \widetilde{\Delta b}_{it+1} + \beta q_{t+1} + (1 - \beta)d_{t+1}\right] + \alpha_5 n_{it} \\
&= a_q(L)a_{it} + \alpha_5\left[M_n(L)\epsilon_{at} + M_n^i(L)\epsilon_{it}\right] \\
&= \left[\frac{a_q(L)}{1 - \rho_a L} + \alpha_5 M_n(L)\right]\epsilon_{at} + \left[a_q(L) + \alpha_5 M_n^i(L)\right]\epsilon_{it},
\end{aligned}$$

where $\widetilde{\Delta b}_{it+1} = \Delta b_{it+1} - \alpha_5 n_{it}$ and $a_q(z)$ can be obtained from the Wiener-Hopf prediction formula.

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Aggregation leads to

$$
\int_I q_{it}^e di = \left[\frac{a_q(L)}{1 - \rho_a L} + \alpha_5 M_n(L) \right] \epsilon_{at} + \left[a_q(L) + \alpha_5 M_n^i(L) \right] \int_I \epsilon_{it} di
$$

$$
= \left[\frac{a_q(L)}{1 - \rho_a L} + \alpha_5 M_n(L) \right] \epsilon_{at}.
$$

Since

$$
q_t = M_q(L) \epsilon_{at}, \quad s_{it+1}^h = M_s^i(L) \epsilon_{it},
$$

it follows from matching coefficients in equation (C.18) that

$$
M_q(z) = \frac{a_q(z)}{1 - \rho_a z} + \alpha_5 M_n(z).
$$

Matching coefficients in equation (C.17) yields

$$
(\alpha_1 - \alpha_2 z) M_s^i(z) = a_q(z) + \alpha_5 M_n^i(z).
$$

It follows from the preceding two equations and (C.19) that

$$
M_q(z) = \alpha_1 \frac{1 - \lambda_s z}{1 - \rho_a z} M_s^i(z),
$$

where $\lambda_s = \alpha_2 / \alpha_1$. Q.E.D.

Proof of Theorem 2: The proof is constructive. We have solved the real side of the model in Proposition 4. We now focus on the finacial side. Plugging (38) and (39) into equation (27) and using (40) and the Wiener-Hopf prediction formula, we can derive the following fixed point equation for $M_q(\cdot)$ (see online Appendix F):

$$
G_q(z) M_q(z) = G_b(z) + (1 - \beta) G_d(z) - \sigma_m \left[\frac{M_q(\lambda_w) \Lambda}{z - \lambda_w} + \frac{M_q(0) \rho_a}{\lambda_w z} \right] - \sigma_\beta \frac{M_q(\lambda_w)}{z - \lambda_w}, \quad \text{(C.20)}
$$

where the analytic functions $G_q(z)$, $G_b(z)$, and $G_d(z)$ satisfy

$$
G_q(z) = \frac{\sigma_m \rho_a P_q(z)}{z(z - \lambda_w)(1 - \lambda_s z)},
$$

$$
G_b(z) = \frac{1}{\sigma_w} \left[\frac{(z - 1) S_{ba}(z)}{z \Gamma(1/z)} \right]_+, \quad G_d(z) = \frac{1}{\sigma_w} \left[\frac{S_{da}(z)}{z \Gamma(1/z)} \right]_+,
$$

S_{ba} and S_{da} are covariance-generating functions, the constants σ_m, σ_β, and Λ are defined as

$$
\sigma_m = \frac{\sigma_i^2}{\sigma_w^2} (1 - \lambda_s), \quad \sigma_\beta = \beta \frac{\sigma_a^2}{\sigma_w^2}, \quad \Lambda = \frac{(1 - \rho_a \lambda_w)(\lambda_w - \rho_a)}{(1 - \lambda_s \lambda_w) \lambda_w},
$$

and the polynomial function $P_q(z)$ is defined as

$$
P_q(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0. \quad \text{(C.21)}
$$

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Here,

$$
\begin{aligned}
a_1 &= -\left[\frac{1}{1-\lambda_s} + \left(\rho_a + \frac{1}{\rho_a}\right) + \frac{\sigma_\beta}{\lambda_w\left(1-\lambda_s\right)}\right], \\
a_2 &= \frac{1+\lambda_s\sigma_\beta}{\lambda_w\left(1-\lambda_s\right)} + \frac{1+\lambda_w}{1-\lambda_s}, \quad a_3 = -\left[\frac{1+\lambda_s\lambda_w}{1-\lambda_s} + \frac{\lambda_s}{\lambda_w\left(1-\lambda_s\right)}\right], \\
a_4 &= \frac{\lambda_s}{1-\lambda_s}, \quad a_0 = 1.
\end{aligned}
$$

Note that the expression for the equity price in (C.20) is not in closed-form due to the presence of unknown constant $M_q\left(\lambda_w\right)$ and $M_q\left(0\right)$. Standard pole-removing procedures can be performed by using the zeros of the rational function $G_q\left(z\right)$ or $P_q\left(z\right)$. The following lemma analyzes the roots of $P_q\left(z\right)$.

Lemma 4 *The univariate polynomial function $P_q\left(z\right)$ has two distinct roots inside the unit circle and two distinct roots outside the unit circle and we can express $P_q\left(z\right)$ as*

$$
P_q\left(z\right) = \left(1 - m_1 z\right)\left(1 - m_2 z\right)\left(1 - m_3 z\right)\left(1 - m_4 z\right),
$$

with $|m_1| \geq |m_2| > 1 > |m_3| \geq |m_4|$.

Denote $m_i^r = 1/m_i$ for $i = 1, 2, 3, 4$. Given the preceding lemma, we can solve for the unique equilibrium by setting $M_q\left(\lambda_w\right)$ and $M_q\left(0\right)$ to remove the poles at $z = m_1^r$ and $z = m_2^r$.

Proof of Lemma 4: Consider the complex quartic polynomial equation in (C.21). Note that $a_0, a_2, a_4 > 0$, while $a_1, a_3 < 0$. By the fundamental theorem of algebra, there exist four roots, counting multiplicities. To show the location of roots, we need the Rouché's theorem (see Brown and Churchill (2009)).

Theorem 4 *(Rouché) Suppose that two functions f and g are analytic inside and on a simple closed contour $\mathbf{C}$. If $|f(z)| > |g(z)|$ at each point on $\mathbf{C}$, then the functions $f(z)$ and $f(z) + g(z)$ have the same numbers of zeros, counting multiplicities, inside $\mathbf{C}$.*

To proceed, let $\mathbf{C} = \mathbb{T}$ be the unit circle and define

$$
P_q^A(z) = a_4 z^4 + a_2 z^2 + a_0, \quad P_q^B(z) = a_3 z^3 + a_1 z.
$$

We need the following lemma on coefficients.

Lemma 5 *We have*

$$
|a_4| + |a_2| + |a_0| > |a_3| + |a_1|.
$$

Proof: The difference between the expressions on the left-hand and right-hand sides of the inequality above is given by

$$1 + \frac{1 + \lambda_s + \lambda_w}{1 - \lambda_s} + \frac{1 + \sigma_\beta \lambda_s}{\lambda_w(1 - \lambda_s)} - \left[\frac{2 + \lambda_s \lambda_w}{1 - \lambda_s} + (\rho_a + \frac{1}{\rho_a}) + \frac{\sigma_\beta + \lambda_s}{\lambda_w(1 - \lambda_s)} \right]$$

$$= \frac{1}{\lambda_w} \left[\lambda_w^2 - \varrho \lambda_w + (1 - \sigma_\beta) \right] = \frac{1}{\lambda_w} \left[\lambda_w^2 - (\varrho + \frac{\beta\tau}{\rho_a})\lambda_w + 1 \right],$$

where we have defined $\varrho \equiv \rho_a + \frac{1}{\rho_a} > 2$ and substituted $\sigma_\beta = \beta \frac{\sigma_\theta^2}{\sigma_w^2} = \frac{\beta\tau\lambda_w}{\rho_a}$. We wish to show that

$$\lambda_w^2 - \left(\varrho + \frac{\beta\tau}{\rho_a} \right) \lambda_w + 1 > 0.$$

By (C.2), we can derive

$$\lambda_w = \frac{h(\tau, \rho_a) - \sqrt{h^2(\tau, \rho_a) - 4\rho_a^2}}{2\rho_a}$$

where $h(\tau, \rho_a) \equiv 1 + \tau + \rho_a^2$. By some tedious algebra, we can show that

$$\lambda_w^2 - \left(\varrho + \frac{\beta\tau}{\rho_a} \right) \lambda_w + 1 = \frac{[h(\tau, \rho_a) - (\varrho\rho_a + \beta\tau)] \left[h(\tau, \rho_a) - \sqrt{h^2(\tau, \rho_a) - 4\rho_a^2} \right]}{2\rho_a^2}.$$

Clearly the term in the second bracket on the numerator is positive by definition. The term in the first bracket can be written as

$$h(\tau, \rho_a) - (\varrho\rho_a + \beta\tau) = 1 + \tau + \rho_a^2 - \left[(\rho_a + \frac{1}{\rho_a})\rho_a + \beta\tau \right] = (1 - \beta)\tau > 0.$$

We then obtain the desired result. $\qquad\square$

Next we show that $P_q^A(z)$ has too roots inside the unit disk and two roots outside the unit disk.

Lemma 6 $P_q^A(z) = a_4 z^4 + a_2 z^2 + a_0$ *has two roots inside the unit disk, and two roots outside the unit disk.*

Proof: We need to apply the Rouché theorem. First, it is trivial to see that $a_2 z^2$ has two roots at $z = 0$ inside the unit disk. Then by the triangle inequality,

$$|a_4 z^4 + a_0| \le |a_4 z^4| + |a_0| = |a_4| + |a_0|, \tag{C.22}$$

where $z \in \mathbb{T}$ is on the unit circle. By the definition of $a_0, a_1, ..., a_4$, we can easily deduce that $|a_2| > |a_4| + |a_0|$. It follows immediately that for all $z \in \mathbb{T}$,

$$|a_2 z^2| = |a_2| > |a_4| + |a_0| \ge |a_4 z^4 + a_0|.$$

Therefore, by the Rouché theorem, $a_2 z^2$ and $a_4 z^4 + a_0 + a_2 z^2$ have the same number of roots (two) inside the unit circle. Use the reverse triangle inequality,

$$\left| P_q^A(z) \right| = \left| a_4 z^4 + a_0 + a_2 z^2 \right| \ge \left| |a_2 z^2| - |a_4 z^4 + a_0| \right| > 0,$$

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for all $z \in \mathbb{T}$. Hence the other two roots of the quartic polynomial $P_q^A(z)$ are located outside the unit circle, by the Fundamental Theorem of Algebra. The proof is complete. □

To apply Rouché's theorem, we need the following lemma.

Lemma 7 *For each $z \in \mathbb{T}$,*

$$\left| P_q^A(z) \right| > \left| P_q^B(z) \right|.$$

Proof: Let $z = e^{-i\omega}$, $\omega \in [-\pi, \pi]$. Without loss of generality, define $\theta = 2\omega \in [-2\pi, 2\pi]$. It follows that

$$
\begin{aligned}
\left| P_q^A(z) \right| &= \left| a_4 e^{-2i\theta} + a_2 e^{-i\theta} + a_0 \right| \\
&= |a_4(\cos 2\theta - i \sin 2\theta) + a_2(\cos \theta - i \sin \theta) + a_0| \\
&= \sqrt{(a_4 \cos 2\theta + a_2 \cos \theta + a_0)^2 + (a_4 \sin 2\theta + a_2 \sin \theta)^2},
\end{aligned}
$$

where we have employed the Euler's formula to expand the expression in trigonometric form. The trigonometric form can be manipulated as

$$
\begin{aligned}
\left| P_q^A(z) \right|^2 &= (a_4 \cos 2\theta + a_2 \cos \theta + a_0)^2 + (a_4 \sin 2\theta + a_2 \sin \theta)^2 \\
&= a_4^2 \cos^2 2\theta + a_2^2 \cos^2 \theta + a_0^2 + 2a_4 a_2 \cos 2\theta \cos \theta + 2a_4 a_0 \cos 2\theta + 2a_2 a_0 \cos \theta \\
&\quad + a_4^2 \sin^2 2\theta + a_2^2 \sin^2 \theta + 2a_2 a_4 \sin 2\theta \sin \theta \\
&= a_4^2 + a_2^2 + a_0^2 + 2a_2 a_4 (\cos 2\theta \cos \theta + \sin 2\theta \sin \theta) + 2a_0 a_4 \cos 2\theta + 2a_0 a_2 \cos \theta \\
&= a_4^2 + a_2^2 + a_0^2 + 2a_2 a_4 \cos \theta + 2a_0 a_4 \cos 2\theta + 2a_0 a_2 \cos \theta,
\end{aligned}
$$

where we have used the standard trigonometric identities

$$\sin^2 n\theta + \cos^2 n\theta = 1,$$

$$\cos 2\theta \cos \theta + \sin 2\theta \sin \theta = \cos(2\theta - \theta) = \cos \theta.$$

On the other hand,

$$
\begin{aligned}
\left| P_q^B(z) \right| &= \left| a_3 z^3 + a_1 z \right| = |z| \cdot \left| a_3 z^2 + a_1 \right| = |a_3(\cos \theta - i \sin \theta) + a_1| \\
&= \sqrt{(a_3 \cos \theta + a_1)^2 + (a_3 \sin \theta)^2} = \sqrt{a_3^2 + a_1^2 + 2|a_1||a_3| \cos \theta},
\end{aligned}
$$

where we have used the trigonometric identity again.

To prove the lemma, it is equivalent to show that

$$
\begin{aligned}
\Delta_q(\theta) &= a_4^2 + a_2^2 + a_0^2 + 2a_2 a_4 \cos \theta + 2a_0 a_4 \cos 2\theta + 2a_0 a_2 \cos \theta \\
&\quad - \left(a_3^2 + a_1^2 + 2|a_1||a_3| \cos \theta \right) > 0, \tag{C.23}
\end{aligned}
$$

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for every $\theta = 2\omega \in [-2\pi, 2\pi]$. By inspection, $\Delta_q(\theta)$ is a periodic, even function with period 2π.[12] We will show analytically that when $\theta \in [-2\pi, 2\pi]$, the minimum of $\Delta_q(\theta)$ is

$$\Delta_q(\theta)_{\min} = (a_4 + a_2 + a_0)^2 - (|a_3| + |a_1|)^2$$

and the minimum is attained at $\theta = 0, -2\pi, 2\pi$ so that $\cos\theta = 1$.

To proceed, we define the trigonometric part of $\Delta_q(\theta)$ as $\overline{\Delta_q}(\theta)$ such that

$$\Delta_q(\theta) = \overline{\Delta_q}(\theta) + a_4^2 + a_2^2 + a_0^2 - (a_1^2 + a_3^2).$$

Then it follows that

$$\begin{aligned}
\overline{\Delta_q}(\theta) &= 2a_2a_4\cos\theta + 2a_0a_4\cos 2\theta + 2a_0a_2\cos\theta - 2|a_1||a_3|\cos\theta \\
&= 2a_0a_4\left(2\cos^2\theta - 1\right) + (2a_2a_4 + 2a_0a_2 - 2|a_1||a_3|)\cos\theta \\
&= 4a_0a_4\cos^2\theta + (2a_2a_4 + 2a_0a_2 - 2|a_1||a_3|)\cos\theta - 2a_0a_4,
\end{aligned}$$

where we have used the trigonometric identity

$$\cos 2\theta = 2\cos^2\theta - 1.$$

Thus $\overline{\Delta_q}(\theta)$ is a quadratic function of $\cos\theta \in [-1, 1]$. It opens upward since $4a_0a_4 > 0$. Therefore, without the trigonometric restriction, the generic quadratic equation

$$\overline{\Delta_q}(t) = 4a_0a_4 t^2 + (2a_2a_4 + 2a_0a_2 - 2|a_1||a_3|)\,t - 2a_0a_4$$

attains its minimum at its vertex (axis of symmetry)

$$t_{\min} = -\frac{2a_2a_4 + 2a_0a_2 - 2|a_1||a_3|}{8a_0a_4} = \frac{|a_1||a_3| - (a_2a_4 + a_0a_2)}{4a_0a_4}.$$

Next we show that $t_{\min} > 1$, given our parameter restrictions. This is equivalent to

$$\Delta = |a_1||a_3| - (a_2a_4 + a_0a_2) - 4a_0a_4 > 0.$$

Expanding Δ, we have

$$\begin{aligned}
(1-\lambda_s)^2\Delta &= \varrho(1-\lambda_s) + (1-\lambda_s)(\varrho\lambda_s - 1)(\lambda_w + \frac{1}{\lambda_w}) \\
&\quad + \sigma_\beta\left[\frac{1}{\lambda_w}(1-\lambda_s) + \lambda_s(1 + \frac{1}{\lambda_w^2})\right] - 4\lambda_s(1-\lambda_s) \\
&= (1-\lambda_s)\left[\varrho + (\varrho\lambda_s - 1)(\lambda_w + \frac{1}{\lambda_w}) - 4\lambda_s\right] + \sigma_\beta\left[\frac{1}{\lambda_w}(1-\lambda_s) + \lambda_s(1 + \frac{1}{\lambda_w^2})\right] \\
&> (1-\lambda_s)\left[\varrho + 2(\varrho\lambda_s - 1) - 4\lambda_s\right] + \sigma_\beta\left[\frac{1}{\lambda_w}(1-\lambda_s) + \lambda_s\left(1 + \frac{1}{\lambda_w^2}\right)\right] \\
&= (1-\lambda_s)(\varrho - 2)(1 + 2\lambda_s) + \sigma_\beta\left[\frac{1}{\lambda_w}(1-\lambda_s) + \lambda_s\left(1 + \frac{1}{\lambda_w^2}\right)\right] \\
&> 0.
\end{aligned}$$

---

[12] Elementary mathematics implies that $\cos 2\theta$ has period π, so that the trigonometric sum has period 2π.

The first inequality follows from the arithmetic inequality $\lambda_w + 1/\lambda_w > 2$ and the fact that $\varrho\lambda_s > \frac{1}{2}\varrho > 1$ (recall $\lambda_s \in (1/2, 1)$). The second inequality follows from the fact that $\varrho = \rho_a + 1/\rho_a > 2$.

Therefore, the unrestricted, global minimum for the quadratic system $\overline{\Delta_q}(t)$ occurs at $t_{\min} > 1$. By definition $t = \cos\theta \in [-1, 1]$, the minimum of $\overline{\Delta_q}(\theta)$ occurs at $\cos\theta = 1$. We can then compute the minimum of $\Delta_q(\theta)$ as

$$\Delta_q(\theta)_{\min} = \overline{\Delta_q}(\theta)_{\min} + a_4^2 + a_2^2 + a_0^2 - (a_1^2 + a_3^2)$$
$$= (a_4 + a_2 + a_0)^2 - (|a_3| + |a_1|)^2 > 0,$$

where the inequality follows from Lemma 5. Therefore, $\Delta_q(\theta) > 0$ for every $\theta \in [-2\pi, 2\pi]$, which is equivalent to $\left|P_q^A(z)\right| > \left|P_q^B(z)\right|$ for each $z \in \mathbb{T}$. $\qquad\square$

Given the preceding three lemmas, we conclude that $P_q^A(z)$ has two roots inside the unit circle and $\left|P_q^A(z)\right| > \left|P_q^B(z)\right|$ for each $z \in \mathbb{T}$. Hence $P_q^A(z)$ and $P_q(z) = P_q^A(z) + P_q^B(z)$ have the same number of roots (two) inside the unit circle by the Rouché theorem. Since $P_q(z)$ cannot have roots on the unit circle, the other two roots are located outside the unit circle. The proof of Lemma 4 is thus complete. Q.E.D.

Using equation (C.20) and Lemma 4, we can derive

$$M_q(z) = \frac{\Delta_q(z)}{\sigma_m \rho_a \prod_{i=1}^{4}(1 - m_i z)}, \qquad (C.24)$$

where $\Delta_q(z)$ is given by

$$\Delta_q(z) = z(z - \lambda_w)(1 - \lambda_s z)\left[G_b(z) + (1 - \beta)G_d(z) - \sigma_m\left[\frac{M_q(\lambda_w)\Lambda}{z - \lambda_w} + \frac{M_q(0)\rho_a}{\lambda_w z}\right] - \sigma_\beta\frac{M_q(\lambda_w)}{z - \lambda_w}\right]$$
$$= z(z - \lambda_w)(1 - \lambda_s z)\left[G_b(z) + (1 - \beta)G_d(z)\right] - z(1 - \lambda_s z)\left[\sigma_m\Lambda + \sigma_\beta\right]M_q(\lambda_w)$$
$$- (z - \lambda_w)(1 - \lambda_s z)\frac{\sigma_m\rho_a}{\lambda_w}M_q(0).$$

By inspection, $\Delta_q(z)$ is a rational analytic function in the closed unit disk ($\mathbb{T} \cup \mathbb{D}$). Thus $\Delta_q(z) \in \mathbf{H}^2(\mathbb{D})$. Note that equation (C.24) does not give a closed-form solution for $M_q(z)$ because the constants $M_q(\lambda_w)$ and $M_q(0)$ on the right side are endogenous. We set these constants to remove the poles inside the unit circle at $m_1^r = 1/m_1$ and $m_2^r = 1/m_2$, ensuring the causal stationarity of the equity price. The pole-removing procedure is equivalent to the following derivative conditions

$$\left.\frac{d^{(0)}\Delta_q(z)}{dz}\right|_{z=m_1^r} = \Delta_q(m_1^r) = 0, \quad \left.\frac{d^{(0)}\Delta_q(z)}{dz}\right|_{z=m_2^r} = \Delta_q(m_2^r) = 0.$$

These conditions also extend to higher-order zeros with multiplicities (see Tan and Walker (2015)). The above condition can be transformed into the following system of linear equations,

$$\left[\sigma_m\Lambda + \sigma_\beta\right]m_1^r M_q(\lambda_w) + \sigma_m(m_1^r - \lambda_w)M_q(0)\frac{\rho_a}{\lambda_w} = g(m_1^r),$$
$$\left[\sigma_m\Lambda + \sigma_\beta\right]m_2^r M_q(\lambda_w) + \sigma_m(m_2^r - \lambda_w)M_q(0)\frac{\rho_a}{\lambda_w} = g(m_2^r),$$

where we define

$$g(z) \equiv z(z - \lambda_w) \left[G_b(z) + (1 - \beta) G_d(z) \right].$$

This is a linear system of equations, which admits a unique solution

$$M_q(0) = \frac{g(m_1^r) m_2^2 - g(m_2^r) m_1^r}{(m_1^2 - m_2^r) \sigma_m \rho_a},$$

$$M_q(\lambda_w) = \frac{(\lambda_w - m_2^r) g(m_1^r) + (m_1^r - \lambda_w) g(m_2^r)}{\lambda_w (m_1^r - m_2^r) [\sigma_m \Lambda + \sigma_\beta]}.$$

Now substituting these expressions back into $\Delta_q(z)$ and simplifying, we obtain

$$\Delta_q(z) = (1 - \lambda_s z) \left[g(z) - \frac{g(m_1^r) - g(m_2^r)}{m_1^r - m_2^r} z + \frac{g(m_1^r) m_2^r - g(m_2^r) m_1^r}{m_1^r - m_2^r} \right].$$

Define the rational function $N_q(z)$ as the expression in the preceding square bracket. By inspection, $N_q(z)$ is analytic in the closed unit disk, and has two zeros at $z = m_1^r$ and $z = m_2^r$. Therefore, following Churchill and Brown (2009), p249, Theorem 1, there exists an analytic function $\hat{N}_q(z)$ that is nonzero at these two points such that

$$N_q(z) = (1 - m_1 z)(1 - m_2 z) \hat{N}_q(z).$$

Finally, we show that the rational polynomial function $N_q(z)$ has real coefficients. To see this, we use Lemma 4 to deduce that $N_q(z)$ has two roots m_1^r and m_2^r inside the unit circle satisfying $|m_1^r| \leq |m_2^r| < 1$. If both roots are real, then the result follows from the definition of $N_q(z)$ as $g(z)$ has real coefficients. Suppose that one of these roots is complex. Since $P_q(z)$ has real coefficients, it follows from the complex conjugate theorem that m_1^r and m_2^r form conjugate pairs, $m_2^r = \overline{m_1^r}$. Since $g(z)$ has real coefficients, it follows that

$$g(m_2^r) = g(\overline{m_1^r}) = \overline{g(m_1^r)},$$

$$g(m_1^r) m_2^r - g(m_2^r) m_1^r = g(m_1^r) \overline{m_1^r} - \overline{g(m_1^r)} \, \overline{m_1^r},$$

where we use $\overline{x}$ to denote conjugation instead of x^* for notational simplicity. It is then clear that the coefficients $\frac{g(m_1^r) - g(m_2^r)}{m_1^r - m_2^r}$ and $\frac{g(m_1^r) m_2^r - g(m_2^r) m_1^r}{m_1^r - m_2^r}$ are real-valued, so that $N_q(z)$ has real coefficients. The proof of Theorem 2 is complete.   Q.E.D.

We need the following lemma to prove Theorem 3.

Lemma 8 *When $\sigma_i \to \infty$, the analytic function $P_q(z)$ converges to*

$$\lim_{\sigma_i \to \infty} P_q(z) = (z - 1) \left(\frac{\lambda_s}{1 - \lambda_s} z - 1 \right) \left[z^2 - \left(\rho_a + \frac{1}{\rho_a} \right) z + 1 \right].$$

There are still two roots inside the unit circle and one root outside the unit circle, but m_3 converges to a unit root as $\sigma_i \to \infty$.

Proof: Since $a_0 = 1$, the Fundamental Theorem of Algebra implies that $P_q(z)$ admits a unique factorization in the form (modulo a constant normalization)

$$P_q(z) = (1 - m_1 z)(1 - m_2 z)(1 - m_3 z)(1 - m_4 z).$$

Moreover, the roots of the polynomials are functions of the coefficients and are continuous with respect to the coefficients. Since the coefficients are continuous functions of τ, all roots are continuous functions of τ.

By the definition of λ_w, σ_w, and $\tau = \sigma_a^2/\sigma_i^2$, we have $\lim_{\sigma_i \to \infty} \tau = 0$, $\lim_{\sigma_i \to \infty} \lambda_w = \rho_a$, and

$$\lim_{\sigma_i \to \infty} \sigma_\beta = \lim_{\sigma_i \to \infty} \beta \frac{\sigma_a^2}{\sigma_w^2} = \lim_{\sigma_i \to \infty} \frac{\beta \sigma_a^2}{\rho_a \sigma_i^2/\lambda_w} = \lim_{\sigma_i \to \infty} \beta \tau = 0.$$

We now compute

$$
\begin{aligned}
\lim_{\sigma_i \to \infty} P_q(z) &= \frac{\lambda_s}{1 - \lambda_s} z^4 - \frac{\lambda_s + \rho_a^2 \lambda_s + \rho_a}{\rho_a (1 - \lambda_s)} z^3 + \left[\frac{1 + (\lambda_s + \rho_a) \rho_a}{\rho_a (1 - \lambda_s)} + 1 \right] z^2 \\
&\quad - \left[\frac{1}{1 - \lambda_s} + \frac{1 + \rho_a^2}{\rho_a} \right] z + 1 \\
&= (z - 1) \left(\frac{\lambda_s}{1 - \lambda_s} z - 1 \right) \left(z^2 - \varrho z + 1 \right),
\end{aligned}
$$

where $\varrho \equiv \frac{1}{\rho_a} + \rho_a > 2$. Since $\lambda_s \in (\frac{1}{2}, 1)$, the equation $\lim_{\sigma_i \to \infty} P_q(z) = 0$ has a root $(1 - \lambda_s)/\lambda_s$ inside the unit circle. The quadratic equation $z^2 - \varrho z + 1 = 0$ obviously has two real roots. By symmetry of its coefficients, one of the roots is inside the unit circle and the other is outside the unit circle. Thus the equation $\lim_{\sigma_i \to \infty} P_q(z) = 0$ has two roots inside the unit circle, one root outside the unit circle, and one root on the unit circle. By the continuity of roots with respect to τ, it must be the case that the smaller roots of $P_q(z) = 0$ outside the unit circle gradually converges to the unit root, i.e., $\lim_{\sigma_i \to \infty} m_3^r = \lim_{\sigma_i \to \infty} 1/m_3 = 1$. The proof is then complete. $\square$

Proof of Theorem 3: We use the Parseval theorem to compute the equity price volatility as

$$
\begin{aligned}
Var(q_t) &= \sigma_a^2 \|M_q(z)\|_{\mathbf{H}^2}^2 = \sigma_a^2 \frac{1}{2\pi i} \oint_{\mathbb{T}} |M_q(z)|^2 \frac{dz}{z} \\
&= \frac{\sigma_a^2}{\sigma_m^2 \rho_a^2} \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{(1 - \lambda_s z)(1 - \lambda_s \frac{1}{z}) \hat{N}_q(z) \hat{N}_q(\frac{1}{z})}{(1 - m_3 z)(1 - m_3 \frac{1}{z})(1 - m_4 z)(1 - m_4 \frac{1}{z})} \frac{dz}{z}.
\end{aligned}
$$

By Lemma 8, $m_1^r = 1/m_1$ and $m_2^r = 1/m_2$ remain inside the unit circle, and m_4^r remains outside the unit circle, while $m_3^r = 1/m_3$ converges to the unit root, as $\sigma_i \to \infty$. Therefore, it suffices to show that $\hat{N}_q(z)$ has no roots on the unit circle when $\sigma_i \to \infty$.

By the definition in Theorem 2 and the derivation of (C.20) in online Appendix F, we have

$$g(z) = z(z - \lambda_w)G_b(z) + z(z - \lambda_w)(1 - \beta)G_d(z)$$
$$= \frac{1}{\sigma_w}\left[z(z - \lambda_w)\psi_b(z) - \varphi_b(\lambda_w)z - \varphi_b(0)(z - \lambda_w)\right]$$
$$+ (1 - \beta)\frac{1}{\sigma_w}\left[z(z - \lambda_w)\psi_d(z) - z\varphi_d(\lambda_w)\right].$$

Using the limiting result

$$\lim_{\sigma_i \to \infty} \lambda_w = \rho_a, \quad \lim_{\sigma_i \to \infty} \frac{\sigma_a^2}{\sigma_w^2} = \lim_{\sigma_i \to \infty} \frac{\sigma_a^2 \lambda_w}{\sigma_i^2 \rho_a} = 0, \quad \lim_{\sigma_i \to \infty} \frac{\sigma_i^2}{\sigma_w^2} = 1,$$

we can show that

$$\lim_{\sigma_i \to \infty} g(z) = (z - 1)(z - \rho_a)M_b^i(z) + M_b^i(0).$$

By Proposition 4 and the derivation of (C.20) in online Appendix F, we can show that

$$M_b^i(z) = \frac{\alpha_5}{\alpha}\left[M_y^i(z) - 1\right] = \frac{\alpha_5}{\alpha}\left[\frac{(\kappa - z)\vartheta}{\xi(1 - \theta z)} - 1\right].$$

Since

$$\lim_{\sigma_i \to \infty} \vartheta = \rho_a, \quad \lim_{\sigma_i \to \infty} \kappa = \frac{1}{\rho_a},$$

it follows that

$$\lim_{\sigma_i \to \infty} M_b^i(z) = \frac{\alpha_5}{\alpha}\left[\frac{(\frac{1}{\rho_a} - z)\rho_a}{\xi(1 - \rho_a z)} - 1\right] = \frac{\alpha_5}{\alpha}\left(\frac{1}{\xi} - 1\right).$$

Given Assumption 1, it follows from Appendix A that $\alpha_5 \neq 0$. Thus

$$\frac{\alpha}{\alpha_5}\lim_{\sigma_i \to \infty} g(z) = (z - 1)(z - \rho_a)\left(\frac{1}{\xi} - 1\right) + \frac{1}{\xi} - 1.$$

In Theorem 2, we have defined

$$N_q(z) = g(z) - \frac{g(m_1^r) - g(m_2^r)}{m_1^r - m_2^r}z + \frac{g(m_1^r)m_2^r - g(m_2^r)m_1^r}{m_1^r - m_2^r}.$$

Moreover, $N_q(z)$ has two roots located at $z = m_1^r$ and $z = m_2^r$. By our proof above, we have shown that $N_q(z)$ becomes a quadratic polynomial function in the limit as $\sigma_i \to \infty$. Thus the limits of m_1^r and m_2^r are the two roots of $\lim_{\sigma_i \to \infty} N_q(z)$. Thus $\lim_{\sigma_i \to \infty} \hat{N}_q(z) = \lim_{\sigma_i \to \infty} \frac{N_q(z)}{(1 - m_1 z)(1 - m_2 z)}$ becomes a constant independent of z. Finally, we have one pole in the limit at $\lim_{\sigma_i \to \infty} m_3^r = 1$ by Lemma 8. Therefore, we have

$$\lim_{\sigma_i \to \infty} ||M_q(z)||_{\mathbf{H}^2} \to \infty.$$

This completes the proof.   Q.E.D.

Online Supplemental Material

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D Proofs of Results in Section 7

We first state an equilibrium existence and uniqueness result.

Theorem 5 *Consider the model under the signal representation (45). Let the analytic functions $A(z)$, $\tilde{A}(z)$, $B(z)$, and $\tilde{B}(z)$ and the signal-noise ratios τ_1, τ_2, τ_3, and τ_4 be defined below. Suppose that*

$$u(z) = \pi_1(z) = \frac{(1 - \lambda_s z)\,[x(z) - x(\gamma_1)]}{-\lambda_s(z - \gamma_1)(z - \gamma_2)}, \tag{D.1}$$

where $x(z) = A(z)a(z) + B(z)$ is an affine transformation of $a(z)$ in $\mathbf{H}^2(\mathbb{D})$ and γ_1 and γ_2 are the two real roots of the quadratic equation $P_1(z) \equiv -\lambda_s z^2 + (\beta \tau_2 \lambda_s + 1)z - [(1 - \lambda_s)\tau_1 + \beta \tau_2]$, with $|\gamma_1| < 1$ and $|\gamma_2| > 1$. If the analytic function

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \kappa(\beta)}{z - \beta}$$

has no roots in the open unit disk $\mathbb{D}$, where

$$\kappa(z) = \tilde{A}(z)a(z) + \tilde{B}(z) - \left[\frac{(1 - \lambda_s)\tau_3}{1 - \lambda_s z} + \beta \tau_4\right]\pi_1(z) + [(1 - \lambda_s)\tau_3 + \beta \tau_4]\,\pi_1(0),$$

then there exists a unique equilibrium with the equity price q_t given by (44).

Proof of Theorem 5: Consider the equilibrium conjecture in (44):

$$q_t = \frac{\pi_1(L)}{1 - \pi_2(L)}\epsilon_{at} + \frac{\pi_2(L)}{1 - \pi_2(L)}u(L)\epsilon_{ut},$$

where $\pi_1(z)$ and $\pi_2(z)$ are rational functions. Given the assumption that $u(z) = \pi_1(z)$, it follows that

$$q_t = \frac{\pi_1(L)}{1 - \pi_2(L)}\epsilon_{at} + \frac{\pi_2(L)\pi_1(L)}{1 - \pi_2(L)}\epsilon_{ut}.$$

For q_t to be a causal stationary process, we need $\frac{\pi_1(z)}{1 - \pi_2(z)}$ and $\pi_1(z)$ to be in the Hardy space $\mathbf{H}^2(\mathbb{D})$.

We need to drive the Wold representation for the signal process $\{X_{it}\}$ given in (45). We can compute the covariance generating function

$$S_x(z) = H(z)\Sigma_{\boldsymbol{\eta}}H(z^{-1})^T = \begin{bmatrix} \sigma_a^2 + \sigma_i^2 & \frac{\pi_1(z^{-1})}{1 - \pi_2(z^{-1})}\sigma_a^2 \\ \frac{\pi_1(z)}{1 - \pi_2(z)}\sigma_a^2 & \frac{\pi_1(z)\pi_1(z^{-1})}{(1 - \pi_2(z))(1 - \pi_2(z^{-1}))}(\sigma_a^2 + \sigma_u^2) \end{bmatrix},$$

where

$$\Sigma_\eta = \begin{bmatrix} \sigma_a^2 & & \\ & \sigma_i^2 & \\ & & \sigma_u^2 \end{bmatrix}$$

is the covariance matrix for the innovation vector η_{it}. We wish to derive the spectral factorization

$$S_x(z) = \Gamma(z)\Gamma(z^{-1})^T.$$

Applying the triangular factorization method in Proposition 1,[13] we obtain

$$\Gamma(z) = \begin{bmatrix} \sigma_e & \frac{\sigma_a^2}{\sigma_p} \\ 0 & \frac{\pi_1(z)}{1-\pi_2(z)}\sigma_p \end{bmatrix},$$

where we define $\sigma_e^2 = \sigma_i^2 + \frac{\sigma_a^2\sigma_u^2}{\sigma_a^2+\sigma_u^2}$ and $\sigma_p^2 = \sigma_a^2 + \sigma_u^2$. Note that

$$\det\Gamma(z) = \sigma_p\sigma_e\frac{\pi_1(z)}{1-\pi_2(z)}.$$

By Theorem 4.6.11 in Lindquist and Picci (2015), $\Gamma(z)$ is an outer spectral factor if and only if $\frac{\pi_1(z)}{1-\pi_2(z)}$ has no roots in the open unit disk. We shall make this assumption and then obtain the Wold representation $X_{it} = \Gamma(L)\,v_{it}$, where v_{it} is a two-dimensional Wold fundamental innovation vector with zero mean and identity covariance matrix. In this case we do not need step 2 when using our two-step procedure.

Now we solve for the equilibrium quantities. We conjecture that $y_{it} = M_y(L)\eta_{it}$, where $M_y(z) = \left[M_y^a(z), M_y^i(z), M_y^u(z)\right]$. Aggregation leads to aggregate output $y_t = M_y^A(L)\,\eta_{it}$, where $M_y^A(z) = M_y(z)I_i$ and

$$I_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Using the Wiener-Hopf prediction formula, we can derive that $\mathbb{E}_{it}[y_t] = \Psi(L)X_{it}$, where

$$\Psi(z) = \left[S_{yx}(z)\left(\Gamma^{-1}(z^{-1})\right)^T\right]_+\Gamma^{-1}(z) = \left[\psi_y(z)\right]_+\Gamma^{-1}(z),$$

and the cross-spectrum is given by

$$S_{yx}(z) = M_y^A(z)\Sigma_s H^T(z^{-1}).$$

Routine algebra then reveals that

$$\psi_y(z) = \left[(\frac{\sigma_a^2}{\sigma_e} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e})M_y^a(z) - \frac{\sigma_a^2\sigma_u^2}{\sigma_p^2\sigma_e}M_y^u(z), \frac{M_y^a(z)\sigma_a^2 + M_y^u(z)\sigma_u^2}{\sigma_p}\right].$$

---

[13]Following Rondina and Walker (2015), we transform the lower-triangular matrix to the upper triangular form by right multiplication of an unitary matrix, which ease the algebra.

By inspection, $\psi_y(z)$ is analytic in the open unit disk and has square-summable Laurent series expansion. Therefore, $[\psi_y(z)]_+ = \psi_y(z)$. In the innovation form, we have

$$\mathbb{E}_{it}[y_t] = [\psi_y(L)]_+ \Gamma^{-1}(L)H(L)\eta_{it}$$
$$= \left[(h_1 + h_3)M_y^a(L) + (h_4 - h_2)M_y^u(L), \ h_1 M_y^a(L) - h_2 M_y^u(L), \ h_3 M_y^a(L) + h_4 M_y^u(L)\right]\eta_{it},$$

where we define

$$h_1 = \frac{\sigma_a^2}{\sigma_e^2} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e^2}, \quad h_2 = \frac{\sigma_a^2\sigma_u^2}{\sigma_p^2\sigma_e^2},$$
$$h_3 = \frac{\sigma_a^2}{\sigma_p^2} + \frac{\sigma_a^6}{\sigma_p^4\sigma_e^2} - \frac{\sigma_a^4}{\sigma_p^2\sigma_e^2}, \quad h_4 = \frac{\sigma_a^4\sigma_u^2}{\sigma_p^4\sigma_e^2} + \frac{\sigma_u^2}{\sigma_p^2}.$$

Plugging $y_{it} = M_y(L)\eta_{it}$ and the preceding conditional expectation $\mathbb{E}_{it}[y_t]$ into (21) and matching coefficients, we obtain a system of linear equations

$$M_y^a(z) = \frac{1}{\xi}a(z) + \theta\left[(h_1 + h_3)M_y^a(z) + (h_4 - h_2)M_y^u(z)\right],$$
$$M_y^u(z) = \theta\left[h_3 M_y^a(z) + h_4 M_y^u(z)\right],$$
$$M_y^i(z) = \frac{1}{\xi} + \theta\left[h_1 M_y^a(z) - h_2 M_y^u(z)\right],$$

which yields the solution

$$M_y^a(z) = \frac{a(z)}{\xi m_1}, \quad M_y^u(z) = \frac{m_2 a(z)}{\xi m_1},$$
$$M_y^i(z) = \frac{1}{\xi} + \theta_\beta \frac{h_1 - h_2 m_2}{\xi m_1}a(z),$$

where $m_1 = 1 - \frac{(h_4 - h_2)h_3\theta}{1 - \theta h_4} - \theta(h_1 + h_3)$, $m_2 = \frac{h_3\theta}{1 - \theta_\beta h_4}$.

Next we proceed to the financial market and construct the informational fixed point. Conjecture the shareholding choice takes the form $s_{it+1}^h = M_s(L)\eta_{it}$, where $M_s(z) = \left[0, M_s^i(z), 0\right]$. By (24) and (43), we obtain

$$(\alpha_1 - \alpha_2 L)M_s(L)\eta_{it} = \pi_1(L)(\epsilon_{at} + \epsilon_{it}) + \pi_2(L)(q_t + u_t) - q_t.$$

Plugging (44) into this equation and using the assumption $u(z) = \pi_1(z)$, we can verify our conjecture by matching coefficients on innovations to get

$$M_s^i(z) = \frac{\pi_1(z)}{\alpha_1 - \alpha_2 z}. \tag{D.2}$$

We can then compute the conditional expectation

$$\mathbb{E}_{it}\left[\alpha_3 s_{it+2}^h\right] = \alpha_3 \left[z^{-1}S_{sx}(L)\left[\Gamma^{-1}(L^{-1})\right]^T\right]_+ \Gamma^{-1}(L)X_{it}$$
$$= \alpha_3 \left[\frac{\sigma_i^2}{\sigma_e^2}\frac{M_s^i(L) - M_s^i(0)}{L}, \quad -\frac{\sigma_a^2\sigma_i^2}{\sigma_p^2\sigma_e^2}\frac{1 - \pi_2(L)}{\pi_1(L)}\frac{M_s^i(L) - M_s^i(0)}{L}\right]X_{it}.$$

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Conjecture that $d_t = M_d(L)\eta_{it}$, $n_t = M_n(L)\eta_{it}$, and $b_t = M_b(L)\eta_{it}$, where

$$
\begin{aligned}
M_d(z) &= \left[M_d^a(z), M_d^i(z), M_d^u(z) \right], \\
M_n(z) &= \left[M_n^a(z), M_n^i(z), M_n^u(z) \right], \\
M_b(z) &= \left[M_b^a(z), M_b^i(z), M_b^u(z) \right].
\end{aligned}
$$

Using equations (25), (26), and (27) and matching coefficients, we can derive that

$$
\begin{aligned}
M_d(z) &= \left[\left(\frac{1}{\alpha_6} - \frac{\alpha_7}{\alpha\alpha_6} \right) M_y^a(z) + \frac{\alpha_7}{\alpha\alpha_6} a(z), \ 0, \ \frac{1}{\alpha_6}(1 - \frac{\alpha_7}{\alpha}) M_y^u(z) \right], \\
M_n(z) &= \frac{1}{\alpha} \left[M_y^a(z) - a(z), \ M_y^i(z) - 1, \ M_y^u(z) \right], \\
M_b(z) &= \alpha_4 M_d(z) + \alpha_5 M_n(z).
\end{aligned}
$$

Therefore, we can compute the rest of the expectational terms in χ_{it}, and then substitute them into our initial conjecture in (43). After some tedious algebra, we construct the following informational fixed point equations

$$
z\pi_1(z) = \alpha_3\tau_1 \left[M_s^i(z) - M_s^i(0) \right] + \beta\tau_2 \left[\pi_1(z) - \pi_1(0) \right] + (1-\beta) \left[G_d^{(1)}(z) - G_d^{(1)}(0) \right] + G_b^{(1)}(z) - G_b^{(1)}(0),
$$
(D.3)

and

$$
\begin{aligned}
z\pi_2(z) = \frac{1 - \pi_2(z)}{\pi_1(z)} &\left[-\alpha_3\tau_3 \left[M_s^i(z) - M_s^i(0) \right] - \beta\tau_4 (\pi_1(z) - \pi_1(0)) - \frac{\beta\pi_1(0)}{1 - \pi_2(0)} \right. \\
&\left. + (1-\beta) \left[G_d^{(2)}(z) - G_d^{(2)}(0) \right] + G_b^{(2)}(z) - G_b^{(2)}(0) \right] + \beta,
\end{aligned}
$$
(D.4)

where we define the relevant signal-noise ratios as

$$
\tau_1 = \frac{\sigma_i^2}{\sigma_e^2}, \quad \tau_2 = \frac{\sigma_a^2\sigma_u^2}{\sigma_e^2\sigma_p^2}, \quad \tau_3 = \frac{\sigma_a^2\sigma_i^2}{\sigma_e^2\sigma_p^2}, \quad \tau_4 = \frac{\sigma_u^2}{\sigma_p^2} \left(1 + \frac{\sigma_a^4}{\sigma_p^2\sigma_e^2} \right),
$$

and the analytic functions as

$$
\begin{aligned}
G_d^{(1)}(z) &= h_1 M_d^a(z) - h_2 M_d^u(z), \\
G_d^{(2)}(z) &= h_3 M_d^a(z) + h_4 M_d^u(z), \\
G_b^{(1)}(z) &= (z-1) \left[h_1 M_b^a(z) - h_2 M_b^u(z) + \tau_1 M_b^i(z) \right], \\
G_b^{(2)}(z) &= (z-1) \left[h_3 M_b^a(z) + h_4 M_b^u(z) - \tau_3 M_b^i(z) \right].
\end{aligned}
$$

We also define the function

$$
x(z) = (1-\beta) \left[G_d^{(1)}(z) - G_d^{(1)}(0) \right] + G_b^{(1)}(z) - G_b^{(1)}(0).
$$

Substituting (D.2) into (D.3), we can show that

$$\pi_1(z) = \frac{(1 - \lambda_s z)\left[x(z) - ((1 - \lambda_s)\tau_1 + \beta\tau_2)\pi_1(0)\right]}{P_1(z)},$$

where

$$P_1(z) \equiv -\lambda_s z^2 + (\beta\tau_2\lambda_s + 1)z - \left[(1 - \lambda_s)\tau_1 + \beta\tau_2\right].$$

We need the following lemma on the distribution of roots.

Lemma 9 *Suppose that $\sigma_i, \sigma_a, \sigma_u \in (0, \infty)$. Then the polynomial function $P_1(z)$ has one real root inside the unit circle and one real root outside the unit circle.*

Proof: We will make use of the Rouché theorem. Notice that $\tau_1 \in (0, 1)$, $\tau_2 \in (0, 1)$, and $\tau_1 + \tau_2 = 1$. It is easy to see that the polynomial function

$$P_1^f(z) \equiv -\lambda_s z^2 + (\beta\tau_2\lambda_s + 1)z$$

has one root inside the unit circle at $z = 0$ and the other outside unit circle at $z = \frac{\beta\tau_2\lambda_s + 1}{\lambda_s} > 1$. Therefore, it suffices to show that

$$\left|P_1^f(z)\right| > |P_1^g(z)| \equiv \left|-\left((1 - \lambda_s)\tau_1 + \beta\tau_2\right)\right| = |(1 - \lambda_s)\tau_1 + \beta\tau_2|$$

for all $z \in \mathbb{T}$. Since $\lambda_s \in (1/2, 1)$, it follows that on the unit circle

$$\left|P_1^f(z)\right| = |\lambda_s z - (1 + \beta\tau_2\lambda_s)| \geq |\lambda_s - (1 + \beta\tau_2\lambda_s)| = (1 + \beta\tau_2\lambda_s) - \lambda_s.$$

Finally,

$$(1 + \beta\tau_2\lambda_s) - \lambda_s - \left[(1 - \lambda_s)\tau_1 + \beta\tau_2\right] = (1 - \lambda_s)(1 - \tau_1 - \beta\tau_2)$$

$$> (1 - \lambda_s)(1 - \tau_1 - \tau_2) = 0.$$

Since $\beta \in (0, 1)$, we have strict inequality. By the Rouché theorem, the proof is complete. $\qquad \square$

Given the preceding lemma, we can write

$$\pi_1(z) = \frac{(1 - \lambda_s z)}{-\lambda_s(z - \gamma_2)(z - \gamma_1)}\left[x(z) - ((1 - \lambda_s)\tau_1 + \beta\tau_2)\pi_1(0)\right],$$

with γ_1 indicating the root inside the unit circle. To remove this pole, we set $\pi_1(0)$ such that

$$x(\gamma_1) - ((1 - \lambda_s)\tau_1 + \beta\tau_2)\pi_1(0) = 0,$$

which implies that

$$\pi_1(0) = \frac{x(\gamma_1)}{(1 - \lambda_s)\tau_1 + \beta\tau_2}.$$

Now we can collect terms and simplify expressions to derive

$$\pi_1(z) = \frac{(1 - \lambda_s z)\,[x(z) - x(\gamma_1)]}{-\lambda_s(z - \gamma_1)(z - \gamma_2)}, \tag{D.5}$$

where $x(z) = A(z)a(z) + B(z)$,

$$\begin{aligned}
A(z) = {}&(1 - \beta)\frac{(h_1 - h_2 m_2)\alpha + [h_1(\xi m_1 - 1) + h_2 m_2]\alpha_7}{\alpha\alpha_6\xi m_1} \\
&+ \{(h_1 - h_2 m_2)\alpha\alpha_4 + h_2 m_2\alpha_7\alpha_4 + [h_1\alpha_5\alpha_6 - h_1\alpha_4\alpha_7](1 - \xi m_1) \\
&+ \alpha_5\alpha_6[\tau_1\theta h_1 - h_2 m_2(1 + \tau_1\theta]\}\frac{(z-1)}{\alpha\alpha_6\xi m_1},
\end{aligned}$$

and

$$\begin{aligned}
B(z) = {}&\Bigg[\frac{(h_1 - h_2 m_2)\alpha\alpha_4 + h_2 m_2\alpha_7\alpha_4 + [h_1\alpha_5\alpha_6 - h_1\alpha_4\alpha_7](1 - \xi m_1) + \alpha_5\alpha_6[\tau_1\theta h_1 - h_2 m_2(1 + \tau_1\theta]}{\alpha\alpha_6\xi m_1} \\
&- (1 - \beta)\frac{(h_1 - h_2 m_2)\alpha + [h_1(\xi m_1 - 1) + h_2 m_2]\alpha_7}{\alpha\alpha_6\xi m_1}\Bigg]a(0) + \frac{\tau_1\alpha_5(1 - \xi)}{\alpha\xi}z.
\end{aligned}$$

Note that $A(z) \in \mathbf{H}^\infty(\mathbb{D})$ defines a linear and bounded multiplication operator in $\mathbb{D}$.[14] Thus $x(z)$ defines an affine transformation of $a(z)$ in $\mathbf{H}^2(\mathbb{D})$.

Next consider the fixed point condition (D.4). After some algebra, we can write the condition in terms of $\frac{\pi_1(z)}{1 - \pi_2(z)}$:

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\tilde{x}(z) + \pi_1(z)z}{z - \beta}, \tag{D.6}$$

where

$$\begin{aligned}
\tilde{x}(z) = {}&-\alpha_3\tau_3\left[M_s^i(z) - M_s^i(0)\right] + \beta\left[-\tau_4\,(\pi_1(z) - \pi_1(0)) - \frac{\pi_1(0)}{1 - \pi_2(0)}\right] \\
&+ (1 - \beta)\left[G_d^{(2)}(z) - G_d^{(2)}(0)\right] + G_b^{(2)}(z) - G_b^{(2)}(0)
\end{aligned}$$

is by construction a rational analytic function in the closed unit disk given the solution of $\pi_1(z)$. Plug (D.2) into the equation above, use the define $\lambda_s = \alpha_2/\alpha_1$ and $1 - \lambda_s = \alpha_3/\alpha_3$, and define

$$\begin{aligned}
\kappa(z) = {}&(1 - \beta)[G_d^{(2)}(z) - G_d^{(2)}(0)] + G_b^{(2)}(z) - G_b^{(2)}(0) - \left[\frac{(1 - \lambda_s)\tau_3}{1 - \lambda_s z} + \beta\tau_4\right]\pi_1(z) + [(1 - \lambda_s)\tau_3 + \beta\tau_4]\,\pi_1(0) \\
\equiv {}&\tilde{A}(z)a(z) + \tilde{B}(z) - \left[\frac{(1 - \lambda_s)\tau_3}{1 - \lambda_s z} + \beta\tau_4\right]\pi_1(z) + [(1 - \lambda_s)\tau_3 + \beta\tau_4]\,\pi_1(0)
\end{aligned}$$

We can easily compute $\tilde{A}(z)$ and $\tilde{B}(z)$ by direct substitution of the expressions for $G_d^{(2)}(\cdot)$ and $G_b^{(2)}(\cdot)$. For simplicity, we omit these expressions.

---

[14]See Conway (1990), p.28, Theorem 1.5 and Lindquist and Picci (2015), Theorem 4.3.3 (Bochner-Chandrasekharan) and Proposition B.2.4.

As we mentioned earlier, we need $\frac{\pi_1(z)}{1-\pi_2(z)}$ to be analytical in the unit disk. Thus we should remove the pole at $z = \beta$ by setting the constant $\pi_2(0)$ such that $\tilde{x}(\beta) + \pi_1(\beta)\beta = 0$. Solving this equation yields

$$\pi_2(0) = 1 - \frac{\pi_1(0)\beta}{\kappa(\beta)}.$$

We can then rewrite (D.6) as

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \kappa(\beta)}{z - \beta}. \tag{D.7}$$

Clearly, the model solution is rational in the frequency domain if $a(z)$ is a rational function. For the solution we presented above to be the unique stationary equilibrium, we need to make sure that the spectral factorization is valid. As discussed earlier, we need the equation

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\kappa(z) - \kappa(\beta)}{z - \beta} = 0$$

to have no roots inside the open unit disk. This condition completes the proof. Q.E.D.

Next we present a limiting result.

Theorem 6 *Suppose the conditions in Theorem 5 hold. Then for any finite $\sigma_a \in (0, \infty)$,*

$$\lim_{\sigma_i \to \infty} \|\pi_1(z)\|_{\mathbf{H}^2} = \infty \quad and \quad \lim_{\sigma_i \to \infty} Var(q_t^f) = \sigma_a^2 \lim_{\sigma_i \to \infty} \left\| \frac{\pi_1(z)}{1 - \pi_2(z)} \right\|_{\mathbf{H}^2}^2 = \infty.$$

Proof of Theorem 6: We first show that a unit root arises in $\pi_1(z)$ as $\sigma_i \to \infty$. Consider the polynomial function,

$$P_1(z) \equiv -\lambda_s z^2 + (\beta \tau_2 \lambda_s + 1)z - [(1 - \lambda_s)\tau_1 + \beta \tau_2].$$

$P_1(z)$ can be written as a continuous function of coefficients, and hence in σ_i, as a parameterized function $P_1(z, \sigma_i)$. Therefore, we know that $\lim_{\sigma_i \to \infty} P_1(z, \sigma_i)$ exists and can be written as

$$\lim_{\sigma_i \to \infty} P_1(z, \sigma_i) = -\lambda_s z^2 + z - (1 - \lambda_s) = -\lambda_s (z - 1)(z - \frac{1 - \lambda_s}{\lambda_s}),$$

where we have used the limits

$$\lim_{\sigma_i \to \infty} \tau_1 = \lim_{\sigma_i \to \infty} \frac{\sigma_i^2}{\sigma_e^2} = 1, \quad \lim_{\sigma_i \to \infty} \tau_2 = \lim_{\sigma_i \to \infty} \frac{\sigma_a^2 \sigma_u^2}{\sigma_e^2 \sigma_p^2} = 0.$$

Since $\lambda_s \in (1/2, 1)$, one of the roots $\gamma_1 = \frac{1 - \lambda_s}{\lambda_s}$ is located strictly inside the unit circle. Lemma 9 shows that $P_1(z)$ always has one root inside the unit circle and the other outside the unit circle when $\sigma_i, \sigma_a, \sigma_u \in (0, \infty)$. Since λ_s and γ_1 are independent of σ_i, by the continuous dependence of roots on coefficients, the larger root γ_2 gradually converges to the unit root as $\sigma_i \to \infty$.

We then show that $\pi_1(z)$ cannot have a zero at $z = 1$. By (D.1), it suffices to show that the analytic function $x(z) - x(\gamma_1)$ does not have a zero at $z = 1$. Using the previous definition, we can show that $\lim_{\sigma_i \to \infty} h_1 = \lim_{\sigma_i \to \infty} h_2 = 0$. Moreover,

$$\lim_{\sigma_i \to \infty} x(z) = \frac{\alpha_5(1 - \xi)}{\alpha\xi} z.$$

Therefore,

$$\lim_{\sigma_i \to \infty} [x(1) - x(\gamma_1)] = \frac{\alpha_5(1 - \xi)}{\alpha\xi}(1 - \gamma_1) = \frac{\alpha_5(1 - \xi)}{\alpha\xi}(1 - \frac{1 - \lambda_s}{\lambda_s}),$$

which is clearly non-zero. It follows from (D.1) that $\lim_{\sigma_i \to \infty} \|\pi_1(z)\|_{\mathbf{H}^2} \to \infty$.

Finally, consider the analytic function derived from (D.7)

$$\frac{\pi_1(z)}{1 - \pi_2(z)} = \frac{\tilde{A}(z)a(z) + \tilde{B}(z) - \left[\frac{(1 - \lambda_s)\tau_3}{1 - \lambda_s z} + \beta\tau_4\right]\pi_1(z) + [(1 - \lambda_s)\tau_3 + \beta\tau_4]\pi_1(0) - \kappa(\beta)}{z - \beta}. \quad \text{(D.8)}$$

We can check that $\tilde{A}(z)$ and $\tilde{B}(z)$ are polynomial functions that are analytic in the entire complex plane, even as $\sigma_i \to \infty$. We then need to show that there is no zero at $z = 1$ as $\sigma_i \to \infty$.

Since $a(z)$ is an exogenous rational analytic function in the closed unit disk, whose form is independent of σ_i. Without loss of generality, we can write $a(z) \equiv \frac{a_p(z)}{a_q(z)}$. We focus on the numerator of (D.8) and plug (D.5) into it to obtain

$$\tilde{A}(z)a(z) + \tilde{B}(z) - \left[\frac{(1 - \lambda_s)\tau_3}{1 - \lambda_s z} + \beta\tau_4\right]\pi_1(z) + [(1 - \lambda_s)\tau_3 + \beta\tau_4]\pi_1(0) - \kappa(\beta)$$

$$= \frac{\lambda_s(z - \gamma_1)(z - \gamma_2)\left[\tilde{A}(z)a_p(z) + \tilde{B}(z)a_q(z) + [((1 - \lambda_s)\tau_3 + \beta\tau_4)\pi_1(0) - \kappa(\beta)]a_q(z)\right]}{\lambda_s(z - \gamma_1)(z - \gamma_2)a_q(z)}$$

$$+ \frac{[x(z) - x(\gamma_1)][(1 - \lambda_s)\tau_3 + \beta\tau_4(1 - \lambda_s z)]a_q(z)}{\lambda_s(z - \gamma_1)(z - \gamma_2)(1 - \lambda_s z)a_q(z)}.$$

Consider the numerators of the two terms on the right side of the equation above. Since $a(z)$ is by construction a rational analytic function on $\mathbb{T} \cup \mathbb{D}$, $a_q(z)$ cannot have zeros on the unit circle. Evaluating the numerator of the first term at $z = \lim_{\sigma_i \to \infty} \gamma_2 = 1$, it vanishes. Consider the second numerator

$$\tilde{\kappa}(z) \equiv [x(z) - x(\gamma_1)][(1 - \lambda_s)\tau_3 + \beta\tau_4(1 - \lambda_s z)]a_q(z).$$

Taking limits as $\sigma_i \to \infty$ yields

$$\lim_{\sigma_i \to \infty} \tau_3 = \frac{\sigma_a^2}{\sigma_p^2} \quad \lim_{\sigma_i \to \infty} \tau_4 = \frac{\sigma_u^2}{\sigma_p^2} = 1 - \frac{\sigma_a^2}{\sigma_p^2}.$$

By continuity, we have

$$\lim_{\sigma_i \to \infty} \tilde{\kappa}(z) = a_q(z)\lim_{\sigma_i \to \infty}[x(z) - x(\gamma_1)]\lim_{\sigma_i \to \infty}[(1 - \lambda_s)\tau_3 + \beta\tau_4(1 - \lambda_s z)]$$

$$= a_q(z)\left[\frac{\alpha_5(1 - \xi)}{\alpha\xi}(z - \frac{1 - \lambda_s}{\lambda_s})\right]\left[(1 - \lambda_s)\frac{\sigma_a^2}{\sigma_p^2} + \beta(1 - \frac{\sigma_a^2}{\sigma_p^2})(1 - \lambda_s z)\right].$$

Since $a(z)$ is by construction a rational analytic function on $\mathbb{T} \cup \mathbb{D}$, $a_q(z)$ cannot have zeros on the unit circle. It is then clear that the above limit cannot have zeros when evaluated at $z = 1$. This completes the proof.   Q.E.D.

Proof of Noise Normalization: We formally prove that

$$\lim_{\sigma_i \to \infty} Var(q_t) = \left\| \frac{\pi_2(z)\pi_1(z)}{1 - \pi_2(z)} \right\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2} < \infty,$$

after we normalize $u(L) = \frac{\pi_1(L)}{\sigma_i}$. Fix a $\sigma_i \in (0, \infty)$. We know that $\frac{\pi_2(z)\pi_1(z)}{1 - \pi_2(z)} = \frac{\pi_1(z)}{1 - \pi_2(z)} - \pi_1(z)$ is in $\mathbf{H}^2(\mathbb{D})$ by Theorem 5. Using the triangle inequality,

$$\left\| \frac{\pi_2(z)\pi_1(z)}{1 - \pi_2(z)} \right\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2} \leq \left\| \frac{\pi_1(z)}{1 - \pi_2(z)} \right\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2} + \|\pi_1(z)\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2}.$$

Without loss of generality, we can write

$$\pi_1(z) = c \frac{\prod_{i=1}^n (1 - z_i z)}{\prod_{j=1}^m (1 - z_j z)}$$

where c is some constant and the degree of m and n depends on the functional form of $a(z)$. In particular, the denominator will contain the roots of the quadratic polynomial $P_1(z)$. For simplicity, we assume $m > n$ and z_j are distinct.[15] Using the residue theorem, we know

$$\|\pi_1(z)\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2} = \frac{\sigma_u^2}{\sigma_i^2} \frac{1}{2\pi i} \oint_{\mathbb{T}} |\pi_1(z)|^2 \frac{dz}{z} = \frac{\sigma_u^2}{\sigma_i^2 c^2} \frac{1}{2\pi i} \oint_{\mathbb{T}} \frac{\prod_{i=1}^n (1 - z_i z) \prod_{i=1}^n (1 - \overline{z_i} z^{-1})}{\prod_{j=1}^m (1 - z_j z) \prod_{j=1}^m (1 - \overline{z_j} z^{-1})} \frac{dz}{z}.$$

Define the function

$$f(z) = \frac{\prod_{i=1}^n (1 - z_i z) \prod_{i=1}^n (1 - \overline{z_i} z^{-1})}{\prod_{j=1}^m (1 - z_j z) \prod_{j=1}^m (1 - \overline{z_j} z^{-1}) z} = \frac{z^{m-n-1} \prod_{i=1}^n (1 - z_i z) \prod_{i=1}^n (z - \overline{z_i})}{\prod_{j=1}^m (1 - z_j z) \prod_{j=1}^m (z - \overline{z_j})}.$$

It is clear that $f(z)$ has isolated singularities insider the unit circle at $z = \overline{z_1}, \overline{z_2} \ldots \overline{z_j}$. The integral can be evaluated as

$$\oint_{\mathbb{T}} f(z) dz = \sum_{j=1}^m Res(f, \overline{z_j}),$$

where

$$Res(f, z_j) = 2\pi i \frac{\overline{z_j}^{m-n-1} \prod_{i=1}^n (1 - z_i \overline{z_j}) \prod_{i=1}^n (\overline{z_j} - \overline{z_i})}{\prod_{k=1}^m (1 - z_k \overline{z_j}) \prod_{k \neq j} (\overline{z_j} - \overline{z_k})}.$$

By the continuous dependence of polynomial roots on coefficients and the partial fraction representation of the complex integral, we know that the norm is continuous in σ_i. Now let $\sigma_i \to \infty$, we know by Theorem 5 and 6 that one real root z_u in the denominator converges to 1. Taking limits, we have

$$\lim_{\sigma_i \to \infty} \|\pi_1(z)\|^2_{\mathbf{H}^2} \frac{\sigma_u^2}{\sigma_i^2} = \lim_{\sigma_i \to \infty} \frac{\sigma_u^2}{\sigma_i^2 c^2} Res(f, z_u).$$

---

[15] In the case of $m \leq n$ or repeated z_j, we need to modify the formula for the residues of higher-order poles. However, the rest of the proof will remain the same.

The rest of residuals are dominated by $\frac{1}{\sigma_i^2}$ and hence vanish. The limit can be written as

$$\lim_{\sigma_i \to \infty} \frac{\sigma_u^2}{\sigma_i^2 c^2} Res(f, z_u) = \lim_{\sigma_i \to \infty} \frac{\sigma_u^2}{\sigma_i^2 c^2 (1 - z_u^2)} \frac{z_u^{m-n-1} \prod_{i=1}^n (1 - z_i z_u) \prod_{i=1}^n (z_u - \overline{z_i})}{\prod_{k \neq u}(1 - z_k z_u) \prod_{k \neq u}(z_u - \overline{z_k})}.$$

The second component is clearly non-zero constant. The root z_u is given by the quadratic equation $P_1(z)$. Elementary asymptotic theory shows that $(1 - z_u^2(\sigma_i)) \sim O(\sigma_i^{-2})$. Therefore by definition,[16]

$$0 < \lim_{\sigma_i^2 \to \infty} \sigma_i^2 (1 - z_u^2) < \infty$$

This ensures that the $0 < \lim_{\sigma_i \to \infty} \|\pi_1(z)\|_{\mathbf{H}^2}^2 \frac{\sigma_u^2}{\sigma_i^2} < \infty$. Similar argument can be used to show that $0 < \lim_{\sigma_i \to \infty} \left\| \frac{\pi_1(z)}{1 - \pi_2(z)} \right\|_{\mathbf{H}^2}^2 \frac{\sigma_u^2}{\sigma_i^2} < \infty$. By continuity, take limit to the triangle inequality derived before. The proof is then complete. In particular, in the limit the model still features persistent dispersed information induced by the noise u_t, but its volatility is normalized and hence finite. Q.E.D.

E   Proofs of Results in Section 4

Proof of Lemma 1:   This is a standard result in the harmonic analysis of stationary time series. Without loss of generality, we normalize the structural innovation to unity. Since $H(z)$ is rational, representing the ARMA (p,q) representation, the signal process is linear regular. By Lindquist and Picci (2015), Theorem 4.2.1, $f_x(\omega)$ is absolutely continuous, and has constant rank ℓ for almost all $\omega \in [-\pi, \pi]$. The proof of the spectral density can be found in standard textbooks such as Sargent (1987) and Brockwell and Davis (2002), which in turn implies that $f_x(\omega)$ is rational. The Hermitian, normal, and non-negative definite property of the spectral density can be found in Rozanov (1967), chapter 1. Brockwell and Davis (2002), p420 contains the covariance matrix generating function formula for real-valued process. For complex-valued processes, the formula for $S_x(z)$ follows from Lippi and Reichlin (1994) and Bernhardt, Seiler, and Taub (2010). Finally, the non-negative definite constant covariance matrix Σ_η can be decomposed, using the unitary eigen-decomposition (spectral theorem), as

$$\Sigma_\eta = Q_\eta D_\eta Q_\eta^*,$$

where Q_η is an unitary matrix ($Q_\eta Q_\eta^* = I$) consisting of orthonormal eigenvectors of Σ_η, D_η is the diagonal matrix containing eigenvalues of Σ_η. By the non-negativeness, all eigenvalues are real and non-negative. Therefore, the normalization

$$\hat{H}\left(e^{-i\omega}\right) = H\left(e^{-i\omega}\right) Q_\eta \sqrt{D_\eta}$$

is well-defined and preserves the spectral density.   Q.E.D.

---

[16]The asymptotic notation means that $(1 - z_u^2)$ and σ_i^{-2} has the same order of convergence rate. The proof of this asymptotic result is lengthy and available upon request.

Proof of Proposition 1: The proof is by construction. Since $f_x(\omega)$ is rational, it has a constant, maximal rank of ℓ except at a finite number of points on the unit circle $\mathbb{T}$. To develop the triangular factorization of the spectral density, we need the following lemma from Rozanov (1967) on rational functions.

Lemma 10 *Every non-negative (real) rational function $f(\omega)$ of $e^{-i\omega}$ can be represented in the form*

$$f(\omega) = \frac{|P(e^{-i\omega})|^2}{|Q(e^{-i\omega})|^2} = \frac{P(e^{-i\omega})\overline{P(e^{-i\omega})}}{Q(e^{-i\omega})\overline{Q(e^{-i\omega})}} = \frac{P(z)\overline{P(z)}}{Q(z)\overline{Q(z)}}$$

for $z \in \mathbb{T}$. The polynomial functions $P(z)$ and $Q(z)$ have no zeros in the open unit disk. If f satisfies

$$f(\omega) = f(-\omega)$$

Then the coefficients of $P(z)$ and $Q(z)$ can be chosen all real.

Proof: See Rozanov (1967), Lemma 10.1. □

If we extend $f(z)$ to be a complex function in the entire complex plane, the preceding lemma implies that it can be factorized in a "symmetric" way such that if λ_i is a root for $f(z)$, so is the conjugate inverse $1/\overline{\lambda_i}$.

Now consider the $\ell \times \ell$ spectral density matrix $f_x(\omega)$, by definition it is Hermitian, normal, and non-negative definite for almost all ω. For simplicity, we drop the x subscript and write the f matrix as

$$f(\omega) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1\ell} \\ f_{21} & f_{22} & \cdots & f_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ f_{\ell 1} & f_{\ell 2} & \cdots & f_{\ell\ell} \end{bmatrix}.$$

Using the Sylvester's criterion for the non-negative definite matrix, define the family of leading principal minors as $M_j(\omega)$, $j = 1, 2, ...\ell$. By definition, $M_j(\omega) \geq 0$ a.e., and $M_1(\omega) = f_{11} \geq 0$ a.e.

Next we implement elementary row operations on the matrix. Adding to the rth row ($r = 2, 3, ...\ell$) the first row, multiplied by $-\frac{f_{r1}}{f_{11}}$, yielding

$$f(\omega) = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1\ell} \\ 0 & f_{22} - f_{12}\frac{f_{21}}{f_{11}} & \cdots & f_{2\ell} - f_{1\ell}\frac{f_{21}}{f_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & f_{\ell 2} - f_{12}\frac{f_{\ell 1}}{f_{11}} & \cdots & f_{\ell\ell} - f_{1\ell}\frac{f_{\ell 1}}{f_{11}} \end{bmatrix}.$$

Similarly, adding to the jth column ($j = 2, 3, ...\ell$) from the first column multiplied by $-\frac{f_{1j}}{f_{11}}$, we have

$$f^{(2)}(\omega) = \begin{bmatrix} f_{11} & 0 & \cdots & 0 \\ 0 & f_{22} - f_{12}\frac{f_{21}}{f_{11}} & \cdots & f_{2\ell} - f_{1\ell}\frac{f_{21}}{f_{11}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & f_{\ell 2} - f_{12}\frac{f_{\ell 1}}{f_{11}} & \cdots & f_{\ell\ell} - f_{1\ell}\frac{f_{\ell 1}}{f_{11}} \end{bmatrix} = \begin{bmatrix} f_{11} & \mathbf{0} \\ \mathbf{0} & g^{(2)} \end{bmatrix},$$

where the elements of matrix $g^{(2)} = [g_{rj}^{(2)}]$ have the form

$$g_{rj}^{(2)} = f_{rj} - \frac{f_{r1}f_{1j}}{f_{11}}.$$

Notice that the diagonal element $g_{22}^{(2)}$ satisfies $g_{22}^{(2)}(\omega) = \frac{M_2(\omega)}{M_1(\omega)}$ a.e. If we denote $g^{(1)} = f^{(1)} = f$, then $f^{(2)}$ is obtained by using the row-column transformations on $f^{(1)}$. Now consider the matrix $g^{(2)}$,

$$g^{(2)} = \begin{bmatrix} f_{22} - f_{12}\frac{f_{21}}{f_{11}} & \cdots & f_{2\ell} - f_{1\ell}\frac{f_{21}}{f_{11}} \\ \vdots & \ddots & \vdots \\ f_{\ell 2} - f_{12}\frac{f_{\ell 1}}{f_{11}} & \cdots & f_{\ell\ell} - f_{1\ell}\frac{f_{\ell 1}}{f_{11}} \end{bmatrix}.$$

we apply the same transformation for $g^{(2)}$ to eliminate its first row and column except the leading coefficient, yielding

$$g^{(2)} = \begin{bmatrix} f_{22} - f_{12}\frac{f_{21}}{f_{11}} & 0 \\ 0 & g^{(3)} \end{bmatrix}$$

it is easy to verify that $g_{33}^{(3)}(\omega) = \frac{M_3(\omega)}{M_2(\omega)}$. We then arrive at a new $\ell \times \ell$ matrix as

$$f^{(3)}(\omega) = \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} - f_{12}\frac{f_{21}}{f_{11}} & 0 \\ 0 & 0 & g^{(3)} \end{bmatrix}.$$

Continue this process until we reach a diagonal matrix $f^{(\ell)}(\omega)$, admitting the following form

$$f^{(\ell)}(\omega) = \begin{bmatrix} h_{11} & & & \\ & h_{22} & & \\ & & \ddots & \\ & & & h_{\ell\ell} \end{bmatrix}.$$

It is easy to see that the diagonal elements are

$$h_{11}(\omega) = M_1(\omega),$$
$$h_{rr}(\omega) = \frac{M_r(\omega)}{M_{r-1}(\omega)}, \quad r = 2, 3, \ldots\ell.$$

It follows that $f(\omega)$ admits the following LDU-like decomposition.

Lemma 11 *The spectral density $f_x(\omega)$ can be decomposed as*

$$f_x = g\, f^{(\ell)}\, g^*$$

where the matrix function $g(\omega)$ is lower triangular with diagonal elements equal to one,

$$g(\omega) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ g_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{\ell 1} & g_{\ell 2} & \dots & 1 \end{bmatrix}.$$

The off-diagonal non-zero elements are defined as

$$g_{rj} = \frac{g_{rj}^{(j)}}{h_{jj}}, \quad r > j,$$

where $g_{rl}^{(l)}$ is determined by the recursion

$$g_{rj}^{(1)} = f_{rj},$$

$$g_{rj}^{(i)} = g_{rj}^{(i-1)} - \frac{g_{r,i-1}^{(i-1)} - g_{i-1,j}^{(i-1)}}{g_{i-1,i-1}^{(i-1)}}, \quad i = 2, 3, \dots j.$$

Proof: Direct matrix multiplication will verify the result. $\qquad\qquad\qquad\qquad\square$

Since the element of $f_x(\omega)$ are rational functions, the matrix transformation implies that elements of g and $f^{(\ell)}$ are rational as well. Next we define

$$g_{rj}(\omega) = \frac{P_{rj}(z)}{Q_{rj}(z)},$$

where $z = e^{-i\omega}$. We extend the definition of z to the entire complex plane, and fix a column $j \in \{1, 2, \dots, \ell\}$. Let $\alpha_p^{(j)}, p = 1, 2, \dots$, denote the roots of the set of polynomials $\{Q_{rj}(z) : r = 1, \dots, \ell\}$ that are located inside the unit circle, counting multiplicities. Define

$$c_j(z) = \prod_p (z - \alpha_p^{(j)}), \quad D_j(z) = \frac{h_{jj}(z)}{|c_j(z)|^2}.$$

Note that $D_j(z)$ is non-negative by construction. We can use Lemma 10 to decompose $D_j(z)$ as

$$D_j(z) = \left| \frac{\Phi_j(z)}{\Psi_j(z)} \right|^2 = \frac{\Phi_j(z)\Phi_j(\frac{1}{z})}{\Psi_j(z)\Psi_j(\frac{1}{z})}$$

on the unit circle, where we can choose $\Phi_j(z)$ and $\Psi_j(z)$ such that they have no zeros inside the unit disk (when extending the definition of z to the entire complex plane). The second equality follows from the real-coefficients assumption. If the polynomials have complex-valued coefficients, we need to conjugate the coefficients accordingly.

Now set

$$\widetilde{\Gamma}_{rj}(z) = g_{rj}(z)c_j(z)\frac{\Phi_j(z)}{\Psi_j(z)}, \quad r = 1, \dots, \ell,$$

where $z = e^{-i\omega}$. Continuing this construction for all columns of g and using Lemma 11, we obtain the desired matrix $\widetilde{\Gamma}(z)$ such that

$$f_x(\omega) = \widetilde{\Gamma}\left(e^{-i\omega}\right) \widetilde{\Gamma}^*\left(e^{-i\omega}\right),$$

where all elements of the matrix function

$$\widetilde{\Gamma}(z) = \begin{bmatrix} \widetilde{\Gamma}_{11}(z) & 0 & \dots & 0 \\ \widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & \dots & \widetilde{\Gamma}_{\ell\ell}(z) \end{bmatrix}$$

are analytic in the closed unit disk and hence in the $\mathbf{H}^2(\mathbb{D})$ space. This completes the proof by construction. Q.E.D.

Proof of Proposition 2: By Proposition 1, we obtain

$$f_x(\omega) = \widetilde{\Gamma}\left(e^{-i\omega}\right) \widetilde{\Gamma}^*\left(e^{-i\omega}\right).$$

The Beurling-Blaschke factorization theorem states that every $\widetilde{\Gamma}(z) \in \mathbf{H}^2(\mathbb{D})$ can be written in the form

$$\widetilde{\Gamma}(z) = \Gamma(z)Q(z), \tag{E.1}$$

where $Q(z)$ is an $\ell \times \ell$ matrix inner function. The proof of this theorem can be found in Rudin (1987), Theorem 17.17. The matrix generalization of this theorem can be found in Lindquist and Picci (2015), Theorem 4.6.5-4.6.8.[17] The factorization is unique up to constant unitary matrices.[18] Since $\widetilde{\Gamma}(z)$ is rational, the outer function $\Gamma(z)$ is also rational as well. A rational outer function is completely characterized by the location of its zeros. That is, a rational function $\Gamma(z)$ is an outer function if and only if $\det(\Gamma(z)) \neq 0$, $\forall |z| < 1$. Hence, the inner function $Q(z)$ can be reduced to the Blaschke matrices satisfying

$$Q(z) = \prod_{j=1}^{n} \widetilde{B_j}(z) V_j, \tag{E.2}$$

where $\widetilde{B_j}$ satisfies

$$\widetilde{B_j}(z) = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \dots & & \frac{z-z_j}{1-\bar{z}_j z} \end{bmatrix} = B_j^{-1}(z),$$

---

[17] Lindquist and Picci (2015) use the engineering definition of $z = e^{i\omega}$ so that the analytic region is reversed comparing with this paper, but all analytic results remain valid.

[18] The conditional uniqueness corresponds only to orthonormal Wold innovations. In fact, given a Wold representation $X_t = \Gamma(L)v_t$, the transformation $X_t = \Gamma(L)\Sigma\Sigma^{-1}v_t$ is also Wold fundamental provided that the constant matrix Σ is invertible. In this case, the Wiener-Hopf formula will be modified to contain Σ.

and z_j are zeros of $\det\left(Q\left(z\right)\right)$ or $\det\left(\widetilde{\Gamma}\left(z\right)\right)$ satisfying $|z_j| < 1$. Here V_j are constant unitary matrices. In other words, the singular part of the rational inner function is absent (see Rudin (1987), Theorem 17.9 and Lindquist and Picci (2015), Theorem 4.6.11). Compared with the general definition of the Blaschke factors, we implicitly assume there are no zeros at $z = 0$ and omit the norm terms $\frac{\bar{z}_j}{|z_j|}$ since finite Blaschke products have no convergence issues.

Combining (E.1) and (E.2), we have

$$\Gamma(z) = \widetilde{\Gamma}\left(z\right) \prod_{j=1}^{n} V_j^{-1} \left[\widetilde{B_j}\left(z\right)\right]^{-1} = \widetilde{\Gamma}\left(z\right) \prod_{j=1}^{n} V_j^{-1} B_j\left(z\right).$$

Note that the Blacheke-inner function satisfies

$$Q(z)Q^*(z) = I, \quad \forall |z| = 1,$$

on the unit circle. The spectral density is preserved under the factorization

$$\Gamma(z)\Gamma^*(z) = \widetilde{\Gamma}\left(z\right) \prod_{j=1}^{n} V_j^{-1} B_j\left(z\right) \prod_{j=1}^{n} B_j^*\left(z\right) (V_j^{-1})^* \widetilde{\Gamma}^*\left(z\right) = f_x(\omega),$$

where $z = e^{-i\omega}$. Moreover, all zeros inside the unit disk are removed because

$$
\begin{aligned}
\det(\Gamma(z)) &= \det(\widetilde{\Gamma}\left(z\right)) \prod_{j=1}^{n} \det(V_j^{-1}) \prod_{j=1}^{n} \frac{1 - \bar{z}_j z}{z - z_j} \\
&= \Upsilon(z) \prod_{j=1}^{n} (z - z_j) \prod_{j=1}^{n} \det(V_j^{-1}) \prod_{j=1}^{n} \frac{1 - \bar{z}_j z}{z - z_j} \\
&= \Upsilon(z) \prod_{j=1}^{n} \det(V_j^{-1}) \prod_{j=1}^{n} (1 - \bar{z}_j z) \\
&\neq 0 \quad \forall |z| < 1
\end{aligned}
$$

where

$$\Upsilon(z) = \frac{\det(\widetilde{\Gamma}\left(z\right))}{\prod_{j=1}^{n}(z - z_j)}$$

has no zeros inside the unit disk by construction. Unfortunately, the right multiplication of the Blaschke matrices also brought poles ($z = z_j$) for the element in the $\widetilde{\Gamma}\left(z\right)$ matrix that has no inside zeros. In order to maintain the analyticity inside the unit disk so that $\Gamma(z) \in \mathbf{H}_{\ell \times \ell}^2(\mathbb{D})$, we need to get rid of these by-product poles. We remove these poles inside the unit disk by setting appropriate constant unitary matrices V_j.

In practice, V_j can be obtained by the singular value decomposition in a sequential procedure. For $j = 1$, we have

$$\Gamma_1(z) = \widetilde{\Gamma}\left(z\right) V_1^{-1} B_1(z)$$

Without the constant unitary matrix V_1, the matrix transformation

$$\widetilde{\Gamma}(z)B_1(z) = \begin{bmatrix} \widetilde{\Gamma}_{11}(z) & 0 & ... & 0 \\ \widetilde{\Gamma}_{21}(z) & \widetilde{\Gamma}_{22}(z) & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Gamma}_{\ell 1}(z) & \widetilde{\Gamma}_{\ell 2}(z) & ... & \widetilde{\Gamma}_{\ell\ell}(z) \end{bmatrix} \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ..... & \frac{1-\bar{z}_1 z}{z-z_1} \end{bmatrix}.$$

It is clear the potential poles can only appear in the last column, if we assume that $\widetilde{\Gamma}_{\ell\ell}(z)$ has no zeros at $z = z_1$. To remove this pole, we follows Rozanov (1967) by employing the singular value decomposition (SVD) for $\widetilde{\Gamma}(z)$ at $z = z_1$

$$\widetilde{\Gamma}(z_1) = U_1 D_1 V_1 = U_1 \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ..... & 0 \end{bmatrix} V_1.$$

By definition, the unitary matrices U_1 and V_1 are given by the (unitary) eigen-decomposition,

$$G_{(z_1)} = \widetilde{\Gamma}(z_1)\widetilde{\Gamma}^*(z_1) = U_1 \bar{D}_1 U_1^*,$$
$$\hat{G}_{(z_1)} = \widetilde{\Gamma}^*(z_1)\widetilde{\Gamma}(z_1) = V_1 \hat{D}_1 V_1^*.$$

Such decomposition always exists as $G_{(z_1)}$ and $\hat{G}_{(z_1)}$ are Hermitian and non-negative definite by construction. The diagonal matrices $\bar{D}_1$ and $\hat{D}_1$ contains eigenvalues of $G_{(z_1)}$ and $\hat{G}_{(z_1)}$, which are not necessarily distinct. The diagonal matrix D_1 in the SVD contains the singular values of $\widetilde{\Gamma}(z)$. The non-zero singular values $\{\lambda_1, \lambda_2, ...\lambda_p\}$ are the square root of the non-zero eigenvalues of $G_{(z_1)}$ and $\hat{G}_{(z_1)}$, which are not necessarily distinct. Since we know that $\det(\widetilde{\Gamma}(z_1)) = 0$,

$$\det(G_{(z_1)}) = \det\left(\widetilde{\Gamma}(z_1)\right)\det\left(\widetilde{\Gamma}(z_1)^*\right) = 0.$$

Therefore, there exists at least one singular value in D_1 that is zero, i.e. $p < d$.[19] Now evaluate $\Gamma_1(z)$ at $z = z_1$,

$$\Gamma_1(z_1) = \widetilde{\Gamma}(z_1)V_1^{-1}B_1(z_1) = U_1 D_1 V_1 V_1^{-1} B_1(z_1)$$
$$= U_1 \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ..... & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ..... & \frac{1-\bar{z}_1 z_1}{z_1-z_1} \end{bmatrix}.$$

Since the last column of D_1 are identically zero, the pole at $\frac{1-\bar{z}_1 z_1}{z_1-z_1}$ vanishes at $z = z_1$. In other words, $\Gamma_1^{(i,j)}(z_1) < \infty$ are all well-defined without poles. On the other hand, condition (E.3) ensures that zeros at $z = z_1$ is removed as well.

---

[19]The rank loss generally depends on the multiplicity of zeros in $\det(\widetilde{\Gamma}(z_1))$.

Now consider the second step $j = 2$,

$$\Gamma_2(z) = \Gamma_1(z) V_2^{-1} B_2(z).$$

Without the constant unitary matrix V_2,

$$\Gamma_1(z) B_2(z) = \Gamma_1(z) \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & \frac{1-\bar{z}_2 z}{z-z_2} \end{bmatrix}$$

would have poles in the last column. Note that $\Gamma_1(z)$ is no longer lower triangular after the first step transformation. To remove these poles at $z = z_2$, we employ the SVD again,

$$\Gamma_2(z_2) = \Gamma_1(z_2) V_2^{-1} B_1(z_2) = U_2 D_2 V_2 V_2^{-1} B_2(z_2)$$

$$= U_2 \begin{bmatrix} \widetilde{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \widetilde{\lambda}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & \frac{1-\bar{z}_2 z_2}{z_2 - z_2} \end{bmatrix},$$

where $\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \ldots \widetilde{\lambda}_{\widetilde{p}}\}$ are the non-zero singular values. Again, there exists at least one zero in the diagonal of D_2 matrix ($\widetilde{p} < d$), since $\det(\Gamma_1(z_2)) = 0$. Arranging the zeros in the last positions of the diagonal, it follows immediately that $\Gamma_2^{(i,j)}(z_1) < \infty$ are all well-defined without poles, since the last column of D_2 are identically zero and the poles introduced by $\frac{1-\bar{z}_2 z_2}{z_2 - z_2}$ vanishes.

Continue this sequential procedure for all z_j, it follows that $\Gamma(z)$ is analytic (component wise) at $z = \{z_1, z_2, \ldots z_n\}$ inside the unit disk. By (E.3), we conclude that $\Gamma(z)$ is indeed Wold (outer) spectral factor.

The above constructive proof can be trivially extended to the case with higher-order zeros, see Rozanov (1967), p47. In particular, the location of the Blaschke factor $\frac{1-\bar{z}_j z}{z-z_j}$ (along the diagonal) is inconsequential, as long as we put the zero in the corresponding diagonal position of D_j.   Q.E.D.

A Working Example of 2×3 Signal System   As a working example of the two-step factorization method, we consider an alternative specification of the 2×3 signal system in Section 7. Let the signal representation be

$$X_{it} = H(L)\eta_{it} \equiv \begin{bmatrix} \frac{1}{1-\rho_a L} & 1 & 0 \\ \frac{\pi_1(L)}{1-\pi_2(L)} & 0 & \frac{\pi_1(L)}{1-\pi_2(L)} \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \\ \epsilon_{ut} \end{bmatrix},$$

where $\frac{\pi_1(L)}{1-\pi_2(L)}$ is an outer function in $\mathbf{H}^2(\mathbb{D})$. In this case the first signal is $a_{it} = a_t + \epsilon_i$.

Step 1: The spectral density $f_x(\omega)$ is given by

$$f_x(\omega) \equiv \begin{bmatrix} \frac{1}{(1-\rho_a z)(1-\rho_a z^{-1})}\sigma_a^2 + \sigma_i^2 & \frac{\pi_1(z^{-1})}{(1-\rho_a z)(1-\pi_2(z^{-1}))}\sigma_a^2 \\ \frac{\pi_1(z)}{(1-\rho_a z^{-1})(1-\pi_2(z))}\sigma_a^2 & \frac{\pi_1(z)\pi_1(z^{-1})}{(1-\pi_2(z))(1-\pi_2(z^{-1}))}[\sigma_a^2 + \sigma_u^2] \end{bmatrix},$$

where $z = e^{-i\omega}$. The leading principal minors are given by

$$M_1(\omega) = f_{11}(\omega) = \frac{(1-\lambda_w z)(1-\lambda_w z^{-1})}{(1-\rho_a z)(1-\rho_a z^{-1})}\sigma_w^2,$$

$$M_2(\omega) = \det(f_x(\omega)) = \frac{\pi_1(z)\pi_1(z^{-1})}{(1-\pi_2(z))(1-\pi_2(z^{-1}))(1-\rho_a z)(1-\rho_a z^{-1})}\left[\sigma_g^2(1-\lambda_w)(1-\lambda_w z^{-1}) - \sigma_a^4\right],$$

where we define $\sigma_p^2 = \sigma_a^2 + \sigma_u^2$ and $\sigma_g^2 = \sigma_w^2\sigma_p^2$. Using Lemma 10,

$$\sigma_g^2(1-\lambda_w)(1-\lambda_w z^{-1}) - \sigma_a^4 = \sigma_h^2(1-\lambda_h z)(1-\lambda_h z^{-1}).$$

The new parameters σ_h and λ_h satisfy

$$\lambda_h = \frac{\lambda_w \sigma_g^2}{\sigma_h^2}, \quad \sigma_h^2(1+\lambda_h^2) = \sigma_g^2(1+\lambda_w^2) - \sigma_a^4.$$

In particular, we can pick a real $\lambda_h \in (0,1)$. Then the spectral density admits the following decomposition,

$$f_x(\omega) = \begin{bmatrix} 1 & 0 \\ g_{21}(\omega) & 1 \end{bmatrix}\begin{bmatrix} h_{11}(\omega) & 0 \\ 0 & h_{22}(\omega) \end{bmatrix}\begin{bmatrix} 1 & g_{21}^*(\omega) \\ 0 & 1 \end{bmatrix}.$$

The diagonal elements h_{11} and h_{22} are given by

$$h_{11}(\omega) = M_1(\omega), \quad h_{22}(\omega) = \frac{M_2(\omega)}{M_1(\omega)}.$$

In addition, we use the recursion formula to get $g_{21}(\omega) = \frac{g_{21}^{(1)}}{h_{11}} = \frac{f_{21}}{h_{11}}$. Therefore,

$$g_{21}(\omega) = \frac{\sigma_a^2}{\sigma_w^2}\frac{\pi_1(z)(1-\rho_a z)}{(1-\pi_2(z))(1-\lambda_w z)(1-\lambda_w z^{-1})}.$$

Now fix the first column $j = 1$, we know the only inside pole is at $z = \lambda_w$ in g_{21}. This implies

$$C_1(z) = (z - \lambda_w), \quad D_1(z) = \frac{h_{11}(z)}{|C_1(z)|^2} = \left|\frac{\Phi_1(z)}{\Psi_1(z)}\right|^2.$$

Hence $\frac{\Phi_1(z)}{\Psi_1(z)} = \frac{\sigma_w}{1-\rho_a z}$. This in turn implies

$$\tilde{\Gamma}_{11}(z) = g_{11}C_1(z)\frac{\Phi_1(z)}{\Psi_1(z)} = \sigma_w\frac{z-\lambda_w}{1-\rho_a z},$$

$$\tilde{\Gamma}_{21}(z) = g_{21}C_1(z)\frac{\Phi_1(z)}{\Psi_1(z)} = \frac{\sigma_a^2}{\sigma_w}\frac{\pi_1(z)z}{(1-\pi_2(z))(1-\lambda_w z)}.$$

We repeat this procedure for the second column. Notice that the second column of g are constants, therefore, $C_2(z) = 1$ and $\frac{\Phi_2(z)}{\Psi_2(z)} = \frac{\sigma_h}{\sigma_w} \frac{\pi_1(z)(1-\lambda_h z)}{(1-\pi_2(z))(1-\lambda_w z)}$. In the end, we obtain the lower-triangular matrix

$$\tilde{\Gamma}(z) = \begin{bmatrix} \sigma_w \frac{z-\lambda_w}{1-\rho_a z} & 0 \\ \frac{\sigma_a^2}{\sigma_w} \frac{\pi_1(z)z}{(1-\pi_2(z))(1-\lambda_w z)} & \frac{\sigma_h}{\sigma_w} \frac{\pi_1(z)(1-\lambda_h z)}{(1-\pi_2(z))(1-\lambda_w z)} \end{bmatrix}.$$

clearly, $\tilde{\Gamma}(z) \in \mathbf{H}^2_{2 \times 2}(\mathbb{D})$.

Step 2: We remove the inside zeros at $z = \lambda_w$ to achieve the Wold fundamentality. Using the Blaschke factorization, we have

$$\Gamma(z) = \tilde{\Gamma}(z) V_1^{-1} B(z),$$

where

$$B(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1-\lambda_w z}{z-\lambda_w} \end{bmatrix}$$

and V_1 satisfies the unitary eigen-decomposition of $\hat{G}(\lambda_w) = \tilde{\Gamma}^*(\lambda_w) \tilde{\Gamma}(\lambda_w) = V_1 \hat{D}_1 V_1^*$. It is easy to check that eigenvalues of Hermitian matrix $\hat{G}(\lambda_w)$ are distinct. Therefore, we can pick two eigenvectors from the two eigenvalues, which are necessarily orthogonal by the spectral theorem. Normalizing these two eigenvectors yields the unitary matrix as desired,

$$V_1 = \begin{bmatrix} \sqrt{\frac{h^2}{1+h^2}} & \sqrt{\frac{1}{1+h^2}} \\ \sqrt{\frac{1}{1+h^2}} & -\sqrt{\frac{h^2}{1+h^2}} \end{bmatrix},$$

where $h = \frac{\sigma_a^2}{\sigma_h} \frac{\lambda_w}{(1-\lambda_h \lambda_w)}$. The resulting matrix $\Gamma(z)$ is the Wold fundamental matrix

$$\Gamma(z) = \begin{bmatrix} \sigma_w \frac{z-\lambda_w}{1-\rho_a z} V_1^{(11)} & \sigma_w \frac{1-\lambda_w z}{1-\rho_a z} V_1^{(12)} \\ \frac{\pi_1(z)}{1-\pi_2(z)} \frac{\sigma_h V_1^{(12)}}{\sigma_w} & \frac{\pi_1(z)}{1-\pi_2(z)} \frac{V_1^{(12)} \sigma_a^2}{\sigma_w(1-\lambda_h \lambda_w)} \end{bmatrix}.$$

Finally, we can transform $\Gamma(z)$ into an upper triangular form by right multiplication of another unitary matrix V_2,

$$V_2 = \begin{bmatrix} \sqrt{\frac{1}{1+x^2}} & \sqrt{\frac{x^2}{1+x^2}} \\ -\sqrt{\frac{x^2}{1+x^2}} & \sqrt{\frac{1}{1+x^2}} \end{bmatrix},$$

where $x = \frac{\sigma_h(1-\lambda_h \lambda_w)}{\sigma_a^2}$. After some algebraic simplifications, we obtain

$$\Gamma(z) = \begin{bmatrix} \frac{\sigma_h}{\sigma_p} \frac{1-\lambda_h z}{1-\lambda_w z} & \frac{\sigma_a^2}{\sigma_p} \frac{1}{1-\rho_a z} \\ 0 & \frac{\pi_1(z)}{1-\pi_2(z)} \sigma_p \end{bmatrix}.$$

Assuming that $\frac{\pi_1(z)}{1-\pi_2(z)}$ has no roots in the open unit disk, we then obtain the Wold representation. Given the signal system in this example, we have shown that the results in Section 7 still hold. The analysis is significantly more complicated and available upon request.

F   Additional Algebra in Section 6

Derivation of Equation (C.1):   We follow the procedure in the proof of Proposition 1 and take the special case of $\ell = 1$. Here

$$X_{it} = a_{it} = \left[\begin{array}{cc} \frac{1}{1-\rho_a L} & 1 \end{array} \right] \left[\begin{array}{c} \epsilon_{at} \\ \epsilon_{it} \end{array} \right].$$

By Lemma 1, the autocovariance generating function of the univariate signal X_{it} is given by

$$S_x(z) = \frac{\sigma_a^2}{(1 - \rho_a z)(1 - \rho_a \frac{1}{z})} + \sigma_i^2 = \frac{\sigma_a^2 + \sigma_i^2 (1 - \rho_a z)(1 - \rho_a \frac{1}{z})}{(1 - \rho_a z)(1 - \rho_a \frac{1}{z})}.$$

By definition, the corresponding spectral density $f_x(\omega)$ on $|z| = 1$ is real-valued and non-negative. Since the signal process is real-valued, we can use Lemma 10 to decompose $f_x(\omega)$ as[20]

$$f_x(\omega) = \sigma_w^2 \frac{(1 - \lambda_w z)(1 - \lambda_w \frac{1}{z})}{(1 - \rho_a z)(1 - \rho_a \frac{1}{z})}, \quad z = e^{-i\omega},$$

where σ_w^2 is the variance-covariance term in the spectral density or the Wold innovation variance. We now determine σ_w^2 and λ_w.

The numerator of $S_x(z)$ is equal to

$$
\begin{aligned}
& \sigma_a^2 + \sigma_i^2 (1 - \rho_a z)(1 - \rho_a \frac{1}{z}) \\
=\ & -\sigma_i^2 \rho_a z - \sigma_i^2 \rho_a z^{-1} + [\sigma_a^2 + \sigma_i^2 (1 + \rho_a^2)] \\
=\ & \sigma_w^2 (1 - \lambda_w z)(1 - \lambda_w \frac{1}{z}) \\
=\ & -\lambda_w \sigma_w^2 z - \sigma_w^2 \lambda_w z^{-1} + \left(1 + \lambda_w^2 \right) \sigma_w^2.
\end{aligned}
$$

Matching coefficients yields

$$\sigma_w^2 \lambda_w = \rho_a \sigma_i^2, \quad \sigma_w^2 (1 + \lambda_w^2) = \sigma_a^2 + \sigma_i^2 (1 + \rho_a^2).$$

Solving these two equations yields (C.2) and (C.3), where we define $\tau = \sigma_a^2 / \sigma_i^2$. It is easy to check that $0 < \lambda_w < \rho_a < 1$.

Now we obtain the Wold fundamental representation $X_{it} = \Gamma(L) v_{it}$, where v_{it} is the Wold innovation with zero mean and unit variance and

$$\Gamma(z) = \sigma_w \frac{(1 - \lambda_w z)}{(1 - \rho_a z)}.$$

Note that the Wold fundamental representation is unique up to constant unitary matrices.   Q.E.D.

---

[20] The real-valued process implies real coefficients for polynomials. For higher-order polynomials, roots can be complex, but complex roots must come in conjugate pairs implied by the complex conjugate theorem. We use this fact in subsequent proofs.

Derivation of Equation (C.6): By the Weiner-Hopf prediction formula, $\mathbb{E}_{it}[y_t] = a_y(L)a_{it}$, where

$$a_y(z) = \left[S_{ya}(z)\,\Gamma^{-1}\left(\frac{1}{z}\right) \right]_+ \Gamma^{-1}(z). \tag{F.1}$$

It follows from (C.1) that

$$\Gamma(z) = \sigma_w \frac{(1-\lambda_w z)}{(1-\rho_a z)}. \tag{F.2}$$

Since

$$\begin{bmatrix} y_t \\ a_{it} \end{bmatrix} = \begin{bmatrix} M_y(L) & 0 \\ \frac{1}{1-\rho_a L} & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{at} \\ \epsilon_{it} \end{bmatrix},$$

we can compute that

$$S_{ya}(z) = M_y(z)\frac{\sigma_a^2}{1-\rho_a/z}.$$

We compute

$$\psi(z) \equiv \frac{S_{yx}(z)}{\Gamma(\frac{1}{z})} = \frac{\sigma_a^2}{\sigma_w}\frac{zM_y(z)}{z-\lambda_w}.$$

The complex function $\psi(z)$ has a first-order pole at $z = \lambda_w < 1$. Following Hansen and Sargent (1980), the annihilation operation is given by

$$[\psi(z)]_+ = \psi(z) - P_1(z),$$

where

$$P_1(z) = \sum_{n=-\infty}^{-1} a_n(z-\lambda_w)^n$$

is the principal part of the Laurent series expansion of $\psi(z)$ in some annulus $0 < |z - \lambda_w| < R^*$ around the point $z = \lambda_w$.[21] Since we only have a first order pole, $P_1(z) = a_{-1}(z-\lambda_w)^{-1} = \frac{a_{-1}}{z-\lambda_w}$. Using the inverse Laurent series formula and the residue theorem in complex analysis

$$a_{-1} = \frac{1}{2\pi i}\oint_{\mathcal{L}} \frac{\psi(z)}{(z-\lambda_w)^{-1+1}}dz = \frac{1}{2\pi i}\oint_{\mathcal{L}} \psi(z)dz$$

$$= Res\,(\psi, \lambda_w) = \lim_{z\to\lambda_w}(z-\lambda_w)\psi(z) = \frac{\sigma_a^2}{\sigma_w}\lambda_w M_y(\lambda_w),$$

where $\mathcal{L}$ refers to any circle centered at $z = \lambda_w$ that is inside the annulus. It follows immediately that

$$[\psi(z)]_+ = \psi(z) - P_1(z) = \psi(z) - \frac{\sigma_a^2}{\sigma_w}\frac{\lambda_w M_y(\lambda_w)}{z-\lambda_w}$$

$$= \frac{\sigma_a^2}{\sigma_w}\left[\frac{zM_y(z)}{z-\lambda_w} - \frac{\lambda_w M_y(\lambda_w)}{z-\lambda_w}\right].$$

Substituting the last equation and (F.2) into (F.1) yields (C.6). By inspection $a_y(z)$ is analytic inside the unit disk, including $z = \lambda_w$. Q.E.D.

---

[21] Hansen and Sargent (1980)'s method requires some restrictions on the isolated singularities of $\psi(z)$. In particular, a common region of convergence (analytic region) is needed for the numerator and denominator of $\psi(z)$. In our model with ARMA (p,q) shocks, these restrictions are satisfied using Corollary 2.4 on rational functions.

Derivation of Equation (C.20): First, we compute the expectation terms in equation (24), using the approach in Section 4. We have shown in (C.1) that $a_{it} = \Gamma(L) v_{it}$, where

$$\Gamma(z) = \sigma_w \frac{1 - \lambda_w z}{1 - \rho_a z}.$$

We can derive the following covariance generating functions

$$S_{sa}(z) = \sigma_i^2 M_s^i(z), \ \ S_{ba}(z) = \sigma_a^2 \frac{M_b(z)}{1 - \rho_a/z} + \sigma_i^2 M_b^i(z),$$

$$S_{qa}(z) = \sigma_a^2 \frac{M_q(z)}{1 - \rho_a/z}, \ \ S_{da}(z) = \sigma_a^2 \frac{M_d(z)}{1 - \rho_a/z},$$

where we have used the frequency domain representations (38), (39),

$$b_{it} = M_b(L)\epsilon_{at} + M_b^i(L)\epsilon_{it}, \ \text{and} \ d_t = M_d(L)\epsilon_{at}.$$

Using equilibrium conditions (22), (25), and (26), we can derive that

$$
\begin{aligned}
M_d(z) &= \left(\frac{1}{\alpha_6} - \frac{\alpha_7}{\alpha\alpha_6}\right) M_y(z) + \frac{\alpha_7}{\alpha\alpha_6} \frac{1}{1 - \rho_a z}, \\
M_b(z) &= \alpha_4 \left[\left(\frac{1}{\alpha_6} - \frac{\alpha_7}{\alpha\alpha_6}\right) M_y(z) + \frac{\alpha_7}{\alpha\alpha_6} \frac{1}{1 - \rho_a z}\right] + \alpha_5 \left[\frac{M_y(z)}{\alpha} - \frac{1}{\alpha(1 - \rho_a z)}\right], \\
M_b^i(z) &= \frac{\alpha_5}{\alpha} \left[M_y^i(z) - 1\right].
\end{aligned}
$$

We compute the conditional expectation about the future shareholdings using the Wiener-Hopf formula, $\mathbb{E}_{it}\left[s_{it+2}^h\right] = a_s(L)a_{it}$, where

$$a_s(z) = \left[\frac{S_{sa}(z)}{z\Gamma(z^{-1})}\right]_+ \frac{1}{\Gamma(z)} \equiv [\psi_s(z)]_+ \frac{1}{\Gamma(z)}.$$

We can compute the annihilation as

$$[\psi_s(z)]_+ = \left[\frac{\sigma_i^2}{\sigma_w} \frac{M_s^i(z)(z - \rho_a)}{z(z - \lambda_w)}\right]_+ = \psi_s(z) - \frac{\varphi_s(\lambda_w)}{z - \lambda_w} - \frac{\varphi_s(0)}{z},$$

where the functional constants are determined using the residual theorem

$$\varphi_s(\lambda_w) = \frac{\sigma_i^2}{\sigma_w} M_s^i(\lambda_w) \frac{(\lambda_w - \rho_a)}{\lambda_w}, \ \ \varphi_s(0) = \frac{\sigma_i^2}{\sigma_w} M_s^i(0) \frac{\rho_a}{\lambda_w}.$$

Similarly, we can compute that

$$\mathbb{E}_{it}\left[q_{t+1}\right] = a_q(L)a_{it}, \ \ \mathbb{E}_{it}\left[d_{t+1}\right] = a_d(L)a_{it},$$

where

$$a_q(z) = \left[\frac{S_{qa}(z)}{z\Gamma(z^{-1})}\right]_+ \frac{1}{\Gamma(z)} \equiv [\psi_q(z)]_+ \frac{1}{\Gamma(z)},$$

$$a_d(z) = \left[\frac{S_{da}(z)}{z\Gamma(z^{-1})}\right]_+ \frac{1}{\Gamma(z)} \equiv [\psi_d(z)]_+ \frac{1}{\Gamma(z)}.$$

The annihilation is given by

$$\left[\psi_q(z)\right]_+ = \left[\frac{\sigma_a^2}{\sigma_w}\frac{M_q(z)}{z-\lambda_w}\right]_+ = \psi_q(z) - \frac{\varphi_q(\lambda_w)}{z-\lambda_w},$$

$$\left[\psi_d(z)\right]_+ = \left[\frac{\sigma_a^2}{\sigma_w}\frac{M_d(z)}{z-\lambda_w}\right]_+ = \psi_d(z) - \frac{\varphi_d(\lambda_w)}{z-\lambda_w},$$

where

$$\varphi_q(\lambda_w) = \frac{\sigma_a^2}{\sigma_w}M_q(\lambda_w), \ \ \varphi_d(\lambda_w) = \frac{\sigma_a^2}{\sigma_w}M_d(\lambda_w).$$

Finally, we derive $\mathbb{E}_{it}\left[\Delta b_{it+1}\right] = a_b(L)a_{it}$, where the Weiner-Hopf prediction formula gives

$$a_b(z) = \left[\frac{(z-1)S_{ba}(z)}{z\Gamma(z^{-1})}\right]_+ \frac{1}{\Gamma(z)} \equiv \left[\psi_b(z)\right]_+ \frac{1}{\Gamma(z)},$$

with

$$\psi_b(z) = \frac{(z-1)\left[\sigma_a^2 z M_b(z) + \sigma_i^2(z-\rho_a)M_b^i(z)\right]}{\sigma_w z(z-\lambda_w)}.$$

The annihilation is given by

$$\left[\psi_b(z)\right]_+ = \psi_b(z) - \frac{\varphi_b(\lambda_w)}{z-\lambda_w} - \frac{\varphi_b(0)}{z},$$

where

$$\varphi_b(\lambda_w) = \frac{(\lambda_w - 1)\left[\sigma_a^2 M_b(\lambda_w)\lambda_w + \sigma_i^2 M_b^i(\lambda_w)(\lambda_w - \rho_a)\right]}{\sigma_w \lambda_w},$$

$$\varphi_b(0) = -\frac{\sigma_i^2 M_b^i(0)\rho_a}{\sigma_w \lambda_w}.$$

Now rewriting (27) in the frequency domain using the preceding expressions and matching coefficients, we obtain

$$(1 - \rho_a z)M_q(z) = \left\{\alpha_3\left[\psi_s(z) - \frac{\varphi_s(\lambda_w)}{z-\lambda_w} - \frac{\varphi_s(0)}{z}\right] + \left[\frac{(z-1)S_{ba}(z)}{z\Gamma(z^{-1})}\right]_+ \right.$$

$$\left. + \beta\left[\psi_q(z) - \frac{\varphi_q(\lambda_w)}{z-\lambda_w}\right] + (1-\beta)\left[\psi_d(z) - \frac{\varphi_d(\lambda_w)}{z-\lambda_w}\right]\right\}\frac{1}{\Gamma(z)},$$

where we have invoked the LLN so that the cross-sectional aggregation eliminates the idiosyncratic innovations. Multiplying $\Gamma(z)$ on the two sides of the above equation, and using (40) and

$$\frac{\alpha_3}{\alpha_1} = 1 - \frac{\alpha_2}{\alpha_1} = 1 - \lambda_s \text{ (by Appendix A)},$$

we can rewrite the preceding equation for $M_q(z)$ as (C.20), where

$$G_q(z) = (1 - \lambda_w z) - \sigma_m \frac{(z - \rho_a)(1 - \rho_a z)}{z(z - \lambda_w)(1 - \lambda_s z)} - \frac{\sigma_\beta}{z - \lambda_w} \equiv \frac{P_q(z)}{Q_q(z)},$$

and the expressions and variables surrounding equation (C.20) are defined in the text. Q.E.D.

G   Frequency Domain Methods

In this section we introduce some mathematical background for the frequency domain methods. We study casual covariance stationary real-valued equilibrium processes that have an MA(∞) representation. For example, the aggregate output process in the model of Section 3 can be written as

$$y_t = \sum_{j=0}^{\infty} M_j \varepsilon_{a,t-j}, \tag{G.1}$$

where $\{M_j\}_{j=0}^{\infty}$ is square summable, i.e., $\sum_{j=0}^{\infty} |M_j|^2 < \infty$. Solving for the infinite sequence of $\{M_j\}_{j=0}^{\infty}$ is a daunting task. The idea of the frequency domain method is to transform this problem into an equivalent problem of solving for an analytical function in the Hardy space. To define this space, we recall that $\mathbb{C}$ denotes the complex plan, $\mathbb{T}$ denotes the unit circle, and $\mathbb{D}$ denotes the open unit disk.

Definition 2 *The Hardy space* $\mathbf{H}^2(\mathbb{D})$ *is the class of analytical functions g on the unit disk $\mathbb{D}$ satisfying*

$$\left\{ \frac{1}{2\pi} \sup_{0 \leq r < 1} \int_{-\pi}^{\pi} \left| g\left(re^{i\omega}\right) \right|^2 d\omega \right\}^{1/2} < \infty.$$

It can be verified that the expression on the preceding inequality defines a norm on $\mathbf{H}^2(\mathbb{D})$, denoted as $\|g\|_{\mathbf{H}^2}$. The Hardy space can also be viewed as a certain closed vector subspace of the complex L^2 space for the unit circle $\mathbb{T}$. This connection is provided by the fact that the radial limit

$$\tilde{g}\left(e^{i\omega}\right) = \lim_{r \uparrow 1} g\left(re^{i\omega}\right)$$

exists for almost all $\omega \in [-\pi, \pi]$. The function $\tilde{g}$ belongs to the space $L^2(\mathbb{T})$ of functions $f : \mathbb{T} \to \mathbb{C}$ with the inner product

$$< f_1, f_2 > = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\omega}\right) \overline{f_2\left(e^{i\omega}\right)} d\omega, \quad f_1, f_2 \in L^2(\mathbb{T}).$$

Then we have

$$\|g\|_{\mathbf{H}^2} = \|\tilde{g}\|_{L^2} = \lim_{r \uparrow 1} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| g\left(re^{i\omega}\right) \right|^2 d\omega \right\}^{1/2} < \infty.$$

Denote by $\mathbf{H}^2(\mathbb{T})$ the vector subspace of $L^2(\mathbb{T})$ consisting of all limit functions $\tilde{g}$, when g varies in $\mathbf{H}^2(\mathbb{D})$.

Theorem 7 *(Katznelson 1976)* $f \in \mathbf{H}^2(\mathbb{T})$ *if and only if* $f \in L^2(\mathbb{T})$ *and* $\hat{f}_n = 0$ *for all* $n < 0$, *where* $\hat{f}_n$ *is the Fourier coefficient of a function f integrable on the unit circle,*

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\omega}\right) e^{-i\omega n} d\omega, \ n = 0, \pm 1, \pm 2, ...$$

Suppose that $\tilde{g} \in \mathbf{H}^2(\mathbb{T})$ and $\tilde{g}$ has Fourier coefficients $\{a_n\}$ with $a_n = 0$ for all $n < 0$. We define

$$g(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < 1.$$

The following theorem ensures $g \in \mathbf{H}^2(\mathbb{D})$. Thus we have a bijection between $\mathbf{H}^2(\mathbb{D})$ and $\mathbf{H}^2(\mathbb{T})$.

Theorem 8 *If $f(z)$ is an analytic function in $\mathbb{D}$ and its Laurent expansion is*

$$f(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then $f \in \mathbf{H}^2(\mathbb{D})$ if and only if $\{b_n\}_{n=0}^{\infty}$ is square summable, i.e., $\sum_{n=0}^{\infty} |b_n|^2 < \infty$. When this condition is satisfied

$$\sum_{n=0}^{\infty} |b_n|^2 = \|f\|_{\mathbf{H}^2}.$$

We call the map from the sequence $\{b_n\}_{n=0}^{\infty}$ to $f(z)$ a z-transform. Theorem 8 also allows us to give an equivalent definition of the Hardy space $\mathbf{H}^2(\mathbb{D})$ as the class of analytical functions $f : \mathbb{D} \to \mathbb{C}$, which are the z-transforms of some square summable sequences. Thus solving for $\{M_j\}_{j=0}^{\infty}$ in (G.1) is equivalent to solving for a function $M(z)$ in the hardy space $\mathbf{H}^2(\mathbb{D})$. In particular, we can write $y_t = M(L)\epsilon_{at}$, where $M(z) \in \mathbf{H}^2(\mathbb{D})$ is the object we will solve for. We can use Theorem 8 to compute the variance of y_t easily because

$$Var(y_t) = \sigma_a^2 \sum_{j=0}^{\infty} M_j^2 = \sigma_a^2 \|M(z)\|_{\mathbf{H}^2}.$$

Finally, a rational function $f(z) \in \mathbf{H}^2(\mathbb{D})$ if and only if $f(z)$ is analytic in the boundary $\mathbb{D}$. In other words, poles are not allowed on the unit circle.