Asset Bubbles and Credit Constraints*

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Abstract

We provide a theory of rational stock price bubbles in production economies with infinitely lived agents. Firms meet stochastic investment opportunities and face endogenous credit constraints. They are not fully committed to repaying debt. Credit constraints are derived from incentive constraints in optimal contracts which ensure default never occurs in equilibrium. Stock price bubbles can emerge through a positive feedback loop mechanism and cannot be ruled out by transversality conditions. These bubbles command a liquidity premium and raise investment by raising the debt limit. Their collapse leads to a recession and a stock market crash.

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1 Introduction

This paper provides a novel theory of rational stock price bubbles in the presence of endogenous credit constraints. Our theory is motivated by two observations. First, fluctuations in observable fundamentals cannot adequately explain stock market booms and busts (Shiller (2005)). Second, stock market booms are often accompanied by credit market booms. For example, overoptimism in the 1990s towards an "East Asian miracle" generated booms in the housing and stock markets in many East Asian countries followed by lending booms and a large expansion of domestic credit (Collyns and Senhadji (2002)). Jordà, Schularick, and Taylor (2014) document empirical evidence on the relation between credit booms and asset price booms in 17 developed countries since 1870. They find that leveraged bubbles are more harmful to the macroeconomy than other types of bubbles, e.g., unleveraged "irrational exuberance" bubbles.

To formalize our theory, we construct a tractable continuous-time general equilibrium model of a production economy with a stock market in which infinitely lived households trade firm stocks in the absence of aggregate uncertainty. In the baseline model households are risk neutral and so the rate of return on any stock is equal to the constant subjective discount rate.[1] A continuum of firms meet uninsured idiosyncratic stochastic investment opportunities as in Kiyotaki and Moore (1997, 2005, 2008). Investment transforms consumption into capital goods, which can be sold in a market for capital. Assume that there is a liquidity mismatch (Jermann and Quadrini (2012)) in the sense that investment must be paid for before capital sales can be realized.[2] Thus, after exhausting internal funds, investing firms must seek external financing. As a starting point, we assume that investing firms only use intratemporal debt borrowed from firms without investment opportunities to finance investment. Investing firms take on debt at the beginning of the period and repay this debt at the end of the period using the proceeds from the sale of newly produced capital. They do not have other sources of financing i.e., they do not own and trade financial assets including the shares of other firms in the stock market, issue new equity, sell capital, or save to accumulate wealth. Some of these assumptions reflect the fact that equity financing is more costly than debt financing due to direct administration and underwriting costs, agency problems, or information asymmetries not explicitly modeled in our paper. Another interpretation following Kiyotaki and Moore (2005, 2008) is that investment opportunities disappear so quickly that firms do not have enough time to raise equity or sell a large amount of capital.

The key assumption of our model is that firms face endogenous credit constraints, which we model in a similar way to Bulow and Rogoff (1989), Kehoe and Levine (1993), Kiyotaki and Moore (1997), Alvarez and Jermann (2000), Albuquerque and Hopenhayn (2004), and Jermann

[1] In Appendix D we show that our key insights also apply to risk-averse households.

[2] We define liquidity as the amount of money that is quickly available for investment. Sometimes we also refer to liquidity as the degree to which an asset can be quickly turned into cash. See Kiyotaki and Moore (2005, 2008), Farhi and Tirole (2012), and Vayanos and Wang (2012) for related studies of liquidity.

and Quadrini (2012). The key idea is that borrowers are not fully committed to repaying debt and repayment is not perfectly enforced. We consider the following lending contract to ensure borrowers never default on their debt in equilibrium. A firm pledges its ownership rights including its physical assets (capital) as collateral. If the firm does not repay its debt, then the lender threatens to seize the firm's collateralized assets and take over the firm. Thus the collateral value to the lender is equal to the market value of the firm with the collateralized assets. The lender and the firm renegotiate the debt such that the debt repayment is limited by this collateral value. For incentive compatibility, the firm chooses not to default. The resulting credit constraint is endogenously derived from the incentive constraint in an optimal contracting problem.

Unlike Kiyotaki and Moore (1997) who assume that the collateral value is equal to the liquidation value of the collateralized assets, we derive the collateral value from the incentive constraint as the *going-concern value* of the reorganized firm. Since the going-concern value is priced in the stock market, it may contain a bubble component. If both the lender and the investing firm optimistically believe that the collateral value is high possibly because it contains a bubble, the firm will borrow more and the lender will not mind lending more because the lender can capture the bubble in the event of default. Thus the firm can finance more investment and make higher profits, making its assets indeed more valuable. This positive feedback loop mechanism makes the beliefs of both the lender and the borrower self-fulfilling and allows a stock price bubble to emerge in equilibrium. We refer to this type of equilibrium as the bubbly equilibrium.

Our credit constraint is equivalent to that endogenously derived from the incentive constraint in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). Suppose that there is no collateral for borrowing.[3] A firm can default on debt by diverting funds. The defaulting firm is shut down and the lender may get nothing in the event of default. The incentive constraint in an optimal contract ensures that the value to the firm of not defaulting is not lower than the outside value of the diverted funds. A stock price bubble can relax the incentive constraint and hence the credit constraint by raising the value to the firm of not defaulting. The firm can then borrow more to finance more investment, supporting a higher firm value. The aforementioned positive feedback loop mechanism still works with a slight modification to support the stock price bubble.

There is a second type of equilibrium in which no one believes in bubbles and hence bubbles do not exist. We call this type the bubbleless equilibrium. We provide explicit conditions to determine which type of equilibrium can exist. We prove that the economy has two steady states: a bubbly one and a bubbleless one. Both steady states are inefficient due to credit constraints and both are local saddle points. The equilibrium around the bubbly steady state is unique and bubbles persist in the long run along a stable manifold, whereas the equilibrium around the bubbleless steady state has indeterminacy of degree one and bubbles eventually burst along a stable manifold. Thus

[3] In Appendix C we show that the self-enforcing contract in which a defaulting firm is punished by being excluded from the credit market can also generate a stock price bubble. In this case the lender gets nothing upon default.

multiple equilibria in our model are not generated by indeterminacy with a unique steady state as in the literature surveyed by Benhabib and Farmer (1999) and Farmer (1999).

Following Blanchard and Watson (1982) and Weil (1987), we construct a third type of equilibrium with stochastic bubbles in which all agents believe that stock price bubbles will burst at each date with a positive probability. When bubbles burst, they cannot reappear. We show that when all agents believe that the probability of bubble bursting is small enough, an equilibrium with stochastic bubbles exists. Once bubbles burst, a recession occurs in that there is a credit crunch and consumption and output fall eventually. In addition, as soon as bubbles burst, investment falls discontinuously and the stock market crashes. All of this happens in the absence of any exogenous shock to economic fundamentals.

After presenting and analyzing our baseline model in Sections 3 through 5, we discuss our model assumptions and study the robustness of our results by analyzing various extensions in Section 6. We find that a stock price bubble can emerge as long as firms use debt financing subject to sufficiently tight credit constraints endogenously derived from optimal contracts with limited commitment, when other sources of finance are limited. First, we show that the usual Kiyotaki and Moore (1997) collateral constraint can generate a pure bubble in intrinsically useless assets (e.g., money), but cannot generate a stock price bubble. By contrast, a pure bubble and a stock price bubble can coexist under our endogenous credit constraints. Second, we allow firms to issue new equity to households or use a fraction of capital sales to finance investment. We show that our insights do not change as long as equity issues or capital sales are sufficiently limited. If they are unlimited, then firms would be able to overcome borrowing constraints and achieve the efficient equilibrium and no bubble could exist.

Finally, we introduce other types of assets such as intertemporal riskfree bonds and assets with exogenous rents (e.g., land). Suppose that firms can trade one of these two types of assets to finance investment. We show that the asset with exogenous rents that grow as fast as the economy can coexist with a stock price bubble, as long as the asset is less liquid than the stock. Otherwise, this asset will dominate the stock price bubble. When intertemporal bonds are available for trade, firms want to save in bonds precautionarily because they anticipate that they will meet uninsured investment opportunity shocks in the future. These bonds and bubbles are perfect substitutes. The equilibrium interest rate is lower than the subjective discount rate so that households prefer to sell bonds. The spread between the stock return and the interest rate reflects the liquidity premium. We introduce market frictions such as short-sale constraints on the additional assets (Kocherlakota (1992)).[4] We also assume that no firm trades the equity shares of other firms to finance investment. Without these frictions, unlimited arbitrage would cause the economy to achieve the efficient equilibrium and no bubble could exist.

[4]Short-sale constraints are widely adopted in the finance literature (e.g., Scheinkman and Xiong (2003)) and can be justified by institutional features such as direct transaction costs and default risk associated with short selling or SEC rules.

2 Basic Intuition and Related literature

To understand the basic intuition behind our model and our contributions to the literature, we begin with the standard asset pricing equation for equity under risk neutrality in a discrete-time deterministic environment

$$V_t = D_t + e^{-r} V_{t+1}, \tag{1}$$

where V_t denotes the cum-dividend stock price, D_t denotes dividends, and r denotes the subjective discount rate. We can write the solution as

$$V_t = V_t^* + B_t,\ V_t^* = \sum_{s=0}^{\infty} e^{-rs} D_{t+s},$$

where V_t^* represents the fundamental component and $B_t \geq 0$ represents the bubble component,

$$B_t = e^{-r} B_{t+1}. \tag{2}$$

In an infinite-horizon model with infinitely lived agents, the transversality condition

$$\lim_{T \to \infty} e^{-rT} V_{t+T} = 0$$

is necessary in equilibrium and rules out bubbles because it implies

$$0 = \lim_{T \to \infty} e^{-rT} B_{t+T} = B_t.$$

The transversality condition can be violated in the overlapping generations (OLG) framework with finitely lived agents. This framework is often used to study bubbles (Samuelson (1958), Diamond (1965), and Tirole (1985)). Giglio, Maggiori, and Stroebel (2016) find no evidence of bubbles that violate the transversality condition in the UK and Singapore housing markets. Abel et al (1989) find no evidence of dynamic inefficiency, which is the condition for the existence of a bubble in Tirole (1985).

Another issue with the standard asset pricing equations (1) and (2) is related to the steady state. If a stock price bubble can exist in the steady state (i.e., $B > 0$), then (1) and (2) imply that $r = 0$ and $D = 0$, where a variable without a time subscript denotes its steady-state value. There are two implications. First, a necessary condition for a bubble to exist is that the growth rate of the bubble must be lower than the growth rate of the economy, i.e., $r \leq 0$ (Tirole (1985) and Santos and Woodford (1997)). Otherwise, the bubble would be growing so fast that no one could afford to buy into the bubble. Second, in order for a stock price bubble to exist in the steady state, the detrended dividend (relative to economic growth) must be equal to zero in that state (Tirole (1985)). On the other hand, if the steady-state detrended dividend is positive, then a stock price bubble cannot exist. Moreover, no bubble can coexist with any infinitely-lived assets with positive (detrended) rents in the steady state. This issue is related to the rate of return dominance puzzle in monetary economics.

The main contribution of our paper is to provide a new theory of stock price bubbles that can overcome the issues discussed above. According to our theory, the asset pricing equation for the stock price bubble is given by

$$B_t = e^{-r} B_{t+1} [1 + LIQ_{t+1}], \tag{3}$$

instead of (2), where LIQ_{t+1} represents the liquidity premium. The key is that a stock price bubble is attached to productive assets (capital) with endogenous payoffs. Our insight is that the stock price bubble has real effects and affects dividends. Although asset pricing equation (1) for equity still holds so that the rate of stock return is equal to the subjective discount rate, the growth rate of the stock price bubble is lower than this rate due to the liquidity premium or "collateral yield." The collateral yield comes from the fact that the stock price bubble helps relax credit constraints and allows firms to make profitable investment, thereby generating more dividends. Consequently, the transversality condition cannot rule out the stock price bubble, which can emerge and sustain in dynamically efficient economies with positive dividends.

Our formulation of the positive feedback loop mechanism that generates a stock price bubble is novel. This mechanism works through credit constraints endogenously derived from incentive constraints in optimal contracts with limited commitment. The critical feature of such contracts is that equity value enters incentive constraints. A stock price bubble raises debt capacity by relaxing incentive constraints and hence raises investment and firm value to support the bubble. We show that a stock price bubble can emerge for several forms of contracts whenever incentive constraints have this feature, e.g., the contract in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011) and the self-enforcing contract (Kehoe and Levine (1993)). By contrast, we show that the usual credit constraints used in the literature (e.g., the Kiyotaki-Moore collateral constraint) can generate a pure bubble, but not a stock price bubble.

Unlike pure bubbles, stock price bubbles are attached to productive firms with positive dividends and are not separately tradable from firm stocks. Stock price bubbles can emerge in different firms or in different sectors, and their emergence or collapse may be unrelated to the emergence or collapse of pure bubbles. Fiat money is a pure bubble supplied by the government. It serves as a store of value and a medium of exchange and has a different nature from stock price bubbles. Thus one must go beyond standard theories of pure bubbles or money to understand stock price bubbles.

We show that firm value consists of a fundamental component and a bubble component. Unlike the extant literature, we explicitly characterize the liquidity premium provided by the bubble component and link the fundamental component to the Q theory of investment (Tobin (1969) and Hayashi (1982)). As in Hayashi (1982), firms are infinitely lived and make investment decisions that maximize their stock market values. The presence of a stock price bubble causes average Q to differ from marginal Q. Thus using average Q to measure marginal Q in empirical studies could be misleading. Our framework of infinite-horizon production economies with bubbles can be easily extended to incorporate many standard ingredients for both theoretical and quantitative analyses of asset prices, business cycles, and economic growth (Miao and Wang (2012, 2014, 2015), Miao, Wang, Xu (2015), Miao, Wang, and Zhou (2015), and Miao, Wang and Xu (2016)). In particular, Miao, Wang and Xu (2015) apply Bayesian estimation methods to study stock market bubbles and business cycles using our framework.

Some studies (e.g., Scheinkman and Weiss (1986), Kocherlakota (1992, 2008), Santos and Woodford (1997), and Hellwig and Lorenzoni (2009)) have found that infinite-horizon models of endowment economies with borrowing constraints can generate rational bubbles. Unlike this literature, our paper analyzes a production economy with stock price bubbles attached to productive firms.[5]

Rather than studying stock price bubbles, the extant literature on production economies typically studies pure bubbles like money that can provide liquidity by raising the borrower's net worth (Woodford (1990), Kiyotaki and Moore (2005, 2008), Caballero and Krishnamurthy (2006), Kocherlakota (2009), Farhi and Tirole (2012), Martin and Ventura (2012), Wang and Wen (2012), and Hirano and Yanagawa (2013)). These studies contain the idea that pure bubbles can relax credit constraints and raise investment. Their credit constraints are different from ours and they do not incorporate an explicit stock market where firms can be valued as in equation (1). Kiyotaki and Moore (2005, 2008) derive an equation similar to (3) for money and emphasize the importance of the liquidity premium for the circulation of money. Martin and Ventura (2012) replicate their baseline OLG model with pure bubbles using stock and credit markets and reinterpret their pure bubble as firm value, which has no fundamental component. In a related to OLG model, Martin and Ventura (2011) assume that an entrepreneur can start a new firm in each period and use its future market value, which may contain bubble and fundamental components, as collateral to borrow.

Unlike in the infinite-horizon models, credit constraints are inessential for the emergence of bubbles in the OLG models because bubbles as pyramid schemes can exist without credit constraints (Tirole (1985)). Their key role is to allow bubbles to have a crowding-in effect and emerge in dynamically efficient OLG economies, instead of providing a positive feedback loop mechanism to support a bubble as in our paper (Farhi and Tirole (2012) and Martin and Ventura (2011, 2012)). None of these three papers studies asset pricing equations like (1) and (3) for stocks and bubbles or the related rate of return dominance discussed earlier. Most OLG models of bubbles are confined to agents that live for two or three periods and thus are not suitable for a quantitative analysis because the time period in these models cannot be tied to the data frequency. If each generation of agents is equally altruistic toward the next generation with intergenerational transfers, then the entire dynasty behaves as a single infinitely lived agent (Barro (1974)). Thus studying models with infinitely lived agents is important and will deepen our understanding of asset bubbles by complementing OLG models.

Finally, our idea that stock price bubbles can provide liquidity is related to the literature on the

[5]See Scheinkman and Xiong (2003) for a model of bubbles based on heterogeneous beliefs and Adam, Marcet, and Nicolini (2015) for an asset-pricing model where agents have subjective beliefs about the pricing function. See Brunnermeier (2009) and Miao (2014) for surveys of various theories of bubbles.

search theory of money (Kiyotaki and Wright (1989), Lagos and Wright (2005), and Gu, Mattesini, and Wright (2016)).[6] This literature emphasizes the role of money and other assets in overcoming trading frictions in economies with decentralized trade. Money commands a liquidity premium and satisfies an equation similar to (3). This literature does not study stock price bubbles attached to firms with endogenous dividends and capital.

3 Baseline Model

We consider an infinite-horizon production economy, consisting of a continuum of identical households of a unit measure and a continuum of ex ante identical, but ex post heterogeneous firms of a unit measure. Firms are subject to independent idiosyncratic shocks and there is no aggregate uncertainty. Time is continuous and denoted by $t \geq 0$. For a better understanding of intuition, we sometimes consider a discrete-time approximation with time denoted by $t = 0,\ \Delta,\ 2\Delta, ...$ We will focus our analysis on the continuous-time limit as $\Delta \to 0$.

Assumption 1 *There are three asset markets. Households are shareholders of all firms and trade firm shares in a stock market without trading frictions. Firms buy and sell capital in a market for capital goods and they do not own or trade the shares of other firms in the stock market. There is also an intratemporal debt market in which firms borrow and lend among themselves.*

The key ingredients of our baseline model are:

- Endogenous credit constraints derived from optimal contracts with limited commitment. The critical feature of this type of contracts is that firm value enters incentive constraints. Under a specific contract form, a firm can borrow against its market value and the lender can seize the stock price bubble in the event of default.
- A liquidity mismatch in the sense that capital sales are realized after investment spending.
- The inability of firms to raise funds to finance investment by issuing new equity, selling capital, or saving to accumulate wealth.

3.1 Households

The representative household is risk neutral and derives utility from a consumption stream $\{C_t\}$ according to the utility function $\sum_{s=0}^{\infty} e^{-rs\Delta} C_{s\Delta}\Delta$. Households supply labor inelastically and aggregate labor supply is normalized to one. They trade firm stocks without any trading frictions. The net supply of each firm's stocks is normalized to one. Since households are identical, they do not trade among themselves and each household holds one unit of shares in equilibrium.

[6] Our paper is also related to the literature on commodity money. Unlike stock price bubbles, commodity money can serve as a consumption good that directly enters a household's utility function (e.g., Sargent (2016)).

The representative household faces the budget constraint during period $[t, t+\Delta]$

$$C_t\Delta + \int \left(V_t^j - D_t^j\Delta\right)\psi_{t+\Delta}^j dj = \int V_t^j \psi_t^j dj + w_t N_t \Delta, \tag{4}$$

where V_t^j denotes firm j's expected cum-dividend equity value, ψ_t^j denotes holdings of firm j's shares, D_t^j denotes firm j's expected dividends determined by its optimization problem, w_t denotes the wage rate, and N_t denotes labor supply.[7] Since there is no aggregate uncertainty, linear utility gives the first-order condition

$$V_t^j = D_t^j\Delta + e^{-r\Delta}V_{t+\Delta}^j, \tag{5}$$

for each firm j. This equation says that the rate of return (or the discount rate) on each stock must be equal to r. Linear utility implies the transversality condition (see, e.g., Ekeland and Scheinkman (1986) and Acemoglu (2009)),

$$\lim_{T\to\infty} e^{-rT}V_T^j\psi_T^j = \lim_{T\to\infty} e^{-rT}V_T^j = 0, \tag{6}$$

where we have used the market-clearing condition $\psi_T^j = 1$ for all T and all j.

3.2 Firms

Each firm $j \in [0,1]$ is endowed with initial capital $K_0^j > 0$ and combines labor $N_t^j \geq 0$ and capital $K_t^j \geq 0$ to produce output at time t according to the Cobb-Douglas production function $Y_t^j = (K_t^j)^\alpha (N_t^j)^{1-\alpha},\ \alpha \in (0,1)$. Capital depreciates at rate δ. After solving the static labor choice problem, we obtain the operating profits

$$R_t K_t^j = \max_{N_t^j}\ (K_t^j)^\alpha (N_t^j)^{1-\alpha} - w_t N_t^j, \tag{7}$$

where w_t is the wage rate and R_t is given by

$$R_t = \alpha\left(\frac{w_t}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}. \tag{8}$$

We will show later that R_t is equal to the marginal product of capital in equilibrium.

Figure 1 illustrates firm j's sequential decision problem during period $[t, t+\Delta]$. The firm hires labor, produces output, and receives profits $R_t K_t^j\Delta$ at time t. It then meets an opportunity to invest in capital with Poisson probability $\pi\Delta$, as in Kiyotaki and Moore (1997, 2005, 2008). Investment transforms consumption into capital goods one for one, which can be sold in the market for capital. With probability $1-\pi\Delta$, no investment opportunity arrives. This assumption captures firm-level investment lumpiness and generates ex post firm heterogeneity. Assume that the arrival of an investment opportunity is independent over time and across firms so that a law of large numbers can be applied for aggregation. This means that only a fraction $\pi\Delta$ of firms have investment opportunities during period $[t, t+\Delta]$.

[7] Households' optimization problem must also satisfy a no-Ponzi-game condition $\lim_{T\to\infty} e^{-rT}\int V_T^j\psi_T^j dj \geq 0$ (Acemoglu (2009)). We use $\dot{z}_t$ to denote dz_t/dt for any variable z_t in continuous time.

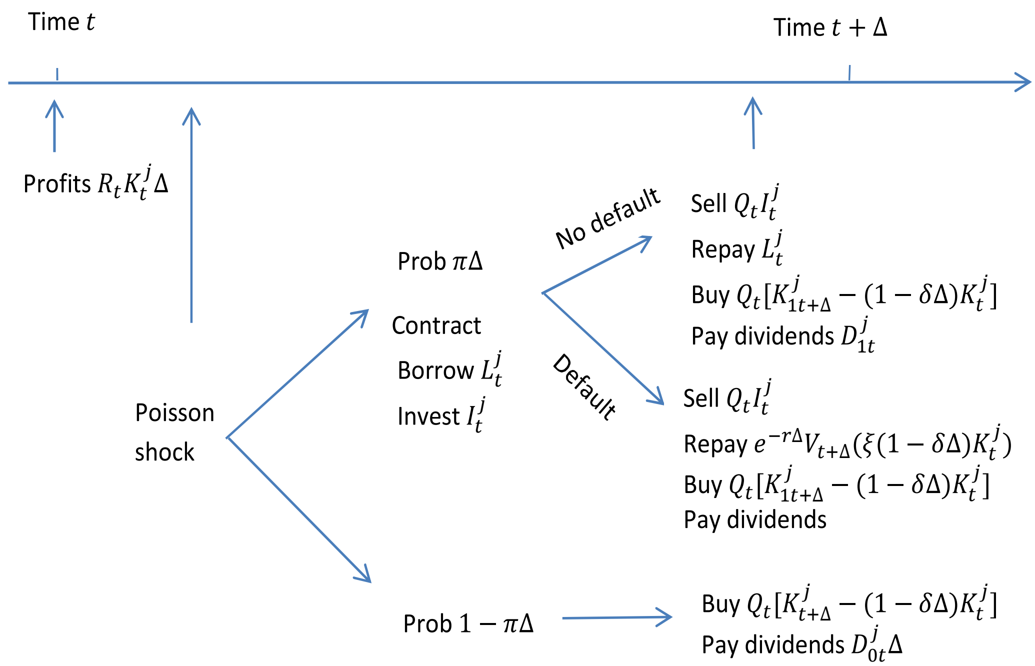

Figure 1: Timeline for firm j's decision process.

Assumption 2 *There is no insurance market against having an investment opportunity.*

When no investment opportunity arrives, firm j buys (sells) additional capital $K_{t+\Delta}^j - (1-\delta\Delta)\, K_t^j >$ $(<) 0$ in the capital goods market at the price Q_t and pays dividends $D_{0t}^j\Delta \geq 0$ at the end of period $[t, t+\Delta]$. When an investment opportunity arrives, firm j invests I_t^j at time t, and then sells its newly produced capital I_t^j and buys (sells) additional capital $K_{1t+\Delta}^j - (1-\delta\Delta)\, K_t^j > (<) 0$ at the price Q_t in the capital goods market at the end of period $[t, t+\Delta]$. Thus capital sales $Q_t I_t^j$ and transactions $Q_t \left[K_{1t+\Delta}^j - (1-\delta\Delta)\, K_t^j\right]$ are realized after investment spending I_t^j. This creates a liquidity mismatch so that firm j must access external funds in addition to its internal funds $R_t K_t^j \Delta$ to finance investment. There is no capital adjustment cost. It is the illiquidity of capital and the associated liquidity mismatch that prevent the use of capital sales to finance investment. Assumption 2 ensures that resources cannot be transferred when they are needed.

Assumption 3 *The only source of external financing for any firm j is intratemporal loans L_t^j. Firms cannot issue new equity, cannot use capital sales for financing due to liquidity mismatch, and do not possess any other financial assets.*

The credit market for the intratemporal debt is operated among firms. Investing firms borrow funds from non-investing firms. The interest rate on the intratemporal debt is zero and its price is one. After capital sales $Q_t I_t^j$ are realized at the end of period $[t, t+\Delta]$, investing firm j repays intratemporal loans L_t^j. It then buys or sells additional capital $K_{1t+\Delta}^j - (1-\delta\Delta) K_t^j$ before paying out dividends $D_{1t}^j \geq 0$.[8] We will show that $Q_t I_t^j > I_t^j = R_t K_t^j \Delta + L_t^j$ (i.e., $Q_t > 1$) in equilibrium so that firm j can fully repay loans after selling newly produced capital I_t^j.

Let the ex ante market value of firm j prior to the realization of an investment opportunity shock be $V_t(K_t^j)$, where we suppress aggregate state variables in the argument. Assume that management acts in the best interest of shareholders (i.e., households) to maximize the market value of the firm (or equity value). It follows from (5) that $V_t(K_t^j)$ satisfies the following Bellman equation:

$$\begin{aligned} V_t\left(K_t^j\right) &= \max_{K_{t+\Delta}^j, K_{1t+\Delta}^j, I_t^j, L_t^j} (1-\pi\Delta)\left[D_{0t}^j \Delta + e^{-r\Delta} V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right] \\ &\quad + \pi\Delta\left[D_{1t}^j + e^{-r\Delta} V_{t+\Delta}\left(K_{1t+\Delta}^j\right)\right] \end{aligned} \tag{9}$$

subject to

$$D_{0t}^j \Delta + Q_t K_{t+\Delta}^j = R_t K_t^j \Delta + Q_t (1-\delta\Delta) K_t^j, \tag{10}$$

$$D_{1t}^j + Q_t K_{1t+\Delta}^j + L_t^j + I_t^j = R_t K_t^j \Delta + L_t^j + Q_t (1-\delta\Delta) K_t^j + Q_t I_t^j, \tag{11}$$

$$I_t^j \leq R_t K_t^j \Delta + L_t^j, \tag{12}$$

and a credit constraint described below. Equations (10) and (11) are the flow-of-funds constraints. Equation (12) is the financing constraint, which means that investment spending I_t^j is limited by internal funds $R_t K_t^j \Delta$ and debt L_t^j.

The most important assumption of our model is as follows:

Assumption 4 *Loans are subject to a credit constraint endogenously derived from an incentive constraint in an optimal contract with limited commitment.*

The contract specifies investment I_t^j and loans L_t^j at time t and repayment L_t^j at the end of period $[t, t+\Delta]$, when an investment opportunity arrives with Poisson probability $\pi\Delta$. Firm j may default on its debt at the end of period $[t, t+\Delta]$. If it defaults, then the firm and the lender will renegotiate the loan repayment in a Nash bargaining problem. The loan repayment is determined by the threat value to the lender. Specifically, the lender threatens to seize a fraction $\xi \in (0,1)$ of depreciated capital $(1-\delta\Delta) K_t^j$ and take over the firm. The remaining fraction represents default costs, which include direct costs of legal expenses and indirect costs resulting from conflicts of interest between the lender and the borrower (Hennessy and Whited (2007)). Alternatively, we may interpret ξ as an efficiency parameter in the sense that the lender may not be able to efficiently use the firm's assets $(1-\delta\Delta) K_t^j$. The lender can run the firm with assets $\xi (1-\delta\Delta) K_t^j$ from time $t+\Delta$ onwards and obtain firm value $e^{-r\Delta} V_{t+\Delta}(\xi (1-\delta\Delta) K_t^j)$ at the end of period $[t, t+\Delta]$. This value is the threat value to the lender.

[8]There is no difference between a flow dividend $D_{0t}^j \Delta$ and a lump-sum dividend D_{1t}^j in discrete time with $\Delta = 1$. But it is important for the convergence to the continuous-time limit as $\Delta \to 0$ due to the nature of Poisson shocks.

Following Jermann and Quadrini (2012), we assume that the firm has all the bargaining power in the renegotiation through Nash bargaining so that the renegotiated repayment is equal to the threat value. After repaying the debt, the firm continues operating its business as usual. The key difference between our model and that of Jermann and Quadrini (2012) is that the threat value to the lender is the going-concern value in our model, while they assume that the lender liquidates the firm's assets and obtains the liquidation value in the event of default.[9]

Enforcement requires that, after an investment opportunity arrives at time t, the continuation value to the firm of not defaulting be no lower than the continuation value of defaulting, that is,

$$-L_t^j + e^{-r\Delta} V_{t+\Delta}(K_{1t+\Delta}^j) \geq -e^{-r\Delta} V_{t+\Delta}(\xi\,(1-\delta\Delta)\,K_t^j) + e^{-r\Delta} V_{t+\Delta}(K_{1t+\Delta}^j),$$

where we have canceled out some common terms on the two sides of the inequality (see Figure 1). This constraint ensures that there is no default in an optimal contract. Simplifying yields the credit constraint

$$L_t^j \leq e^{-r\Delta} V_{t+\Delta}(\xi\,(1-\delta\Delta)\,K_t^j). \tag{13}$$

The continuous-time limit of the previous dynamic programming problem as $\Delta \to 0$ becomes

$$\begin{aligned} rV_t\left(K_t^j\right) \quad = \quad \max_{\dot{K}_t^j, K_{1t}, I_t^j, L_t^j} \quad & D_{0t}^j + \dot{V}_t\left(K_t^j\right) + \pi\left(Q_t - 1\right) I_t^j \\ & + \pi\left[Q_t K_t^j - Q_t K_{1t}^j + V_t\left(K_{1t}^j\right) - V_t\left(K_t^j\right)\right] \end{aligned} \tag{14}$$

subject to

$$D_{0t}^j = R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right), \tag{15}$$

$$I_t^j \leq L_t^j, \tag{16}$$

$$L_t^j \leq V_t(\xi K_t^j). \tag{17}$$

Since internal funds $R_t K_t^j \Delta$ come as flows, the limit vanishes as $\Delta \to 0$ so that (12) converges to (16). Thus internal cash flows do not help finance lumpy investment. The continuous-time limit of (11) becomes $D_{1t}^j = Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j$. Total expected dividends are $D_t^j = D_{0t}^j + \pi D_{1t}^j$. Capital may jump from K_t^j to K_{1t}^j at the time of investment. In Section 4 we will show that this jump does not affect the solution given Assumption 3 and constant-returns-to-scale technology.

3.3 Competitive Equilibrium

Let $K_t = \int_0^1 K_t^j dj$, $I_t = \int_0^1 I_t^j dj$, and $Y_t = \int_0^1 Y_t^j dj$ denote the aggregate capital stock, aggregate investment of firms with investment opportunities, and aggregate output, respectively. Then a

[9] U.S. bankruptcy law has recognized the need to preserve the going-concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. 1101 et seq.). Bankruptcy laws seek to preserve the going-concern value whenever possible by promoting the reorganization, as opposed to the liquidation, of businesses. Bris, Welch and Zhu (2006) find empirical evidence that Chapter 11 reorganizations are less costly and more widely observed than Chapter 7 liquidations.

competitive equilibrium is defined as the paths of $\{Y_t\}$, $\{C_t\}$, $\{K_t\}$, $\{I_t\}$, $\{N_t\}$, $\{w_t\}$, $\{R_t\}$, $\{V_t(K_t^j)\}$, $\{I_t^j\}$, $\{K_t^j\}$, $\{N_t^j\}$ such that households and firms optimize and markets clear, i.e., $\psi_t^j = 1$, $N_t = \int_0^1 N_t^j dj = 1$, $C_t + \pi I_t = Y_t$, and $\dot{K}_t = -\delta K_t + \pi I_t$. The last equation is the continuous-time limit of the following market-clearing condition for capital goods as $\Delta \to 0$:

$$K_{t+\Delta} \equiv (1-\pi\Delta)\int K_{t+\Delta}^j dj + \pi\Delta \int K_{1t+\Delta}^j dj = \int (1-\delta\Delta)\, K_t^j dj + \pi\Delta \int I_t^j dj,$$

where the right-hand (left-hand) side of the last equality gives the aggregate supply (demand) of capital.

4 Equilibrium System

We first solve an individual firm's dynamic programming problem (14) subject to (15), (16), and (17) when the wage rate w_t or R_t in (8) is taken as given. This problem does not give a contraction mapping and hence may admit multiple solutions. We conjecture and verify that the ex ante firm value takes the following form:

$$V_t(K_t^j) = Q_t K_t^j + B_t, \tag{18}$$

where B_t is a variable to be determined. Since firm value $V_t(K_t^j)$ is always nonnegative, we must have $B_t \geq 0$. Note that $B_t = 0$ is a possible solution in general equilibrium. In this case we interpret $Q_t K_t^j$ as the fundamental value of the firm. The fundamental value is proportional to the firm's physical assets K_t^j, and has the same form as in Hayashi (1982). There may be another solution in which $B_t > 0$ due to optimistic beliefs. In this case, we interpret B_t as a bubble component since the firm is still valued at B_t even when there is no fundamental, i.e., $K_t^j = 0$. In Section 6.1 we will show that when an intrinsically useless asset is traded in the market, its price and B_t follow the same asset pricing equation (i.e., they are perfect substitutes), further justifying our interpretation of B_t as a bubble component.[10]

The following result characterizes firm j's optimization problem and its proof along with proofs of other results in the baseline model are given in Appendix A.

Proposition 1 *Suppose that $Q_t > 1$. Then the optimal investment level when an investment opportunity arrives is given by*

$$I_t^j = \xi Q_t K_t^j + B_t, \tag{19}$$

where

$$\dot{B}_t = rB_t - B_t \pi (Q_t - 1), \tag{20}$$

[10]According to the standard definition for exchange economies, a bubble is equal to the difference between the market value of an asset and the present value of the asset's *exogenously* given dividends. It is subtle to apply this definition to our model since dividends are *endogenously* generated through investment and production. Bubbles can help firms make more investment and hence generate additional dividends. One criticism of the standard test for bubbles is that it is hard to separate bubbles from fundamentals in the data (see Gurkaynak (2008) and Galí and Gambetti (2013)). If one prefers not to use the term "bubbles," one can call B_t a sunspot, self-fulfilling or speculative component without affecting our results.

$$\dot{Q}_t = (r+\delta) Q_t - R_t - \pi\xi Q_t(Q_t - 1), \tag{21}$$

and R_t is given by (8). Moreover, $\dot{K}_t^j$ and K_{1t}^j are indeterminate and the following transversality conditions hold:

$$\lim_{T\to\infty} e^{-rT} Q_T K_T^j = 0, \quad \lim_{T\to\infty} e^{-rT} B_T = 0. \tag{22}$$

To better understand the intuition behind this proposition, we consider the discrete-time problem (9) and conjecture $V_t(K_t^j) = a_t K_t^j + b_t$, where $b_t \geq 0$ is a bubble component. Substituting this conjecture and equations (10) and (11) into (9), we can rewrite the firm's dynamic programming problem as

$$\begin{aligned} a_t K_t^j + b_t \quad = \quad & \max_{K_{t+\Delta}^j, K_{1t+\Delta}^j, I_t^j, L_t^j} R_t K_t^j \Delta + Q_t (1-\delta\Delta) K_t^j + e^{-r\Delta} b_{t+\Delta} \\ & + (1-\pi\Delta) \left[-Q_t K_{t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{t+\Delta}^j \right] \\ & + \pi\Delta \left[(Q_t - 1) I_t^j - Q_t K_{1t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{1t+\Delta}^j \right] \end{aligned} \tag{23}$$

subject to

$$I_t^j \leq R_t K_t^j \Delta + L_t^j \leq R_t K_t^j \Delta + e^{-r\Delta} \left(a_{t+\Delta} (1-\delta\Delta) \xi K_t^j + b_{t+\Delta} \right), \tag{24}$$

where the last inequality follows from (13).

Constant-returns-to-scale technology implies that the objective function in (23) is linear in $K_{t+\Delta}^j$ and $K_{1t+\Delta}^j$. Optimization gives $Q_t = e^{-r\Delta} a_{t+\Delta}$ so that the capital price Q_t is equal to the marginal value of capital or Tobin's marginal Q. Thus firm j is indifferent between buying and selling capital, as it cannot use capital sales to finance investment anyway due to Assumption 3. It is possible that some firms grow slower and others grow faster. The firm size is bounded by the aggregate capital stock. The indeterminacy of firm dynamics at the micro-level will not affect the aggregate equilibrium dynamics as shown in Proposition 2 below, which is our focus.

When an investment opportunity arrives at the beginning of period $[t, t+\Delta]$, one unit of investment transforms one unit of consumption good into one unit of new capital, which is sold at the price Q_t at the end of period $[t, t+\Delta]$. If $Q_t > 1$, the firm will make as much investment as possible so that the financing constraint (12) and the credit constraint (13) bind. If $Q_t = 1$, the investment level is indeterminate. If $Q_t < 1$, the firm will make as little investment as possible. This investment choice is similar to Tobin's Q theory (Tobin (1969) and Hayashi (1982)). In what follows, we impose assumptions to ensure $Q_t > 1$ in the neighborhood of the steady state so that optimal investment is given by

$$I_t^j = R_t K_t^j \Delta + Q_t (1-\delta\Delta) \xi K_t^j + e^{-r\Delta} b_{t+\Delta}. \tag{25}$$

An optimistic belief about the stock market value of the firm due to a bubble component b_t costs the representative household b_t additional units of consumption good to buy one unit of the stock. The bubble generates a discounted resale value $e^{-r\Delta}b_{t+\Delta}$. The bubble also relaxes the credit constraint (13) and raises investment by $e^{-r\Delta}b_{t+\Delta}$ as (25) shows. This investment generates additional dividends $(Q_t - 1)$ with probability $\pi\Delta$ as (23) shows. Thus the total discounted benefit of the bubble is $[\pi\Delta(Q_t - 1) + 1]\, e^{-r\Delta}b_{t+\Delta}$. Equating the benefit with the cost yields

$$b_t = [\pi\Delta(Q_t - 1) + 1]\, e^{-r\Delta}b_{t+\Delta}. \tag{26}$$

This is the positive feedback loop mechanism supporting bubbles in our model.

We define $B_t = e^{-r\Delta}b_{t+\Delta}$ and take the continuous-time limit as $\Delta \to 0$ to derive (18), (19), and (20). We call $\pi(Q_t - 1)$ the liquidity premium of the bubble, which reflects the additional dividends generated by the stock price bubble. By substituting (25) back into (23), matching coefficients of K_t^j, and then taking the continuous-time limit as $\Delta \to 0$, we obtain (21). This equation shows that the return on capital is given by

$$\frac{R_t - \delta Q_t + \dot{Q}_t}{Q_t} = r - \xi\pi(Q_t - 1).$$

Since a fraction ξ of capital can be used as collateral to borrow, one unit of capital can finance ξQ_t units of investment by (19), thereby generating $\xi\pi Q_t(Q_t - 1)$ units of additional dividends. The term $\xi\pi(Q_t - 1)$ represents the liquidity premium of capital.

Through the firm's decision problem (23), we can understand the difference between our mechanism and that of Martin and Ventura (2011, 2012).[11] In their OLG models a young productive entrepreneur can create a new firm at each date and use its future value as collateral to borrow from unproductive entrepreneurs (savers). The new bubble attached to this firm can relax credit constraints and raise investment. This crowding-in effect is similar to that described in (24). However the new bubble is not supported by the positive feedback loop mechanism as in (26) because productive entrepreneurs do not solve a dynamic programming problem like (23). Moreover old bubbles created by the previous generations crowd out investment and can also emerge in equilibrium. All new and old bubbles in their models are supported by pyramid schemes like $b_t = e^{-r\Delta}b_{t+\Delta}$ so that the growth rate of bubbles equals the stock return (discount rate). Thus bubbles can be ruled out by transversality conditions. Bubbles serve as a store of value and can be sold from old agents to young agents as in Tirole (1985). By contrast, in our model a stock price bubble can emerge only when it can relax credit constraints and provide a liquidity premium.

We can reinterpret our credit constraint (17) as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). In particular, in the discrete-time approximation, (13) is equivalent to

$$Q_t I_t^j - L_t^j - Q_t(K_{1t+\Delta}^j - (1-\delta\Delta)K_t^j) + e^{-r\Delta}V_{t+\Delta}(K_{1t+\Delta}^j) \geq Q_t I_t^j + (1-\xi)(1-\delta\Delta)\, Q_t K_t^j,$$

where $e^{-r\Delta}V_{t+\Delta}(K_{1t+\Delta}^j) = Q_t K_{1t+\Delta}^j + B_t$. The left-hand side of the inequality above is the continuation value of the firm if it chooses to repay the debt L_t^j. The right-hand side is the value if the

[11] There are many other differences in model setups and predictions, not discussed here.

firm chooses to default by stealing the selling value of new capital $Q_t I_t^j$ and a fraction $1-\xi$ of the selling value of depreciated capital. The defaulting firm is shut down and the lender gets nothing. The stock price bubble B_t can still relax the incentive constraint by raising the value to the firm of not defaulting. It plays the role of maintaining reputations of the firm to repay its debt.

Although our model features a constant-returns-to-scale technology, marginal Q is not equal to average Q in the presence of bubbles, because (18) implies that average Q is equal to

$$\frac{V_t(K_t^j)}{K_t^j} = Q_t + \frac{B_t}{K_t^j} \quad \text{for } B_t > 0.$$

Thus the existence of stock price bubbles invalidates Hayashi's (1982) result. In the empirical investment literature, researchers typically use average Q to measure marginal Q under the constant-returns-to-scale assumption because marginal Q is not observable. Our analysis shows that this method may be misleading.

Now we aggregate individual firms' decision rules and impose market-clearing conditions. We then characterize a competitive equilibrium by a system of nonlinear differential equations.

Proposition 2 *Suppose that $Q_t > 1$. Then the equilibrium variables (B_t, Q_t, K_t) satisfy the system of differential equations, (20), (21), and*

$$\dot{K}_t = -\delta K_t + \pi(\xi Q_t K_t + B_t), \; K_0 \text{ given,} \tag{27}$$

where $R_t = \alpha K_t^{\alpha-1}$. The usual transversality conditions hold.

Equation (27) gives the law of motion for the aggregate capital stock derived from the market-clearing condition for capital. We use the market-clearing condition for labor and (8) to derive $R_t = \alpha K_t^{\alpha-1}$. The system of differential equations (20), (21), and (27) provides a tractable way to analyze equilibrium.

If we just focus on the firm's optimization problem in partial equilibrium taking Q_t and w_t as given, then $V_t(K_t^j) = Q_t K_t^j + B_t$ with $B_t > 0$ gives the maximal firm value. However, since $V_t(K_t^j)$ is the stock price, it is prone to speculation in general equilibrium. We will show later that both $B_t = 0$ and $B_t > 0$ can be supported in general equilibrium under certain conditions. That is, our model has multiple equilibria. This reflects the usual notion of a competitive equilibrium: Given a price system, individuals optimize. If this price system also clears all markets, then it is an equilibrium system. There could be multiple equilibria with different price systems and different price systems would generate different optimization problems with different sets of constraints.

After obtaining the solution for (B_t, Q_t, K_t), we can derive the equilibrium wage rate $w_t = (1-\alpha) K_t^{\alpha}$, aggregate output $Y_t = K_t^{\alpha}$, aggregate investment $\pi I_t = \pi(\xi Q_t K_t + B_t)$, and aggregate consumption $C_t = Y_t - \pi I_t$.

5 Analysis of Multiple Equilibria

We study three types of equilibria.[12] The first type is bubbleless in which $B_t = 0$ for all t. The second type is bubbly in which $B_t > 0$ for all t. For the third type the economy switches from a bubbly equilibrium to a bubbleless equilibrium. All three types of equilibria can exist due to self-fulfilling beliefs.

5.1 Bubbleless Equilibrium

In a bubbleless equilibrium $B_t = 0$ for all t. Equation (20) becomes an identity. We only need to focus on (Q_t, K_t) as determined by the differential equations (21) and (27) in which $B_t = 0$ for all t. We first analyze the steady state, in which all aggregate variables are constant over time so that $\dot{Q}_t = \dot{K}_t = 0$. We use a variable without a time subscript to denote its steady-state value and use a variable with an asterisk to denote its value in the bubbleless equilibrium.

Proposition 3 *(i) If*

$$\xi \geq \frac{\delta}{\pi}, \tag{28}$$

then there exists a unique bubbleless steady-state equilibrium with $Q^* = Q_E \equiv 1$ *and* $K^* = K_E$, *where* K_E *is the efficient capital stock satisfying* $\alpha (K_E)^{\alpha-1} = r + \delta$.

(ii) If

$$0 < \xi < \frac{\delta}{\pi}, \tag{29}$$

then there exists a unique bubbleless steady-state equilibrium with

$$Q^* = \frac{\delta}{\pi\xi} > 1, \tag{30}$$

$$\alpha \left(K^*\right)^{\alpha-1} = \frac{r\delta}{\pi\xi} + \delta. \tag{31}$$

In addition, $K^* < K_E$.

Assumption (28) says that if firms pledge sufficient assets as collateral, then the credit constraint will not bind in equilibrium. The competitive equilibrium allocation is the same as the efficient allocation. The latter is achieved by solving a social planner's problem in which the social planner maximizes the representative household's utility subject to the resource constraint only. Note that we assume that the social planner also faces stochastic investment opportunities, similar to firms in a competitive equilibrium. Unlike firms in a competitive equilibrium, the social planner is not subject to credit constraints.

Assumption (29) says that if firms cannot pledge sufficient assets as collateral, then the credit constraint will be sufficiently tight so that firms are credit constrained in the neighborhood of the

[12] We focus on the case where either all firms have bubbles of the same size in their stock prices or no firms have bubbles. It is possible to have another type of equilibrium in which different firms have bubbles of different sizes in their stock prices. See Appendix D.

steady-state equilibrium in which $Q^* > 1$. We can then apply Proposition 2 in this neighborhood. Proposition 3 also shows that the steady-state capital stock for the bubbleless equilibrium is less than the efficient steady-state capital stock. This reflects the fact that not enough resources are transferred from savers to investors due to financial frictions.

We can verify that $R^*K^* > \pi I^* = \delta K^*$ so that firms without investment opportunities have enough funds to lend to firms with investment opportunities in the bubbleless steady state and hence in the neighborhood of the bubbleless steady state. More intuitively, during period $[t, t+\Delta]$, investing firms need a total of $I_t \pi \Delta$ in funds to finance investment. Firms without investment opportunities possess a total of $(1-\pi\Delta) R_t K_t \Delta$ in cash. In a neighborhood of the bubbleless steady state, $(1-\pi\Delta) R_t K_t \Delta > I_t \pi \Delta$ for a sufficiently small Δ.

For (29) to hold, the arrival rate π of investment opportunities must be sufficiently small, holding everything else constant. The intuition is that if π is too high, then too many firms will have investment opportunities, which would make the accumulated aggregate capital stock so large as to lower the capital price Q to the efficient level as shown in part (i) of Proposition 3. Condition (29) requires that technological constraints at the firm level be sufficiently tight.

To study the local dynamics around the bubbleless steady state (Q^*, K^*), we linearize the system of differential equations (21) and (27) around (Q^*, K^*) for $B_t = 0$ for all t. We can easily show that the linearized system has a positive eigenvalue and a negative eigenvalue so that (Q^*, K^*) is a saddle point. Thus, in the neighborhood of (Q^*, K^*), for any given initial value K_0, there is a unique initial value Q_0 such that (Q_t, K_t) converges to the bubbleless steady state (Q^*, K^*) along a unique saddle path as $t \to \infty$.

5.2 Bubbly Equilibrium

In this section we study the bubbly equilibrium in which $B_t > 0$ for all t. We will analyze the dynamic system for (B_t, Q_t, K_t) given in (20), (21), and (27). We first rewrite (20) as

$$\frac{\dot{B}_t}{B_t} = r - \pi(Q_t - 1) \text{ for } B_t > 0. \tag{32}$$

This equation shows that the return on the stock price bubble $\dot{B}_t/B_t$ is equal to the discount rate minus the liquidity premium. As discussed in Section 4, stock price bubbles in our model can influence dividends due to the positive feedback loop effect through our credit constraint (17) or (24). The liquidity premium $\pi(Q_t - 1)$ makes the growth rate of bubbles lower than the discount rate r. Thus transversality conditions cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically efficient in our model. Specifically, the golden rule capital stock is given by $K_{GR} = (\delta/\alpha)^{\frac{1}{\alpha-1}}$. One can verify that $K^* < K_{GR}$. Thus the condition that the economy must be dynamically inefficient in Tirole (1985) cannot ensure the existence of bubbles in our model. Next we will give our new conditions.

5.2.1 Steady State

We first study the existence of a bubbly steady state in which $B > 0$. We use a variable with a subscript b to denote its bubbly steady state value.

Proposition 4 *There exists a bubbly steady state satisfying*

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \xi(\frac{r}{\pi} + 1) > 0, \tag{33}$$

$$Q_b = \frac{r}{\pi} + 1 > 1, \tag{34}$$

$$R_b = \alpha \left(K_b\right)^{\alpha-1} = [(1-\xi)r + \delta](\frac{r}{\pi} + 1), \tag{35}$$

if and only if the following condition holds:

$$0 < \xi < \frac{\delta}{r+\pi}. \tag{36}$$

In addition, (i) $Q_b < Q^*$, (ii) $K_{GR} > K_E > K_b > K^*$, (iii) $C_E > C_b > C^*$, and (iv) the bubble-asset ratio B/K_b decreases with ξ.

Condition (36) reveals that bubbles emerge when ξ is sufficiently small, *ceteris paribus*. The intuition is as follows. When the degree of pledgeability is sufficiently low, the credit constraint is too tight. A bubble can help relax this constraint and allows firms to borrow more and invest more. If the credit constraint is not tight enough, firms would be able to borrow sufficient funds to finance investment. In this case a bubble serves no function.

Note that condition (36) implies condition (29). Thus, if condition (36) holds, then there exist two steady state equilibria: one bubbleless and the other bubbly. The bubbleless steady state has been analyzed in Proposition 3. Propositions 3 and 4 reveal that the steady-state capital price is lower in the bubbly equilibrium than in the bubbleless equilibrium, i.e., $Q_b < Q^*$. The intuition is as follows. Bubbles help relax credit constraints and induce firms to make more investment than in the case without bubbles. The increased capital stock in the bubbly equilibrium lowers the marginal product of capital. Since the capital price partly reflects the present value of the marginal product of capital by (21), it is lower in the bubbly steady state than in the bubbleless steady state.

We can verify that $R_b K_b > \pi I_b = \delta K_b$ in the bubbly steady state. By a similar analysis to that in Section 5.1, we deduce that firms without investment opportunities have enough funds to lend to investing firms to finance investment in a neighborhood of the bubbly steady state.

As mentioned in Section 2, an important implication of our model is that stocks with positive dividends and stock price bubbles can coexist in the steady state. To see this point, we can show that aggregate dividends in the bubbly steady state are given by

$$\int D_0^j dj + \pi \int D_1^j dj = R_b K_b - \pi I_b = \left(R_b - \delta\right) K_b > 0.$$

This is consistent with the dynamic efficiency criterion in Abel et al (1989).

Do stock price bubbles crowd out capital in the steady state? In Tirole's (1985) OLG model, households may use part of their savings to buy bubble assets instead of accumulating capital. Thus bubbles crowd out capital in the steady state. In our model, bubbles are attached to productive assets. If the capital price were the same in the bubbly and bubbleless steady states, then bubbles would induce firms to invest more and hence to accumulate more capital stock. On the other hand, there is a general equilibrium price feedback effect as discussed earlier. The lower capital price in the bubbly steady state discourages firms from investing. The net effect is that bubbles lead to higher capital accumulation, contrary to Tirole's (1985) result.

The stock price bubble improves resource allocation even if it does not bring the economy to the first-best allocation. As Proposition 4 shows, the bubbly steady-state capital stock K_b is higher than the bubbleless steady-state level K^*, but lower than the first-best steady-state level K_E, which in turn is lower than the golden rule level K_{GR}. Moreover the bubble helps improve welfare in terms of consumption, i.e., $C_E > C_b > C^*$. By contrast, bubbles overcome dynamic inefficiency by crowding out capital in Tirole's (1985) OLG model. Introducing credit constraints and recurrent bubbles to Tirole's (1985) model, Martin and Venture (2012) show that new bubbles raise investment and this effect can dominate the crowding-out effect of old bubbles.

How does the parameter ξ affect the size of bubbles? Proposition 4 shows that a smaller ξ leads to a larger bubble relative to capital in the steady state. This is intuitive. If firms can only pledge a smaller amount of capital, they will face a tighter credit constraint so that a larger bubble will emerge to relax this constraint.

5.2.2 Dynamics

Now we study the stability of the bubbleless and bubbly steady states and their local dynamics. We linearize the equilibrium system (20), (21), and (27) around the two steady states. We then compute the eigenvalues of the linearized system and compare the number of stable eigenvalues with the number of predetermined variables (Coddington and Levinson (1955)). The equilibrium system has only one predetermined variable K_t and two nonpredetermined variables, B_t and Q_t.

Proposition 5 *Suppose that condition (36) holds. Then there exists a unique local equilibrium around the bubbly steady state* (B, Q_b, K_b) *and the local equilibrium around the bubbleless steady state* $(0, Q^*, K^*)$ *has indeterminacy of degree one.*

We prove that there is a unique stable eigenvalue for the linearized system around the bubbly steady state. Thus there is a neighborhood $\mathcal{N} \subset \mathbb{R}^3_+$ of the bubbly steady state (B, Q_b, K_b) and a continuously differentiable function $\phi : \mathcal{N} \to \mathbb{R}^2$ such that given any K_0 there exists a unique solution (B_0, Q_0) to the equation $\phi(B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}$, and (B_t, Q_t, K_t) converges to (B, Q_b, K_b) starting at (B_0, Q_0, K_0) as t approaches infinity. The set of points (B, Q, K)

satisfying the equation $\phi(B, Q, K) = 0$ is a one-dimensional stable manifold of the system. If the initial value (B_0, Q_0, K_0) is on the stable manifold, then the solution to the nonlinear system (20), (21), and (27) is also on the stable manifold and converges to (B, Q_b, K_b) as t approaches infinity.

Although the bubbleless steady state $(0, Q^*, K^*)$ is also a local saddle point, the local dynamics around this steady state are different. In Appendix A we prove that the stable manifold for the bubbleless steady state is two dimensional because there are two stable eigenvalues for the linearized system around the bubbleless steady state. Thus the local equilibrium has indeterminacy of degree one. Formally, there is a neighborhood $\mathcal{N}^* \subset \mathbb{R}^3_+$ of $(0, Q^*, K^*)$ and a continuously differentiable function $\phi^* : \mathcal{N}^* \to \mathbb{R}$ such that given K_0 for any $B_0 > 0$ there exists a unique solution Q_0 to the equation $\phi^*(B_0, Q_0, K_0) = 0$ with $(B_0, Q_0, K_0) \in \mathcal{N}^*$, and (B_t, Q_t, K_t) converges to $(0, Q^*, K^*)$ starting at (B_0, Q_0, K_0) as t approaches infinity. Intuitively, along the two-dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually. There exist multiple bubbly equilibrium paths converging to the bubbleless steady state and the initial value $B_0 > 0$ is indeterminate. This feature suggests that self-fulfilling beliefs can generate economic fluctuations without any shocks to economic fundamentals.

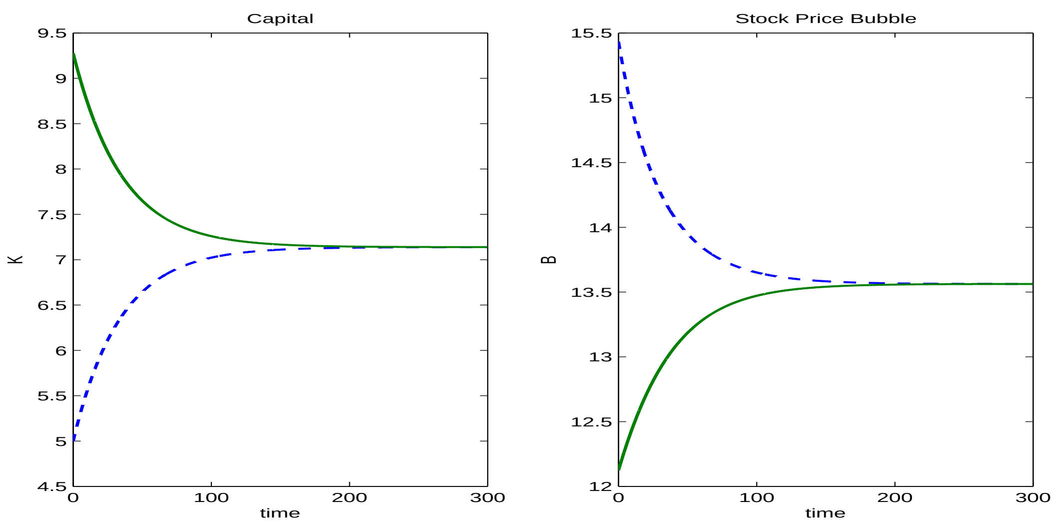

Figure 2: Transition paths for capital and the stock price bubble. The parameter values are $r = 0.02$, $\alpha = 0.4$, $\delta = 0.025$, $\pi = 0.01$, and $\xi = 0.2$.

Figure 2 illustrates the transition paths of capital and the stock price bubble around the bubbly steady state, given two initial values of capital. For larger initial capital (corresponding to solid lines), the capital price is lower so that investment is less profitable and the liquidity premium is lower. Thus the initial size of the bubble is smaller. The bubble then gradually expands to the bubbly steady state and the capital stock gradually decreases to the bubbly steady state. The opposite is true for the case with lower initial capital.

5.3 Equilibrium with Stochastic Bubbles

So far we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now introduce stochastic bubbles to the baseline model in Section 3 with intratemporal loans. Suppose that a stock price bubble exists initially, i.e., $B_0 > 0$. At the beginning of period $[t, t+\Delta]$ before production, the bubble bursts with Poisson probability $\theta\Delta$ (see Figure 1). Once it bursts, it will never have value again and the economy is at the bubbleless equilibrium studied in Section 5.1.[13] This event is independent of the Poisson investment opportunity shock.

We use a variable with an asterisk (except for aggregate capital K_t) to denote its value in the bubbleless equilibrium. In particular, $V_t^*(K_t^j) = Q_t^* K_t^j$ denotes firm j's value function, where $Q_t^* = G(K_t)$ for some function G. Let $V_t\left(K_t^j\right)$ denote the value function prior to the two Poisson shocks. Then firm j's dynamic programming problem in continuous time becomes

$$\begin{aligned} rV_t\left(K_t^j\right) \quad = \quad & \max_{I_t^j, \dot{K}_t^j, L_t^j} \; R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \dot{V}_t\left(K_t^j\right) + \pi\left(Q_t - 1\right) I_t^j \\ & +\pi\left[Q_t K_t^j - Q_t K_{1t}^j + V_t\left(K_{1t}^j\right) - V_t\left(K_t^j\right)\right] + \theta\left[V_t^*\left(K_t^j\right) - V_t\left(K_t^j\right)\right] \end{aligned}$$

subject to (16) and (17). The last expression on the second line reflects the fact that once the bubble bursts, firm value changes from $V_t(K_t^j)$ to $V_t^*(K_t^j)$. In Appendix A we show that $V_t\left(K_t^j\right) = Q_t K_t^j + B_t$.

Proposition 6 *Suppose $Q_t > 1$. Before the bubble bursts, the equilibrium with stochastic bubbles (B_t, Q_t, K_t) satisfies the following system of differential equations:*

$$\dot{B}_t = (r+\theta)B_t - \pi(Q_t - 1)B_t, \tag{37}$$

$$\dot{Q}_t = (r+\delta+\theta)Q_t - \theta Q_t^* - R_t - \pi(Q_t - 1)\xi Q_t, \tag{38}$$

and (27), where $R_t = \alpha K_t^{\alpha-1}$ and $Q_t^* = G(K_t)$ is the capital price after the bubble bursts.

Equation (37) is an asset pricing equation for the bubble and reflects the possibility of its collapse. In general, it is difficult to characterize the full set of equilibria with stochastic bubbles. In order to transparently illustrate the adverse impact of the collapse of a bubble on the economy, we consider a simple type of equilibrium. Following Weil (1987) and Kocherlakota (2009), we study a stationary equilibrium with stochastic bubbles that has the following properties: The capital stock, the stock price bubble, and the capital price are constant at K_s, B_s, and Q_s before the bubble collapses. Immediately after the bubble collapses, the capital stock gradually moves to the bubbleless steady-state value K^*, the bubble drops to zero and stays there forever, and the capital price jumps to Q_t^* before gradually moving to the bubbleless steady-state value Q^* given in (30).

[13] If a bubble reemerges in the future, it would have value today by its asset pricing equation. See Martin and Ventura (2012), Wang and Wen (2012), Gali (2014), and Miao, Wang, and Xu (2015) for models of recurrent bubbles.

Proposition 7 *Let condition (36) hold. If*

$$0 < \theta < \theta^* \equiv \frac{\delta}{\xi} - \pi - r,$$

then there exists a stationary equilibrium (B_s, Q_s, K_s) with stochastic bubbles such that $K_s > K^*$. In addition, if θ is sufficiently small, then consumption falls eventually after the bubble bursts.

As in Weil (1987), a stationary equilibrium with stochastic bubbles exists if the probability that the bubble will burst is sufficiently small. In Weil's (1987) OLG model, the capital stock and output eventually rise after the bubble collapses. In contrast to his result, in our model consumption, capital and output all fall eventually and the economy enters a recession after the bubble bursts. The intuition is that the collapse of the bubble tightens the credit constraint and impairs investment efficiency.

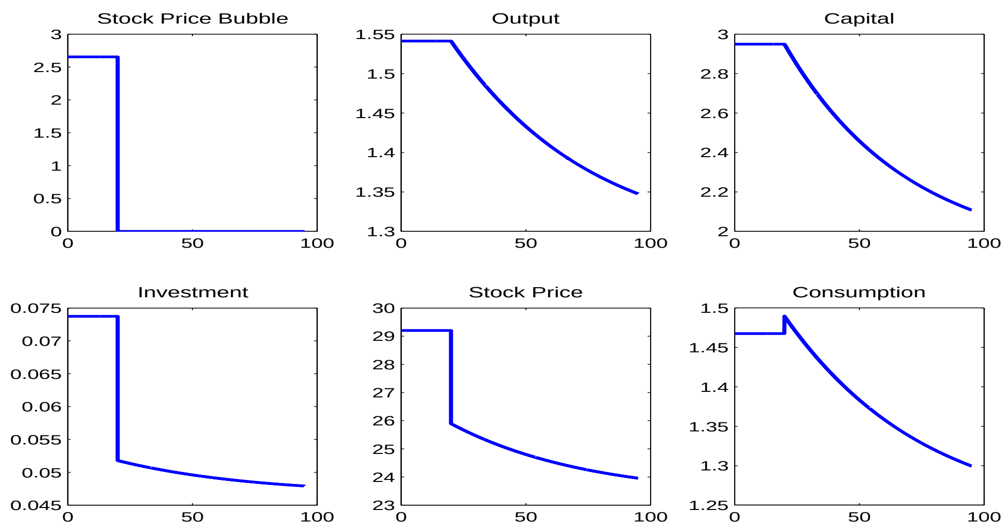

Figure 3: Dynamics of the stationary equilibrium with stochastic bubbles. The bubble bursts at $t = 20$. The parameter values are $r = 0.02$, $\alpha = 0.4$, $\delta = 0.025$, $\theta = 0.05$, $\pi = 0.01$, and $\xi = 0.2$.

Proposition 7 compares the economies before and after the bubble collapses only in the steady state. We now solve for the transition path numerically and present the results in Figure 3. In this numerical example we assume that the bubble collapses at time $t = 20$. Immediately after the bubble collapses, investment falls discontinuously and then gradually decreases to its bubbleless steady-state level. But output and capital decrease continuously to their bubbleless steady-state levels since capital is predetermined and labor is exogenous. Consumption rises initially because of the fall in investment,[14] but it quickly falls and then decreases to its bubbleless steady-state level.

[14]One way to generate the fall in consumption and output on impact is to introduce endogenous capacity utilization. Following the collapse of bubbles, the capacity utilization rate falls because the price of installed capital rises. Then both output and consumption would fall on impact.

Importantly, stock prices drop discontinuously and the stock market crashes immediately after the bubble collapses.

The existing macroeconomic models typically study dynamics around a unique deterministic steady state. These models introduce large shocks to economic fundamentals to generate a recession. For example, motivated by the recent Great Recession, Gertler and Kiyotaki (2010) and Gertler and Karadi (2011) introduce large capital quality shocks or net worth shocks. This literature is typically silent on the stock market behavior. In contrast to this literature, our model features two steady states. A change in beliefs or confidence can shift the economy from a "good" steady state to a "bad" steady state. A recession and a stock market crash can occur without any shocks to the fundamentals.

6 Discussions and Extensions

In this section we discuss our model assumptions and study the robustness of our results by analyzing several extensions. Our main message is that a stock price bubble can emerge as long as firms need debt financing because other sources of financing are insufficient to cover investment spending. And our modeling of credit constraints is critical for the emergence of a stock price bubble.

6.1 Endogenous Credit Constraints

A key feature of our model is that credit constraints are endogenously derived from optimal contracts with limited commitment. As a result equity value enters this type of constraints. To see the critical role of this feature, we show that the widely adopted Kiyotaki-Moore collateral constraint can generate a pure bubble like money, but cannot generate a stock price bubble.[15] This feature distinguishes our model from the literature on pure bubbles.

We write the Kiyotaki-Moore collateral constraint in discrete time as

$$L_t^j \leq \xi Q_t \left(1 - \delta \Delta\right) K_t^j, \tag{39}$$

where $\xi Q_t \left(1 - \delta \Delta\right) K_t^j$ is the liquidation value of the collateralized assets. We may reinterpret this constraint as an incentive constraint as in (13), where $e^{-r\Delta} V_{t+\Delta}(\xi \left(1 - \delta \Delta\right) K_t^j)$ is replaced by $\xi Q_t \left(1 - \delta \Delta\right) K_t^j$. The continuous-time limit of (39) as $\Delta \to 0$ is

$$L_t^j \leq \xi Q_t K_t^j. \tag{40}$$

Now we replace our credit constraint (13) with the Kiyotaki-Moore collateral constraint (39) in the baseline model of Section 3. Consider firm j's dynamic programming problem (9) or (23). It

[15] In Chapter 14 of his textbook, Tirole's (2006) shows that there may exist multiple equilibria in a simplified variant of the Kiyotaki and Moore (1997) model. In contrast to ours, these equilibria are characterized by a one-dimensional nonlinear dynamical system. Some equilibria may exhibit cycles. We would like to thank Jean Tirole for a helpful discussion on this point.

follows from (12) and (39) that optimal investment satisfies $I_t^j = R_t K_t^j \Delta + \xi Q_t (1-\delta\Delta) K_t^j$ when $Q_t > 1$. Substituting this investment rule back into (23), we deduce that bubbles grow at the rate r, i.e., $b_t = e^{-r\Delta} b_{t+\Delta}$. In this case bubbles do not help finance investment and hence there is no liquidity premium. The transversality condition implies that $\lim_{T\to\infty} e^{-r\Delta T} b_t e^{r\Delta T} = b_t = 0$ and thus no stock price bubble can emerge.

Next we show that a pure bubble can emerge by introducing an intrinsically useless asset (e.g., money) with a unit supply for firms and households to trade in the baseline model of Section 3 under the Kiyotaki-Moore collateral constraint (39) or (40).

Assumption 5 *Neither firms nor households can short the intrinsically useless asset (e.g., money).*

If firms or households could hold unlimited short positions, a pure bubble could not emerge due to unlimited arbitrage (Kocherlakota (1992)). Let $V_t(K_t^j, M_t^j)$ denote the ex ante market value of firm j when its capital stock and asset holdings at time t are K_t^j and $M_t^j \geq 0$, respectively. Let P_t denote the market price of the intrinsically useless asset. Then firm j chooses $K_{t+\Delta}^j$, $K_{1t+\Delta}^j$, $M_{t+\Delta}^j$, $M_{1t+\Delta}^j$, I_t^j, and L_t^j to maximize its market value by solving the following Bellman equation:

$$\begin{aligned} V_t\left(K_t^j, M_t^j\right) &= \max \ (1-\pi\Delta)\left[D_{0t}^j \Delta + e^{-r\Delta} V_{t+\Delta}\left(K_{t+\Delta}^j, M_{t+\Delta}^j\right)\right] \\ &\quad + \pi\Delta\left[D_{1t}^j + e^{-r\Delta} V_{t+\Delta}\left(K_{1t+\Delta}^j, M_{1t+\Delta}^j\right)\right] \end{aligned}$$

subject to (39),

$$D_{0t}^j \Delta + Q_t K_{t+\Delta}^j = R_t K_t^j \Delta + Q_t (1-\delta\Delta) K_t^j + P_t \left(M_t^j - M_{t+\Delta}^j\right), \tag{41}$$

$$D_{1t}^j + Q_t K_{1t+\Delta}^j + L_t^j + I_t^j = R_t K_t^j \Delta + L_t^j + P_t\left(M_t^j - M_{1t+\Delta}^j\right) + Q_t I_t^j + Q_t (1-\delta\Delta) K_t^j, \tag{42}$$

$$I_t^j \leq R_t K_t^j \Delta + L_t^j + P_t\left(M_t^j - M_{1t+\Delta}^j\right), \tag{43}$$

where $M_{t+\Delta}^j$ ($M_{1t+\Delta}^j$) are the asset holdings chosen at time t when no investment arrives (an investment opportunity arrives). Equations (41) and (42) are the flow-of-funds constraints. Inequality (43) says that firm j can sell the intrinsically useless asset to finance investment.

In Appendix B.1 we derive the continuous-time limit and show that firm value is given by

$$V_t\left(K_t^j, M_t^j\right) = Q_t K_t^j + P_t M_t^j. \tag{44}$$

The following proposition characterizes the equilibrium system.

Proposition 8 *Suppose that there is an intrinsically useless asset available for households and firms to trade in the baseline model under Assumption 5 and the credit constraint in (40). If $Q_t > 1$, then the continuous-time equilibrium system for (K_t, Q_t, P_t) is given by (21),*

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + P_t), \tag{45}$$

$$\dot{P}_t = r P_t - \pi(Q_t - 1) P_t, \tag{46}$$

where $R_t = \alpha K_t^{\alpha-1}$, and the usual transversality conditions hold. In equilibrium households do not hold the intrinsically useless asset and investing firms sell this asset to non-investing firms. The steady states are characterized by Propositions 3 and 4 where B is replaced by P.

A pure bubble is generated through the net worth channel: an investing firm sells the intrinsically useless asset to non-investing firms to raise its net worth so that the financing constraint (43), instead of the collateral constraint (39) or (40), is relaxed. The intrinsically useless asset provides a liquidity premium $\pi(Q_t - 1)$ and raises investment and dividends to support a bubble. Non-investing firms are willing to buy the asset for precautionary reasons because they anticipate that they will be credit constrained when a future investment opportunity arrives. This mechanism also works for general exogenous or endogenous credit constraints, e.g., the constraint that no firms can borrow ($\xi = 0$) and the constraint that firms can borrow against a fraction of future investment payoffs (Kiyotaki and Moore (2005, 2008) and Hirano and Yanagawa (2013)). Kiyotaki and Moore (2005, 2008) argue that the existence of a liquidity premium is critical for the circulation of fiat money.

Comparing Propositions 2 and 8, we find that the stock price bubble and the pure bubble satisfy the same equilibrium asset pricing equation. However, a pure bubble cannot be interpreted as a stock price bubble, because they are attached to different types of assets. A stock price bubble cannot be attached to a separately traded asset different from firm stocks. If a firm does not hold the intrinsically useless asset at some time t, i.e., $M_t^j = 0$, then its value is equal to $V_t(K_t^j, 0) = Q_t K_t^j$, implying that this firm does not contain a stock price bubble, even though there is a pure bubble in the economy.

By contrast, if we adopt a credit constraint similar to (17) based on optimal contracts with limited commitment in continuous time

$$L_t^j \leq V_t\left(\xi K_t^j, 0\right), \tag{47}$$

we show in Appendix B.1 that firm value is equal to

$$V_t\left(K_t^j, M_t^j\right) = Q_t K_t^j + B_t + P_t M_t^j. \tag{48}$$

Thus firm value consists of a fundamental component $Q_t K_t^j$, a bubble component B_t, and an asset value component $P_t M_t^j$. Constraint (47) says that firm j does not use its intrinsically useless asset M_t^j as collateral, because it has already sold to finance investment when an investment opportunity arrives so that the lender cannot seize M_t^j on default.

Proposition 9 *Suppose that there is an intrinsically useless asset for trading in the baseline model under Assumption 5 and the credit constraint in (47). If $Q_t > 1$, then the continuous-time equilibrium system for (K_t, Q_t, B_t, P_t) is given by (20), (21), (46), and*

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + P_t + B_t),$$

where $R_t = \alpha K_t^{\alpha-1}$, and the usual transversality conditions hold. An equilibrium can only determine the total size of bubbles $P_t + B_t$, but not P_t and B_t independently. The steady states are characterized by Propositions 3 and 4 where B is replaced by $P + B$.

Since the pure bubble P_t and the stock price bubble B_t can help raise investment to the same extent, they are perfect substitutes. However, the mechanisms generating these two types of bubbles are different. A pure bubble is generated when investing and non-investing firms trade for the purpose of financing investment. The stock price bubble is not sold to finance investment as there is no trade in stocks in equilibrium. It is in firm value, which is used as collateral to borrow. Unlike the pure bubble, the stock price bubble directly raises firm value and hence relaxes the credit constraint (47) and the debt limit. This feature provides a positive feedback loop to support the stock price bubble. This intuition suggests that other types of credit constraints endogenously derived from incentive constraints in optimal contracts may generate a stock price bubble as long as firm value enters incentive constraints.

6.2 Liquidity Mismatch

In the baseline model we have assumed that capital sales are realized after investment spending, causing a liquidity mismatch (Assumption 3). The interpretation is that selling capital may take time so that the proceeds from sales may not be available at the time of investment (Kiyotaki and Moore (2005, 2008)). We now relax this assumption by allowing at most a fraction λ of the proceeds from the sales of existing capital to be used to finance investment. Then the financing constraint (12) becomes

$$I_t^j \leq R_t K_t^j \Delta + L_t^j + Q_t \left[(1 - \delta\Delta) K_t^j - K_{1t+\Delta}^j \right], \tag{49}$$

where $K_{1t+\Delta}^j$ satisfies

$$K_{1t+\Delta}^j \geq (1 - \lambda)(1 - \delta\Delta) K_t^j. \tag{50}$$

In Appendix B.2 we derive the continuous-time equilibrium system and show that the bubbly and bubbleless steady states coexist if and only if

$$0 < \xi + \lambda < \frac{\delta}{r + \pi}.$$

This implies that as long as λ is sufficiently small, firms cannot overcome credit constraints and a stock price bubble can emerge. Thus our main insights do not change as long as not enough capital can be sold to finance investment due to the illiquidity of capital. Our baseline model corresponds to the extreme case with $\lambda = 0$.

6.3 Equity Issues

In the baseline model we have assumed that firms cannot issue new equity by selling new shares to finance investment (Assumption 3). This assumption is typically adopted in the literature on models with financial frictions (e.g., Carlstrom and Fuerst (1997) and Bernanke, Gertler, and Gilchrist (1999)). If firms could issue unlimited amount of new equity, then they would not be financially constrained and a stock price bubble could not emerge. What is critical for our results is that equity issues are limited so that debt financing is needed.

Based on the Flow of Funds Accounts of the Federal Reserve Board, Figure 1 in Jermann and Quadrini (2012) shows that equity payouts in the US nonfinancial business sector are almost always positive between 1952 Q1 and 2010 Q2.[16] This figure suggests that nonfinancial firms on average pay out dividends or repurchase shares instead of issuing new equity during that period. Using the same source of data for the sample period from 1945 to 2002, Table 2 of Frank and Goyal (2008) shows that net debt issues finance a large part of financing deficit (defined as investment and dividends minus internal funds). Equity issues are negative and debt issues exceed the financing deficit during the last two decades. At the firm level, Hennessy and Whited (2007) find that the average ratio of equity issuance to total assets is 8.9% for US nonfinancial and unregulated firms during the period from 1988 to 2001 using the Compustat database. One explanation for the preceding evidence of limited equity issues is that issuing equity incurs direct administrative and underwriting costs and may also risk the loss of control. In terms of theory, Myers (1984) and Myers and Majluf (1984) argue that firms prefer internal to external financing and debt to equity if external financing is used because of adverse selection. Issuing new equity may signal bad news to outside shareholders when there is information asymmetry between managers and outside shareholders.

We can relax our extreme assumption by allowing firms to issue new equity.

Assumption 6 *No firm holds the shares of other firms so that new equity is issued to households as shareholders subject to external equity financing costs.*

We first show that under this assumption the normalization of the stock supply to one is innocuous. We use a discrete-time setup to illustrate this point as in Miller and Modigliani (1961). Let n_t, d_t, and v_t denote the number of existing outstanding shares, dividends per share, and the cum-dividend stock price per share, respectively. Then the stock price per share satisfies the asset pricing equation

$$v_t = d_t + e^{-r\Delta} v_{t+\Delta}. \tag{51}$$

Let $V_t = n_t v_t$ denote the total market value of the enterprise and $D_t = n_t d_t$ denote total dividends. Suppose that the firm sells the number m_t of new shares at the closing price $v_{t+\Delta}$ at date t. Then we have $n_{t+\Delta} = n_t + m_t$. Multiplying both sides of equation (51) by n_t gives

$$\begin{aligned} V_t &= D_t + e^{-r\Delta} n_t v_{t+\Delta} = D_t + e^{-r\Delta} \left[n_{t+\Delta} v_{t+\Delta} - (n_{t+\Delta} - n_t) v_{t+\Delta} \right] \\ &= D_t - e^{-r\Delta} m_t v_{t+\Delta} + e^{-r\Delta} n_{t+\Delta} v_{t+\Delta} = (D_t - S_t) + e^{-r\Delta} V_{t+\Delta}, \end{aligned}$$

[16]Equity payout is defined as dividends and share repurchases minus equity issues of nonfinancial corporate businesses, minus the net proprietor's investment in noncorporate businesses.

where $S_t = e^{-r\Delta} m_t v_{t+\Delta}$ is the value of new equity. The macroeconomics and finance literature often interprets $D_t - S_t$ as "dividends" and negative dividends represent new equity (e.g., Hennessy and Whited (2007) and Jermann and Quadrini (2012)). By normalizing the total stock supply of the enterprise to one, we can interpret the stock price as the stock market value of the enterprise. The asset pricing equations remains the same as before. Thus we do not need to explicitly model the change in the number of shares.

Now suppose that firm j can issue new equity to households (shareholders) in the discrete-time setup of Section 3. Its objective is to maximize the equity value of existing shareholders. We can describe firm j's decision problem by dynamic programming:

$$\begin{aligned} V_t\left(K_t^j\right) = & \max_{K_{t+\Delta}^j, K_{1t+\Delta}^j, I_t^j, L_t^j, S_{0t}^j, S_{1t}^j} (1-\pi\Delta)\left[\left(D_{0t}^j - S_{0t}^j\right)\Delta + e^{-r\Delta} V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right] \\ & + \pi\Delta\left[D_{1t}^j - S_{1t}^j + e^{-r\Delta} V_{t+\Delta}\left(K_{1t+\Delta}^j\right)\right] \end{aligned}$$

subject to (13),

$$D_{0t}^j\Delta + Q_t K_{t+\Delta}^j = R_t K_t^j \Delta + Q_t\left(1-\delta\Delta\right)K_t^j + S_{0t}^j\Delta - \frac{\varphi}{2}\frac{(S_{0t}^j)^2}{K_t^j}\Delta,$$

$$D_{1t}^j + Q_t K_{1t+\Delta}^j + L_t^j + I_t^j = R_t K_t^j \Delta + L_t^j + S_{1t}^j - \frac{\varphi}{2}\frac{(S_{1t}^j)^2}{K_t^j} + Q_t\left(1-\delta\Delta\right)K_t^j + Q_t I_t^j,$$

$$I_t^j \le R_t K_t^j \Delta + L_t^j + S_{1t}^j,$$

where S_{1t}^j (S_{0t}^j) represents new equity issues when an (no) investment opportunity arrives, and $\varphi(S_{0t}^j)^2/(2K_t^j)$ and $\varphi(S_{1t}^j)^2/(2K_t^j)$ represent external equity financing costs.[17] The parameter $\varphi > 0$ represents the size of the equity financing cost. The preceding two equations are the flow-of-funds constraints. The inequality is the financing constraint, which says that investment is financed by internal funds $R_t K_t^j \Delta$, debt L_t^j, and new equity S_{1t}^j.

In Appendix B.3 we study the continuous-time limit and show that the firm will not issue new equity, i.e., $S_{0t}^j = 0$, when no investment opportunity arrives due to the equity financing cost. When an investment opportunity arrives and $Q_t > 1$, the firm issues equity

$$S_{1t}^j = \frac{1}{\varphi}(Q_t - 1)K_t^j.$$

The following proposition characterizes the conditions for the existence of a bubbly steady state.

[17] Using a structural model based on the Compustat database, Hennessy and Whited (2007) estimate a quadratic function of the external equity financing cost.

Proposition 10 *Given Assumption 6 in the baseline model, there exists a unique bubbly steady state satisfying*

$$\begin{aligned}
Q_b &= 1 + \frac{r}{\pi} > 1, \\
\frac{B}{K_b} &= \frac{\delta}{\pi} - \xi(\frac{r}{\pi} + 1) - \frac{r}{\varphi\pi} > 0, \\
R_b &= \alpha (K_b)^{\alpha-1} = [(1-\xi)r + \delta](\frac{r}{\pi} + 1) - \frac{1}{2}\frac{r^2}{\varphi\pi} > 0,
\end{aligned}$$

if and only if $0 < \xi(r+\pi) + r/\varphi < \delta$.

If the equity financing cost is too large, i.e., $\varphi \to \infty$, firms will not issue any equity and the proposition reduces to Proposition 4. If the equity financing cost (i.e., φ) is too small, then the conditions in the proposition are violated. In this case firms can issue sufficient new equity to overcome the credit constraints so that a stock price bubble could not exist. In the extreme case without equity financing cost (i.e., $\varphi = 0$), firms can issue sufficient new equity to finance investment at the efficient level so that the economy attains the efficient equilibrium (Miller and Modigliani (1961)). Thus our key insights will not change as long as new equity issues are sufficiently limited due to external equity financing costs.

6.4 Additional Asset with Exogenous Rents

In Section 2 we have discussed the issue of the rate of return dominance. We have shown that our model can generate a stock price bubble in firms with positive dividends. One may wonder whether a stock price bubble can still exist if there is another asset with exogenous rents that grow as fast as the economy. Tirole (1985) shows that this is possible in an OLG model by assuming that rents are not capitalized before their creation. In this subsection we show that this is also possible in our infinite-horizon model if the asset with exogenous rents is less liquid than the stock price bubble for financing investment. To this end, we introduce an asset with exogenous rents $X_t = xe^{gt} > 0$ paid at each time t to the baseline model of Section 3. The supply of the asset is normalized to one. To prevent unlimited arbitrage, we make the following assumption.

Assumption 7 *Neither households nor firms can short the asset with exogenous rents (e.g., land).*

We also introduce economic growth by setting the production function as $Y_t^j = (K_t^j)^\alpha (A_t N_t^j)^{1-\alpha}$, where $A_t = e^{gt}$ ($g \geq 0$) represents technical progress. A simple way to model the illiquidity of the asset is to impose a resaleability constraint in continuous time (Kiyotaki and Moore (2008)):

$$M_{1t}^j \geq (1-\zeta)\, M_t^j, \tag{52}$$

where $M_t^j \geq 0$ denotes firm j's existing asset holdings and $M_{1t}^j \geq 0$ denotes firm j's new asset holdings when an investment opportunity arrives. The interpretation is that firm j can sell at most a fraction $\zeta \in (0,1)$ of the asset to finance its investment. In this case the asset is less liquid than the bubble. For simplicity suppose that firm j does not use the asset with rents as collateral and we adopt the credit constraint in (47). We still use $V_t(K_t^j, M_t^j)$ to denote firm j's value function.

In Appendix B.4 we show that $V_t(K_t^j, M_t^j)$ takes the form in (48). For $Q_t > 1$, the resaleability constraint (52) binds when an investment opportunity arrives, because firm j will sell the asset to non-investing firms as much as possible to finance investment. The aggregate capital stock K_t, asset price P_t, and stock price bubble B_t will all grow at the rate g. But the capital price Q_t will not grow. Moreover, B_t, Q_t, and P_t satisfy the asset pricing equations (20), (21), and

$$\dot{P}_t = rP_t - X_t - \pi(Q_t - 1)\zeta P_t. \tag{53}$$

Thus the return on the asset with rents is higher than the return on the bubble and

$$\frac{\dot{P}_t + X_t}{P_t} = r - \pi(Q_t - 1)\zeta > r - \pi(Q_t - 1) = \frac{\dot{B}_t}{B_t} \text{ for } B_t > 0.$$

If the asset is fully liquid, i.e., $\zeta = 1$, then the two returns are the same and equal to zero in the bubbly steady state without growth $(g = 0)$. This is impossible if rents X_t are positive, generating the rate of return dominance puzzle discussed in Section 2.

We solve this puzzle by assuming that the asset with rents is less liquid than the stock price bubble, i.e., $\zeta \in (0,1)$. In this case the return on the asset with rents is higher than the return on the bubble because the asset with rents commands a lower liquidity premium than the bubble. Non-investing firms buy the asset with rents for a precautionary motive because they anticipate being credit constrained when an investment opportunity arrives in the future. Since the return on the asset is lower than the discount rate r, households want to sell the asset until their short-sale constraints bind. In equilibrium households do not hold any of the asset.

We also consider the more general case with growth $g > 0$. In the bubbly steady state, Q_t and the detrended variables $k_t = K_t/A_t$, $p_t = P_t/A_t$, and $b_t = B_t/A_t$ are constant over time. The following proposition gives the conditions such that the stock price bubble and the asset with growing rents X_t can coexist in the steady state.

Proposition 11 *Suppose that there is an asset with growing rents X_t available for households and firms to trade in the baseline model under Assumption 7. There exists a unique bubbly steady state (Q_b, k_b, b, p) satisfying*

$$\begin{aligned}
b &= \frac{\delta + g}{\pi} k_b - Q_b \xi k_b - \zeta p > 0, \\
p &= \frac{x}{(r-g)(1-\zeta)} > 0, \quad Q_b = \frac{r-g}{\pi} + 1 > 1. \\
\alpha k_b^{\alpha - 1} &= [r + \delta - (r-g)\xi] \left(\frac{r-g}{\pi} + 1 \right) > 0,
\end{aligned}$$

if and only if

$$r > g \geq 0, \; 0 < \xi < \frac{\delta + g}{r + \pi - g}, \tag{54}$$

$$0 < x < (r-g)\frac{1-\zeta}{\zeta}\left[\frac{\delta+g}{\pi} - \left(\frac{r-g}{\pi}+1\right)\xi\right] k_b. \tag{55}$$

The interpretation of condition (54) is similar to that of (36). The intuition behind condition (55) is that the asset with detrended rents x must be sufficiently illiquid (i.e., ζ must be sufficiently small), or the detrended dividend x must be sufficiently small. Otherwise, this asset will dominate the stock price bubble and rule out the latter in the steady state.

6.5 Intertemporal Borrowing and Savings

In this subsection we replace intratemporal debt with riskfree *intertemporal* bonds in the baseline model of Section 3.[18] With intertemporal bonds, firms can raise new debt to payoff old debt. Firms with investment opportunities can use these bonds to finance investment subject to credit constraints. Anticipating being credit constrained in the future, firms without investment opportunities will save in the bonds precautionarily. The bonds are in zero net supply. Households can also trade them, but are subject to short-sale constraints.

Assumption 8 *Households cannot short intertemporal bonds and firms cannot long each other's stocks (equity shares).*

One may interpret the bonds here as corporate bonds issued by firms and households cannot borrow by issuing corporate bonds. We will show that households will never hold any intertemporal bonds in equilibrium, because the interest rate r_{ft} is lower than the discount rate r. A similar result is derived in Kiyotaki and Moore (2005, 2008).

Let $L_t^h \geq 0$ denote the representative household's bond holdings. Let $L_t^j > (<) 0$ denote firm j's debt level (saving). The market-clearing condition for the bonds is $\int L_t^j dj = L_t^h$. Let $V_t(K_t^j, L_t^j)$ denote the ex ante market value of firm j when its capital stock and debt level at time t are K_t^j and L_t^j, respectively, prior to the realization of the Poisson shock. We suppress the aggregate state variables in the argument. Assume that firm j maximizes its market value and hence it solves the following dynamic programming problem in discrete time:

$$\begin{aligned} V_t\left(K_t^j, L_t^j\right) = & \max_{I_t^j, K_{t+\Delta}^j, K_{1t+\Delta}^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} (1-\pi\Delta)\left[D_{0t}^j \Delta + e^{-r\Delta} V_{t+\Delta}\left(K_{t+\Delta}^j, L_{t+\Delta}^j\right)\right] \\ & + \pi\Delta\left[D_{1t}^j + e^{-r\Delta} V_{t+\Delta}\left(K_{1t+\Delta}^j, L_{1t+\Delta}^j\right)\right] \end{aligned}$$

[18] In Appendix F we study a model in which there is no market for capital goods. Firms make investment and accumulate capital on their own. They can use new capital or future capital as collateral to borrow. We show that our key insights do not change. See Miao, Wang, and Xu (2015) for a related discrete-time model.

subject to

$$D_{0t}^{j}\Delta+Q_{t}K_{t+\Delta}^{j}=R_{t}K_{t}^{j}\Delta+e^{-r_{ft}\Delta}L_{t+\Delta}^{j}-L_{t}^{j}+Q_{t}\left(1-\delta\Delta\right)K_{t}^{j}, \tag{56}$$

$$D_{1t}^{j}+Q_{t}K_{1t+\Delta}^{j}+I_{t}^{j}=R_{t}K_{t}^{j}\Delta+e^{-r_{ft}\Delta}L_{1t+\Delta}^{j}-L_{t}^{j}+Q_{t}I_{t}^{j}+Q_{t}\left(1-\delta\Delta\right)K_{t}^{j}, \tag{57}$$

$$I_{t}^{j}\leq R_{t}K_{t}^{j}\Delta+e^{-r_{ft}\Delta}L_{1t+\Delta}^{j}-L_{t}^{j}, \tag{58}$$

$$V_{t+\Delta}(K_{1t+\Delta}^{j},L_{1t+\Delta}^{j})\geq V_{t+\Delta}\left(K_{1t+\Delta}^{j},0\right)-V_{t+\Delta}(\xi\left(1-\delta\Delta\right)K_{t}^{j},0), \tag{59}$$

where $L_{1t+\Delta}^{j}$ ($L_{t+\Delta}^{j}$) represents the new debt level or saving when an investment opportunity arrives (no investment opportunity arrives). The price of the debt at time t that pays off one unit of consumption good at time $t+\Delta$ is $e^{-r_{ft}\Delta}$. Equations (56) and (57) are the flow-of-funds constraints. Inequality (58) gives the financing constraint, which says that investment is financed by internal funds and new debt.

Debt is subject to the credit constraint (59), which is interpreted in a similar way to (13). When an investment opportunity arrives at time t, firm j borrows $e^{-r_{ft}\Delta}L_{1t+\Delta}^{j}>0$ from other firms without investment opportunities. If it does not default on debt $L_{1t+\Delta}^{j}$ at time $t+\Delta$, it obtains continuation value $V_{t+\Delta}\left(K_{1t+\Delta}^{j},L_{1t+\Delta}^{j}\right)$. If it defaults, debt is renegotiated and the repayment $L_{1t+\Delta}^{j}$ is relieved. The new repayment is determined by Nash bargaining. Assume that firm j has a full bargaining power. Then the new repayment is given by the threat value to the lender, which is equal to the market value of the firm $V_{t+\Delta}(\xi\left(1-\delta\Delta\right)K_{t}^{j},0)$ when the lender takes over the firm and keeps it running by recovering a fraction ξ of depreciated capital $\left(1-\delta\Delta\right)K_{t}^{j}$. The expression on the right-hand side of (59) is the value to the firm if it chooses to default on the previous debt and repay $V_{t+\Delta}(\xi\left(1-\delta\Delta\right)K_{t}^{j},0)$. We then have the incentive constraint in (59).

In Appendix B.5 we derive equilibria in the continuous-time limit. We show that the value function takes the form

$$V_{t}\left(K_{t}^{j},L_{t}^{j}\right)=Q_{t}K_{t}^{j}-L_{t}^{j}+B_{t}, \tag{60}$$

and the continuous-time limit of the credit constraint (59) becomes

$$L_{1t}^{j}\leq Q_{t}\xi K_{t}^{j}+B_{t}, \tag{61}$$

where $B_{t}\geq0$ is the bubble component of equity value.

Proposition 12 *For the model in this subsection with intertemporal bonds under Assumption 8, if $Q_t>1$, then the continuous-time equilibrium system for (K_t,Q_t,B_t,r_{ft}) is given by (20), (21), (27), and*

$$r_{ft}=r-\pi\left(Q_{t}-1\right)<r, \tag{62}$$

where $R_t=\alpha K_t^{\alpha-1}$, and the usual transversality conditions hold.

The credit constraint (61) and the continuous-time limit of the financing constraint (58) together imply that

$$I_t^j \le L_{1t}^j - L_t^j \le \xi Q_t K_t^j + B_t - L_t^j.$$

When $Q_t > 1$, it is profitable for firm j to invest and both constraints bind. We then have

$$I_t^j = \xi Q_t K_t^j + B_t - L_t^j.$$

With intertemporal bonds, firm j can use both its savings when $L_t^j < 0$ and new debt $\xi Q_t K_t^j + B_t$ to finance investment.

Equation (62) shows that the equilibrium interest rate r_{ft} is equal to the subjective discount rate r minus a liquidity premium $\pi(Q_t - 1)$. The liquidity premium exists because bonds can provide liquidity to investing firms by raising their net worth. Since the stock price bubble and the bonds can be used to finance investment to the same extent, they command the same amount of liquidity premium.

Firms without investment opportunities are willing to save and lend even though $r_{ft} < r$ because they anticipate that they will be credit constrained when an investment opportunity arrives in the future. Their demand for bonds pushes up the bond price and lowers the interest rate, which reflects a precautionary saving motive as in the incomplete markets models (e.g., Aiyagari (1994)). Unlike in Aiyagari (1994), however, firms in our model are subject to uninsured idiosyncratic investment opportunity shocks and credit constraints.[19] To better understand the intuition, we consider the discrete-time approximation (see Appendix B.5). Buying one unit of bonds at time t costs $e^{-r_{ft}\Delta}$ dollars. At time $t+\Delta$, the bond pays off one dollar. When firm j meets an investment opportunity with probability $\pi\Delta$, it uses the bond payoff to finance one dollar of investment, which generates $(Q_{t+\Delta} - 1)$ dollars of dividends. Thus the total discounted marginal benefit from the bond is given by $e^{-r\Delta}[1 + \pi\Delta(Q_{t+\Delta} - 1)]$. Equating this marginal benefit with the marginal cost $e^{-r_{ft}\Delta}$ and taking the continuous-time limit as $\Delta \to 0$ give (62).

Since $r > r_{ft}$, households want to borrow by selling bonds until their short-sale constraints bind, i.e., $L_t^h = 0$. We then have the bond market-clearing condition $\int L_t^j dj = 0$. Without a short-sale constraint, households would keep shorting bonds (or effectively borrowing) until $r_{ft} = r$. In this case firms would be able to accumulate enough savings in bonds so that their credit constraints would no longer bind. The liquidity premium would be zero so that $Q_t = 1$ and the economy would reach the efficient equilibrium and no bubble could exist.

More generally, as long as households are subject to sufficiently tight borrowing limits (or short-sale constraints) in the sense that they cannot issue sufficiently many bonds, the efficient

[19] Carlstrom and Fuerst (1997) and Bernanke, Gertler, and Gilchrist (1999) adopt the costly state verification model of debt contracts between entrepreneurs and lenders, in which entrepreneurs can default on debt. In this case debt is risky and internal funds earn higher returns than external funds due to agency costs. To prevent entrepreneurs from saving to overcome borrowing constraints, one can assume either that entrepreneurs discount the future more heavily than households or that entrepreneurs die randomly. By contrast, debt is riskfree in our model and the interest rate is lower than the discount rate due to firms' precautionary saving motives.

equilibrium cannot be attained and there is a scarcity of financial assets for savers (firms).[20] The existence of a stock price bubble is effectively a way of increasing the supply of financial assets that can be held by firms. The government can also play a role in supplying financial assets to firms by issuing government bonds, thereby improving efficiency.

Proposition 13 *For the model in this subsection with intertemporal bonds, if condition (36) holds, then the bubbly and bubbleless steady states with* $Q > 1$ *coexist. Moreover, the interest rates in the bubbleless and bubbly steady states are given by* $r_f^* = r + \pi - \delta/\xi < 0$ *and* $r_f = 0$*, respectively.*

Under condition (36), $Q_t > 1$ in a neighborhood of either the bubbleless or the bubbly steady state so that Proposition 12 applies. This condition is equivalent to the standard condition (Tirole (1985) and Santos and Woodford (1997)) requiring that the interest rate on bonds in the bubbleless steady state be lower than the rate of economic growth ($r_f^* < 0$). Unlike in Tirole (1985), the economy is dynamically efficient in our model. Proposition 13 shows that the interest rate on bonds must be equal to zero in the bubbly steady state ($r_f = 0$), because the steady-state return on the stock price bubble $\dot{B}_t/B_t$ is equal to zero. To generate a positive steady-state interest rate, we can introduce economic growth as in Section 6.4.

6.6 Cross-Holdings of Shares

We have assumed that no firm can hold the shares of other firms and trade these shares to finance investment. This assumption is justified by the US aggregate and firm-level data. From Table F103 of the Flow of Funds Accounts, we find that between 2005 and 2015 the average ratio of the net acquisition of mutual fund shares (line 30) to the net acquisition of financial assets (line 16) in the US nonfinancial corporate business sector is 1.74%. In terms of levels, Frank and Goyal (2008) find that in the 1990s corporate equity was held heavily by households (39% of the aggregate equity outstanding), pension and mutual funds (20%), insurance firms (28%), the rest of the world (10%), and banks and the government (3%). Thus cross-holdings of other firms' shares by nonfinancial corporations account for a negligible fraction in the aggregate data. And trading of other firms' shares is not a major source of external financing for nonfinancial corporations.

We also investigate the firm-level data from 2000 to 2016 using the Compustat database. The item ISEQ (investment securities – equity) reports the holdings of other firms' equity. We find that this item is missing for most nonfinancial and non-utility firms in our sample. Moreover, for those firms with ISEQ entries, the average ratio of ISEQ to total assets in each year ranges from 0 to 1.5%, and the sample mean is 0.6% from 2000 to 2016. By contrast, a large literature has found that firms hold a sizable amount of cash (e.g., Bates, Kahle and Stulz (2009)). We find that the average ratio of cash holdings (item CH in Compustat) to total assets in each year during 2000-2016 ranges from 14.3% to 20.1%, and the sample mean is 17.5%. This evidence shows that

[20] See Appendix E for the analysis of a general short-sale constraint (or borrowing constraint).

US nonfinancial firms hold a large amount of cash with a low return and very little of other firms' equity with a high return.

Some nonfinancial firms may have cross-holdings for reasons such as mergers and acquisitions, corporate governance, diversification, and strategic alliance. However, they typically do not trade other firms' shares for regular investment financing. One reason is that such trading incurs large administration, filing, and monitoring costs, and may signal takeover interest to other firms. Such trading is risky and may lead to fire sales, loss of control over upstream suppliers, or competition.

If we relax Assumption 8 by allowing firms to trade each other's shares to finance investment in the model of Section 6.5 with intertemporal bonds,[21] then each firm can earn a return r higher than the interest rate r_{ft} by holding other firms' shares. If there is no market friction, firms may end up holding too many shares of other firms and eventually overcome credit constraints. Unlimited arbitrage would cause the economy to attain the efficient equilibrium with $Q_t = 1$ and $r = r_{ft}$ in which no bubble could exist. Assumption 8 supports the equilibrium with $r > r_{ft}$. This assumption prevents unlimited arbitrage and is justified by the empirical evidence discussed above (also see footnote 4).

The critical feature of Assumption 8 is not the restriction that no firm can long equity shares of other firms, but is the restriction that this source of finance is limited. In Appendix G we show that, even if each firm can hold a market portfolio of firm stocks and earn the return r in the model of Section 6.5, a stock price bubble can still emerge and Proposition 13 still holds as long as firms do not use the market portfolio to finance investment due to the reasons discussed above. In this case the stock price bubble and the bonds can coexist with the market portfolio because the bubble and bonds provide a liquidity service, while the market portfolio does not. In equilibrium the sum of the interest rate r_{ft} and the liquidity premium $\pi (Q_t - 1)$ is equal to r.

7 Conclusion

We have developed a theory of stock price bubbles in the presence of endogenous credit constraints in production economies with infinitely lived agents. Bubbles emerge through a positive feedback loop mechanism in which credit constraints derived from optimal contracts with limited commitment play an essential role. Our analysis differs from most studies in the literature that analyze bubbles in intrinsically useless assets or in assets with exogenous payoffs in an endowment economy or an OLG framework. Our model can incorporate this type of bubbles and thus provides a unified framework to study asset bubbles. Our theory can be integrated into the dynamic stochastic general equilibrium framework and has important implications for empirical studies. First, using average Q to measure marginal Q may be misleading even for constant-returns-to-scale technology because

[21] Cross-holdings of shares lead to the well-known problem of inflating market values (e.g., Fedenia, Hodder, and Triantis (1994)). Elliott, Golub, and Jackson (2014) show that the interdependence through cross-holdings of financial firms can generate financial contagions and cascades of failures in a static model.

they are not identical in the presence of stock price bubbles. Second, using the present value of dividends to measure the fundamental value of a stock may also be misleading because dividends and bubbles cannot be separated. Third, tests based on transversality conditions can rule out rational bubbles in OLG models, but not in our model.

In future research it would be interesting to study how bubbles can explain asset pricing puzzles, how bubbles contribute to business cycles in a quantitative dynamic stochastic general equilibrium model (Miao, Wang and Xu (2015)), how bubbles affect long-run growth (Caballero, Farhi and Hammour (2006), Martin and Venture (2012), Hirano and Yanagawa (2013), and Miao and Wang (2014)), and what the implications of asset price bubbles are for monetary policy (Galí (2014)).

References

Abel, Andrew B., N. Gregory Mankiw, Lawrence H. Summers, and Richard J. Zeckhauser, 1989, Assessing Dynamic Efficiency: Theory and Evidence, *Review of Economic Studies* 56, 1-19.

Acemoglu, Daron, *Introduction to Modern Economic Growth*, Princeton University Press, Princeton and Oxford.

Adam, Klaus, Albert Marcet and Juan Pablo Nicolini, 2015, Stock Market Volatility and Learning, forthcoming in *Journal of Finance*.

Aiyagari, S. Rao, 1994, Uninsured Idiosyncratic Risk and Aggregate Saving, *Quarterly Journal of Economics* 109, 659-684.

Albuquerque, Rui and Hugo A. Hopenhayn, 2004, Optimal Lending Contracts and Firm Dynamics, *Review of Economic Studies* 71, 285-315.

Alvarez, Fernando and Urban J. Jermann, 2000, Efficiency, Equilibrium, and Asset Pricing with Risk of Default, *Econometrica* 68, 775-798.

Barro, Robert J., 1974, Are Government Bonds Net Worth, *Journal of Political Economy* 82, 1095-1117.

Bates, Thomas W., Kathleen M. Kahle, René M. Stulz, 2009, Why Do U.S. Firms Hold So Much More Cash than They Used to? *Journal of Finance* 64, 1985-2021.

Benhabib, Jess and Farmer, Roger, 1999. Indeterminacy and Sunspots in Macroeconomics, in: Taylor, J., Woodford, M. (Eds.), *Handbook of Macroeconomics*, Vol. 1A, North-Holland, Amsterdam, 387–448.

Bernanke, Ben, Mark Gertler and Simon Gilchrist, 1999, Financial Accelerator in a Quantitative Business Cycle Framework, In: J. Taylor and M. Woodford (eds.), *Handbook of Macroeconomics*, Elsevier Science, Amsterdam.

Blanchard, Olivier, and Mark Watson, 1982, Bubbles, Rational Expectations and Financial Markets, Harvard Institute of Economic Research Working Paper No. 945.

Bris, Arturo, Ivo Welch, and Ning Zhu, 2006, The Costs of Bankruptcy: Chapter 7 Liquidation versus Chapter 11 Reorganization, *Journal of Finance* 61, 1253–1303.

Brunnermeier, Markus, 2009, Bubbles, entry in *The New Palgrave Dictionary of Economics*, edited by Steven Durlauf and Lawrence Blume, 2nd edition.

Bulow, Jeremy and Kenneth Rogoff, 1989, Sovereign Debt: Is to Forgive to Forget?, *American Economic Review* 79, 43-50.

Caballero, Ricardo, Emmanuel Farhi and Mohamad Hammour, 2006, Speculative Growth: Hints from the US Economy, *American Economic Review* 96, 1159-1192

Caballero, Ricardo J. and Arvind Krishnamurthy, 2006, Bubbles and Capital Flow Volatility: Causes and Risk Management, *Journal of Monetary Economics* 53, 35-53.

Carlstrom, Charles T. and Timothy S. Fuerst, 1997, Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis, *American Economic Review* 87, 893-910.

Coddington, Earl A., and Norman Levinson, 1955, *Theory of Ordinary Differential Equations*, New York: McGraw-Hill.

Collyns, Charles and Abdelhak Senhadji, 2002, Lending Booms, Real Estate Bubbles, and The Asian Crisis, IMF Working Paper No. 02/20.

Diamond, Peter, 1965, National Debt in a Neoclassical Growth Model, *American Economic Review* 55, 1126-1150.

Ekeland, Ivar and Jose A. Scheinkman, 1986, Transversality Conditions for Some Infinite Horizon Discrete Time Optimization Problems, *Mathematics of Operations Research* 11, 216-229.

Elliott, Matthew, Benjamin Golub, and Matthew O. Jackson, 2014, Financial Networks and Contagion, *American Economic Review* 104, 3115–3153.

Farhi, Emmanuel and Jean Tirole, 2012, Bubbly Liquidity, *Review of Economic Studies* 79, 678-706.

Farmer, Roger E. A., 1999, *The Macroeconomics of Self-Fulfilling Prophecies*, 2nd ed., MIT Press, Cambridge.

Fedenia, Mark, James E. Hodder, and Alexander J. Triantis, 1994, Cross-Holdings: Estimation Issues, Biases, and Distortions, *Review of Financial Studies* 7, 61-96.

Frank, Murray, and Vidhan K. Goyal, 2008, Tradeoff and Pecking Order Theories of Debt, in B. Espen Eckbo (ed), *Handbook of Empirical Corporate Finance* Vol. 2, Amsterdam: Elsevier/North Holland, 135-197.

Galí, Jordi, 2014, Monetary Policy and Rational Asset Price Bubbles, *American Economic Review* 104, 721-752.

Galí, Jordi and Luca Gambetti, 2013, The Effects of Monetary Policy on Stock Market Bubbles: Some Evidence, forthcoming in *American Economic Journal: Macroeconomics.*

Gertler, Mark and Peter Karadi, 2011, A Model of Unconventional Monetary Policy, *Journal of Monetary Economics* 58, 17-34.

Gertler, Mark and Nobuhiro Kiyotaki, 2010, Financial Intermediation and Credit Policy in Business Cycle Analysis, in Ben Friedman and Michael Woodford (eds), *Handbook of Monetary Economics*, Volume 3A, 547-599.

Giglio, Stefano, Matteo Maggiori, and Johannes Stroebel, 2016, No-Bubble Condition: Model-Free Tests in Housing Markets, *Econometrica* 84, 1047-1091.

Gu, Chao, Fabrizio Mattesini, and Randall Wright, 2016, Money and Credit Redux, *Econometrica* 84, 1-32.

Gurkaynak, Refet S., 2008, Econometric Test of Asset Price Bubbles: Taking Stock, *Journal of Economic Surveys* 22, 166-186.

Hayashi, Fumio, 1982, Tobin's Marginal q and Average q: A Neoclassical Interpretation, *Econometrica* 50, 213-224.

Hellwig, Christian, and Guido Lorenzoni, 2009, Bubbles and Self-Enforcing Debt, *Econometrica* 77, 1137-1164.

Hennessy, Christopher A., and Toni M. Whited, 2007, How Costly Is External Financing? Evidence from a Structural Estimation, *Journal of Finance* 62, 1705-1745.

Hirano, Tomohiro and Noriyuki Yanagawa, 2013, Asset Bubbles, Endogenous Growth and Financial Frictions, working paper, University of Tokyo.

Jermann, Urban, and Vincenzo Quadrini, 2012, Macroeconomic Effects of Financial Shocks, *American Economic Review* 102, 238-271.

Jordà, Òscar, Mortiz Schularick, and Alan M. Taylor, 2014, Leveraged Bubbles, working paper, University of California at Davis.

Kehoe, Timothy J. and David K. Levine, 1993, Debt-Constrained Asset Markets, *Review of Economic Studies* 60, 865-888.

Kiyotaki, Nobuhiro, and John Moore, 1997, Credit Cycles, *Journal of Political Economy* 105, 211-248.

Kiyotaki, Nobuhiro, and John Moore, 2005, Liquidity and Asset Prices, *International Economic Review* 46, 317-349.

Kiyotaki, Nobuhiro, and John Moore, 2008, Liquidity, Business Cycles and Monetary Policy, working paper, Princeton University.

Kiyotaki, Nobuhiro, and Randall Wright, 1989, On Money as a Medium of Exchange, Journal of Political Economy 97, 927-954.

Kocherlakota, Narayana, 1992, Bubbles and Constraints on Debt Accumulation, *Journal of Economic Theory* 57, 245-256.

Kocherlakota, Narayana, 2008, Injecting Rational Bubbles, *Journal of Economic Theory* 142, 218-232.

Kocherlakota, Narayana, 2009, Bursting Bubbles: Consequences and Cures, working paper, University of Minnesota.

Lagos, Ricardo, Wright, Randall, 2005, A Unified Framework for Monetary Theory and Policy Analysis, *Journal of Political Economy* 113, 463–484.

Martin, Alberto and Jaume Ventura, 2011, Theoretical Notes on Bubbles and the Current Crisis, IMF Economic Review 59, 6-40.

Martin, Alberto and Jaume Ventura, 2012, Economic Growth with Bubbles, *American Economic Review* 102, 3033-3058.

Miao, Jianjun, 2014, Introduction to Economic Theory of Bubbles, *Journal of Mathematical Economics* 53, 130-136.

Miao, Jianjun and Pengfei Wang, 2012, Bubbles and Total Factor Productivity, *American Economic Review* 102, 82-87.

Miao, Jianjun and Pengfei Wang, 2014, Sectoral Bubbles, Misallocation, and Endogenous Growth, *Journal of Mathematical Economics* 53, 153-163.

Miao, Jianjun and Pengfei Wang, 2015, Banking Bubbles and Financial Crises, *Journal of Economic Theory* 157, 763-792.

Miao, Jianjun, Pengfei Wang, and Zhiwei Xu, 2015, A Bayesian DSGE Model of Stock Market Bubbles and Business Cycles, *Quantitative Economics* 6, 599-635.

Miao, Jianjun, Pengfei Wang, and Lifang Xu, 2016, Stock Market Bubbles and Unemployment, *Economic Theory* 61, 273-307.

Miao, Jianjun, Pengfei Wang, and Jing Zhou, 2015, Asset Bubbles, Collateral, and Policy Analysis, *Journal of Monetary Economics* 76, S57-S70.

Miller, H. Merton, and Franco Modigliani, 1961, Dividend Policy, Growth, and the Valuation of Shares, *Journal of Business* 34, 411-433.

Myers, Stewart C., 1984, The Capital Structure Puzzle, *Journal of Finance* 39, 575-592.

Myers, Stewart C., and Nicholas S. Majluf, 1984. Corporate Financing and Investment Decisions when Firms Have Information That Investors Do Not Have, *Journal of Financial Economics* 13, 187-221.

Samuelson, P., 1958, An Exact Consumption-loan Model of Interest with or without the Social Contrivance of Money, *Journal of Political Economy* 66, 467-482.

Santos, Manuel S. and Michael Woodford, 1997, Rational Asset Pricing Bubbles, *Econometrica* 65, 19-58.

Sargent, Thomas J., 2016, Commodity and Token Monies, working paper, New York University.

Scheinkman, Jose A. and Laurence Weiss, 1986, Borrowing Constraints and Aggregate Economic Activity, *Econometrica* 54, 23-46.

Scheinkman, Jose, A. and Wei Xiong, 2003, Overconfidence and Speculative Bubbles, *Journal of Political Economy* 111, 1183-1219.

Shiller, Robert, J., 2005, *Irrational Exuberance*, 2nd Edition, Princeton University Press.

Tirole, Jean, 1985, Asset Bubbles and Overlapping Generations, *Econometrica* 53, 1499-1528.

Tirole, Jean, 2006, *The Theory of Corporate Finance*, Princeton University Press, Princeton and Oxford.

Tobin, James, 1969, A General Equilibrium Approach to Monetary Policy, *Journal of Money, Credit, and Banking* 1, 15-29.

Vayanos, Dimitri and Jiang Wang, 2012, Theories of Liquidity, *Foundations and Trends in Finance* 6, 221-317.

Wang, Pengfei and Yi Wen, 2012, Speculative Bubbles and Financial Crisis, *American Economic Journal: Macroeconomics* 4, 184-221.

Weil, Philippe, 1987, Confidence and the Real Value of Money in an Overlapping Generations Economy, *Quarterly Journal of Economics* 102, 1-22.

Woodford, Michael, 1990, Public Debt as Private Liquidity, *American Economic Review* 80, 382-388.

For Online Publication Appendices

A Proofs of Results in the Baseline Model

Proof of Proposition 1: We first derive the solution in the discrete-time setup and then take the continuous-time limit. Conjecture that the value function is given by $V_t\left(K_t^j\right) = a_t K_t^j + b_t$. Substituting this conjecture and the flow-of-funds constraints (10) and (11) into the Bellman equation (9) yields

$$\begin{aligned} a_t K_t^j + b_t &= \max_{K_{t+\Delta}^j, K_{1t+\Delta}^j, I_t^j, L_t^j} R_t K_t^j \Delta + Q_t (1-\delta\Delta) K_t^j + e^{-r\Delta} b_{t+\Delta} \\ &\quad + (1-\pi\Delta)\left[-Q_t K_{t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{t+\Delta}^j\right] \\ &\quad + \pi\Delta\left[(Q_t - 1) I_t^j - Q_t K_{1t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{1t+\Delta}^j\right] \end{aligned} \tag{A.1}$$

subject to

$$I_t^j \le R_t K_t^j \Delta + L_t^j \le R_t K_t^j \Delta + e^{-r\Delta}\left(a_{t+\Delta}(1-\delta\Delta)\xi K_t^j + b_{t+\Delta}\right).$$

The first-order condition for $K_{t+\Delta}^j$ yields

$$Q_t = e^{-r\Delta} a_{t+\Delta}, \tag{A.2}$$

and hence $K_{t+\Delta}^j$ and $K_{1t+\Delta}^j$ are indeterminate. This implies that firm j is indifferent between buying and selling its existing capital. Under the assumption $Q_t > 1$, the financing constraint and the credit constraint bind so that optimal investment is given by

$$I_t^j = R_t K_t^j \Delta + Q_t (1-\delta\Delta) \xi K_t^j + B_t, \tag{A.3}$$

where we define

$$B_t \equiv e^{-r\Delta} b_{t+\Delta}. \tag{A.4}$$

Substituting the investment rule back into the preceding Bellman equation and matching coefficients, we can derive

$$b_t = \left[\pi\Delta(Q_t - 1) + 1\right] e^{-r\Delta} b_{t+\Delta},$$

$$a_t = R_t\Delta + Q_t(1-\delta\Delta) + \pi\Delta(Q_t - 1)\left[\xi Q_t(1-\delta\Delta) + R_t\Delta\right].$$

Using (A.2) and (A.4), we obtain

$$B_t = e^{-r\Delta} B_{t+\Delta}[1 + \pi\Delta(Q_{t+\Delta} - 1)], \tag{A.5}$$

$$Q_t = e^{-r\Delta}\left[R_{t+\Delta}\Delta + (1-\delta\Delta)Q_{t+\Delta} + \pi\Delta(Q_{t+\Delta} - 1)(\xi Q_{t+\Delta}(1-\delta\Delta) + R_{t+\Delta}\Delta)\right]. \tag{A.6}$$

Taking the continuous-time limit as $\Delta \to 0$ yields (20), (21), and (19).

We can also derive the continuous-time limit of the Bellman equation (9). Note that we can replace $e^{-r\Delta}$ with $1/(1+r\Delta)$ up to first-order approximation. Multiplying the two sides of (9) by $1+r\Delta$ gives

$$\begin{aligned}(1+r\Delta)V_t\left(K_t^j\right) &= \max\ (1-\pi\Delta)\left[(1+r\Delta)D_{0t}^j\Delta+V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right]\\ &\quad+\pi\Delta\left[(1+r\Delta)D_{1t}^j+V_{t+\Delta}\left(K_{1t+\Delta}^j\right)\right]\\ &= \max\ (1-\pi\Delta)(1+r\Delta)D_{0t}^j\Delta+V_{t+\Delta}\left(K_{t+\Delta}^j\right)+\pi\Delta(1+r\Delta)D_{1t}^j\\ &\quad+\pi\Delta\left[V_{t+\Delta}\left(K_{1t+\Delta}^j\right)-V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right].\end{aligned}$$

Eliminating terms of orders higher than Δ gives

$$\begin{aligned}(1+r\Delta)V_t\left(K_t^j\right) &= \max D_{0t}^j\Delta+V_{t+\Delta}\left(K_{t+\Delta}^j\right)+\pi\Delta D_{1t}^j\\ &\quad+\pi\Delta\left[V_{t+\Delta}\left(K_{1t+\Delta}^j\right)-V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right].\end{aligned}$$

Manipulating yields

$$rV_t\left(K_t^j\right)=\max\ D_{0t}^j+\frac{V_{t+\Delta}\left(K_{t+\Delta}^j\right)-V_t\left(K_t^j\right)}{\Delta}+\pi D_{1t}^j+\pi\left[V_{t+\Delta}\left(K_{1t+\Delta}^j\right)-V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right].$$

Now we take limits as $\Delta\to 0$ to obtain the continuous-time Bellman equation in (14), where we notice that

$$D_{1t}^j=Q_tI_t^j-I_t^j+Q_tK_t^j-Q_tK_{1t}^j$$

in continuous time. Moreover, (10), (12), and (13) converge to (15), (16), and (17), respectively, as $\Delta\to 0$.

We can prove proposition 1 in continuous time directly. Given the conjecture (18), we can rewrite the dynamic programming (14) as

$$\begin{aligned}rQ_tK_t^j+rB_t &= \max_{I_t^j,\dot{K}_t^j,K_{1t}^j,L_t^j}\ R_tK_t^j-Q_t\left(\dot{K}_t^j+\delta K_t^j\right)+\dot{Q}_tK_t^j+Q_t\dot{K}_t^j+\dot{B}_t\\ &\quad+\pi(Q_t-1)I_t^j+\pi\left[Q_tK_t^j-Q_tK_{1t}^j+Q_tK_{1t}^j+B_t-\left(Q_tK_t^j+B_t\right)\right]\end{aligned}\tag{A.7}$$

subject to

$$I_t^j\le L_t^j\le\xi Q_tK_t^j+B_t.\tag{A.8}$$

Given the assumption $Q_t>1$, (16) and (A.8) bind. We then obtain (19). Substituting this equation back into (A.7) and matching coefficients, we obtain (20) and (21). By the transversality condition (6) and the form of the value function,

$$\lim_{T\to\infty}e^{-rT}\left(Q_TK_T^j+B_T\right)=0.$$

We thus obtain (22). Q.E.D.

Proof of Proposition 2: Using the optimal investment rule in (19), we derive the aggregate capital accumulation equation (27). The first-order condition for the static labor choice problem (7) gives $w_t = (1-\alpha)\left(K_t^j/N_t^j\right)^{\alpha}$. We then obtain (8) and $K_t^j = N_t^j \left(w_t/\left(1-\alpha\right)\right)^{1/\alpha}$. Thus the capital-labor ratio is identical for all firms. Aggregating yields $K_t = N_t \left(w_t/\left(1-\alpha\right)\right)^{1/\alpha}$ so that $K_t^j/N_t^j = K_t/N_t$ for all $j \in [0,1]$. Substituting out w_t in (8) yields $R_t = \alpha K_t^{\alpha-1} N_t^{1-\alpha} = \alpha K_t^{\alpha-1}$ since $N_t = 1$ in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_t^j)^{\alpha} (N_t^j)^{1-\alpha} dj = \int (K_t^j/N_t^j)^{\alpha} N_t^j dj = (K_t^j/N_t^j)^{\alpha} \int N_t^j dj = K_t^{\alpha} N_t^{1-\alpha}.$$

This completes the proof. Q.E.D.

Proof of Proposition 3: (i) The social planner solves the following problem:

$$\max_{I_t} \int_0^{\infty} e^{-rt} \left(K_t^{\alpha} - \pi I_t\right) dt,$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t, \; K_0 \text{ given},$$

where K_t is the aggregate capital stock and I_t is the investment level for a firm with an investment opportunity. From this problem, we can derive the efficient capital stock K_E, which satisfies $\alpha \left(K_E\right)^{\alpha-1} = r + \delta$. The efficient output, investment and consumption levels are given by $Y_E = \left(K_E\right)^{\alpha}$, $I_E = \delta/\pi K_E$, and $C_E = \left(K_E\right)^{\alpha} - \delta K_E$, respectively.

Suppose that assumption (28) holds. We conjecture that $Q^* = Q_t = 1$ in the steady state. In this case, firm value is given by $V(K_t^j) = K_t^j$. The optimal investment rule for each firm satisfies $R_t = r + \delta = \alpha K_t^{*\alpha-1}$. Thus $K_t^* = K_E$ for $t > 0$. Given this constant capital stock for all firms, we must have $\delta K_t^* = \pi I_t^*$ for $t > 0$. Let each firm's optimal investment level satisfy $I_t^j = \delta K_t^j/\pi$. Then, when assumption (28) holds, the investment and credit constraints, $I_t^j = \delta K_t^j/\pi \leq \xi K_t^j = V(\xi K_t^j)$, are satisfied. We conclude that, under assumption (28), the solutions $Q_t = 1$, $K_t^* = K_E$, and $I_t^*/K_t^* = \delta/\pi$ give the bubbleless equilibrium, which also achieves the efficient allocation.

(ii) Suppose that (29) holds. Conjecture that $Q_t > 1$ in some neighborhood of the bubbleless steady state in which $B_t = 0$ for all t. We can then apply Proposition 2 and derive the steady-state equations for (21) and (27) as

$$\dot{Q} = 0 = \left(r + \delta\right) Q - R - \pi \xi Q (Q-1), \tag{A.9}$$

$$\dot{K} = 0 = -\delta K + \pi(\xi Q K), \tag{A.10}$$

where $R = \alpha K^{\alpha-1}$. From these equations, we obtain the steady-state solutions Q^* and K^* in (30) and (31), respectively. Assumption (29) implies that $Q^* > 1$. By continuity, $Q_t > 1$ in some neighborhood of $\left(Q^*, K^*\right)$. This verifies our conjecture. Q.E.D.

Proof of Proposition 4: In the bubbly steady state, (20) and (27) imply that

$$0 = rB - B\pi(Q-1), \text{ and} \tag{A.11}$$

$$0 = -\delta K + [\xi QK + B]\pi, \tag{A.12}$$

where $R = \alpha K^{\alpha-1}$. Solving equations (A.9), (A.11), and (A.12) yields equations (33), (34), and (35). By (33), $B > 0$ if and only if (36) holds. From (30) and (34), we deduce that $Q_b < Q^*$. Using condition (36), it is straightforward to check that $K_{GR} > K_E > K_b > K^*$. By the resource constraint, steady-state consumption satisfies $C = Y - \pi I = K^\alpha - \delta K$. Substituting the expressions for K_E, K_b, and K^* in Propositions 3 and 4, we can show that $C_E > C_b > C^*$. From (33), it is also straightforward to verify that the bubble-asset ratio B/K_b decreases with ξ. Q.E.D.

Proof of Proposition 5: First, we consider the log-linearized system around the bubbly steady state (B, Q_b, K_b). We use $\hat{X}_t$ to denote the percentage deviation from the steady state value for any variable X_t, i.e., $\hat{X}_t = \ln X_t - \ln X$. We can show that the log-linearized system is given by

$$\begin{bmatrix} d\hat{B}_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = A \begin{bmatrix} \hat{B}_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -(r+\pi) & 0 \\ 0 & \delta + r - \xi(2r+\pi) & [(1-\xi)r+\delta](1-\alpha) \\ \pi B/K_b & \xi(r+\pi) & -\pi B/K_b \end{bmatrix}. \tag{A.13}$$

We denote this matrix by

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix},$$

where we deduce from (A.13) that $a < 0$, $c > 0$, $d > 0$, $e > 0$, and $f < 0$. Since $\xi < \frac{\delta}{r+\pi}$, we have $b = (1-\xi)r + \delta - \xi(r+\pi) > 0$. The characteristic equation for the matrix A is

$$F(x) \equiv x^3 - (b+f)x^2 + (bf - ce)x - acd = 0. \tag{A.14}$$

We observe that $F(0) = -acd > 0$ and $F(-\infty) = -\infty$. Thus, there exists a negative root to the above equation, denoted by $\lambda_1 < 0$. Let the other two roots be λ_2 and λ_3. We rewrite $F(x)$ as

$$\begin{aligned} F(x) &= (x-\lambda_1)(x-\lambda_2)(x-\lambda_3) \\ &= x^3 - (\lambda_1+\lambda_2+\lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \end{aligned} \tag{A.15}$$

Matching terms in equations (A.14) and (A.15) yields $\lambda_1\lambda_2\lambda_3 = acd < 0$ and

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0. \tag{A.16}$$

We consider two cases. (i) If λ_2 and λ_3 are two real roots, then it follows from $\lambda_1 < 0$ that λ_2 and λ_3 must have the same sign. Suppose $\lambda_2 < 0$ and $\lambda_3 < 0$. We then have $\lambda_1\lambda_2 > 0$ and $\lambda_1\lambda_3 > 0$. This implies that $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$, which contradicts equation (A.16). Thus we must have $\lambda_2 > 0$ and $\lambda_3 > 0$.

(ii) If either λ_2 or λ_3 is complex, then the other must also be complex. Let

$$\lambda_2 = a_1 + a_2 i \text{ and } \lambda_3 = a_1 - a_2 i,$$

where a_1 and a_2 are some real numbers and $i = \sqrt{-1}$. We can show that

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2a_1\lambda_1 + a_1^2 + a_2^2.$$

Since $\lambda_1 < 0$, the above equation and equation (A.16) imply that $a_1 > 0$.

From the above analysis, we conclude that the matrix A has one negative eigenvalue and the other two eigenvalues are either positive real numbers or complex numbers with a positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state $(0, Q^*, K^*)$. We linearize B_t around zero and log-linearize Q_t and K_t and obtain the following linearized system:

$$\begin{bmatrix} dB_t/dt \\ d\hat{Q}_t/dt \\ d\hat{K}_t/dt \end{bmatrix} = \begin{bmatrix} r - \pi(Q^* - 1) & 0 & 0 \\ 0 & a & b \\ \frac{\pi}{K^*} & c & d \end{bmatrix} \begin{bmatrix} B_t \\ \hat{Q}_t \\ \hat{K}_t \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{R^*}{Q^*} - \xi\pi Q^*, \ b = \frac{R^*}{Q^*}(1-\alpha) > 0, \\ c &= \pi\xi Q^* > 0, \ d = 0. \end{aligned}$$

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix J. One eigenvalue, denoted by λ_1, is equal to $r - \pi(Q^* - 1) < 0$ and the other two, denoted by λ_2 and λ_3, satisfy

$$\lambda_2\lambda_3 = ad - bc = 0 - bc < 0. \tag{A.17}$$

It follows from (A.17) that λ_2 and λ_3 must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

Proof of Proposition 6: The discrete-time Bellman equation is given by

$$\begin{aligned} V_t\left(K_t^j\right) &= \max \ (1-\theta\Delta)(1-\pi\Delta)\left[D_{0t}^j\Delta + e^{-r\Delta}V_{t+\Delta}\left(K_{t+\Delta}^j\right)\right] \\ &\quad + (1-\theta\Delta)\pi\Delta\left[D_{1t}^j + e^{-r\Delta}V_{t+\Delta}\left(K_{1t+\Delta}^j\right)\right] + \theta\Delta V_t^*\left(K_t^j\right). \end{aligned}$$

As in the proof of Proposition 1, taking the continuous-time limit as $\Delta \to 0$ and substituting the flow-of-funds constraints yield the Bellman equation in Section 5.3. Substituting the conjectured value function $V_t(K_t^j) = Q_t K_t^j + B_t$ into this equation yields

$$\begin{aligned} r\left(Q_t K_t^j + B_t\right) &= \max_{I_t^j,\, \dot{K}_t^j} \; R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t \\ &\quad + \pi\left(Q_t - 1\right) I_t^j + \theta\left[Q_t^* K_t^j - \left(Q_t K_t^j + B_t\right)\right] \end{aligned}$$

subject to

$$I_t^j \leq \xi Q_t K_t^j + B_t.$$

When $Q_t > 1$, optimal investment is given by $I_t^j = \xi Q_t K_t^j + B_t$. Substituting this rule back into the preceding Bellman equation and matching coefficients yield (37) and (38). Equation (27) follows from aggregation and the market-clearing condition. Q.E.D.

Proof of Proposition 7: By (37), we can show that

$$Q_s = \frac{r+\theta}{\pi} + 1. \tag{A.18}$$

Since $Q_s > 1$, we can apply Proposition 6 in some neighborhood of Q_s. Equation (38) implies that

$$0 = (r + \delta + \theta) Q_s - \theta G\left(K\right) - R - \pi (Q_s - 1) \xi Q_s, \tag{A.19}$$

where $R = \alpha K^{\alpha - 1}$. The solution to this equation gives K_s. Once we have obtained K_s and Q_s, we use equation (27) to determine B_s.

The difficult part is to solve for K_s since $G\left(K\right)$ is not an explicit function. To show the existence of K_s, we define θ^* as

$$\frac{r+\theta^*}{\pi} + 1 = \frac{\delta}{\pi\xi} = Q^*.$$

That is, θ^* is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

Let $Q\left(\theta\right)$ be the expression on the right-hand side of equation (A.18). We then use this equation to rewrite equation (A.19) as

$$\alpha K^{\alpha-1} - (r + \delta + \theta) Q(\theta) + \theta G(K) + (r + \theta) \xi Q(\theta) = 0.$$

Define the function $F\left(K; \theta\right)$ as the expression on the left-hand side of the equation above. Notice that $Q(\theta^*) = Q^* = G(K^*)$ by definition and $Q(0) = Q_b$ where Q_b is given in (34). Condition (36) ensures the existence of the bubbly steady-state value Q_b and the bubbleless steady-state values Q^* and K^*.

Define

$$K_{\max} = \max_{0\leq\theta\leq\theta^*} \left[\frac{(r+\delta+\theta-(r+\theta)\xi)Q(\theta)-\theta Q^*}{\alpha}\right]^{\frac{1}{\alpha-1}}.$$

By (35), we can show that

$$K_b = \left[\frac{(r+\delta-r\xi)Q(0)}{\alpha}\right]^{\frac{1}{\alpha-1}}.$$

Thus we have $K_{\max} \geq K_b$ and hence $K_{\max} > K^*$. We want to prove that

$$F\left(K^*;\theta\right) > 0,\ F\left(K_{\max};\theta\right) < 0,$$

for $\theta \in (0,\theta^*)$. If this is true, then it follows from the intermediate value theorem that there exists a solution K_s to $F(K;\theta) = 0$ such that $K_s \in (K^*, K_{\max})$.

First, notice that

$$F\left(K^*;0\right) = \alpha K^{*\alpha-1} - r(1-\xi)Q_b - \delta Q_b > \alpha K_b^{\alpha-1} - r(1-\xi)Q_b - \delta Q_b = 0,$$

and $F(K^*;\theta^*) = 0$. We can verify that $F(K;\theta)$ is concave in θ for any fixed K. Thus, for all $0 < \theta < \theta^*$,

$$\begin{aligned} F\left(K^*;\theta\right) &= F\left(K^*, (1-\frac{\theta}{\theta^*})0 + \frac{\theta}{\theta^*}\theta^*\right) \\ &> (1-\frac{\theta}{\theta^*})F(K^*,0) + \frac{\theta}{\theta^*}F(K^*,\theta^*) > 0. \end{aligned}$$

Next we can derive

$$\begin{aligned} F\left(K_{\max};\theta\right) &= \alpha K_{\max}^{\alpha-1} - (r+\delta+\theta)Q(\theta) + \theta G(K_{\max}) + (r+\theta)\xi Q(\theta) \\ &< \alpha K_{\max}^{\alpha-1} - (r+\delta+\theta)Q(\theta) + \theta G(K^*) + (r+\theta)\xi Q(\theta) < 0, \end{aligned}$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping by inspecting the phase diagram for (K_t, Q_t) so that $G(K_{\max}) < G(K^*)$, and the second inequality follows from the definition of $K_{\max}$ and the fact that $G(K^*) = Q^*$.

Finally, note that $Q(\theta) < Q^*$ for $0 < \theta < \theta^*$. We use equation (A.12) and $K_s > K^*$ to deduce that

$$\frac{B_s}{K_s} = \frac{\delta}{\pi} - \xi Q(\theta) > \frac{\delta}{\pi} - \xi Q^* = 0.$$

This completes the proof of the existence of a stationary equilibrium with stochastic bubbles (B_s, Q_s, K_s).

When $\theta = 0$, the bubble never bursts and hence $K_s = K_b$. When θ is sufficiently small, K_s is close to K_b by continuity. Since K_b is smaller than the golden rule capital stock K_{GR}, $K_s < K_{GR}$ when θ is sufficiently small. Since $K^\alpha - \delta K$ is increasing for all $K < K_{GR}$, we deduce that $K_s^\alpha - \delta K_s > K^{*\alpha} - \delta K^*$. This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

B Proofs of Results in Section 6

B. 1 Endogenous Credit Constraints

Proof of Proposition 8: As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$\begin{aligned} rV_t\left(K_t^j, M_t^j\right) &= \max_{\dot{M}_t^j, \dot{K}_t^j, M_{1t}^j, K_{1t}^j, I_t^j, L_t^j} \quad D_{0t}^j + \dot{V}_t\left(K_t^j, M_t^j\right) \\ &\quad +\pi\left[D_{1t}^j + V_t\left(K_{1t}^j, M_{1t}^j\right) - V_t\left(K_t^j, M_t^j\right)\right] \end{aligned} \tag{B.1}$$

subject to (40),

$$D_{0t}^j = R_t K_t^j - P_t \dot{M}_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right), \tag{B.2}$$

$$D_{1t}^j = P_t\left(M_t^j - M_{1t}^j\right) + Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j, \tag{B.3}$$

$$I_t^j \leq P_t\left(M_t^j - M_{1t}^j\right) + L_t^j. \tag{B.4}$$

When an investment opportunity arrives with the Poisson rate π, firm j's asset holdings jump to $M_{1t} \geq 0$ and its value function changes from $V_t\left(K_t^j, M_t^j\right)$ to $V_t\left(K_{1t}^j, M_{1t}^j\right)$. This explains the Bellman equation in (B.1). The interpretations of constraints are similar to those in Sections 3. In particular, equation (B.4) is the financing constraint. Firm j can sell assets $(M_t^j - M_{1t}^j)$ and borrow L_t^j to finance investment. According to the collateral constraint (40), firm j uses capital as collateral only.

Substituting the conjectured value function in (44) and the flow-of-funds constraints (B.2) and (B.3) into the dynamic programming problem (B.1) yields

$$\begin{aligned} & r\left(Q_t K_t^j + P_t M_t^j\right) \\ = & \max_{\dot{M}_t^j, \dot{K}_t^j, M_{1t}^j, K_{1t}^j, I_t^j, L_t^j} \quad R_t K_t - P_t \dot{M}_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) \\ & +Q_t \dot{K}_t^j + K_t^j \dot{Q}_t + \dot{P}_t M_t^j + P_t \dot{M}_t^j \\ & +\pi\left[P_t\left(M_t^j - M_{1t}^j\right) + Q_t I_t^j - I_t^j + Q_t K_t - Q_t K_{1t}\right] \\ & +\pi\left[Q_t K_{1t} + P_t M_{1t}^j - \left(Q_t K_t + P_t M_t^j\right)\right] \end{aligned}$$

subject to (B.4) and

$$L_t^j \leq \xi Q_t K_t^j.$$

Thus $P_t \dot{M}_t^j$ and $Q_t \dot{K}_t^j$ cancel themselves out in the Bellman equation so that firm j is indifferent between buying and selling any amount of the intrinsically useless asset and indifferent between buying and selling any amount of capital, when no investment opportunity arrives. Moreover, $Q_t K_{1t}^j$ also cancels itself out and hence K_{1t}^j is indeterminate.

When an investment opportunity arrives with Poisson rate π, under the assumption $Q_t > 1$, it is profitable to invest as much as possible. In this case firm j sells all its asset holdings to non-investing firms, i.e., $M_{1t}^j = 0$, and borrows as much as possible so that $L_t^j = \xi Q_t K_t^j$. The optimal investment level is

$$I_t^j = \xi Q_t K_t^j + P_t M_t^j.$$

Substituting this solution back into the preceding Bellman equation and matching coefficients, we obtain equations (21) and (46).

It follows from (46) that $rP_t > \dot{P}_t$. Thus households will not hold the bubble asset and their short-sale constraints bind. This means that the market-clearing condition for the asset is given by $\int M_t^j dj = 1$. By a law of large numbers, aggregate capital satisfies

$$\dot{K}_t = \delta K_t + \pi \left(\xi Q_t K_t + P_t \int M_t^j dj \right).$$

We then obtain (45). Since the equilibrium system is the same as that in Proposition 2 once we set $P_t = B_t$, we can use Proposition 4 to study the steady state with a bubble $P > 0$. Thus the existence condition is (36). Note that $\xi = 0$ also permits the existence of a bubble. Q.E.D.

Proof of Proposition 9: The proof follows from that of Proposition 11 in Appendix B.4 by setting $X_t = 0$, $\zeta = 1$, and $g = 0$. We omit the details. Q.E.D.

B. 2 Liquidity Mismatch

We now relax the liquidity mismatch assumption and suppose that at most a fraction λ of the proceeds from the sale of old capital is available to finance investment. Then the financing constraint in continuous time becomes

$$I_t^j \leq L_t^j + Q_t \left(K_t^j - K_{1t}^j \right), \tag{B.5}$$

and K_{1t}^j satisfies

$$K_{1t}^j \geq (1 - \lambda) K_t^j. \tag{B.6}$$

Firm j's decision problem is given by the Bellman equation (14) subject to (15), (17), (B.5), and (B.6). We conjecture that the value function takes the form $V_t \left(K_t^j \right) = Q_t K_t^j + B_t$. Substitute this conjecture into the Bellman equation. When an investment opportunity arrives, under the assumption $Q_t > 1$, firm j wants to invest as much as possible so that the financing constraint and the credit constraint bind. Moreover, the firm chooses $K_{1t}^j = (1 - \lambda) K_t^j$ and optimal investment is given by

$$I_t^j = (\xi + \lambda) Q_t K_t^j + B_t.$$

Substituting these decision rules into the Bellman equation and matching coefficients, we deduce that B_t still satisfies equation (20), and Q_t satisfies

$$\dot{Q}_t = (r+\delta)\,Q_t - R_t - \pi\,(\xi+\lambda)\,Q_t(Q_t-1). \tag{B.7}$$

Aggregate investment is given by

$$\pi I_t = \pi\left[(\xi+\lambda)\,Q_t K_t + B_t\right],$$

and aggregate capital satisfies

$$\dot{K}_t = -\delta K_t + \pi\left[(\xi+\lambda)\,Q_t K_t + B_t\right]. \tag{B.8}$$

The equilibrium system for (Q_t, K_t, B_t) is given by (B.7), (B.8) and (20). Thus the analysis in Sections 3 and 4 still applies except that ξ is replaced by $\xi+\lambda$. In particular, by Proposition 4, the bubbly and bubbleless steady states coexist if and only if

$$0 < \xi+\lambda < \frac{\delta}{r+\pi}.$$

This implies that as long as λ is sufficiently small, a bubbly equilibrium exists.

B. 3 Equity Issues

Proof of Proposition 10: As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$\begin{aligned} rV_t\left(K_t^j\right) &= \max_{\dot{K}_t^j, K_{1t}^j, I_t^j, L_t^j, S_{0t}^j, S_{1t}^j} \left(D_{0t}^j - S_{0t}^j\right) + \dot{V}_t\left(K_t^j\right) + \pi\left(D_{1t}^j - S_{1t}^j\right) \\ &\quad + \pi\left[V_t\left(K_{1t}^j\right) - V_t\left(K_t^j\right)\right] \end{aligned}$$

subject to (17),

$$D_{0t}^j = R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + S_{0t}^j - \frac{\varphi}{2}\frac{(S_{0t}^j)^2}{K_t^j}, \tag{B.9}$$

$$D_{1t}^j + I_t^j + L_t^j = Q_t I_t^j + L_t^j + Q_t K_t^j - Q_t K_{1t}^j + S_{1t}^j - \frac{\varphi}{2}\frac{(S_{1t}^j)^2}{K_t^j}, \tag{B.10}$$

$$I_t^j \le L_t^j + S_{1t}^j. \tag{B.11}$$

Substituting (B.10) into the Bellman equation yields

$$\begin{aligned} rV_t\left(K_t^j\right) &= \max_{\dot{K}_t^j, K_{1t}^j, I_t^j, L_t^j, S_{0t}^j, S_{1t}^j} \left(D_{0t}^j - S_{0t}^j\right) + \dot{V}_t\left(K_t^j\right) + \pi\left[(Q_t-1)\,I_t^j - \frac{\varphi}{2}\frac{(S_{1t}^j)^2}{K_t^j}\right] \\ &\quad + \pi\left[Q_t K_t^j - Q_t K_{1t}^j + V_t\left(K_{1t}^j\right) - V_t\left(K_t^j\right)\right]. \end{aligned}$$

Conjecture that V_t is given by (18). Using (B.9), we can show that $S_{0t}^j = 0$.

When an investment opportunity arrives, under the assumption $Q_t > 1$, firm j invests as much as possible so that the credit constraint (17) and the financing constraint (B.11) bind. Using the first-order condition for S_{1t}^j, we derive

$$S_{1t}^j = \frac{1}{\varphi}\left(Q_t - 1\right) K_t^j, \quad I_t^j = \xi Q_t K_t^j + B_t + \frac{1}{\varphi}\left(Q_t - 1\right) K_t^j.$$

Substituting the conjectured value function $V_t\left(K_t^j\right) = Q_t K_t^j + B_t$ and the above decision rules into the Bellman equation and matching coefficients, we obtain (20) and

$$\dot{Q}_t = \left(r + \delta\right) Q_t - R_t - \pi\left[\xi Q_t + \frac{1}{2\varphi}(Q_t - 1)\right](Q_t - 1). \tag{B.12}$$

Aggregate capital satisfies

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + B_t + \frac{1}{\varphi}(Q_t - 1)K_t). \tag{B.13}$$

As in the proof of Proposition 1, we can show that $R_t = \alpha K_t^{\alpha - 1}$.

In the bubbly steady state, we use equation (20) to derive

$$Q_b = 1 + \frac{r}{\pi} > 1.$$

Thus $Q_t > 1$ in a neighborhood of the bubbly steady state. Using (B.13), we derive

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \xi Q_b - \frac{1}{\varphi}(Q_b - 1).$$

Given the condition in the proposition we have $B > 0$. Finally, we use (B.12) to derive

$$\begin{aligned} R_b &= \alpha\left(K_b\right)^{\alpha-1} = \left(r+\delta\right)Q_b - \pi\left[\xi Q_b + \frac{1}{2\varphi}(Q_b - 1)\right](Q_b - 1) \\ &= [(1-\xi)r + \delta](\frac{r}{\pi} + 1) - \frac{1}{2\varphi}\frac{r^2}{\pi}. \end{aligned}$$

Given the condition in the proposition we can check that $R_b > 0$. From the proof above we can see that the condition is also necessary. Q.E.D.

B. 4 Additional Asset with Exogenous Rents

Proof of Proposition 11: With technical progress, firm j's static labor choice problem is

$$R_t K_t^j = \max_{N_t^j} \; (K_t^j)^\alpha (A_t N_t^j)^{1-\alpha} - w_t N_t^j, \tag{B.14}$$

where w_t is the wage rate and R_t is given by

$$R_t = \alpha\left(\frac{w_t / A_t}{1 - \alpha}\right)^{\frac{\alpha - 1}{\alpha}}. \tag{B.15}$$

Firm j's dynamic programming problem in continuous time is given by (B.1) subject to (B.3), (B.4), (47), (52), and

$$D_{0t}^j = R_t K_t^j + X_t M_t^j - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right).$$

Since one unit of the asset pays X_t rents, $X_t M_t^j$ enters the above flow-of-funds constraint.

Conjecture that the value function takes the following form:

$$V_t \left(K_t^j, M_t^j \right) = Q_t K_t^j + P_t M_t^j + B_t.$$

Substituting this conjectured function and the flow-of-funds constraints into the dynamic programming problem (B.1) yields

$$\begin{aligned} & r\left(Q_t K_t^j + P_t M_t^j + B_t\right) \\ = & \max_{\dot{M}_t^j, \dot{K}_t^j, M_{1t}^j, K_{1t}^j, I_t^j, L_t^j} R_t K_t + X_t M_t^j - P_t \dot{M}_t^j - Q_t \left(\dot{K}_t^j + \delta K_t^j \right) \\ & + Q_t \dot{K}_t^j + K_t^j \dot{Q}_t + \dot{P}_t M_t^j + P_t \dot{M}_t^j + \dot{B}_t \\ & + \pi \left[P_t \left(M_t^j - M_{1t}^j \right) + Q_t I_t^j - I_t^j + P_t \left(M_{1t}^j - M_t^j \right) \right] \end{aligned}$$

subject to (B.4), (52), and

$$L_t^j \leq \xi Q_t K_t^j + B_t. \tag{B.16}$$

Thus $P_t \dot{M}_t^j$ and $Q_t \dot{K}_t^j$ cancel themselves out in the Bellman equation so that firm j is indifferent between buying and selling any amount of the intrinsically useless asset and indifferent between buying and selling any amount of capital, when no investment opportunity arrives. Moreover, $Q_t K_{1t}^j$ cancels itself out and hence K_{1t}^j is indeterminate.

When an investment opportunity arrives with Poisson rate π, under the assumption $Q_t > 1$, the firm will invest as much as possible. It follows from (B.4), (52), and (B.16) that $M_{1t}^j = (1 - \zeta) M_t^j$ and optimal investment is given by

$$I_t^j = \xi Q_t K_t^j + \zeta P_t M_t^j + B_t.$$

Substituting this solution back into the preceding Bellman equation and matching coefficients, we obtain equations

$$\dot{P}_t = r P_t - X_t - \pi (Q_t - 1) \zeta P_t, \tag{B.17}$$

$$\dot{B}_t = r B_t - \pi (Q_t - 1) B_t, \tag{B.18}$$

$$\dot{Q}_t = (r + \delta) Q_t - R_t - \pi (Q_t - 1) Q_t \xi. \tag{B.19}$$

It follows from (B.17) that $r P_t > \dot{P}_t + X_t$. Thus households will not hold the asset and their short-sale constraints bind. This means that the market-clearing condition for the asset is given by

$\int M_t^j dj = 1$. By a law of large numbers, aggregate capital satisfies

$$\dot{K}_t = \delta K_t + \pi \left(\xi Q_t K_t + P_t \zeta \int M_t^j dj + B_t \right).$$

We then obtain

$$\dot{K}_t = -\delta K_t + \pi (Q_t \xi K_t + \zeta P_t + B_t).$$

As in the proof of Proposition 2, the labor-market clearing condition gives $R_t = \alpha \left(K_t / A_t \right)^{\alpha - 1}$ and $Y_t = K_t^{\alpha} A_t^{1-\alpha}$.

Then aggregate capital K_t, the asset price P_t, and the stock price bubble B_t will all grow at the rate g in the steady state. However, the capital price Q_t and the rental rate R_t will not grow. The detrended equilibrium system becomes

$$\begin{aligned}
\dot{k}_t &= -(\delta + g) k_t + \pi (Q_t \xi k_t + \zeta p_t + b_t), \\
\dot{p}_t &= (r - g) p_t - x - \pi (Q_t - 1) \zeta p_t, \\
\dot{b}_t &= (r - g) b_t - \pi (Q_t - 1) b_t, \\
\dot{Q}_t &= (r + \delta) Q_t - \alpha k_t^{\alpha - 1} - \pi (Q_t - 1) Q_t \xi,
\end{aligned}$$

where $k_t = K_t / A_t$, $p_t = P_t / A_t$, $b_t = B_t / A_t$, and $x = X_t / A_t$. In the bubbly steady state these variables and Q_t are all constant over time. Suppressing the time subscript in the steady state gives

$$0 = -(\delta + g) k + \pi (Q \xi k + \zeta p + b), \tag{B.20}$$

$$0 = (r - g) p - x - \pi (Q - 1) \zeta p, \tag{B.21}$$

$$0 = (r - g) b - \pi (Q - 1) b, \tag{B.22}$$

$$0 = (r + \delta) Q - \alpha k^{\alpha - 1} - \pi (Q - 1) Q \xi. \tag{B.23}$$

In the bubbly steady state $b > 0$, we can use (B.22) to compute

$$Q_b = \frac{r - g}{\pi} + 1.$$

Assume that $r > g$ so that $Q_b > 1$ and hence $Q_t > 1$ in the neighborhood of the bubbly steady state. Using (B.21) and (B.23), we can compute

$$\begin{aligned}
p &= \frac{x}{(r - g)(1 - \zeta)}, \\
R &= \alpha k^{\alpha - 1} = [(r + \delta) - (r - g) \xi] \left(\frac{r - g}{\pi} + 1 \right).
\end{aligned}$$

Thus the bubbly steady-state detrended capital stock is given by

$$k_b = \left\{ \frac{1}{\alpha} \left[(r + \delta) - (r - g) \xi \right] \left(\frac{r - g}{\pi} + 1 \right) \right\}^{\frac{1}{\alpha - 1}}.$$

After solving for Q_b, k_b, and p, we use equation (B.20) to solve for b described in the proposition. We need $b > 0$. We then have the second inequality in condition (55). For $x > 0$ in (55), we need

$$\frac{\delta + g}{\pi} - \left(\frac{r-g}{\pi} + 1\right)\xi > 0.$$

We then obtain the condition in (54). This condition also implies that $(r+\delta) - (r-g)\xi > 0$ so that $k_b > 0$. The conditions in the propositions are also necessary. Q.E.D.

B. 5 Intertemporal Debt

Proof of Proposition 12: We first derive the discrete-time solution and then take the continuous-time limit. Conjecture that the value function takes the form

$$V_t\left(K_t^j, L_t^j\right) = a_t K_t^j - a_t^L L_t^j + b_t.$$

Substituting this conjecture and the flow-of-funds constraints (56) and (57) into the Bellman equation yields

$$\begin{aligned}
a_t K_t^j - a_t^L L_t^j + b_t \quad = \quad & \max_{I_t^j, K_{t+\Delta}^j, K_{1t+\Delta}^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} \quad R_t K_t^j \Delta - L_t^j + Q_t\left(1-\delta\Delta\right) K_t^j + e^{-r\Delta} b_{t+\Delta} \\
& + \left(1-\pi\Delta\right)\left[e^{-r_{ft}\Delta} L_{t+\Delta}^j - Q_t K_{t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{t+\Delta}^j - e^{-r\Delta} a_{t+\Delta}^L L_{t+\Delta}^j\right] \\
& + \pi\Delta\left[e^{-r_{ft}\Delta} L_{1t+\Delta}^j - Q_t K_{1t+\Delta}^j + e^{-r\Delta} a_{t+\Delta} K_{1t+\Delta}^j - e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j\right] \\
& + \pi\Delta\left(Q_t - 1\right) I_t^j
\end{aligned}$$

subject to

$$I_t^j \le R_t K_t^j \Delta + e^{-r_{ft}\Delta} L_{1t+\Delta}^j - L_t^j, \tag{B.24}$$

$$a_{t+\Delta}^L L_{1t+\Delta}^j \le b_{t+\Delta} + a_{t+\Delta}\xi\left(1-\delta\Delta\right) K_t^j, \tag{B.25}$$

where (B.25) is the credit constraint derived from (59) using the conjectured value function.

By the linear property of the Bellman equation above, the first-order conditions for $L_{t+\Delta}^j$ and $K_{t+\Delta}^j$ yield

$$e^{-r_{ft}\Delta} = e^{-r\Delta} a_{t+\Delta}^L, \quad Q_t = e^{-r\Delta} a_{t+\Delta}. \tag{B.26}$$

and hence $L_{t+\Delta}^j$, $K_{t+\Delta}^j$, and $K_{1t+\Delta}^j$ are indeterminate. This implies that firm j is indifferent between saving and borrowing when no investment opportunity arrives, and is also indifferent between buying and selling capital. When $Q_t > 1$, it is profitable for firm j to invest as much as possible so that the financing constraint (B.24) and the credit constraint (B.25) bind. Thus optimal investment is given by

$$I_t^j = R_t K_t^j \Delta + B_t + Q_t \xi\left(1-\delta\Delta\right) K_t^j - L_t^j,$$

where we have used (B.26) and defined

$$B_t \equiv e^{-r\Delta} b_{t+\Delta}. \tag{B.27}$$

Substituting the investment rule back into the Bellman equation and matching coefficients, we can derive

$$a_t = R_t\Delta + Q_t(1-\delta\Delta) + \pi\Delta(Q_t-1)(R_t\Delta + Q_t\xi(1-\delta\Delta)),$$

$$a_t^L = 1 + \pi\Delta(Q_t-1),$$

$$b_t = e^{-r\Delta} b_{t+\Delta} + \pi\Delta(Q_t-1)B_t.$$

Using (B.26) and (B.27) and the preceding three equations, we derive

$$Q_t = e^{-r\Delta}\left[R_{t+\Delta}\Delta + Q_{t+\Delta}(1-\delta\Delta) + \pi\Delta(Q_{t+\Delta}-1)(R_{t+\Delta}\Delta + Q_{t+\Delta}\xi(1-\delta\Delta))\right], \tag{B.28}$$

$$e^{-r_{ft}\Delta} = e^{-r\Delta}\left[1 + \pi\Delta(Q_{t+\Delta}-1)\right], \tag{B.29}$$

$$B_t = e^{-r\Delta}\left[1 + \pi\Delta(Q_{t+\Delta}-1)\right] B_{t+\Delta}. \tag{B.30}$$

Taking the continuous-time limit as $\Delta \to 0$ yields the equations in Proposition 12.

As in the proof of Proposition 1, we derive the continuous-time limit of the dynamic programming problem as

$$\begin{aligned} rV_t\left(K_t^j, L_t^j\right) \quad &= \max_{D_{0t}^j, D_{1t}^j, I_t^j, L_{1t}^j} \quad D_{0t}^j + \dot{V}_t\left(K_t^j, L_t^j\right) \\ &\quad + \pi\left[D_{1t}^j + V_t\left(K_{1t}^j, L_{1t}^j\right) - V_t\left(K_t^j, L_t^j\right)\right] \end{aligned} \tag{B.31}$$

subject to

$$\dot{L}_t^j = r_{ft}L_t^j + D_{0t}^j - R_tK_t^j + Q_t\left(\dot{K}_t^j + \delta K_t^j\right), \tag{B.32}$$

$$D_{1t}^j = Q_tI_t^j + L_{1t}^j - L_t^j - I_t^j + Q_tK_t^j - Q_tK_{1t}^j, \tag{B.33}$$

$$I_t^j \le L_{1t}^j - L_t^j, \tag{B.34}$$

$$V_t\left(K_{1t}^j, L_{1t}^j\right) \ge V_t\left(K_{1t}^j, 0\right) - V_t(\xi K_t^j, 0). \tag{B.35}$$

Conjecture that the value function takes the form

$$V_t\left(K_t^j, L_t^j\right) = Q_tK_t^j - L_t^j + B_t. \tag{B.36}$$

Substituting this conjecture into the Bellman equation yields

$$\begin{aligned} r\left(Q_tK_t^j - L_t^j + B_t\right) \quad &= \quad \max \ \dot{L}_t^j - r_{ft}L_t^j + R_tK_t^j - Q_t\dot{K}_t^j - Q_t\delta K_t^j + Q_t\dot{K}_t^j + \dot{Q}_tK_t^j - \dot{L}_t^j + \dot{B}_t \\ &\quad + \pi\left[(Q_t-1)I_t^j + L_{1t}^j - L_t^j + Q_tK_t^j - Q_tK_{1t}^j\right] \\ &\quad + \pi\left[Q_tK_{1t}^j - L_{1t}^j + B_t - \left(Q_tK_t^j - L_t^j + B_t\right)\right]. \end{aligned}$$

Thus $\dot{K}_t^j$ and $\dot{L}_t^j$ cancel themselves out so that firm j is indifferent between saving and borrowing and between buying and selling capital, when no investment opportunity arrives. Moreover, $Q_t K_{1t}^j$ also cancels itself out so that firm j is indifferent between buying and selling capital when an investment opportunity arrives. Simplifying yields

$$\begin{aligned} r\left(Q_t K_t^j - L_t^j + B_t\right) &= \max \; -r_{ft} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j + \dot{B}_t \\ &\quad + \pi \left(Q_t - 1\right) I_t^j. \end{aligned} \tag{B.37}$$

Given the conjectured value function, the credit constraint (B.35) becomes

$$L_{1t}^j \leq Q_t \xi K_t^j + B_t.$$

Using the financing constraint (B.34), we obtain

$$I_t^j \leq L_{1t}^j - L_t^j \leq \xi Q_t K_t^j + B_t - L_t^j.$$

When an investment opportunity arrives, under the assumption $Q_t > 1$, it is profitable for firm j to invest as much as possible so that both the financing and credit constraints bind. We then have

$$I_t^j = \xi Q_t K_t^j + B_t - L_t^j.$$

Substituting this investment rule back into the Bellman equation (B.37) and matching coefficients, we derive the equations for Q_t, B_t, and r_{ft} given in Proposition 12.

We now compute

$$I_t = \int I_t^j dj = \xi Q_t K_t + B_t - \int L_t^j dj.$$

Since $r_{ft} < r$, households short-sale constraints bind so that $L_t^h = 0$ and the bond market-clearing condition becomes $\int L_t^j dj = 0$. Thus

$$I_t = \xi Q_t K_t + B_t. \tag{B.38}$$

Substituting (B.38) into the law of motion for aggregate capital yields the equation for K_t given in Proposition 12. Finally, we can use the same procedure as in the proof of Proposition 2 to derive $R_t = \alpha K_t^{\alpha-1}$. Q.E.D.

Proof of Proposition 13: The proof follows from those of Propositions 3 and 4. Since $r_{ft} = \dot{B}_t / B_t$ in the bubbly equilibrium, $r_f = 0$ in the bubbly steady state as $\dot{B}_t = 0$.

In the bubbleless steady state in which $B = 0$, we have $Q^* = \delta / (\pi \xi)$ and

$$r_f^* = r - \pi \left(Q^* - 1\right) = r + \pi - \delta / \xi < 0,$$

where the inequality follows from condition (36). Q.E.D.

C Self-Enforcing Debt Contracts

Consider a type of credit constraint which is popular in the self-enforcing debt literature (see, e.g., Bulow and Rogoff (1989), Kehoe and Levine (1993), Alvarez and Jermann (2000), Albuquerque and Hopenhayn (2004), Kocherlakota (2008), and Hellwig and Lorenzoni (2009)).[22] There is no collateral. Suppose that the only penalty on the firm for defaulting is that it will be excluded from the financial market forever. Since internal funds $R_t K_t^j$ come as flows, the firm has no funds with which to make a lumpy investment I_t^j. Denote by $V_t^a(K_t^j)$ the autarky value of firm j that cannot access the financial market. $V_t^a(K_t^j)$ satisfies the Bellman equation

$$rV_t^a\left(K_t^j\right) = \max_{\dot{K}_t^j} \; R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \dot{V}_t^a\left(K_t^j\right).$$

This is a standard dynamic programming problem and no bubble can exist in V_t^a by the usual transversality condition. Conjecture that $V_t^a\left(K_t^a\right) = Q_t^a K_t^j$. Substituting this conjecture into the Bellman equation above yields

$$rQ_t^a K_t^j = \max_{\dot{K}_t^j} \; R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \dot{Q}_t^a K_t^j + Q_t^a \dot{K}_t^j.$$

Optimizing with respect to $\dot{K}_t^j$, we deduce $Q_t = Q_t^a$. Matching the coefficients of K_t^j gives

$$\dot{Q}_t = \left(r+\delta\right) Q_t - R_t. \tag{C.1}$$

We now turn to firm j's decision problem before defaulting. Firm value $V_t(K_t^j)$ satisfies the Bellman equation

$$\begin{aligned} rV_t\left(K_t^j\right) \quad = \quad & \max_{\dot{K}_t^j,\, I_t^j, K_{1t}^j} \quad R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \dot{V}_t\left(K_t^j\right) \\ & +\pi\left[Q_t I_t^j - I_t^j + Q_t K_t^j - Q_t K_{1t}^j + V_t\left(K_{1t}^j\right) - V_t\left(K_t^j\right)\right] \end{aligned} \tag{C.2}$$

subject to the financing constraint $I_t^j \leq L_t^j$ and the following credit constraint

$$-L_t^j + V_t(K_{1t}^j) \geq V_t^a(K_{1t}^j). \tag{C.3}$$

This credit constraint is an incentive constraint which can be interpreted as follows. Write the discrete-time approximation to (C.3) as

$$-L_t^j + e^{-r\Delta} V_{t+\Delta}(K_{1t+\Delta}^j) \geq e^{-r\Delta} V_{t+\Delta}^a(K_{1t+\Delta}^j). \tag{C.4}$$

When an investment opportunity arrives at time t, firm j takes on debt L_t^j to finance investment I_t^j. At the end of period $[t, t+\Delta]$, the firm's capital sales $Q_t I_t^j$ are realized. If it repays the debt,

[22] Kocherlakota (2008) and Hellwig and Lorenzoni (2009) show that a bubble can exist with self-enforcing debt constraints while leaving consumption allocation unchanged in a pure exchange economy.

its continuation value is given by the expression on the left-hand side of (C.4). If it defaults on the debt, it will be excluded from the financial market forever and its continuation value is given by the expression on the right-hand side of (C.4). Inequality (C.4) ensures that the firm has no incentive to default. The constraint (C.3) is the continuous time limit as $\Delta \to 0$.

Conjecture that

$$V_t\left(K_t^j\right) = Q_t K_t^j + B_t. \tag{C.5}$$

Then (C.3) becomes $L_t^j \leq B_t$. This constraint is similar to that in Martin and Ventura (2012). Substituting (C.5) into (C.2) yields

$$\begin{aligned} rQ_tK_t^j + rB_t &= \max_{I_t^j, \dot{K}_t^j} R_tK_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \pi\left(Q_t - 1\right)I_t^j \\ &\quad + \dot{Q}_tK_t^j + Q_t\dot{K}_t^j + \dot{B}_t \end{aligned} \tag{C.6}$$

subject to

$$I_t^j \leq B_t. \tag{C.7}$$

When $Q_t > 1$, the optimal investment level is $I_t^j = B_t$. Substituting this investment rule back into the Bellman equation and matching coefficients, we obtain (C.1) and

$$rB_t = \dot{B}_t + \pi\left(Q_t - 1\right)B_t. \tag{C.8}$$

The law of motion for aggregate capital is

$$\dot{K}_t = -\delta K_t + \pi B_t, \;\; K_0 \text{ given.} \tag{C.9}$$

The equilibrium system is given by three differential equations (C.1), (C.8), and (C.9) for (Q_t, B_t, K_t) together with the usual transversality condition.

This equilibrium system is the same as that for the baseline model in Section 2 when $\xi = 0$. Thus the analysis in Sections 4 and 5 for $\xi = 0$ applies here. Both bubbleless and bubbly equilibria exist and their steady states are unique.

D Risk-Averse Households

We replace risk-neutral households with risk-averse households in the baseline model. Suppose that the representative household has the following utility function:

$$\int_0^\infty e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}dt,$$

where ρ is the subjective discount rate and γ is the risk aversion parameter. The household faces the budget constraint (4) subject to the no-Ponzi-game condition. Then we derive the consumption Euler equation

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma}\left(r_t - \rho\right), \tag{D.1}$$

where r_t is equal to the return on any stock j in the absence of aggregate uncertainty and is also called the discount rate. Equation (5) holds where r is replaced by r_t. Firm j solves the following dynamic programming problem:

$$r_t V_t\left(K_t^j\right) = \max_{D_t^j,\, I_t^j} \quad D_t^j + \dot{V}_t\left(K_t^j\right) + \pi\left[\left(L_t^j - I_t^j\right) + \left(Q_t I_t^j - L_t^j\right)\right] \tag{D.2}$$

subject to (15), (16), and (17). For tractability, we assume that capital does not jump at the time when an investment opportunity arrives. As we show earlier, this assumption is without loss of generality due to the liquidity mismatch assumption.

The aggregate state variables of the economy are B_t, Q_t, and K_t, where B_t represents the aggregate size of the bubble. The discount rate r_t is a function of the aggregate state variables. Conjecture that

$$V_t\left(K_t^j\right) = Q_t K_t^j + B_t^j,$$

where B_t^j is the bubble component in firm j's stock price. Substituting this conjecture into the preceding dynamic programming problem yields

$$\begin{aligned} r_t Q_t K_t^j + r_t B_t^j \quad = \quad & \max_{I_t^j, \dot{K}_t^j} \; R_t K_t^j - Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + \pi\left(Q_t - 1\right) I_t^j \\ & + \dot{Q}_t K_t^j + Q_t \dot{K}_t^j + \dot{B}_t^j, \end{aligned} \tag{D.3}$$

subject to

$$I_t^j \leq \xi Q_t K_t^j + B_t^j. \tag{D.4}$$

When $Q_t > 1$, the constraint (D.4) binds so that the optimal investment level is $I_t^j = \xi Q_t K_t^j + B_t^j$. Substituting this rule back into the Bellman equation and matching the coefficients of K_t^j, we obtain

$$\dot{Q}_t = \left(r_t + \delta\right) Q_t - R_t - \pi \xi Q_t (Q_t - 1), \tag{D.5}$$

$$\dot{B}_t^j = r_t B_t^j - B_t^j \pi (Q_t - 1). \tag{D.6}$$

The usual transversality conditions must hold.

Since $B_t = \int B_t^j dj$, it follows from (D.6) that the aggregate bubble satisfies

$$\dot{B}_t = r_t B_t - B_t \pi (Q_t - 1). \tag{D.7}$$

The law of motion for aggregate capital still satisfies (27). The resource constraint is given by

$$C_t + \pi(\xi Q_t K_t + B_t) = Y_t. \tag{D.8}$$

The equilibrium system consists of five equations (27), (D.1), (D.5), (D.7), and (D.8) for five aggregate variables $\left(C_t, r_t, K_t, Q_t, B_t\right)$. The transversality condition also holds

$$\lim_{T\to\infty} e^{-\int_0^T r_s ds} Q_T K_T = 0, \quad \lim_{T\to\infty} e^{-\int_0^T r_s ds} B_T = 0. \tag{D.9}$$

Note that an equilibrium only determines the size B_t of the aggregate bubble, but an individual firm's bubble size B_t^j is indeterminate. Thus it is possible that some firms have no bubbles, while others have bubbles of different sizes.

We use a variable without the time subscript to denote its steady state value. Then (D.1) implies $r = \rho$ and hence the steady-state system is the same as that in the baseline model of Section 3. Our analysis of steady states in Sections 4 and 5 still applies to the case of risk-averse households. We are unable to derive analytical results for local dynamics because the equilibrium system contains five equations, but it is straightforward to derive numerical solutions.

E General Short-Sale Constraints

In Section 6.5 we have assumed that households cannot short intertemporal bonds, or effectively they cannot borrow. We now relax this assumption and allow households to borrow a proportion of their labor income.

Assumption 9 *The representative household can borrow or short intertemporal bonds up to a proportion χ of its wage income, i.e., $L_t^h \geq -\chi w_t$, $\chi \geq 0$. Firms cannot hold or trade each other's stocks.*

We follow the same steps as before to derive the equilibrium system. From the firm's decision problem we show that the value function takes the form

$$V_t\left(K_t^j, L_t^j\right) = Q_t K_t^j - L_t^j + B_t. \tag{E.1}$$

When $Q_t > 1$, optimal investment is given by

$$I_t^j = \xi Q_t K_t^j + B_t - L_t^j.$$

We can also show that the equations for Q_t, B_t, and r_{ft} are given in Proposition 12. We need to derive the law of motion for aggregate capital.

Since $r_{ft} < r$, households will borrow by short-selling bonds until their short-sale constraints bind, i.e.,

$$L_t^h = -\chi w_t = -\chi(1-\alpha)K_t^\alpha.$$

The last equality follows from the wage equation in equilibrium. By the bond market-clearing condition

$$\int L_t^j dj = L_t^h = -(1-\alpha)\chi Y_t.$$

Aggregating the law of motion for an individual firm's capital, we obtain

$$\begin{aligned} \dot{K}_t &= -\delta K_t + \pi\left(\xi Q_t K_t + B_t - \int L_t^j dj\right) \\ &= -\delta K_t + \pi\left(\xi Q_t K_t + B_t + (1-\alpha)\chi K_t^\alpha\right). \end{aligned} \tag{E.2}$$

We now derive the bubbly steady state. Using equations for Q_t, K_t and r_{ft} in Proposition 12, we can show that

$$Q_b = \frac{r+\pi}{\pi} > 1, \ r_f = 0, \ R_b = \frac{r+\pi}{\pi}[(1-\xi)r+\delta].$$

Using (E.2), we can show that

$$\begin{aligned} \frac{B}{K_b} &= \frac{\delta}{\pi} - \xi Q_b - (1-\alpha)\chi K_b^{\alpha-1} \\ &= \frac{\delta}{\pi} - \xi Q_b - \chi \frac{(1-\alpha)}{\alpha} R_b. \end{aligned}$$

The bubbly equilibrium requires $B > 0$. Using the preceding equations, we then obtain the necessary and sufficient conditions

$$0 \leq \chi < \frac{\alpha}{1-\alpha} \frac{1}{r(1-\xi)+\delta} \left[\frac{\delta}{r+\pi} - \xi \right].$$

This result shows that a stock price bubble can exist as long as the short-sale constraint for households is sufficiently tight. The analysis of Section 6.5 corresponds to the case of $\chi = 0$.

F Intertemporal Debt without a Market for Capital

In this appendix we show that the equilibrium system analyzed in Section 6.5 is equivalent to a setup where there is no market for capital goods. We replace intratemporal debt in the baseline model with intertemporal bonds with zero net supply. With intertemporal bonds, firms can raise new debt to payoff old debt. Let r_{ft} denote the interest rate on the bonds. Suppose that firms can invest and accumulate capital on their own. We allow the lender to seize both a fraction ξ of the defaulting firm's existing capital and a fraction η of its newly installed capital in the event of default.[23] The solution in Section 6.5 corresponds to the special case with $\eta = 0$.

Assumption 10 *Households cannot short intertemporal bonds. Firms do not own or trade each other's shares and do not issue new equity to finance investment. The only sources of finance are internal funds, savings, and intertemporal debt.*

We will derive equilibria in which investing firms borrow from non-investing firms and households do not hold any bonds. Let $L_t^h \geq 0$ denote the representative household's bond holdings. Let $L_t^j > (<) 0$ denote firm j's debt level (saving). The market-clearing condition for the bonds is $\int L_t^j dj = L_t^h$. Let $V_t(K_t^j, L_t^j)$ denote the ex ante equity value of firm j when its capital stock and debt level at time t are K_t^j and L_t^j, respectively, prior to the realization of the Poisson shock. We

[23] If we introduce this assumption in Section 6.5, then the resulting equilibrium system is equivalent to that studied in this appendix.

suppress the aggregate state variables in the argument. Then V_t satisfies the following Bellman equation in discrete time:

$$\begin{aligned} V_t\left(K_t^j, L_t^j\right) &= \max_{I_t^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} (1-\pi\Delta)\left[D_{0t}^j\Delta + e^{-r\Delta}V_{t+\Delta}\left((1-\delta\Delta)K_t^j, L_{t+\Delta}^j\right)\right] \\ &\quad +\pi\Delta\left[D_{1t}^j + e^{-r\Delta}V_{t+\Delta}\left(K_{t+\Delta}^j, L_{1t+\Delta}^j\right)\right] \end{aligned}$$

subject to

$$0 \le D_{0t}^j\Delta = R_t K_t^j\Delta + e^{-r_{ft}\Delta}L_{t+\Delta}^j - L_t^j, \tag{F.1}$$

$$0 \le D_{1t}^j = R_t K_t^j\Delta + e^{-r_{ft}\Delta}L_{1t+\Delta}^j - L_t^j - I_t^j, \tag{F.2}$$

$$K_{t+\Delta}^j = (1-\delta\Delta)K_t^j + I_t^j, \tag{F.3}$$

$$V_{t+\Delta}(K_{t+\Delta}^j, L_{1t+\Delta}^j) \ge V_{t+\Delta}\left(K_{t+\Delta}^j, 0\right) - V_{t+\Delta}(\xi(1-\delta\Delta)K_t^j + \eta I_t^j, 0). \tag{F.4}$$

where $L_{1t+\Delta}^j$ ($L_{t+\Delta}^j$) represents the new debt level or saving when an investment opportunity arrives (no investment opportunity arrives). The price of the debt at time t that pays off one unit of consumption good at time $t+\Delta$ is $e^{-r_{ft}\Delta}$. By assumption, firm j cannot issue new equity to finance investment when an investment opportunity arrives so that $D_{1t}^j \ge 0$. Since there is no market for capital goods, the flow-of-funds constraints are different from those in the model of Section 6.5. When firm j invests I_t^j with Poisson probability $\pi\Delta$, its capital stock jumps to $K_{t+\Delta}^j$ as shown in (F.3).

Debt is subject to the credit constraint (F.4). Firm j borrows $L_{1t+\Delta}^j$ at time t when an investment opportunity arrives. It may default on debt $L_{1t+\Delta}^j$ at time $t+\Delta$. If it does not default, it obtains continuation value $V_{t+\Delta}\left(K_{t+\Delta}^j, L_{1t+\Delta}^j\right)$. If it defaults, debt is renegotiated and the repayment $L_{1t+\Delta}^j$ is relieved. The lender can seize a fraction ξ of depreciated capital $(1-\delta\Delta)K_t^j$ and a fraction η of newly installed capital I_t^j.[24] The lender keeps the firm running with these assets by reorganizing the firm. Thus the threat value to the lender is $V_t(\xi(1-\delta\Delta)K_t^j + \eta I_t^j, 0)$. Assume that firm j has a full bargaining power so that the renegotiated repayment is given by the threat value to the lender. The expression on the right-hand side of (F.4) is the value to the firm if it chooses to default. We then have the incentive constraint given in (F.4).

Conjecture that

$$V_t\left(K_t^j, L_t^j\right) = a_t K_t^j - a_t^L L_t^j + b_t.$$

Define $Q_t = e^{-r\Delta}a_{t+\Delta}$. Here Q_t is Tobin's marginal Q or the shadow price of capital, instead of the market price of capital. Substituting this conjecture and equations (F.1), (F.2), and (F.3) into

[24] If $\xi = \eta$, the lender effectively seizes the firm's future capital $\xi K_{t+\Delta}^j$.

the Bellman equation yields

$$
\begin{aligned}
a_t K_t^j - a_t^L L_t^j + b_t \quad = & \max_{I_t^j, K_{t+\Delta}^j, K_{1t+\Delta}^j, L_{t+\Delta}^j, L_{1t+\Delta}^j} R_t K_t^j \Delta - L_t^j + e^{-r\Delta} b_{t+\Delta} \\
& + (1-\pi\Delta) \left[e^{-r_f t \Delta} L_{t+\Delta}^j + Q_t (1-\delta\Delta) K_t^j - e^{-r\Delta} a_{t+\Delta}^L L_{t+\Delta}^j \right] \\
& + \pi\Delta \left[e^{-r_f t \Delta} L_{1t+\Delta}^j + Q_t (1-\delta\Delta) K_t^j - e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j \right] \\
& + \pi\Delta (Q_t - 1) I_t^j
\end{aligned}
$$

subject to

$$I_t^j \leq R_t K_t^j \Delta + e^{-r_f t \Delta} L_{1t+\Delta}^j - L_t^j, \tag{F.5}$$

$$a_{t+\Delta}^L L_{1t+\Delta}^j \leq b_{t+\Delta} + a_{t+\Delta} \xi (1-\delta\Delta) K_t^j + a_{t+\Delta} \eta I_t^j, \tag{F.6}$$

where (F.5) follows from $D_{1t}^j \geq 0$ and says that investment is financed by internal funds, savings, and debt only. Credit constraint (F.6) follows from (F.4).

By the linear property of the Bellman function, the first-order condition for $L_{t+\Delta}^j$ yields

$$e^{-r_f t \Delta} = e^{-r\Delta} a_{t+\Delta}^L, \tag{F.7}$$

and hence $L_{t+\Delta}^j$ is indeterminate. This implies that firm j is indifferent between saving and borrowing when no investment opportunity arrives. Multiplying the two sides of inequality (F.6) by $e^{-r\Delta}$ and using (F.7), we obtain

$$e^{-r_f t \Delta} L_{1t+\Delta}^j = e^{-r\Delta} a_{t+\Delta}^L L_{1t+\Delta}^j \leq B_t + Q_t \xi (1-\delta\Delta) K_t^j + Q_t \eta I_t^j, \tag{F.8}$$

where we have used $Q_t = e^{-r\Delta} a_{t+\Delta}$ and the definition

$$B_t \equiv e^{-r\Delta} b_{t+\Delta}. \tag{F.9}$$

When $1 < Q_t < 1/\eta$, the financing constraint (F.5) and the credit constraint (F.8) bind so that optimal investment is given by

$$I_t^j = \frac{1}{1-\eta Q_t} \left[R_t K_t^j \Delta + B_t + Q_t \xi (1-\delta\Delta) K_t^j - L_t^j \right],$$

where the multiplier $1/(1-\eta Q_t)$ reflects the leverage effect.

Substituting the investment rule back into the Bellman equation and matching coefficients, we derive

$$a_t = R_t \Delta + Q_t (1-\delta\Delta) + \pi\Delta (Q_t - 1) \frac{R_t \Delta + Q_t \xi (1-\delta\Delta)}{1-\eta Q_t},$$

$$a_t^L = 1 + \pi\Delta \frac{Q_t - 1}{1-\eta Q_t},$$

$$b_t = e^{-r\Delta} b_{t+\Delta} + \pi\Delta \frac{(Q_t - 1) B_t}{1-\eta Q_t}.$$

Using (F.7) and (F.9) and the preceding three equations, we can derive

$$Q_t = e^{-r\Delta}\left[R_{t+\Delta}\Delta + Q_{t+\Delta}(1-\delta\Delta) + \pi\Delta(Q_{t+\Delta}-1)\frac{R_{t+\Delta}\Delta + Q_{t+\Delta}\xi(1-\delta\Delta)}{1-\eta Q_t}\right],$$

$$e^{-r_{ft}\Delta} = e^{-r\Delta}\left[1+\pi\Delta\frac{Q_{t+\Delta}-1}{1-\eta Q_t}\right],$$

$$B_t = e^{-r\Delta}\left[1+\pi\Delta\frac{Q_{t+\Delta}-1}{1-\eta Q_t}\right]B_{t+\Delta}.$$

Taking the continuous-time limit as $\Delta \to 0$ yields

$$\dot{Q}_t = (r+\delta)Q_t - R_t - \frac{\pi(Q_t-1)Q_t\xi}{1-\eta Q_t}, \tag{F.10}$$

$$\dot{B}_t = rB_t - \frac{\pi(Q_t-1)}{1-\eta Q_t}B_t, \tag{F.11}$$

$$r_{ft} = r - \frac{\pi(Q_t-1)}{1-\eta Q_t} < r. \tag{F.12}$$

We now show that this solution is the same as that in the continuous-time setup. We derive the continuous-time limit of the dynamic programming problem as

$$\begin{aligned} rV_t\left(K_t^j, L_t^j\right) = \max_{D_{0t}^j, D_{1t}^j, I_t^j, L_{1t}^j} \quad & D_{0t}^j + \dot{V}_t\left(K_t^j, L_t^j\right) \\ & +\pi\left[D_{1t}^j + V_t\left(K_t^j + I_t^j, L_{1t}^j\right) - V_t\left(K_t^j, L_t^j\right)\right] \end{aligned} \tag{F.13}$$

subject to

$$\dot{L}_t^j = r_{ft}L_t^j + D_{0t}^j - R_tK_t^j, \tag{F.14}$$

$$D_{1t}^j = L_{1t}^j - L_t^j - I_t^j, \tag{F.15}$$

$$I_t^j \leq L_{1t}^j - L_t^j, \tag{F.16}$$

$$V_t(K_t^j + I_t^j, L_{1t}^j) \geq V_t\left(K_t^j + I_t^j, 0\right) - V_t(\xi K_t^j + \eta I_t^j, 0). \tag{F.17}$$

When no investment opportunity arrives, capital simply depreciates so that $\dot{K}_t^j = -\delta K_t^j$. Whenever an investment opportunity arrives, capital jumps to $K_t^j + I_t^j$.

Conjecture the value function takes the form

$$V_t\left(K_t^j, L_t^j\right) = Q_tK_t^j - L_t^j + B_t, \tag{F.18}$$

and hence the credit constraint (F.17) becomes

$$L_{1t}^j \leq Q_t\xi K_t^j + \eta Q_t I_t^j + B_t, \tag{F.19}$$

where $B_t \geq 0$ is the bubble component of equity value.

Substituting the conjectured value function into the Bellman equation yields

$$\begin{aligned} r\left(Q_t K_t^j - L_t^j + B_t\right) &= \max\ \dot{L}_t^j - r_{ft} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j - \dot{L}_t^j + \dot{B}_t \\ &\quad + \pi\left[\left(L_{1t}^j - L_t^j - I_t^j\right) + \left(Q_t I_t^j - L_{1t}^j + L_t^j\right)\right]. \end{aligned}$$

Thus $\dot{L}_t^j$ cancels itself out so that firm j is indifferent between saving and borrowing when no investment opportunity arrives. Simplifying yields

$$\begin{aligned} r\left(Q_t K_t^j - L_t^j + B_t\right) &= \max\ - r_{ft} L_t^j + R_t K_t^j + \dot{Q}_t K_t^j - Q_t \delta K_t^j + \dot{B}_t \\ &\quad + \pi \left(Q_t - 1\right) I_t^j. \end{aligned} \tag{F.20}$$

Using the credit constraint (F.19) and the financing constraint (F.16), we obtain

$$I_t^j \leq L_{1t}^j - L_t^j \leq \xi Q_t K_t^j + \eta Q_t I_t^j + B_t - L_t^j.$$

If $1 < Q_t < 1/\eta$, it is profitable for firm j to invest as much as possible and both constraints bind. In this case firm j borrows by selling bonds. We then have

$$I_t^j = \frac{\xi Q_t K_t^j + B_t - L_t^j}{1 - \eta Q_t}.$$

Substituting this investment rule back into the Bellman equation (F.13) and matching coefficients, we derive the equations for Q_t, B_t, and r_{ft} given above.

We now compute aggregate investment

$$I_t = \int I_t^j dj = \frac{\xi Q_t K_t + B_t - \int L_t^j dj}{1 - \eta Q_t}.$$

Since $r_{ft} < r$, households' short-sale constraints bind so that $L_t^h = 0$ and the bond market-clearing condition becomes $\int L_t^j dj = 0$. Thus

$$I_t = \frac{\xi Q_t K_t + B_t}{1 - \eta Q_t}. \tag{F.21}$$

We can then derive the law of motion for aggregate capital

$$\int K_{t+\Delta}^j dj = \int \left(1 - \delta\Delta\right) K_t^j dj + \pi\Delta \int I_t^j dj.$$

Taking the limit as $\Delta \to 0$ yields

$$\dot{K}_t = -\delta K_t + \pi I_t.$$

Substituting (F.21) into the above equation yields the equation for K_t

$$\dot{K}_t = -\delta K_t + \pi \frac{\xi Q_t K_t + B_t}{1 - \eta Q_t}, \tag{F.22}$$

Finally, we can use the same procedure in the proof of Proposition 2 to derive $R_t = \alpha K_t^{\alpha-1}$. The equilibrium system for (Q_t, B_t, r_{ft}, K_t) consists of (F.10), (F.11), (F.12), and (F.22) when $1 < Q_t < 1/\eta$. The usual transversality conditions must be satisfied. We can see that the equilibrium system presented in Proposition 12 is the special case with $\eta = 0$.

We can also prove the following result.

Proposition 14 *For the model in this subsection with intertemporal bonds, if*

$$0 < \xi < \frac{\delta(1-\eta)}{r+\pi}, \tag{F.23}$$

then the bubbly and bubbleless steady states with $1 < Q < 1/\eta$ coexist. Moreover, the interest rates in the bubbleless and bubbly steady states are given by $r_f^* = r + \pi - \delta(1-\eta)/\xi < 0$ and $r_f = 0$, respectively.

Proof. We first derive the bubbly steady state in which $B > 0$. Using the equilibrium system derived above, we can show that

$$Q_b = \frac{r+\pi}{\eta r + \pi}, \quad r_f = 0, \tag{F.24}$$

$$R_b = \alpha K_b^{\alpha-1} = \frac{r+\pi}{\eta r+\pi}[(1-\xi)r+\delta], \tag{F.25}$$

$$\frac{B}{K_b} = \frac{\delta}{\pi} - \frac{\xi(r+\pi)}{\pi(1-\eta)}. \tag{F.26}$$

Since $\eta \in (0,1)$, we have $1 < Q_b < 1/\eta$. Given condition (F.23), we have $B > 0$ and hence a bubbly steady state exists.

We next derive the bubbleless steady state in which $B = 0$. Using the equilibrium system derived above, we can show that

$$Q^* = \frac{\delta}{\pi\xi + \eta\delta},$$

$$R^* = \alpha K^{*\alpha-1} = \frac{\delta r}{\pi\xi+\eta\delta} + \delta,$$

$$r_f^* = r + \pi - \delta(1-\eta)/\xi.$$

Under condition (F.23), we have $1 < Q^* < 1/\eta$. Thus a bubbleless steady state exists. ■

G Cross-Holdings

In this appendix we assume that households hold a fraction $1-H$ shares of a market porfolio of all firm stocks and firms hold $H \in (0,1)$ shares of the market porfolio in the model of Section 6.5. For technical convenience we consider the continuous-time setup. Assume that firms do not use the market portfolio to finance investment for the reasons discussed in Section 6.6.

Let $V_t\left(K_t^j, L_t^j, H_t^j\right)$ denote the ex ante market value of firm j, where H_t^j denotes firm j's holdings of the market portfolio prior to the investment opportunity shock. Then V_t satisfies the continuous-time Bellman equation

$$\begin{aligned} rV_t\left(K_t^j, L_t^j, H_t^j\right) &= \max \quad D_{0t}^j + \dot{V}_t\left(K_t^j, L_t^j, H_t^j\right) \\ &\quad +\pi\left[D_{1t}^j + V_t\left(K_{1t}^j, L_{1t}^j, H_{1t}^j\right) - V_t\left(K_t^j, L_t^j, H_t^j\right)\right] \end{aligned} \tag{G.1}$$

subject to the flow-of-funds constraints

$$\dot{L}_t^j = r_{ft}L_t^j + D_{0t}^j - R_tK_t^j + Q_t\left(\dot{K}_t^j + \delta K_t^j\right) + P_t\dot{H}_t^j - X_tH_t^j, \tag{G.2}$$

$$D_{1t}^j = Q_tI_t^j + L_{1t}^j - L_t^j - I_t^j + Q_tK_t^j - Q_tK_{1t}^j + P_t\left(H_t^j - H_{1t}^j\right), \tag{G.3}$$

the financing constraint

$$I_t^j \leq L_{1t}^j - L_t^j, \tag{G.4}$$

and the credit constraint

$$V_t\left(K_{1t}^j, L_{1t}^j, H_{1t}^j\right) \geq V_t\left(K_{1t}^j, 0, H_{1t}^j\right) - V_t(\xi K_t^j, 0, 0), \tag{G.5}$$

where H_{1t}^j denotes firm j's holdings of the market portfolio when an investment opportunity arrives. Here P_t denotes the value of the market portfolio,

$$P_t = \int V_t\left(K_t^j, L_t^j, H_t^j\right)dj,$$

and X_t denotes the total dividends of the portfolio

$$X_t = \int D_t^j dj = \int D_{0t}^j dj + \pi \int D_{1t}^j dj.$$

Note that the value of the market portfolio does not jump even if the value of an individual firm can jump when an investment opportunity arrives. This is because

$$P_{t+\Delta} = (1-\pi\Delta)\int V_{t+\Delta}\left(K_{t+\Delta}^j, L_{t+\Delta}^j, H_{t+\Delta}^j\right)dj + \pi\Delta\int V_{t+\Delta}\left(K_{1t+\Delta}^j, L_{1t+\Delta}^j, H_{1t+\Delta}^j\right)dj$$

so that $P_{t+\Delta} \to P_t$ as $\Delta \to 0$.

The financing constraint (G.4) means that firm j only uses debt and savings to finance investment. The interpretation of the credit constraint (G.5) is similar to that in Section 6.5. In particular, the lender can only recover a fraction ξ of capital and take over the firm in the event of default.

Conjecture that the value function takes the form

$$V_t\left(K_t^j, L_t^j, H_t^j\right) = Q_tK_t^j + B_t - L_t^j + P_tH_t^j. \tag{G.6}$$

Substituting this conjecture and the flow-of-funds constraints into the preceding Bellman equation yields

$$\begin{aligned}
& r\left(Q_t K_t^j - L_t^j + B_t + P_t H_t^j\right) \\
= \quad & \max \ \dot{L}_t^j - r_{ft} L_t^j + R_t K_t^j - Q_t \dot{K}_t^j - Q_t \delta K_t^j - P_t \dot{H}_t^j + X_t H_t^j \\
& + Q_t \dot{K}_t^j + \dot{Q}_t K_t^j - \dot{L}_t^j + \dot{B}_t + \dot{P}_t H_t^j + P_t \dot{H}_t^j \\
& + \pi \left[(Q_t - 1) I_t^j + L_{1t}^j - L_t^j + Q_t K_t^j - Q_t K_{1t}^j + P_t H_t^j - P_t H_{1t}^j \right] \\
& + \pi \left[Q_t K_{1t}^j - L_{1t}^j + B_t + P_t H_{1t}^j - \left(Q_t K_t^j - L_t^j + B_t \right) - P_t H_t^j \right].
\end{aligned}$$

Given the conjectured value function, the credit constraint becomes

$$L_{1t}^j \leq Q_t \xi K_t^j + B_t.$$

If $Q_t > 1$, the financing constraint and the credit constraint bind so that optimal investment is given by

$$I_t^j = Q_t \xi K_t^j + B_t - L_t^j.$$

Substituting this investment rule back into the Bellman equation and matching coefficients, we obtain (20), (21), (62), and

$$r P_t = X_t + \dot{P}_t.$$

Thus the rate of return on the market portfolio is equal to r. Aggregation yields the law of motion for aggregate capital (27). Thus the equilibrium system for (Q_t, K_t, B_t, r_{ft}) is the same as that in Section 6.5 and Appendix B.5 and hence Proposition 13 still holds. The only difference lies in the valuation of the firm.

Since $\int H_t^j = H$, aggregation of (G.6) yields

$$P_t = \frac{Q_t K_t + B_t}{1 - H}.$$

As discussed in Fedenia, Hodder, and Triantis (1994) and Elliott, Golub, and Jackson (2014), the equation above and equation (G.6) show that cross-holdings inflate the market capitalization. Since households hold $1 - H$ shares of all firms, the portfolio value to the households is $Q_t K_t + B_t$. Thus cross-holdings do not have any effects on welfare and real allocation as long as cross-holdings do not help finance investment.