Bubbles and Credit Constraints*

Jianjun Miao‡  Pengfei Wang‡

January 15, 2015

Abstract

We provide a theory of credit-driven stock price bubbles in production economies with infinitely lived agents. Firms meet stochastic investment opportunities and face endogenous credit constraints. Firms have limited commitment to repay debt. Credit constraints are derived from incentive constraints in optimal contracts which ensure default never occurs in equilibrium. A stock price bubble can emerge through a positive feedback loop mechanism. It commands a liquidity premium and improves investment efficiency because it raises debt capacity by relaxing incentive constraints. We provide conditions under which bubbles can coexist with other types of assets. We show that the collapse of stock price bubbles leads to a recession and a stock market crash. There is a government policy that can eliminate bubbles and achieve efficient allocation.

Keywords: Credit-Driven Bubbles, Credit Constraints, Asset Price, Arbitrage, Q Theory, Liquidity, Multiple Equilibria

JEL codes: E2, E44, G1

*We thank Bruno Biais, Jess Benhabib, Toni Braun, Markus Brunnermeier, Henry Cao, Christophe Chamley, Tim Cogley, Russell Cooper, Douglas Gale, Jordi Gali, Mark Gertler, Simon Gilchrist, Christian Hellwig, Hugo Hoppenhaun, Andreas Hornstein, Boyan Jovanovic, Bob King, Nobu Kiyotaki, Anton Korinek, Felix Kubler, Kevin Lansing, John Leahy, Eric Leeper, Zheng Liu, Gustavo Manso, Ramon Marimon, Erwan Morellec, Fabrizio Perri, Jean-Charles Rochet, Tom Sargent, Jean Tirole, Jon Willis, Mike Woodford, Tao Zha, Lin Zhang, and, especially, Wei Xiong and Yi Wen for helpful discussions. We have also benefitted from comments from seminar and conference participants at the BU macro lunch workshop, Cheung Kong Graduate School of Business, European University Institute, Indiana University, New York University, Toulouse School of Economics, CREI at University of Pompeu Fabra, University of Mannheim, University of Lausanne, University of Southern Denmark, University of Zurich, Zhejiang University, the 2011 Econometric Society Summer Meeting, the 2011 International Workshop of Macroeconomics and Financial Economics at the Southwestern University of Finance and Economics, the Federal Reserve Banks of Atlanta, Boston, San Francisco, Richmond, and Kansas, the Theory Workshop on Corporate Finance and Financial Markets at Stanford, Shanghai University of Finance and Economics, the 2011 SED conference in Ghent, and the 7th Chinese Finance Annual Meeting. First version: December 2010.

‡Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215. Tel.: 617-353-6675. Email: miaoj@bu.edu. Homepage: http://people.bu.edu/miaoj.

‡Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Hong Kong. Tel: (+852) 2358 7612. Email: pfwang@ust.hk
1 Introduction

This paper provides a theory of credit-driven stock price bubbles. Our theory is motivated by two observations. First, it is difficult to explain the stock market booms and busts entirely by fundamentals (Shiller (2005)). Second, the stock market booms are often accompanied by the credit market booms. For example, overoptimism in the 1990s towards an “East Asian miracle” generated booms in the housing and stock markets in some east Asian countries which were accompanied by lending booms and a large expansion of domestic credit (Collyns and Senhadji (2002)). Jordà, Schularick, and Taylor (2014) document empirical evidence on credit-driven bubbles in 17 developed countries since 1870. They find that credit-driven bubbles are more dangerous to the macroeconomy than other types of bubbles, e.g., unleveraged “irrational exuberance” bubbles.

To formalize our theory, we construct a tractable continuous-time model of a production economy in which identical households are infinitely lived and trade firm stocks. There is no aggregate uncertainty. In the baseline model households are risk neutral so that the rate of return on any stock is equal to the constant subjective discount rate. A continuum of firms meet idiosyncratic stochastic investment opportunities as in Kiyotaki and Moore (1997, 2005, 2008). Firms use intratemporal debt to finance investment because of a liquidity mismatch (Jermann and Quadrini (2012)): investment spending must be paid before the realization of investment returns. In addition, firms cannot raise equity or sell a lump-sum amount of capital to finance investment. This assumption reflects the fact that equity financing is more costly than debt financing. Another interpretation following Kiyotaki and Moore (2005, 2008) is that investment opportunities disappear so quickly that firms do not have enough time to raise equity or sell a large amount of capital.

Firms face endogenous credit constraints, which are modeled in a similar way to Bulow and Rogoff (1989), Kehoe and Levine (1993), Kiyotaki and Moore (1997), Alvarez and Jermann (2000), Albuquerque and Hopenhayn (2004), and Jermann and Quadrini (2012). The key idea is that borrowers (firms) have limited commitment and debt repayments are imperfectly enforced. We consider the following lending contract to ensure the repayment of debt. A firm pledges its physical assets (capital) as collateral. If the firm does not repay its debt, then it loses its collateralized assets and the right to run the firm all to the lender. Thus the collateral value to the lender is equal to the market value of the firm with the collateralized assets. The lender and the firm renegotiate the debt such that the debt is limited by this collateral value. The resulting credit constraint is endogenously derived from the incentive constraint in an optimal contracting problem.

Unlike Kiyotaki and Moore (1997) who assume that the collateral value is equal to the liquidation value of the collateralized assets, we derive the collateral value from the incentive constraint as the going-concern value of the reorganized firm. Because the going-concern value is priced in the
The aforementioned mechanism still works. There is another type of equilibrium in which no one believes in bubbles and hence bubbles do not appear. We call this type the bubbleless equilibrium. We provide explicit conditions to determine which type of equilibrium can exist. We prove that the economy has two steady states: a bubbly one and a bubbleless one. Both steady states are inefficient due to credit constraints and both are local saddle points. The equilibrium around the bubbly steady state is unique and bubbles persist in the long run along a stable manifold. But the equilibrium around the bubbleless steady state has indeterminacy of degree one and bubbles eventually burst along a stable manifold. Thus multiple equilibria in our model are not generated by indeterminacy with a unique steady state as in the literature surveyed by Benhabib and Farmer (1999) and Farmer (1999).

It is difficult to generate rational bubbles for economies with infinitely lived agents (Tirole (1982) and Santos and Woodford (1997)). A necessary condition for bubbles to exist is that the growth rate of bubbles cannot exceed the growth rate of the economy. Otherwise, investors cannot afford to buy into bubbles. In standard deterministic models bubbles on assets with exogenous payoffs or on intrinsically useless assets must grow at the interest rate by the no-arbitrage principle. Thus the interest rate cannot exceed the growth rate of the economy. This implies that the present value of aggregate endowments must be infinity. In an overlapping generations (OLG) economy, this condition implies that the bubbleless equilibrium must be dynamically inefficient (Tirole (1985)).

For infinitely lived agents, utility maximization implies a transversality condition for bubbles, which requires that the present value of bubbles be zero in the limit. This condition rules out bubbles in standard models with infinitely lived agents when bubbles grow at the interest rate.

---

1 See the online appendix for another type of credit constraints endogenously derived from optimal contracts that can generate a stock price bubble.
How does our model generate bubbles and how do we reconcile our result with that in Santos and Woodford (1997) or Tirole (1985)? The key is that stock price bubbles in our model are attached to productive assets (capital) with endogenous payoffs. Our novel insight is that stock price bubbles have real effects and affect firm dividends. Although a no-arbitrage equation still holds in that the discount rate on bubbles is equal to the negative growth rate of the discounted marginal utility or the subjective discount rate for risk-neutral utility, the growth rate of bubbles is not equal to this rate. Rather, it is equal to the discount rate minus a “collateral yield” or “liquidity premium.” The collateral yield comes from the fact that stock price bubbles help relax credit constraints and allow firms to make more investment.

We extend our baseline model to include other types of assets such as intertemporal bonds, assets with rents/dividends (e.g., tree), and assets without rents (pure bubble, e.g., tulip). Suppose that firms can trade one of these assets to finance investment. We study the conditions under which stock price bubbles or firm bubbles can coexist with other types of assets. If an asset can play the same role as a firm bubble in helping firms finance investment, then this asset will generate additional dividends to the firms, which are identical to the collateral yield. If, in addition, this asset delivers positive rents, then it dominates a bubble and hence they cannot coexist in equilibrium. But if this asset is a pure bubble, then it is a perfect substitute for the firm bubble. Only the total size of the bubble can be determined in equilibrium. For a firm bubble to coexist with intertemporal bonds, the equilibrium interest rate on the bonds must be zero in the steady state. We also need to introduce market frictions such as short-sales constraints on the additional assets (Kocherlakota (1992, 2009) and Kiyotaki and Moore (2005, 2008)). Without market frictions, the economy would achieve the efficient equilibrium and no bubble would exist.

So far, we have only considered deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we construct a third type of equilibrium with stochastic bubbles in the baseline model. In this equilibrium all agents believe that there is a positive probability that bubbles will burst at each date. When bubbles burst, they cannot reappear. We show that when all agents believe that the probability of bubble bursting is small enough, an equilibrium with stochastic bubbles exists. In contrast to Weil (1987), we show that after a bubble bursts, a recession occurs in that there is a credit crunch and consumption and output fall eventually. In addition, immediately after the bubble bursts, investment falls discontinuously and the stock market crashes. The recession and the stock market crash occur without any exogenous shock to the fundamentals of the economy.

What is an appropriate government policy in the wake of a bubble collapse? The inefficiency in our model comes from the firms’ credit constraints. The collapse of bubbles tightens these constraints and impairs investment efficiency. To overcome this inefficiency, the government can issue public bonds backed by lump-sum taxes. Public bonds can provide liquidity to firms. Thus public
bonds can help relax credit constraints and play the same role as bubbles do. If the government constantly retires public bonds at the interest rate to maintain a constant total bond value and pays the interest payments of these bonds by levying lump-sum taxes, then this policy will eliminate bubbles and allow the economy to achieve the efficient equilibrium.

**Related literature** Some papers in the literature (e.g., Scheinkman and Weiss (1986), Kocherlakota (1992, 2008), Santos and Woodford (1997) and Hellwig and Lorenzoni (2009)) find that infinite-horizon models with borrowing constraints may generate rational bubbles. Unlike these papers which study pure exchange economies, our paper analyzes a production economy with bubbles in stock prices whose payoffs are endogenously determined by investment and affected by bubbles.²

Our paper is closely related to some recent studies of production economies with bubbles, that introduce credit constraints to OLG models (Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Martin and Ventura (2011, 2012)) and to infinite-horizon models (Woodford (1990), Kiyotaki and Moore (2008), Kocherlakota (2009), Wang and Wen (2012), and Hirano and Yanagawa (2013)). These papers contain the idea that bubbles can help relax credit constraints and raise investment. Based on optimal contracts with limited commitment, our modeling of credit constraints is different from theirs. Our novel insight is that some types of credit constraints beyond the usual collateral constraint can generate a stock price bubble because the bubble can raise firm dividends by relaxing incentive constraints in optimal contracts and raising debt capacity.

Rather than studying stock price bubbles, the extant literature typically studies pure bubbles on intrinsically useless assets (e.g., money) that can provide liquidity by raising the borrower’s net worth.³ For example, Kiyotaki and Moore (2008) assume that entrepreneurs can pledge a fraction of investment returns as collateral to borrow. Equity (capital) is illiquid and only a fraction can be sold to finance investment. They show that money as a pure bubble asset can circulate because it is more liquid than other assets and raises net worth. Building on their insights, Kocherlakota (2009) assumes that entrepreneurs use land as collateral. Land is traded as an intrinsically useless asset. Land bubbles can emerge and burst. Unlike these papers, our main contribution is to provide a theory of credit-driven stock price bubbles in production economies with infinitely lived agents. The distinction between a pure bubble and a stock price bubble is important because the stock price bubble is not directly observable or tradable, but can affect dividends endogenously.

Building on Samuelson (1958), Diamond (1965), and Tirole (1985), Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Martin and Ventura (2012) study pure bubbles in OLG models with credit constraints. Credit constraints are not essential for the emergence of

---

² See Scheinkman and Xiong (2003) for a model of bubbles based on heterogeneous beliefs and Brunnermeier (2009) and Miao (2014) for surveys of various theories of bubbles.

bubbles in the traditional OLG models, unlike in infinite-horizon models. But they allow bubbles to emerge in dynamically efficient OLG economies. An important issue specific to infinite-horizon models is that firms may eventually overcome credit constraints by saving over time if they can buy sufficiently many (intertemporal) bonds from households. Thus one must impose borrowing constraints or short-sales constraints on households so that they cannot sell too many bonds. This generates a low interest rate on bonds. The spread between the stock return and the interest rate reflects the liquidity premium. By contrast, the stock return is equal to the interest rate in Farhi and Tirole (2011) and Martin and Ventura (2011, 2012).

Caballero and Krishnamurthy (2006) show that stochastic bubbles are beneficial because they provide domestic stores of value, thereby reducing capital outflows while increasing investment. But they come at a cost, as they expose the country to bubble crashes and capital flow reversals. Farhi and Tirole (2012) study the interplay between inside and outside liquidity by assuming that only a fraction of investment returns can be pledgeable. Bubbles and outside liquidity do not raise debt capacity directly, but they can raise entrepreneurs’ net worth used to finance more investment. The rise in investment then affects debt capacity indirectly through a leverage effect, which depends on the interest rate. The impact of bubbles on investment depends on the relative potency of a liquidity effect and a leverage effect.

Martin and Ventura (2012) introduce investor sentiment shocks and study stochastic equilibria with bubble creation and destruction. They assume that productive investors cannot borrow from unproductive ones. When a market for pure bubbles opens, productive investors raise their net worth by selling bubbles to unproductive ones. Thus productive investors can make more investment, thereby raising investment efficiency. Sentiment shocks can generate macroeconomic fluctuations without shocks to the fundamentals. Since pure bubbles (except for money, arts, etc.) are rarely traded in reality, Martin and Ventura (2012) reinterpret their benchmark model by allowing entrepreneurs to trade firms in the stock market. Entrepreneurs can pledge an empty firm as collateral. This modified model is equivalent to their benchmark when the market value of the firm is equal to the pure bubble value. In a similar model Martin and Ventura (2011) assume that entrepreneurs can start new firms and non-entrepreneurs trade old firms and give credit to new firms. A firm can use a fraction of its investment returns and market value as collateral to borrow. A bubble in firm value can emerge in addition to a fundamental component.

We also show that firm value consists of a fundamental component and a bubble component. Unlike Farhi and Tirole (2011) and Martin and Ventura (2011, 2012), we explicitly characterize the collateral yield provided by the bubble component and link the fundamental component to the Q theory of investment (Tobin (1969) and Hayashi (1982)). As in Hayashi (1982), firms are infinitely lived and make investment decisions that maximize their stock market values. Our framework of
infinite-horizon production economies with bubbles can be easily extended to incorporate many standard ingredients for both theoretical and quantitative analyses of asset prices, business cycles, and economic growth (Miao and Wang (2012, 2013, 2014), Miao, Wang, Xu (2012), Miao, Wang and Xu (2013), and Miao, Wang, and Zhou (2014)). For example, Miao, Wang and Xu (2013) apply Bayesian estimation methods to study stock market bubbles and business cycles using our framework. By contrast, existing OLG models of bubbles are confined to two- or three-period lived agents. Although these models are tractable to deliver economic insights, they are not suitable for a quantitative analysis because the time period in these models cannot be tied to the data frequency. Moreover, if each generation is equally altruistic toward the next generation with intergenerational transfers, then the entire dynasty behaves as a single infinitely lived agent (Barro (1974)). Thus studying models with infinitely lived agents will deepen our understanding of asset bubbles and complement OLG models.

2 The Baseline Model

We consider an infinite-horizon production economy, consisting of a continuum of identical households and a continuum of ex ante identical firms indexed by $j \in [0, 1]$. There is no aggregate uncertainty. Time is continuous and denoted by $t \geq 0$.

2.1 Households

There is a continuum of identical households of a unit measure. The representative household is risk neutral and derives utility from a consumption stream $\{C_t\}$ according to the utility function $\int_0^\infty e^{-rt}C_t dt$, where $r$ is the subjective discount rate (or rate of time preference).\footnote{In Appendix B we will show that our key insights hold true for the case of risk averse households.} Households supply labor inelastically and aggregate labor supply is normalized to one. They trade firm stocks without any trading frictions. The net supply of each firm’s stock is normalized to one.

The representative household faces the budget constraint

$$C_t + \int V_t^j \psi_t^j dj = \int \Pi_t^j \psi_t^j dj + w_t N_t,$$

where $V_t^j$ denotes firm $j$’s stock price, $\psi_t^j$ denotes holdings of firm $j$’s stocks, $\Pi_t^j$ denotes firm $j$’s dividends determined by its optimization problem, $w_t$ denotes the wage rate, and $N_t$ denotes labor supply.\footnote{We use $\dot{X}_t$ to denote $dX_t/dt$ for any variable $X_t$. Households’ optimization problem must also satisfy a no-Ponzi-game condition $\lim_{T \to \infty} e^{-rt} \int V_t^j \psi_t^j dj \geq 0$ (see Acemoglu (2009)).} Because there is no aggregate uncertainty, linear utility gives the first-order condition

$$r V_t^j = \Pi_t^j + \dot{V}_t^j,$$
for each firm \( j \). This equation says that the rate of return (or the discount rate) on each stock must be equal to \( r \). Linear utility implies the transversality condition (see, e.g., Ekeland and Scheinkman (1986) and Acemoglu (2009)),

\[
\lim_{T \to \infty} e^{-rT} V^j_T \psi_T = \lim_{T \to \infty} e^{-rT} V^j_T = 0,
\]

where we have used the market-clearing condition \( \psi_T = 1 \) for all \( T \) and all \( j \).

2.2 Firms

Each firm \( j \in [0, 1] \) combines labor \( N^j_t \) and capital \( K^j_t \) to produce output according to the Cobb-Douglas production function \( Y^j_t = (K^j_t)^\alpha (N^j_t)^{1-\alpha}, \ \alpha \in (0, 1) \). After solving the static labor choice problem, we obtain the operating profits

\[
R^j_t K^j_t = \max_{N^j_t} (K^j_t)^\alpha (N^j_t)^{1-\alpha} - w_t N^j_t,
\]

where \( w_t \) is the wage rate and \( R^j_t \) is given by

\[
R^j_t = \alpha \left( \frac{w_t}{1-\alpha} \right)^ {\frac{\alpha-1}{\alpha}}.
\]

We will show later that \( R^j_t \) is equal to the marginal product of capital in equilibrium.

Following Kiyotaki and Moore (1997, 2005, 2008), we assume that each firm \( j \) meets an opportunity to invest in capital over the small time interval \([t, t + \Delta t]\) with Poisson probability \( \pi \Delta t \). With probability \( 1 - \pi \Delta t \), no investment opportunity arrives. This assumption captures firm-level investment lumpiness and generates ex post firm heterogeneity. Assume that the arrival of an investment opportunity is independent across firms so that a law of large numbers can be applied to aggregation. This means that only a fraction \( \pi \Delta t \) of firms have investment opportunities at each time \( t \). There is no insurance market against having an investment opportunity. Investment transforms one unit of consumption goods into one unit of capital. Firms can buy or sell capital at a flow rate in a capital good market at the price \( Q_t \) at each time \( t \).

For simplicity, suppose that firms finance investment \( I^j_t \) using intratemporal loans (Jermann and Quadrini (2012)) and cannot use equity financing. This assumption reflects the fact that equity financing is more costly than debt financing and that selling a lump-sum amount of capital stock takes time.\(^6\) Intratemporal loans have no interest and the credit market for these loans are operated among firms. The interest rate on the intratemporal debt is zero and its price is one. Following Kiyotaki and Moore (2005, 2008) and Jermann and Quadrini (2012), we assume that there is a

\(^6\)In the online appendix we allow firms to sell a fraction \( \lambda \) of their capital to finance investment as in Kiyotaki and Moore (2005, 2008). We show that our results carry over except that \( \xi \) is replaced with \( \lambda + \xi \).
liquidity mismatch in that investment spending \( I_t^j \) must be paid immediately after the arrival of an investment opportunity at the beginning of time \( t \) and before the realization of investment returns \( Q_t I_t^j \). At the end of time \( t \) firms repay their debt after the realization of the returns. Firms without investment opportunities have cash and are the lenders. Assume that firms cannot borrow or save by trading intertemporal debt. We will relax this assumption in Section 6.1.

Let the ex ante market value of the firm prior to the realization of an investment opportunity shock be \( V_t(K_t^j) \), where we suppress aggregate state variables in the argument. Management acts in the best interest of shareholders so that \( V_t(K_t^j) = V_t^j \) satisfies the following Bellman equation by (2):

\[
rv_t(K_t^j) = \max_{K_{t-1}^j, I_{t-1}^j} \left( D_t^j + V_t(K_{t-1}^j) + \pi \left[ (L_t^j - I_t^j) + (Q_t I_t^j - L_t^j) \right] \right)
\]

subject to

\[
D_t^j = R_t K_t^j - Q_t \left( K_t^j + \delta K_t^j \right),
\]

and some additional constraints to be specified shortly. Here \( \delta \) represents the depreciation rate of capital and \( D_t^j \) represents dividends excluding net investment returns. Firm \( j \) receives internal funds \( R_t K_t^j \) and purchases (or sell) capital \( K_t^j \) at price \( Q_t \). When an investment opportunity arrives, the firm borrows \( L_t^j \) from other firms that do not have investment opportunities, and makes investment \( I_t^j \) before receiving investment returns \( Q_t I_t^j \). There is no jump in capital because we assume that the firm cannot sell a lump-sum amount of capital immediately after the Poisson arrival of an investment opportunity. Investment is subject to a financing constraint

\[
I_t^j \leq L_t^j.
\]

Loans \( L_t^j \) are fully repaid after the realization of investment returns \( Q_t I_t^j \) when \( Q_t I_t^j \geq L_t^j \).

The key assumption of our model is that loans are subject to credit constraints. In the baseline model we consider the following collateral constraint:

\[
L_t^j \leq V_t(\xi K_t^j).
\]

This constraint is endogenously derived from an incentive constraint in an optimal contract between firm \( j \) and the lender with limited commitment (Jermann and Quadrini (2012)). To best understand it, we consider a discrete time approximation. In the time interval \([t, t + \Delta t]\), the contract specifies investment \( I_t^j \) and loans \( L_t^j \) at the beginning of period \( t \), and repayments \( L_t^j \) at the end of period \( t \), only when an investment opportunity arrives with Poisson probability \( \pi \Delta t \). When no investment opportunity arrives, the firm does not invest and hence does not borrow. Firm \( j \) may default on debt at the end of period \( t \). If it defaults, then the firm and the lender will renegotiate the
loan repayment. In addition, the lender has the right to reorganize the firm. Because of default costs, the lender can only seize a fraction $\xi$ of capital $K^j_t$. Alternatively, we may interpret $\xi$ as an efficiency parameter in that the lender may not be able to efficiently use the firm’s assets $K^j_t$. The lender can run the firm with these assets at the beginning of period $t + \Delta t$ and obtain firm value $e^{-r\Delta t}V_{t+\Delta t}(\xi K^j_t)$ at time $t$. Or it can sell these assets to a third party at the going-concern value $e^{-r\Delta t}V_{t+\Delta t}(\xi K^j_t)$ if the third party can run the firm using assets $\xi K^j_t$ at the beginning of period $t + \Delta t$. This value is the threat value (or the collateral value) to the lender at the end of period $t$. Following Jermann and Quadrini (2012), we assume that the firm has all the bargaining power in the renegotiation through Nash bargaining and the lender obtains only the threat value. The key difference from Jermann and Quadrini (2012) is that the threat value to the lender is the going-concern value in our model, while Jermann and Quadrini (2012) assume that the lender liquidates the firm’s assets and obtains the liquidation value in the event of default.\footnote{U.S. bankruptcy law has recognized the need to preserve the going-concern value when reorganizing businesses in order to maximize recoveries by creditors and shareholders (see 11 U.S.C. 1101 et seq.). Bankruptcy laws seek to preserve going concern value whenever possible by promoting the reorganization, as opposed to the liquidation, of businesses. Bris, Welch and Zhu (2006) find empirical evidence that Chapter 11 reorganizations are less costly and more widely observed than Chapter 7 liquidations.}

Enforcement requires that, after an investment opportunity arrives at date $t$, the continuation value to the firm of not defaulting be no smaller than the continuation value of defaulting, that is,

$$-L^j_t + e^{-r\Delta t}V_{t+\Delta t}(K^j_{t+\Delta t}) \geq e^{-r\Delta t}V_{t+\Delta t}(K^j_{t+\Delta t}) - e^{-r\Delta t}V_{t+\Delta t}(\xi K^j_t).$$ \hfill (10)

This constraint ensures that there is no default in an optimal contract. Simplifying yields

$$L^j_t \leq e^{-r\Delta t}V_{t+\Delta t}(\xi K^j_t).$$

Taking the continuous-time limit as $\Delta t \to 0$, we obtain the credit constraint in (9).

Note that our modeling of the collateral constraint is different from that of Kiyotaki and Moore (1997):

$$L^j_t \leq \xi Q_t K^j_t.$$ \hfill (11)

where $\xi Q_t K^j_t$ is the liquidation value of the collateralized assets. We may reinterpret this constraint as an incentive constraint as in (10) where $e^{-r\Delta t}V_{t+\Delta t}(\xi K^j_t)$ is replaced with $\xi Q_t K^j_t$. In Section 5 we will show that this type of collateral constraint will rule out stock price bubbles.\footnote{In Chapter 14 of Tirole’s (2006) textbook, he shows that there may exist multiple equilibria in a simplified variant of the Kiyotaki and Moore (1997) model. In contrast to ours, these equilibria are characterized by a one-dimensional nonlinear dynamical system. Some equilibria may exhibit cycles. We would like to thank Jean Tirole for a helpful discussion on this point.} By contrast, according to (9), we allow the collateralized assets to be valued in the stock market as the going-concern value when the firm is reorganized and kept running using the collateralized assets after...
default. If both the firm and the lender believe that the firm’s assets are overvalued due to stock market bubbles, then these bubbles will help relax the collateral constraint, providing a positive feedback loop mechanism.

### 2.3 Competitive Equilibrium

Let \( K_t = \int_0^1 K_t^j d j \), \( I_t = \int_0^1 I_t^j d j \), and \( Y_t = \int_0^1 Y_t^j d j \) denote the aggregate capital stock, average investment of firms with investment opportunities, the aggregate labor demand, and aggregate output, respectively. Then a competitive equilibrium is defined as sequences of \( \{ Y_t \}, \{ C_t \}, \{ K_t \}, \{ I_t \}, \{ N_t \}, \{ w_t \}, \{ R_t \}, \{ V_t(K_t^j) \}, \{ I_t^j \}, \{ K_t^j \}, \{ N_t^j \} \) such that households and firms optimize and markets clear in that

\[
\psi_t^j = 1, \quad N_t = \int_0^1 N_t^j d j = 1,
\]

\[
C_t + \pi I_t = Y_t, \quad \dot{K}_t = -\delta K_t + \pi I_t.
\]

### 3 Equilibrium System

We first solve an individual firm’s optimal contracting problem (6) subject to (7), (8), and (9) when the wage rate \( w_t \) or \( R_t \) in (5) is taken as given. This problem does not give a contraction mapping and hence may admit multiple solutions. We conjecture that the ex ante firm value takes the following form:

\[
V_t(K_t^j) = Q_t K_t^j + B_t, \quad \text{(12)}
\]

where \( B_t \) is a nonpredetermined variable. Since \( V_t(K_t^j) \) must be always nonnegative due to limited liability, we must have \( B_t \geq 0 \). Note that \( B_t = 0 \) is a possible solution in general equilibrium. In this case we interpret \( Q_t K_t^j \) as the fundamental value of the firm. The fundamental value is proportional to the firm’s assets \( K_t^j \), which has the same form as that in Hayashi (1982). Intuitively, the firm has no fundamental value if it has no assets \( (K_t^j = 0) \). There may be another solution in which \( B_t > 0 \) due to optimistic beliefs. In this case, we interpret \( B_t \) as a bubble component since the firm is still valued at \( B_t \) even when there is no market fundamental, i.e., \( K_t^j = 0 \). In Section 6.2 we will show that when an intrinsically useless asset is traded in the market, its price and \( B_t \) follow the same asset-pricing equation (i.e., they are perfect substitutes), further justifying our interpretation of \( B_t \) as a bubble component.\(^9\)

\(^9\)According to the standard definition for exchange economies, a bubble is equal to the difference between the market value of an asset and the present value of the asset’s exogenously given dividends. It is subtle to apply this definition to our model since dividends are endogenously generated through investment and production. Bubbles can help firms make more investment and hence generate additional dividends. One criticism of the standard test for bubbles is that it is hard to separate bubbles from fundamentals in the data (see Gurkaynak (2008) and Galí and Gambetti (2013)). If one prefers not to use the term “bubbles,” one can call \( B_t \) a sunspot, self-fulfilling or speculative
The following result characterizes firm $j$’s optimization problem and its proof along with proofs of other results in the paper is given in Appendix A.

**Proposition 1** Suppose that $Q_t > 1$. Then the optimal investment level when an investment opportunity arrives is given by

$$I^j_t = \xi Q_t K^j_t + B_t,$$

where

$$\dot{B}_t = rB_t - B_t \pi (Q_t - 1),$$

$$\dot{Q}_t = (r + \delta) Q_t - R_t - \pi \xi Q_t (Q_t - 1),$$

and $R_t$ is given by (5). Moreover, the following transversality conditions hold:

$$\lim_{T \to \infty} e^{-rT} Q_T K_T = 0, \quad \lim_{T \to \infty} e^{-rT} B_T = 0.$$  \hfill (16)

To better understand the intuition behind this proposition, we substitute (12) and (7) into (6) to rewrite the firm’s problem explicitly as

$$rQ_t K^j_t + B_t = \max_{I^j_t, K^j_t} R_t K^j_t - Q_t \left( K^j_t + \delta K^j_t \right) + \pi (Q_t - 1) I^j_t$$

$$+ \dot{Q}_t K^j_t + Q_t \dot{K}^j_t + \dot{B}_t$$

subject to

$$I^j_t \leq \xi Q_t K^j_t + B_t,$$  \hfill (18)

where (18) follows from (8), (9), and (12). In the special case where $\xi = 0$, (18) reduces to the credit constraint studied by Martin and Ventura (2012) who show that firm value is equal to the bubble in a bubbly equilibrium in their OLG model. By contrast, we will show in Section 5 that firm value consists of a bubble component $B_t$ and a fundamental component $Q_t K_t$ in a bubbly equilibrium.

Given (12), we can reinterpret our credit constraint (9) following Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). In particular, (9) is equivalent to

$$-L^j_t + V_t(K^j_t) \geq (1 - \xi) Q_t K^j_t.$$  \hfill (19)

The left-hand side is the continuation value of the firm if the firm chooses to repay the debt $L^j_t$. The right-hand side is the value if the firm chooses to default. It steals a fraction $1 - \xi$ of firm assets and runs away. In an optimal contract, the preceding incentive constraint must hold. A notable component without affecting our results.
feature of this incentive constraint is that the value on default does not contain a bubble. But the stock price bubble $B_t$ can still relax the incentive constraint.

When an investment opportunity arrives, an additional unit of investment costs the firm one unit of the consumption good, but generates an additional value of $Q_t$. Since $Q_t = \partial V_t(K^j_t)/\partial K^j_t$ by (12), $Q_t$ represents the marginal value of the firm following a unit increase in installed capital, i.e., Tobin’s marginal $Q$. If $Q_t > 1$, the firm will make the maximal possible level of investment so that the credit constraint (9) or (18) binds. If $Q_t = 1$, the investment level is indeterminate. If $Q_t < 1$, the firm will make the minimal possible level of investment. This investment choice is similar to Tobin’s $Q$ theory (Tobin (1969) and Hayashi (1982)). In what follows, we impose assumptions to ensure $Q_t > 1$ in the neighborhood of the steady state equilibrium. We thus obtain the investment rule given in (13). Substituting this rule into the Bellman equation (17) and matching coefficients, we obtain equations (14) and (15).

Since $Q_t K^j_t$ cancels out in (17) due to the constant-returns-to-scale technology, the amount of capital purchases $K^j_t$ is indeterminate at the firm level. Thus firm dynamics are indeterminate. It is possible that some firms grow slower and others grow faster. The firm size is bounded by the aggregate capital stock. The indeterminacy of firm dynamics at the micro-level will not affect the aggregate equilibrium dynamics as shown in Proposition 2 below, which are our focus.

Equation (15) is an asset-pricing equation for capital. It says that the return on a unit of capital $rQ_t$ is equal to the sum of the marginal product of capital $R_t$, the additional value generated from new investment $\pi \xi Q_t (Q_t - 1)$, and capital gains, minus the depreciation $\delta Q_t$. Equation (14) is an asset-pricing equation for the bubble. Note that the stock price bubble is not attached to a traded intrinsically useless asset. Only stocks are directly traded in the market. Shareholders’ portfolio choice problem studied in Section 2.1 delivers the no-arbitrage equation (2). Management acts in the best interest of shareholders and solves the dynamic programming problem (6) or (17). Both (14) and (15) are derived from this optimization problem.

To interpret (14), we observe that the expectation of a higher firm value due to a bubble $B_t > 0$ allows the borrowing constraint to be relaxed as revealed by (18). Thus, bubbles are accompanied by a credit boom, leading the firm to make more investments by $B_t$. This raises firm value by $\pi (Q_t - 1) B_t$ if $Q_t > 1$ by (17), justifying the initial optimistic beliefs. We can interpret $\pi (Q_t - 1)$ as a “collateral yield” or a “liquidity premium.” The sum of the collateral yield and the capital gain of the bubble $\dot{B}_t$ is equal to the total return $rB_t$ as shown in (14). We will provide a further discussion of equation (14) in Section 5.

Although our model features a constant-returns-to-scale technology, marginal $Q$ is not equal to
average $Q$ in the presence of bubbles, because average $Q$ is equal to

$$\frac{V_t(K_t)}{K_t} = Q_t + \frac{B_t}{K_t}, \text{ for } B_t \neq 0.$$ 

Thus the existence of stock price bubbles invalidates Hayashi’s (1982) result. In the empirical investment literature, researchers typically use average $Q$ to replace marginal $Q$ under the constant-returns-to-scale assumption because marginal $Q$ is not observable. Our analysis demonstrates that the existence of collateral constraints implies that stock prices may contain a bubble component that makes marginal $Q$ not equal to average $Q$.

Next we aggregate individual firm’s decision rules and impose market-clearing conditions. We then characterize a competitive equilibrium by a system of nonlinear differential equations:

**Proposition 2** Suppose that $Q_t > 1$. Then the equilibrium variables $(B_t, Q_t, K_t)$ satisfy the following system of differential equations (14), (15), and

$$\dot{K}_t = -\delta K_t + \pi(\xi Q_t K_t + B_t), ~ K_0 \text{ given}, \tag{20}$$

and the transversality conditions (16), where $R_t = \alpha K_t^{\alpha - 1}$.

Equation (20) gives the law of motion for the aggregate capital stock derived from the market-clearing condition for capital. We use the market-clearing condition for labor and (5) to derive $R_t = \alpha K_t^{\alpha - 1}$. The system of differential equations (14), (15), and (20) provides us a tractable way to analyze equilibrium.

If we just focus on the firm’s optimization problem in partial equilibrium taking $Q_t$ and $w_t$ as given, then $V_t(K_t^0) = Q_t K_t^0 + B_t$ with $B_t > 0$ gives the maximal firm value. However, since $V_t(K_t^0)$ is the stock price, it is prone to speculation in general equilibrium. We will show later that both $B_t = 0$ and $B_t > 0$ can be supported in general equilibrium under some conditions. That is, our model has multiple equilibria. This reflects the usual notion of a competitive equilibrium: Given a price system, individuals optimize. If this price system also clears all markets, then it is an equilibrium system. There could be multiple equilibria with different price systems. And different price systems would generate different optimization problems with different sets of constraints.

After obtaining the solution for $(B_t, Q_t, K_t)$, we can derive the equilibrium wage rate $w_t = (1 - \alpha) K_t^\alpha$, aggregate output $Y_t = K_t^\alpha$, aggregate investment,

$$\pi I_t = \pi (\xi Q_t K_t + B_t), \tag{21}$$

and aggregate consumption $C_t = Y_t - \pi I_t$. We focus on two types of equilibrium.\(^{10}\) The first type

\(^{10}\) We focus on the case where either all firms have the same size of bubbles in their stock prices or no firms have
is bubbleless, for which $B_t = 0$ for all $t$. In this case, the market value of firm $j$ is equal to its fundamental value in that $V_t(K^j_t) = Q_tK^j_t$. The second type is bubbly, for which $B_t > 0$ for some $t$ and $V_t(K^j_t) = Q_tK^j_t + B_t$. Both types can exist due to self-fulfilling beliefs. We next study these two types of equilibrium.

4 Bubbleless Equilibrium

In a bubbleless equilibrium, $B_t = 0$ for all $t$. Equation (14) becomes an identity. We only need to focus on $(Q_t, K_t)$ determined by the differential equations (15) and (20) in which $B_t = 0$ for all $t$.

We first analyze the steady state, in which all aggregate variables are constant over time so that $\dot{Q}_t = \dot{K}_t = 0$. We use $X$ to denote the steady-state value of any variable $X_t$. We use a variable with an asterisk to denote its value in the bubbleless equilibrium.

**Proposition 3** (i) If

$$\xi \geq \frac{\delta}{\pi},$$

then there exists a unique bubbleless steady-state equilibrium with $Q^* = Q_E \equiv 1$ and $K^* = K_E$, where $K_E$ is the efficient capital stock satisfying $\alpha(K_E)^{a-1} = r + \delta$.

(ii) If

$$0 < \xi < \frac{\delta}{\pi},$$

then there exists a unique bubbleless steady-state equilibrium with

$$Q^* = \frac{\delta}{\pi \xi} > 1,$$

$$\alpha (K^*)^{a-1} = \frac{r \delta}{\pi \xi} + \delta.$$

In addition, $K^* < K_E$.

Assumption (22) says that if firms pledge sufficient assets as collateral, then the collateral constraints will not bind in equilibrium. The competitive equilibrium allocation is the same as the efficient allocation. The efficient allocation is achieved by solving a social planner’s problem in which the social planner maximizes the representative household’s utility subject to the resource constraint only. Note that we assume that the social planner also faces stochastic investment opportunities, like firms in a competitive equilibrium. Thus one may view our definition of the efficient allocation as the constrained efficient allocation. Unlike firms in a competitive equilibrium, the social planner is not subject to collateral constraints.

It is possible to have another type of equilibrium in which only a fraction of firms have different sizes of bubbles in their stock prices (see Appendix B).
Assumption (23) says that if firms do not pledge sufficient assets as collateral, then the collateral constraints will be sufficiently tight so that firms are credit constrained in the neighborhood of the steady-state equilibrium in which \( Q^* > 1 \). We can then apply Proposition 2 in this neighborhood. Proposition 3 also shows that the steady-state capital stock for the bubbleless equilibrium is less than the efficient steady-state capital stock. This reflects the fact that not enough resources are transferred from savers to investors due to the collateral constraints.

We can verify that \( R^* K^* > \pi I^* = \delta K^* \) so that firms without investment opportunities have enough funds to lend to firms with investment opportunities in the bubbleless steady state and hence in the neighborhood of the bubbleless steady state. More intuitively, in the small time interval \([t, t + \Delta t]\), the total funds needed to finance investment for all firms with investment opportunities are \( I_t \pi \Delta t \). The total cash owned by all firms without investment opportunities is \((1 - \pi \Delta t) R_t K_t \Delta t\). In a neighborhood of the bubbleless steady state \((1 - \pi \Delta t) R_t K_t \Delta t > I_t \pi \Delta t\) for a sufficiently small \( \Delta t \).

For (23) to hold, the arrival rate \( \pi \) of the investment opportunity must be sufficiently small, holding everything else constant. The intuition is that if \( \pi \) is too high, then too many firms will have investment opportunities so that the accumulated aggregate capital stock will be sufficiently large, thereby lowering the capital price \( Q \) to the efficient level as shown in part (i) of Proposition 3. Condition (23) requires that technological constraints at the firm level be sufficiently tight.

To study the local dynamics around the bubbleless steady state \((Q^*, K^*)\), we linearize the system of differential equations (15) and (20) around \((Q^*, K^*)\) for \( B_t = 0 \) for all \( t \). In the online appendix we prove that the linearized system has a positive eigenvalue and a negative eigenvalue so that \((Q^*, K^*)\) is a saddle point. Thus, in the neighborhood of \((Q^*, K^*)\), for any given initial value \( K_0 \), there is a unique initial value \( Q_0 \) such that \((Q_t, K_t)\) converges to the bubbleless steady state \((Q^*, K^*)\) along a unique saddle path as \( t \to \infty \).

5 Bubbly Equilibrium

In this section we study the bubbly equilibrium in which \( B_t > 0 \) for all \( t \). We will analyze the dynamic system for \((B_t, Q_t, K_t)\) given in (14), (15), and (20). Before we conduct a formal analysis later, we first explain why bubbles can exist in our infinite-horizon model. The key lies in understanding equation (14), rewritten here as

\[
\frac{\dot{Q}}{Q} = \frac{\dot{B}}{B} + \pi (Q - 1), \quad \text{for } B \neq 0. \tag{26}
\]

The first term on the right-hand side is the rate of capital gains of bubbles. The second term represents the collateral yield. Equation (14) or (26) reflects a no-arbitrage relation in that the
discount rate \( r \) on the bubble is equal to the sum of the rate of capital gains and the collateral yield. The existing literature on bubbles (e.g., Blanchard and Watson (1982), Tirole (1985), and Weil (1987)) typically studies bubbles on zero-payoff assets or unproductive assets with exogenously given payoffs. In this case the second term on the left-hand side of (26) vanishes and bubbles grow at the discount rate \( r \).

If we adopt the Kiyotaki and Moore (1997) collateral constraint (11), then it follows from (8) and (11) that optimal investment satisfies \( I_t^* = \xi Q_t K_t^2 \) when \( Q_t > 1 \). Substituting this investment rule and (12) into (17), we deduce that bubbles grow at the rate \( r \), i.e., \( r B_t = \dot{B}_t \). In this case there is no collateral yield. The transversality condition (16) implies that \( \lim_{T \to 0} e^{-rT} B_0 e^{rT} = B_0 = 0 \) and thus a stock price bubble cannot emerge. The transversality condition is not needed in OLG models and a bubble can emerge in dynamically inefficient OLG economies (Tirole (1985)).

By contrast, stock price bubbles in our model can influence fundamentals (dividends) due to the positive feedback effect through our collateral constraint (9) or (18). Specifically, one dollar bubble raises the collateral value by one dollar and allows the firm to borrow an additional dollar. The firm then makes one more dollar of investment when an investment opportunity arrives with the Poisson arrival rate \( \pi \). The investment raises firm value by \( Q_t > 1 \). Subtracting one dollar of costs, we then deduce that the second term on the right-hand side of (26) represents the net increase in firm value for each dollar of a bubble. This term causes the growth rate of bubbles to be lower than the discount rate \( r \). Thus the transversality conditions cannot rule out bubbles in our model. We can also show that the bubbleless equilibrium is dynamically efficient in our model. Specifically, the golden rule capital stock is given by \( K_{GR} = (\delta/\alpha)^{\frac{1}{\alpha-1}} \). One can verify that \( K^* < K_{GR} \). Thus one cannot use the condition for the OLG economies in Tirole (1985) to ensure the existence of bubbles. Next we will give our new conditions.

5.1 Steady State

We first study the existence of a bubbly steady state in which \( B > 0 \). We use a variable with a subscript \( b \) to denote this variable’s bubbly steady state value.

**Proposition 4** There exists a bubbly steady state satisfying

\[
\frac{B}{K_b} = \frac{\delta}{\pi} - \xi \left( \frac{r}{\pi} + 1 \right) > 0, \tag{27}
\]

\[
Q_b = \frac{r}{\pi} + 1 > 1, \tag{28}
\]

\[
\alpha (K_b)^{\alpha-1} = [(1-\xi)r + \delta] \left( \frac{r}{\pi} + 1 \right), \tag{29}
\]
if and only if the following condition holds:

\[ 0 < \xi < \frac{\delta}{r + \pi}. \]  

(30)

In addition, (i) \( Q_b < Q^* \), (ii) \( K_{GR} > K_E > K_b > K^* \), and (iii) the bubble-asset ratio \( B/K_b \) decreases with \( \xi \).

Condition (30) reveals that bubbles occur when \( \xi \) is sufficiently small, *ceteris paribus*. The intuition is as follows. When the degree of pledgeability is sufficiently low, the credit constraint is too tight and a bubble can help relax this constraint. This allows firms to borrow more and invest more. If the collateral constraint is not tight enough, firms can borrow sufficient funds to finance investment. In this case a bubble serves no function.

Note that condition (30) implies condition (23). Thus, if condition (30) holds, then there exist two steady state equilibria: one bubbleless and the other bubbly. The bubbleless steady state is analyzed in Proposition 3. Propositions 4 and 3 reveal that the steady-state capital price is lower in the bubbly equilibrium than in the bubbleless equilibrium, i.e., \( Q_b < Q^* \). The intuition is as follows. Bubbles help relax credit constraints and induce firms to make more investment than in the case without bubbles. The increased capital stock in the bubbly equilibrium lowers the marginal product of capital. Since the capital price partly reflects the present value of the marginal product of capital by (15), it is lower in the bubbly steady state than in the bubbleless steady state.

We can verify that \( R_b K_b > \pi I_b = \delta K_b \) in the bubbly steady state. By a similar discussion in Section 4, we deduce that firms without investment opportunities have enough funds to lend to firms with investment opportunities to finance investment in a neighborhood of the bubbly steady state.

Do bubbles crowd out capital in the steady state? In Tirole’s (1985) OLG model, households may use part of their savings to buy bubble assets instead of accumulating capital. Thus bubbles crowd out capital in the steady state. In our model, bubbles are attached to productive assets. If the capital price were the same in both bubbly and bubbleless steady states, then bubbles would induce firms to invest more and hence to accumulate more capital stock. However, there is a general equilibrium price feedback effect as discussed earlier. The lower capital price in the bubbly steady state discourages firms to accumulate more capital stock. The net effect is that bubbles lead to higher capital accumulation, unlike Tirole’s (1985) result. Note that bubbles do not lead to efficient allocation. The capital stock in the bubbly steady state is still lower than that in the efficient allocation.

How does the parameter \( \xi \) affect the size of bubbles? Proposition 4 shows that a smaller \( \xi \) leads to a larger bubble relative to capital in the steady state. This is intuitive. If firms can only pledge
a smaller amount of assets, they will face a tighter collateral constraint so that a larger bubble is needed to relax this constraint.

5.2 Dynamics

Now we study the stability of the bubbleless and bubbly steady states and the local dynamics around them. We linearize the equilibrium system (14), (15), and (20) around the two steady states. We then compute the eigenvalues of the linearized system and compare the number of stable eigenvalues with the number of predetermined variables. The equilibrium system has only one predetermined variable (\( K_t \)) and two nonpredetermined variables (\( B_t \) and \( Q_t \)).

**Proposition 5** Suppose that condition (30) holds. Then there exists a unique local equilibrium around the bubbly steady state \((B, Q_b, K_b)\) and the local equilibrium around the bubbleless steady state \((0, Q^*, K^*)\) has indeterminacy of degree one.

We prove that there is a unique stable eigenvalue for the linearized system around the bubbly steady state. Thus there is a neighborhood \( N \subset \mathbb{R}^4 \) of the bubbly steady state \((B, Q_b, K_b)\) and a continuously differentiable function \( \phi : N \rightarrow \mathbb{R}^2 \) such that given any \( K_0 \) there exists a unique solution \((B_0, Q_0)\) to the equation \( \phi(B_0, Q_0, K_0) = 0 \) with \((B_0, Q_0, K_0) \in N\), and \((B_t, Q_t, K_t)\) converges to \((B, Q_b, K_b)\) starting at \((B_0, Q_0, K_0)\) as \( t \) approaches infinity. The set of points \((B, Q, K)\) satisfying the equation \( \phi(B, Q, K) = 0 \) is a one-dimensional stable manifold of the system. If the initial value \((B_0, Q_0, K_0)\) is on the stable manifold, then the solution to the nonlinear system (14), (15), and (20) is also on the stable manifold and converges to \((B, Q_b, K_b)\) as \( t \) approaches infinity.

Although the bubbleless steady state \((0, Q^*, K^*)\) is also a local saddle point, the local dynamics around this steady state are different. In Appendix A we prove that the stable manifold for the bubbleless steady state is two dimensional because there are two stable eigenvalues for the linearized system around the bubbleless steady state. Thus the local equilibrium has indeterminacy of degree one. Formally, there is a neighborhood \( N^* \subset \mathbb{R}^4 \) of \((0, Q^*, K^*)\) and a continuously differentiable function \( \phi^* : N^* \rightarrow \mathbb{R} \) such that given \( K_0 \) for any \( B_0 > 0 \) there exists a unique solution \( Q_0 \) to the equation \( \phi^*(B_0, Q_0, K_0) = 0 \) with \((B_0, Q_0, K_0) \in N^*\), and \((B_t, Q_t, K_t)\) converges to \((0, Q^*, K^*)\) starting at \((B_0, Q_0, K_0)\) as \( t \) approaches infinity. Intuitively, along the two-dimensional stable manifold, the bubbly equilibrium is asymptotically bubbleless in that bubbles will burst eventually. There exist multiple bubbly equilibrium paths converging to the bubbleless steady state and the initial value \( B_0 > 0 \) is indeterminate. This feature suggests that self-fulfilling beliefs can generate economic fluctuations without any shocks to economic fundamentals.

---

6 Additional Assets

So far, we have assumed that households can trade stocks only without trading frictions. Firms can only take intratemporal debt without interest among themselves subject to credit constraints. In this section we introduce other types of assets for both households and firms to trade and study the conditions under which a bubble can exist in the presence of different types of assets. We first consider intertemporal private bonds with interest in Section 6.1. We then consider assets with rents and assets with a zero market fundamental in Section 6.2. The latter assets can be thought of as pieces of paper (money) or pure bubbles. We will show that both households and firms must face trading constraints for a bubble to exist in an infinite-horizon economy. A similar point is made by Kocherlakota (1992) and Santos and Woodford (1997) for pure-exchange economies.

6.1 Intertemporal Borrowing and Savings

Suppose that there is no intratemporal debt. But firms can borrow or save by selling or buying intertemporal bonds. Firms can use these bonds to finance investments subject to credit constraints. These bonds are in zero net supply. Households can also trade them, but are subject to short-sales constraints (Kocherlakota (1992, 2009) and Kiyotaki and Moore (2008)). One may interpret the bonds here as corporate bonds issued by firms and households cannot borrow by issuing bonds.

We will derive an equilibrium in which firms with investment opportunities choose to borrow by selling bonds, firms without investment opportunities choose to save by buying bonds, and households do not hold any bonds. We will also show that the equilibrium interest rate \( r_{ft} \) on the bonds is lower than the subjective discount rate \( r \) because these bonds provide liquidity to the firms and demand a liquidity premium. This is similar to the role of money studied by Kiyotaki and Moore (2008).

Let \( L^h_t \) denote the representative household’s bond holdings. It faces the budget constraint

\[
C_t + \int V_t^j \psi_t^j dj + \dot{L}^h_t = \int \Pi_t^j \psi_t^j dj + w_t N_t + r_{ft} L^h_t,
\]

and the short-sales constraint \( L^h_t \geq 0 \) for all \( t \). Let \( L^j_t \) denote firm \( j \)'s debt level. When \( L^j_t < 0 \), \( L^j_t \) means savings by firm \( j \). The market-clearing condition for the bonds is \( \int L^j_t dj = L^h_t \). Let \( V_t(K^j_t, L^j_t) \) denote the ex ante equity value of a typical firm \( j \) when its capital stock and debt level at time \( t \) are \( K^j_t \) and \( L^j_t \), respectively, prior to the realization of the Poisson shock. We suppress
the aggregate state variables in the argument. Then $V_t$ satisfies the following Bellman equation:

$$rV_t\left(K^j_t, L^j_t\right) = \max_{D_{0t}^j, D_{1t}^j} D_{0t}^j + V_t\left(K^j_t, L^j_t\right)$$

$$+ \pi \left[D_{1t}^j + V_t\left(K^j_t, L^j_{1t}\right) - V_t\left(K^j_t, L^j_t\right)\right]$$

subject to

$$\dot{L}^j_t = r_f L^j_t + D_{0t}^j - R_t K^j_t + Q_t \left(K^j_t + \delta K^j_t\right),$$

$$D_{1t}^j = Q_t I^j_t + \left(L^j_{1t} - L^j_t - I^j_t\right),$$

$$I^j_t \leq L^j_{1t} - L^j_t,$$

$$V_t\left(K^j_t, L^j_{1t}\right) \geq V_t\left(K^j_t, 0\right) - V_t(\xi K^j_t, 0),$$

where $D_{0t}^j$ and $D_{1t}^j$ denote dividends and $L^j_{1t}$ represents the new debt level when an investment opportunity arrives.

The interpretation of the Bellman equation (31) is as follows. When no investment opportunity arrives, firm $j$ pays out dividends $D_{0t}^j$, purchases new capital, and borrows or saves $L^j_t$, resulting in the flow-of-funds constraint (32). When an investment opportunity arrives with Poisson arrival rate $\pi$, firm $j$ borrows $L^j_{1t} \geq 0$, pays dividends $D_{1t}^j$, and makes investments $I_t^j$. The last two terms in (31) reflect the fact that equity value changes from $V_t(K^j_t, L^j_t)$ to $V_t(K^j_t, L^j_{1t})$ when the debt level jumps from $L^j_t$ to $L^j_{1t}$. In this case the flow-of-funds constraint is (33) and investment is subject to the financing constraint (34). Because we assume that firms cannot raise equity or sell capital to finance investment immediately after the arrival of an investment opportunity, there is no jump in the capital stock $K^j_t$.

Debt is subject to the credit constraint (35). The interpretation of (35) is similar to that of (9). Consider the discrete time approximation. Suppose that at time $t$, firm $j$ pledges a fraction $\xi$ of its capital $K^j_{t+\Delta t}$ as collateral. It may default on debt $L^j_{1t+\Delta t}$ at the beginning of period $t + \Delta t$. If it does not default, it obtains continuation value $V_{t+\Delta t}\left(K^j_{t+\Delta t}, L^j_{1t+\Delta t}\right)$. If it defaults, debt is renegotiated and the repayment $L^j_{1t+\Delta t}$ is relieved. The lender can seize the collateralized assets $\xi K^j_{t+\Delta t}$ and keep the firm running with these assets by reorganizing the firm. Thus the threat value to the lender is $V_{t+\Delta t}(\xi K^j_{t+\Delta t}, 0)$. Assume that firms have full bargaining power. We then have the incentive constraint

$$V_{t+\Delta t}(K^j_{t+\Delta t}, L^j_{1t+\Delta t}) \geq V_{t+\Delta t}\left(K^j_{t+\Delta t}, 0\right) - V_{t+\Delta t}(\xi K^j_{t+\Delta t}, 0).$$

The expression on the right-hand side of (36) is the value to the firm if it chooses to default. This constraint ensures that firm $j$ does not have an incentive to default. Its continuous-time limit as
\[ \Delta t \to 0 \text{ is (35).} \]

We conjecture and verify that equity value takes the following form:

\[ V_t(K^j_t, L^j_t) = Q_t K^j_t - L^j_t + B_t, \]  

(37)

and thus the credit constraint (35) becomes

\[ L^j_t \leq \xi Q_t K^j_t + B_t, \]  

(38)

where \( B_t \geq 0 \) is the bubble component of equity value. Given the value function in (37), the credit constraint (35) is equivalent to

\[ V_t(K^j_t, L^j_t) \geq (1 - \xi) Q_t K^j_t. \]  

(39)

This constraint admits the following interpretation as in Gertler and Kiyotaki (2010) and Gertler and Karadi (2011). At each time \( t \), firm \( j \) can default on the debt \( L^j_t \) and run away by stealing a fraction \((1 - \xi)\) of assets. The outside value on default is \((1 - \xi) Q_t K^j_t\). The inequality above ensures that the firm has no incentive to default.

**Proposition 6** Suppose that \( Q_t > 1 \). Then the equilibrium system for \((K_t, Q_t, B_t, r_{ft})\) is given by (14), (15), (20), and

\[ r_{ft} = r - \pi (Q_t - 1) < r, \]  

(40)

where \( R_t = \alpha K^\alpha_t \), and the usual transversality conditions hold.

When \( Q_t > 1 \), both (34) and (38) bind. Thus, with intertemporal bonds, firms can use both savings when \( L^j_t < 0 \) and new debt \( L^j_t = \xi Q_t K^j_t + B_t \) to finance investment. Because bonds are in zero net supply, to clear the bond market, some firms must be borrowers and repay previous debt \( (L^j_t > 0) \) before raising new debt to finance investment. Thus aggregate investment is the same as that in the case of intratemporal debt studied in Section 2 and the equilibrium system for aggregate variables \((K_t, Q_t, B_t)\) is also the same.

Equation (40) shows that the interest rate \( r_{ft} \) is equal to the subjective discount rate \( r \) minus a liquidity premium \( \pi (Q_t - 1) \). The liquidity premium exists because bonds can provide liquidity to firms and help them finance investment. Because both the bubble and the bonds can help firms finance investment to the same extent, they command the same amount of the liquidity premium. Since \( r > r_{ft} \), households prefer to borrow by selling bonds until their short-sales constraints bind. This property is similar to that for money studied by Kiyotaki and Moore (2008). Without a short-sales constraint, households would keep shorting these bonds (or effectively borrowing) until
In this case the liquidity premium would be zero so that \( Q_t = 1 \) and the economy would reach the efficient equilibrium and no bubble could exist. Next we characterize the steady state.

**Proposition 7**  
(i) If condition (23) holds, then there exists a unique bubbleless steady state \((K^*, Q^*, r_f^*)\) given by (24), (25), and \( r_f^* = r + \pi - \delta/\xi \).  
(ii) If condition (30) holds, then there exists a unique bubbleless steady state \((K^*, Q^*, r_f^*)\) given in part (i) and a unique bubbly steady state \((K_b, Q_b, B, r_f)\) given by (27), (28), (29), and \( r_f = 0 \).

Conditions (23) and (30) ensure that the assumption \( Q_t > 1 \) in Proposition 6 is satisfied in a neighborhood of both the bubbleless and bubbly steady states so that this proposition can be applied. Condition (30) is equivalent to the standard condition that the interest rate on bonds in the bubbleless steady state be less than the rate of growth of the economy \( (r_f^* < 0) \). Proposition 7 shows that the interest rate on bonds must be equal to zero in the bubbly steady state \( (r_f = 0) \). This is because both bonds and bubbles can be used to finance investment. But the bubble is intrinsically useless. If the steady-state interest rate on bonds is positive, then bonds dominate bubbles and a bubble cannot exist. This result is related to the rate-of-return-dominance puzzle in monetary economics.

One may introduce economic growth to generate the coexistence of a bubble and a positive-interest-rate bond as in Tirole (1985). Formally, let the production function be \( Y_t^j = (K_t^j)^{\alpha}(A_t^jN_t^j)^{1-\alpha} \), where \( A_t = e^{g_t} \) \( (g > 0) \) represents technical progress. We can then show that aggregate capital and the firm bubble grow at the same rate \( g \). Proposition 6 still characterizes the equilibrium system except that now \( R_t = \alpha (K_t^j/A_t^j)^{\alpha-1} \). It follows from (14) and (40) that the steady-state interest rate on the bond is equal to \( g > 0 \).

### 6.2 Assets with or without Rents

We now follow Tirole (1985) and introduce an asset that brings a real rent (or dividend) to the baseline model in Section 2. For simplicity, suppose that the rent is given by a constant \( X \geq 0 \) each time. If \( X = 0 \), then this asset has no market fundamental and is called a pure bubble (e.g., tulip). If \( X > 0 \), we call it a “tree.” Firms can invest in the asset and sell it to finance its investment when an investment opportunity arrives. We normalize the total supply of the asset to unity. We follow Kiyotaki and Moore (2008) and Kocherlakota (1992, 2009) and assume that both households and firms face short-sales constraints.

The representative household faces the budget constraint

\[
C_t + \int V_t^j \psi_t^j d\tilde{j} + P_t \tilde{M}_t^h = \int \Pi_t^j \psi_t^j d\tilde{j} + XM_t^h + w_t N_t,
\]

22
and the short-sales constraint \( M^h_t \geq 0 \), where \( M^h_t \) denotes the household’s asset holdings. We will show that households will sell the asset until the short-sales constraint binds in equilibrium.

Since we have shown that intertemporal borrowing is not essential for the existence of a bubble, we will focus on the case of intratemporal loans for simplicity. Let \( V_t(K^j_t, M^j_t) \) denote the ex ante market value of a typical firm \( j \) when its capital stock and asset holdings at time \( t \) are \( K^j_t \) and \( M^j_t \geq 0 \), respectively. Let \( P_t \) denote the market price of the asset. Then \( V_t \) satisfies the following Bellman equation:

\[
r V_t \left( K^j_t, M^j_t \right) = \max_{D^j_{0t}, D^j_{1t}, M^j_{1t}, I^j_t, L^j_t} \left[ D^j_{0t} + D^j_{1t} + V_t \left( K^j_t, M^j_{1t} \right) - V_t \left( K^j_t, M^j_t \right) \right]
\]

subject to

\[
D^j_{0t} = R_t K^j_t + X M^j_t - P_t M^j_{1t} - Q_t \left( K^j_t + \delta K^j_t \right),
\]

\[
D^j_{1t} = P_t \left( M^j_t - M^j_{1t} \right) + L^j_t - I^j_t + Q_t I^j_t - L^j_t,
\]

\[
I^j_t \leq P_t \left( M^j_t - M^j_{1t} \right) + L^j_t,
\]

\[
L^j_t \leq V_t \left( \xi K^j_t, 0 \right),
\]

where \( L^j_t \) and \( M^j_{1t} \geq 0 \) denote the intratemporal debt and the new asset holdings when an investment opportunity arrives.

The interpretations of the Bellman equation and the constraints are similar to those in Sections 2 and 6.1. In particular, equation (42) is the financing constraint. Firm \( j \) can sell assets \( (M^j_t - M^j_{1t}) \) and borrow \( L^j_t \) to finance investment. Equation (43) gives the credit constraints. Since firm \( j \) can sell the asset directly to finance investment when an investment opportunity arrives, it uses \( \xi K^j_t \) as collateral only.

An alternative formulation is to assume that the firm cannot sell the asset \( M^j_t \) but can use it as collateral to borrow. The borrowed funds can finance investment \( I^j_t \) and asset purchases \( M^j_{1t} \). The flow-of-funds constraint, the financing constraint, and the credit constraint become

\[
D^j_{1t} = L^j_t - I^j_t - P_t M^j_{1t} + \left( Q_t I^j_t - L^j_t \right),
\]

\[
I^j_t + P_t M^j_{1t} \leq L^j_t,
\]

\[
L^j_t \leq V_t \left( \xi K^j_t, M^j_t \right).
\]

By a similar analysis in Section 2.2, the right-hand side of (46) gives the threat value or the recovery
value to the lender if the firm defaults.

In Appendix A we show that $V_t$ takes the following form:

$$ V_t \left( K_t^j, M_t^j \right) = Q_t K_t^j + P_t M_t^j + B_t. $$

Given this value function, we can verify that the preceding two ways of financing are equivalent and give an identical equilibrium outcome.

**Proposition 8** Suppose $Q_t > 1$. Then the equilibrium system for $(K_t, Q_t, B_t, P_t)$ is given by

\begin{align}
\dot{K}_t &= -\delta K_t + \pi(Q_t \xi K_t + P_t + B_t), \\
\dot{Q}_t &= (r + \delta)Q_t - R_t - \pi(Q_t - 1)Q_t \xi, \\
\dot{B}_t &= r B_t - \pi(Q_t - 1)B_t, \\
\dot{P}_t &= r P_t - X - \pi(Q_t - 1)P_t,
\end{align}

where $R_t = \alpha K_t^\alpha - 1$, and the usual transversality conditions hold.

When $Q_t > 1$, firms with investment opportunities sell all their assets ($M_t^j = 0$) and take the maximal level of debt to finance investments. Firms without investment opportunities buy the assets and make lending. The interpretations of equations (47)-(49) are similar to those in Section 2. We interpret $\pi(Q_t - 1)$ as the collateral yield or the liquidity premium. Equation (50) is an asset-pricing equation for the asset. We rewrite it as

$$ \frac{\dot{P}_t}{P_t} + \frac{X}{P_t} = r - \pi(Q_t - 1) \text{ for } P_t > 0, $$

which implies that the return on the asset is equal to the subjective discount rate $r$ minus the liquidity premium.

Since the return on the asset is lower than the subjective discount rate, households have no incentive to hold the asset and want to sell it until their short-sales constraints bind. If there were no household short-sales constraint, then no arbitrage would force the liquidity premium to vanish so that $Q_t = 1$. In this case, there would be no investment friction and hence the economy would reach the efficient equilibrium in which no bubble could exist.

In the special case of $X = 0$, the asset is intrinsically useless and becomes a pure bubble with a zero market fundamental (e.g., money). Equation (50) reduces to

$$ \dot{P}_t = r P_t - \pi(Q_t - 1)P_t, $$

which implies that the return on the pure bubble $\dot{P}_t/P_t$ is less than the subjective discount rate.
when \( Q_t > 1 \). A similar point for money is made by Kiyotaki and Moore (2008). Comparing the equation above with (49) reveals that the bubble value \( P_t \) and the component \( B_t \) in firm value follow the same asset-pricing equation. By (47), we can determine the total value \( P_t + B_t \) only, but not \( B_t \) or \( P_t \) separately. That is, the bubble asset and the component \( B_t \) in firm value are perfect substitutes. This justifies our interpretation of \( B_t \) as a bubble component. In this case equilibria can still be characterized as in Sections 4 and 5. The bubbly equilibrium determines the total size of the bubble \( P_t + B_t \), but the decomposition of the total bubble is indeterminate. The equilibrium real allocation is independent of the decomposition. This result is analogous to that discussed in Section 5 of Tirole (1985).

In the case of \( X > 0 \), a bubbly equilibrium cannot exist because the tree with dividends dominates the bubble. To see this, suppose that a bubble exists in the steady state. Then (49) and (50) imply the steady-state relation

\[
0 = r - \pi(Q - 1) = X/P > 0,
\]

which is a contradiction. The intuition is that the tree with positive dividends plays the same role as a firm bubble in that both can be used to finance investment. The tree dominates the bubble since it delivers positive dividends, but the bubble has a zero market fundamental.

For a bubble and a tree to coexist, we may introduce economic growth to the model by allowing technical progress as in Section 6.1 (Tirole (1985)). In this case Proposition 8 still applies except that \( R_t = \alpha (K_t/A_t)^{\alpha-1} \). We then obtain

\[
\dot{b}_t = (r - g)b_t - \pi(Q_t - 1)b_t,
\]

\[
\dot{p}_t = (r - g)p_t - \pi(Q_t - 1)p_t - X/e^{gt},
\]

where \( b_t = B_t/A_t \) and \( p_t = P_t/A_t \). Since the detrended dividend \( X/e^{gt} \) vanishes in the long run, the firm bubble and the tree can coexist in the steady state. More generally, if dividends grow at a lower rate than the economy does, then dividends normalized by the trend disappears in the long-run. In this case the tree can still coexist with the firm bubble.

7 Stochastic Bubbles and Policy Implications

So far, we have focused on deterministic bubbles. Following Blanchard and Watson (1982) and Weil (1987), we now introduce stochastic bubbles to the baseline model in Section 2 with intratemporal loans. Suppose that a bubble exists initially, i.e., \( B_0 > 0 \). The bubble may burst in the future with
Poisson arrival rate $\theta$. Once it bursts, it will never be valued again.\footnote{If a bubble reemerged in the future, it would have value today by the no-arbitrage asset-pricing equation. To generate recurrent bubbles and crashes, Miao, Wang, and Xu (2013) introduce firm entry and exit. See Martin and Ventura (2012) and Wang and Wen (2012) for other approaches.}

### 7.1 Equilibrium with Stochastic Bubbles

First, we consider the case in which the bubble has collapsed. This corresponds to the bubbleless equilibrium studied in Section 4. We use a variable with an asterisk (except for $K_t$) to denote its value in the bubbleless equilibrium. In particular, $V^*(K_t^j, Q_t^j)$ denotes firm $j$’s value function. It is given by $V^*(K_t^j, Q_t^j) = Q_t^j K_t^j$, where $(Q_t^j, K_t)$ satisfies the equilibrium system (15) and (20) with $B_t = 0$. We may express the solution for $Q_t^j$ in a feedback form in that $Q_t^j = g(K_t)$ for some function $g$.

Next we consider the case in which the bubble has not collapsed. We assume that the debt contract is not contingent on extraneous beliefs or sunspots described earlier. Firms borrow only when an investment opportunity arrives. The credit constraint is still given by (9). We write firm $j$’s dynamic programming problem as follows:

$$
\tau V\left(K_t^j, Q_t, B_t\right) = \max_{I_t^j} R_t K_t^j - Q_t \left(\dot{K}_t + \delta K_t\right) + \dot{V} \left(K_t^j, Q_t, B_t\right) + \pi (Q_t - 1) I_t^j + \theta \left[V^* \left(K_t^j, Q_t^*\right) - V \left(K_t^j, Q_t, B_t\right)\right]
$$

subject to (8) and (9). The expression on the second line reflects the fact that once the bubble bursts, firm value changes from $V(K_t^j, Q_t, B_t)$ to $V^*(K_t^j, Q_t^*)$.

We can conjecture and verify that the value function takes the form $V(K_t^j, Q_t, B_t) = Q_t K_t^j + B_t$.

**Proposition 9** Suppose $Q_t > 1$. Before the bubble bursts, the equilibrium with stochastic bubbles $(B_t, Q_t, K_t)$ satisfies the following system of differential equations:

$$
\dot{B}_t = (r + \theta) B_t - \pi (Q_t - 1) B_t, \quad (51)
$$

$$
\dot{Q}_t = (r + \delta + \theta) Q_t - \theta Q_t^* - R_t - \pi (Q_t - 1) \xi Q_t, \quad (52)
$$

and (20), where $R_t = \alpha K_t^{\alpha - 1}$ and $Q_t^* = g(K_t)$ is the capital price after the bubble bursts.

Equation (51) is an asset-pricing equation for the bubble and reflects the possibility of the collapse of the bubble. In general, it is difficult to characterize the full set of equilibria with stochastic bubbles. In order to transparently illustrate the adverse impact of bubble bursting on the economy, we consider a simple type of equilibrium. Following Weil (1987) and Kocherlakota (2009), we study a stationary equilibrium with stochastic bubbles that has the following properties: The capital
stock is constant at the value $K_s$ over time before the bubble collapses. It continuously moves to the bubbleless steady-state value $K^*$ after the bubble collapses. The bubble is also constant at the value $B_s > 0$ before it collapses. It jumps to zero and then stays at this value after collapsing. The capital price is constant at the value $Q_s$ before the bubble collapses. It jumps to the value $g(K)$ after the bubble collapses and then converges to the bubbleless steady-state value $Q^*$ given in equation (24).

Our objective is to show the existence of $(B_s, Q_s, K_s)$. By (51), we can show that

$$Q_s = \frac{r + \theta}{\pi} + 1.$$  

Since $Q_s > 1$, we can apply Proposition 9 in some neighborhood of $Q_s$. Equation (52) implies that

$$0 = (r + \delta + \theta)Q_s - \theta g(K) - R - \pi(Q_s - 1)\xi Q_s,$$  

where $R = \alpha K^{\alpha - 1}$. The solution to this equation gives $K_s$. Once we have obtained $K_s$ and $Q_s$, we use equation (20) to determine $B_s$.

The difficult part is to solve for $K_s$ since $g(K)$ is not an explicit function. To show the existence of $K_s$, we define $\theta^*$ as

$$\frac{r + \theta^*}{\pi} + 1 = \frac{\delta}{\pi \xi} = Q^*.$$  

That is, $\theta^*$ is the bursting probability such that the capital price in the stationary equilibrium with stochastic bubbles is the same as that in the bubbleless equilibrium.

**Proposition 10** Let condition (30) hold. If $0 < \theta < \theta^*$, then there exists a stationary equilibrium $(B_s, Q_s, K_s)$ with stochastic bubbles such that $K_s > K^*$. In addition, if $\theta$ is sufficiently small, then consumption falls eventually after the bubble bursts.

As in Weil (1987), a stationary equilibrium with stochastic bubbles exists if the probability that the bubble will burst is sufficiently small. In Weil’s (1987) OLG model, the capital stock and output eventually rise after the bubble collapses. In contrast to his result, in our model the economy enters a recession after the bubble bursts in that consumption, capital and output all fall eventually. The intuition is that the collapse of the bubble tightens the collateral constraint and impairs investment efficiency.

Proposition 10 compares the economies before and after the bubble collapses only in the steady state. It would be interesting to see what happens along the transition path. Since analytical results are not available, we solve the transition path numerically and present the results in Figure 1. In this numerical example, we assume that the bubble collapses at time $t = 20$. Immediately after the bubble collapses, investment falls discontinuously and then gradually decreases to its bubbleless
steady-state level. But output and capital decrease continuously to their bubbleless steady-state levels since capital is predetermined and labor is exogenous. Consumption rises initially because of the fall in investment. But it quickly falls and then decreases to its bubbleless steady-state level. Importantly, the stock market crashes immediately after the bubble collapses in that the stock price drops discontinuously. One way to generate the fall in consumption and output on impact is to introduce endogenous capacity utilization (see the online appendix). Following the collapse of bubbles, the capacity utilization rate falls because the price of installed capital rises. Then both output and consumption fall on impact.

The existing macroeconomic models typically study dynamics around a unique deterministic steady state. These models need large shocks to fundamentals of the economy to generate a recession. For example, motivated by the recent Great Recession, Gertler and Kiyotaki (2010) and Gertler and Karadi (2011) introduce large capital quality shocks or net worth shocks. This literature is typically silent on the stock market behavior. In contrast to this literature, our model features two steady states. A change in beliefs or confidence can shift the economy from one good steady state to another bad steady state. A recession and a stock market crash can occur without any shocks to the fundamentals.
7.2 Policy Implications

We have shown that the collapse of bubbles generates a recession. Is there a government policy that can restore economic efficiency? The inefficiency in our model comes from the credit constraints. Bubbles help relax these constraints, while the collapse of bubbles tightens them. Suppose that the government can supply liquidity to firms by issuing public bonds in the baseline model of Section 2. These bonds are backed by lump-sum taxes.\footnote{As an idealized benchmark, we ignore the issues of moral hazard and distortional taxes.} Firms can use public bonds as collateral to relax their credit constraints. They can also buy and sell these bonds to finance investment. Firms and households cannot issue public bonds, i.e., they face short-sales constraints on public bonds.

Let the total issued quantity of government bonds be $M_t$ and the bond price be $P_t$. Let $T_t$ denote lump-sum taxes net of government spending. Then the government budget constraint is given by $\dot{M}_t P_t = -T_t$. Defining the government debt as $D_t = P_t M_t$, we can use the fact that $\dot{D}_t = \dot{P}_t M_t + \dot{M}_t P_t$ to derive

$$
\dot{D}_t - \dot{P}_t M_t = \dot{D}_t - \frac{\dot{P}_t}{P_t} D_t = -T_t \text{ if } P_t > 0.
$$

(55)

By a similar analysis to that in Section 6.2, we can derive that the price of the government bond satisfies the asset-pricing equation:

$$
r P_t = \dot{P}_t + \pi (Q_t - 1) P_t.
$$

If the government bond is not backed by taxes (i.e., $T_t = 0$), then it is a pure bubble (Diamond (1965)) and it can coexist with firm bubbles in equilibrium. As long as the government bond is backed by taxes (i.e., $T_t > 0$), then equation (55) implies that $\dot{D}_t = 0$ and $\dot{P}_t > 0$ in the steady state. Hence it dominates firm bubbles since $\dot{B}_t = 0$ in the steady state. The government can then choose the amount of debt by adjusting the size of lump-sum taxes so that firms can overcome credit constraints.

**Proposition 11** Suppose that assumption (30) holds. Let the government issue a constant value $D$ of government debt given by

$$
D_t = D \equiv K_E \frac{\delta - \pi \xi}{\pi} > 0.
$$

(56)

which is backed by lump-sum taxes $T_t = T \equiv r D$ for all $t$. Firms can trade government bonds or use them as collateral to finance investment. Then this credit policy will eliminate the bubble in firm value and enable the economy to achieve efficient allocation.

This proposition indicates that the government can design a policy that eliminates bubbles and achieves efficient allocation. The key intuition is that the government may provide sufficient
liquidity to firms so that firms do not need to rely on bubbles to relax credit constraints. The government plays the role of financial intermediaries by transferring funds from households to firms directly so that firms can overcome credit constraints. The government bond is a store of value and can also generate dividends for firms. The liquidity premium comes from the net benefit from new investment. Households prefer to sell bonds to firms as much as possible until the short-sales constraints bind when the interest rate on the government bond is lower than the subjective discount rate. Under the government policy in the proposition, the government can make the interest rate on the bond exactly equal to the subjective discount rate. Thus the liquidity premium is zero so that Tobin’s marginal $Q$ is equal to 1.

To implement the policy above, the government constantly retires the public bonds at the interest rate $r$ in order to keep the total bond value constant. To back the government bonds, the government levies constant lump-sum taxes equal to the interest payments of bonds.

Our discussion of government policy is related to that in Caballero and Krishnamurthy (2006) and Kocherlakota (2009). As in these studies, government bonds can serve as collateral to help relax credit constraints in our model. Unlike their proposed policies, our proposed policy can make the economy achieve the efficient allocation when government bonds are backed by lump-sum taxes. Unbacked public assets are intrinsically useless and may have positive value (a pure bubble). Issuing unbacked public assets can boost the economy after the collapse of stock price bubbles. But the real allocation is still inefficient and the bubble on unbacked public assets can also burst, causing the economy to enter a recession again.

8 Conclusion

We have developed a theory of credit-driven stock price bubbles in production economies with infinitely lived agents. Bubbles emerge through a positive feedback loop mechanism in which credit constraints derived from optimal contracts with limited commitment play an essential role. Our analysis differs from most studies in the literature that analyze bubbles on intrinsically useless assets or on assets with exogenously given rents or dividends in an endowment economy framework or an OLG framework. Our model can incorporate this type of bubbles and thus provides a unified framework to study asset bubbles. In future research it would be interesting to study how bubbles contribute to business cycles in a quantitative dynamic stochastic general equilibrium model (Miao, Wang and Xu (2013)), how bubbles affect long-run growth (Caballero, Farhi and Hammour (2006), Martin and Venture (2012), Hirano and Yanagawa (2013) and Miao and Wang (2014)), and what are the implications of asset price bubbles for monetary policy (Galí (2014)).

14This policy is analogous to the Friedman rule in monetary economics.
Appendices

A Proofs

**Proof of Proposition 1:** As described in the main text, we can rewrite the dynamic programming (6) as (17). Given the assumption $Q_t > 1$, (8) and (18) bind. We then obtains (13). Substituting this equation back into (17) and matching coefficients, we obtain (14) and (15). By the transversality condition (3) and the form of the value function,

$$
\lim_{T \to \infty} e^{-rt} (Q_T K_T + B_T) = 0.
$$

We then obtain (16). Q.E.D.

**Proof of Proposition 2:** Using the optimal investment rule in (13), we can derive the aggregate capital accumulation equation (20) and the aggregate investment equation (21). The first-order condition for the static labor choice problem (4) gives

$$w_t = (1 - \alpha) (K_j^1/N_j^1)^\alpha.\] We then obtain (5) and $K_j^1 = N_j^1 (w_t/(1 - \alpha))^{1/\alpha}$. Thus the capital-labor ratio is identical for each firm. Aggregating yields $K_t = N_t (w_t/(1 - \alpha))^{1/\alpha}$ so that $K_j^1/N_j^1 = K_t/N_t$ for all $j \in [0, 1]$. Substituting out $w_t$ in (5) yields $R_t = \alpha K_t^{1-\alpha} N_t^{1-\alpha} = \alpha K_t^{1-\alpha}$ since $N_t = 1$ in equilibrium. Aggregate output satisfies

$$Y_t = \int (K_j^1)^\alpha (N_j^1)^{1-\alpha} dj = \int (K_t^1/N_t^1)^\alpha N_t^1 dj = (K_t^1/N_t^1)^\alpha \int N_t^1 dj = K_t^1 N_t^{1-\alpha}.$$ This completes the proof. Q.E.D.

**Proof of Proposition 3:** (i) The social planner solves the following problem:

$$\max_{I_t} \int_0^\infty e^{-rt} (K_t^\alpha - \pi I_t) dt,$$

subject to

$$\dot{K}_t = -\delta K_t + \pi I_t, \ K_0 \text{ given},$$

where $K_t$ is the aggregate capital stock and $I_t$ is the investment level for a firm with an investment opportunity. From this problem, we can derive the efficient capital stock $K_E$, which satisfies $\alpha (K_E)^{\alpha-1} = r + \delta$. The efficient output, investment and consumption levels are given by $Y_E = (K_E)^\alpha$, $I_E = \delta/\pi K_E$, and $C_E = (K_E)^\alpha - \delta K_E$, respectively.

Suppose that assumption (22) holds. We conjecture that $Q^* = Q_t = 1$ in the steady state. In this case, firm value is given by $V \left( K_j^1 \right) = K_j^1$. The optimal investment rule for each firm satisfies $R_t = r + \delta = \alpha K_t^{1-\alpha}$. Thus $K_t^1 = K_E$. Given this constant capital stock for all firms, we must
have \( \delta K^*_t = \pi I^*_t \) for all \( t \). Let each firm’s optimal investment level satisfy \( I^*_t = \delta K^*_t / \pi \). Then, when assumption (22) holds, the investment and credit constraints, \( I^*_t = \delta K^*_t / \pi \leq \xi K^*_t = V \left( \xi K^*_t \right) \), are satisfied for all \( t \). We conclude that, under assumption (22), the solutions \( Q_t = 1, K^*_t = K_E \), and \( I^*_t = \delta / \pi \) give the bubbleless equilibrium, which also delivers the efficient allocation.

(ii) Suppose that (23) holds. Conjecture that \( Q_t > 1 \) in some neighborhood of the bubbleless steady state in which \( B_t = 0 \) for all \( t \). We can then apply Proposition 2 and derive the steady-state equations for (15) and (20) as

\[
\dot{Q} = 0 = (r + \delta) Q - R - \pi \xi Q (Q - 1), \quad (A.1)
\]

\[
\dot{K} = 0 = -\delta K + \pi (\xi Q K), \quad (A.2)
\]

where \( R = \alpha K^{-1} \). From these equations, we obtain the steady-state solutions \( Q^* \) and \( K^* \) in (24) and (25), respectively. Assumption (23) implies that \( Q^* > 1 \). By continuity, \( Q_t > 1 \) in some neighborhood of \((Q^*, K^*)\). This verifies our conjecture. Q.E.D.

**Proof of Proposition 4:** In the bubbly steady state, (14) and (20) imply that

\[
0 = r B - B \pi (Q - 1), \quad (A.3)
\]

\[
0 = -\delta K + [\xi Q K + B] \pi, \quad (A.4)
\]

where \( R = \alpha K^{-1} \). Solving equations (A.1), (A.3), and (A.4) yields equations (27), (28), and (29). By (27), \( B > 0 \) if and only if (30) holds. From (24) and (28), we deduce that \( Q_b < Q^* \). Using condition (30), it is straightforward to check that \( K_{GR} > K_E > K_b > K^* \). From (27), it is also straightforward to verify that the bubble-asset ratio \( B/K_b \) decreases with \( \xi \). Q.E.D.

**Proof of Proposition 5:** First, we consider the log-linearized system around the bubbly steady state \((B, Q_b, K_b)\). We use \( \hat{X}_t \) to denote the percentage deviation from the steady state value for any variable \( X_t \), i.e., \( \hat{X}_t = \ln X_t - \ln X \). We can show that the log-linearized system is given by

\[
\begin{bmatrix}
\frac{d\hat{B}_t}{dt} \\
\frac{d\hat{Q}_t}{dt} \\
\frac{d\hat{K}_t}{dt}
\end{bmatrix} = A
\begin{bmatrix}
\hat{B}_t \\
\hat{Q}_t \\
\hat{K}_t
\end{bmatrix},
\]

where

\[
A =
\begin{bmatrix}
0 & -(r + \pi) & 0 \\
0 & \delta + r - \xi (2r + \pi) & [(1 - \xi) r + \delta] (1 - \alpha) \\
\pi B / K_b & \xi (r + \pi) & -\pi B / K_b
\end{bmatrix}.
\]

(A.5)
We denote this matrix by
\[ A = \begin{bmatrix} 0 & a & 0 \\ 0 & b & c \\ d & e & f \end{bmatrix}, \]
where we deduce from (A.5) that \( a < 0, \ C > 0, \ D > 0, \ E > 0 \), and \( F < 0 \). Since \( \xi < \frac{\delta}{r+\xi} \), we have \( b = (1 - \xi)r + \delta - \xi(r + \xi) > 0 \). The characteristic equation for the matrix \( A \) is
\[ F(x) = x^3 - (b + f)x^2 + (bf - ce)x - acd = 0. \tag{A.6} \]
We observe that \( F(0) = -acd > 0 \) and \( F(-\infty) = -\infty \). Thus, there exists a negative root to the above equation, denoted by \( \lambda_1 < 0 \). Let the other two roots be \( \lambda_2 \) and \( \lambda_3 \). We rewrite \( F(x) \) as
\[ F(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3) = x^3 - (\lambda_1 + \lambda_2 + \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x - \lambda_1\lambda_2\lambda_3. \tag{A.7} \]
Matching terms in equations (A.6) and (A.7) yields \( \lambda_1\lambda_2\lambda_3 = acd < 0 \) and
\[ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = bf - cd < 0. \tag{A.8} \]

We consider two cases. (i) If \( \lambda_2 \) and \( \lambda_3 \) are two real roots, then it follows from \( \lambda_1 < 0 \) that \( \lambda_2 \) and \( \lambda_3 \) must have the same sign. Suppose \( \lambda_2 < 0 \) and \( \lambda_3 < 0 \). We then have \( \lambda_1\lambda_2 > 0 \) and \( \lambda_1\lambda_3 > 0 \). This implies that \( \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0 \), which contradicts equation (A.8). Thus we must have \( \lambda_2 > 0 \) and \( \lambda_3 > 0 \).

(ii) If either \( \lambda_2 \) or \( \lambda_3 \) is complex, then the other must also be complex. Let
\[ \lambda_2 = g + hi \text{ and } \lambda_3 = g - hi, \]
where \( g \) and \( h \) are some real numbers and \( i = \sqrt{-1} \). We can show that
\[ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 2g\lambda_1 + g^2 + h^2. \]
Since \( \lambda_1 < 0 \), the above equation and equation (A.8) imply that \( g > 0 \).

From the above analysis, we conclude that the matrix \( A \) has one negative eigenvalue and the other two eigenvalues are either positive real numbers or complex numbers with a positive real part. As a result, the bubbly steady state is a local saddle point and the stable manifold is one dimensional.

Next, we consider the local dynamics around the bubbleless steady state \((0, Q^*, K^*)\). We lin-
earize $B_t$ around zero and log-linearize $Q_t$ and $K_t$ and obtain the following linearized system:

$$\begin{bmatrix}
\frac{dB_t}{dt} \\
\frac{dQ_t}{dt} \\
\frac{dK_t}{dt}
\end{bmatrix} = \begin{bmatrix}
r - \pi(Q^* - 1) & 0 & 0 \\
0 & a & b \\
\frac{\pi}{K^*} & c & d
\end{bmatrix} \begin{bmatrix}
B_t \\
Q_t \\
K_t
\end{bmatrix},$$

where

$$a = \frac{R^*}{Q^*} - \xi Q^*, \quad b = \frac{R^*}{Q^*}(1 - \alpha) > 0,$$
$$c = \pi \xi Q^* > 0, \quad d = 0.$$ 

Using a similar method for the bubbly steady state, we analyze the three eigenvalues of the matrix $J$. One eigenvalue, denoted by $\lambda_1$, is equal to $r - \pi(Q^* - 1) < 0$ and the other two, denoted by $\lambda_2$ and $\lambda_3$, satisfy

$$\lambda_2 \lambda_3 = ad - bc = 0 - bc < 0. \quad (A.9)$$

It follows from (A.9) that $\lambda_2$ and $\lambda_3$ must be two real numbers with opposite signs. We conclude that the bubbleless steady state is a local saddle point and the stable manifold is two dimensional. Q.E.D.

**Proof of Proposition 6:** Conjecture that the value function takes the form in (37). Then the credit constraint (35) becomes (38). Substituting (37), (32), and (33) into (31) yields

$$r \left( Q_t K_i^j - L_t^j + B_t \right) = \max_{K_t^j, L_t^j} R_t K_t^j - Q_t \left( \dot{K}_t^j + \delta K_t^j \right) - r_f L_t^j + \dot{L}_t^j + \dot{Q}_t K_t^j + \dot{K}_t^j Q_t - \dot{L}_t^j + \dot{B}_t$$
$$+ \max_{I_t^j, L_t^j} \pi \left[ Q_t I_t^j + L_t^j - L_i^j - I_t^j - \left( L_{i_t}^j - L_t^j \right) \right].$$

subject to (34) and (38). Simplifying yields

$$r \left( Q_t K_i^j - L_t^j + B_t \right) = \max_{I_t^j, L_t^j} R_t K_t^j - Q_t \delta K_t^j - r_f L_t^j + \dot{Q}_t K_t^j + \dot{B}_t + \pi (Q_t - 1) I_t^j \quad (A.10)$$

subject to (34) and (38). When $Q_t > 1$, (34) and (38) bind. Thus optimal investment is given by

$$I_t^j = \xi Q_t K_t^j + B_t - L_t^j. \quad (A.11)$$

Substituting (A.11) back into (A.10) and matching coefficients of $K_t^j$, $L_t^j$, and $B_t$, we can derive (14), (15), and (40). Because $\dot{K}_t^j$ and $\dot{L}_t^j$ cancel out in the preceding dynamic programming problem, these values are indeterminate at the firm level.
When $Q_t > 1$, equation (40) implies that $rf_t < r$ so that households will not hold any bonds $L^h_t = 0$. The market-clearing condition for the bonds become $\int L^h_t \, dj = 0$. It follows from this condition and a law of large numbers, aggregate investment is given by $\pi (\xi Q_t K_t + B_t)$. We then obtain (20). Q.E.D.

**Proof of Proposition 7:** The proof follows from that for Propositions 3 and 4. Q.E.D.

**Proof of Proposition 8:** Let $V \left( K^j_t, M^j_t, B_t, Q_t, P_t \right)$ denote the value function. Conjecture that

$$ V \left( K^j_t, M^j_t, B_t, Q_t, P_t \right) = Q_t K^j_t + P_t M^j_t + B_t. $$

Substituting this conjectured function and the flow-of-funds constraints into the dynamic programming problem (41) yields

$$ r \left( Q_t K^j_t + P_t M^j_t + B_t \right) = \max_{M^j_t, I^j_t, L^j_t} R_t K_t + X M^j_t - P_t M^j_t - Q_t \left( \dot{K}^j_t + \delta K^j_t \right) + Q_t \dot{K}^j_t + K^j_t \dot{Q}_t + \dot{P}_t M^j_t + P_t \dot{M}^j_t + \dot{B}_t + \pi \left[ P_t \left( M^j_t - M^j_{1t} \right) + L^j_t - I^j_t + \left( Q_t I^j_t - L^j_t \right) + P_t \left( M^j_{1t} - M^j_t \right) \right] $$

subject to (42) and

$$ L^j_t \leq \xi Q_t K^j_t + B_t. $$

Under the assumption $Q_t > 1$, we have $M^j_{1t} = 0$ and the optimal investment level

$$ I^j_t = \xi Q_t K^j_t + P_t M^j_t + B_t. $$

Substituting this solution back into the preceding Bellman equation and matching coefficients, we obtain equations (48), (49), and (50).

It follows from (50) that $rP_t > \dot{P}_t + X$. Thus households will not hold the asset and their short-sales constraint binds. This means that the market-clearing condition for the asset is given by $\int M^j_t \, dj = 1$. By a law of large numbers, aggregate capital satisfies

$$ \dot{K}_t = \delta K_t + \pi \left( \xi Q_t K_t + P_t \int M^j_t \, dj + B_t \right). $$

We then obtain (47). Q.E.D.
Proof of Proposition 9: Substituting the conjectured value function \( V(Q_t, K^j_t, B_t) = Q_tK^j_t + B_t \) into the dynamic programming problem yields

\[
r \left(Q_tK^j_t + B_t\right) = \max_{I^j_t, K^j_t} R_t K^j_t - Q_t \left(\dot{K}^j_t + \delta K^j_t\right) + 
\dot{Q}_t K^j_t + Q_t \dot{K}^j_t + \dot{B}_t + 
\pi (Q_t - 1) I^j_t + \theta \left[Q_t K^j_t - \left(Q_t K^j_t + B_t\right)\right]
\]

subject to

\[
I^j_t \leq \xi Q_t K^j_t + B_t.
\]

When \( Q_t > 1 \), optimal investment is given by \( I^j_t = \xi Q_t K^j_t + B_t \). Substituting this rule back into the preceding Bellman equation and matching coefficients yields (51) and (52). Equation (20) follows from aggregation and the market-clearing condition. Q.E.D.

Proof of Proposition 10: Let \( Q(\theta) \) be the expression on the right-hand side of equation (53). We then use this equation to rewrite equation (54) as

\[
\alpha K^{\alpha - 1} - (r + \delta + \theta)Q(\theta) + \theta g(K) + (r + \theta)\xi Q(\theta) = 0.
\]

Define the function \( F(K; \theta) \) as the expression on the left-hand side of the equation above. Notice \( Q(\theta^*) = Q^* = g(K^*) \) by definition and \( Q(0) = Q_b \) where \( Q_b \) is given in (28). Condition (30) ensures the existence of the bubbly steady-state value \( Q_b \) and the bubbleless steady-state values \( Q^* \) and \( K^* \).

Define

\[
K_{\max} = \max_{0 \leq \theta \leq \theta^*} \left[\frac{(r + \delta + \theta - (r + \theta)\xi)Q(\theta) - \theta Q^*}{\alpha}\right]^{\frac{1}{\alpha - 1}}.
\]

By (29), we can show that

\[
K_b = \left[\frac{(r + \delta - r\xi)Q(0)}{\alpha}\right]^{\frac{1}{\alpha - 1}}.
\]

Thus we have \( K_{\max} \geq K_b \) and hence \( K_{\max} > K^* \). We want to prove that

\[
F(K^*; \theta) > 0, \quad F(K_{\max}; \theta) < 0,
\]

for \( \theta \in (0, \theta^*) \). If this is true, then it follows from the intermediate value theorem that there exists a solution \( K_s \) to \( F(K; \theta) = 0 \) such that \( K_s \in (K^*, K_{\max}) \).

First, notice that

\[
F(K^*, 0) = \alpha K^{\alpha - 1} - r(1 - \xi)Q_b - \delta Q_b > \alpha K_b^{\alpha - 1} - r(1 - \xi)Q_b - \delta Q_b = 0,
\]

36
and $F(K^*, \theta^*) = 0$. We can verify that $F(K; \theta)$ is concave in $\theta$ for any fixed $K$. Thus, for all $0 < \theta < \theta^*$,

$$F(K^*; \theta) = F\left(K^*, (1 - \frac{\theta}{\theta^*})0 + \frac{\theta}{\theta^*} \theta^*\right)$$

$$> (1 - \frac{\theta}{\theta^*})F(K^*, 0) + \frac{\theta}{\theta^*} F(K^*, \theta^*) > 0.$$  

Next we can derive

$$F(K_{\text{max}}; \theta) = \alpha K_{\text{max}}^{\alpha - 1} - (r + \delta + \theta)Q(\theta) + \theta g(K_{\text{max}}) + (r + \theta)\xi Q(\theta)$$

$$< \alpha K_{\text{max}}^{\alpha - 1} - (r + \delta + \theta)Q(\theta) + \theta g(K^*) + (r + \theta)\xi Q(\theta) < 0,$$

where the first inequality follows from the fact that the saddle path for the bubbleless equilibrium is downward sloping as shown in the online appendix so that $g(K_{\text{max}}) < g(K^*),$ and the second inequality follows from the definition of $K_{\text{max}}$ and the fact that $g(K^*) = Q^*.$

Finally, note that $Q(\theta) < Q^*$ for $0 < \theta < \theta^*.$ We use equation (A.4) and $K_s > K^*$ to deduce that

$$\frac{B_s}{K_s} = \frac{\delta}{\pi} - \xi Q(\theta) > \frac{\delta}{\pi} - \xi Q^* = 0.$$  

This completes the proof of the existence of a stationary equilibrium with stochastic bubbles $(B_s, Q_s, K_s).$

When $\theta = 0,$ the bubble never bursts and hence $K_s = K_b.$ When $\theta$ is sufficiently small, $K_s$ is close to $K_b$ by continuity. Since $K_b$ is smaller than the golden rule capital stock $K_{GR},$ $K_s < K_{GR}$ when $\theta$ is sufficiently small. Since $K^\alpha - \delta K$ is increasing for all $K < K_{GR},$ we deduce that $K_s^\alpha - \delta K_s > K^\alpha - \delta K^*.$ This implies that the consumption level before the bubble collapses is higher than the consumption level in the steady state after the bubble collapses. Q.E.D.

**Proof of Proposition 11:** The equilibrium with government debt can be characterized similarly to that in Proposition 8. In particular, $(K_t, Q_t, B_t, P_t)$ satisfies

$$\dot{K}_t = -\delta K_t + \pi(Q_t \xi K_t + P_t M_t + B_t),$$  

$$\dot{Q}_t = (r + \delta)Q_t - R_t - \pi(Q_t - 1)Q_t \xi,$$  

$$\dot{B}_t = r B_t - \pi(Q_t - 1)B_t,$$  

$$\dot{P}_t = r P_t - \pi(Q_t - 1)P_t,$$

and the usual transversality condition. The difference from Proposition 8 is that (i) the total supply of the government bond is $M_t$ instead of 1, and (ii) $X = 0.$
Under the policy in the proposition, equation (55) implies that \( \dot{P}_t = rP_t \). It follows from (A.15) that \( Q_t = 1 \). Substituting it into equation (A.13) reveals that \( R_t = r + \delta \). This equation gives the efficient capital stock \( K_E \) for all time \( t \) after the collapse of the bubble. Let \( K_t = K_E \) and \( P_t M_t = D \) in (A.12), where \( D \) is given by (56). We can show that \( B_t = 0 \) for all \( t \). Q.E.D.

**B Risk Averse Households**

Suppose that the representative household has the following utility function

\[
\int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt,
\]

where \( \rho \) is the subjective discount rate and \( \gamma \) is the risk aversion parameter. The household faces the budget constraint (1) subject to a no-Ponzi-game condition similar to that in footnote 5. Then we can derive the consumption Euler equation

\[
\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma} (r_t - \rho),
\]

where \( r_t \) is equal to the return on any stock \( j \) in absence of aggregate uncertainty and is also called the discount rate. Equation (2) holds where \( r \) is replaced with \( r_t \). Firm \( j \) solves the following dynamic programming problem:

\[
 r_t V_t \left(K^j_t\right) = \max_{I_t^j, L_t^j} \left(D_t^j + \dot{V}_t \left(K^j_t\right) + \pi \left[ \left(L_t^j - I_t^j\right) + \left(Q_t I_t^j - L_t^j\right)\right]\right)
\]

subject to (7), (8), and (9).

The aggregate state variables of the economy are \( B_t, Q_t \), and \( K_t \), where \( B_t \) represents the aggregate size of the bubble. The discount rate \( r_t \) is a function of the aggregate state variables. Conjecture that

\[
V_t \left(K^j_t\right) = Q_t K^j_t + B_t^j,
\]

where \( B_t^j \) is the bubble component in firm \( j \)'s stock price. Substituting this conjecture into the preceding dynamic programming problem yields

\[
r_t Q_t K^j_t + r_t B_t^j = \max_{I_t^j, K_t^j} R_t K^j_t - Q_t \left(\dot{K}_t^j + \delta K^j_t\right) + \pi (Q_t - 1) I_t^j
\]

\[
+ Q_t K^j_t + \dot{Q}_t K_t^j + \dot{B}_t^j,
\]

subject to

\[
I_t^j \leq \xi Q_t K_t^j + B_t^j.
\]

38
When $Q_t > 1$, the constraint (B.4) binds so that the optimal investment level is $I^j_t = \xi Q_t K^j_t + B^j_t$. Substituting this rule back into the Bellman equation and matching coefficients of $K^j_t$, we obtain

$$
\dot{Q}_t = (r_t + \delta) Q_t - R_t - \pi \xi Q_t (Q_t - 1),
$$
(B.5)

$$
\dot{B}^j_t = r_t B^j_t - B^j_t \pi (Q_t - 1).
$$
(B.6)

The usual transversality conditions must hold.

Since $B_t = \int B^j_t dj$, it follows from (B.6) that the aggregate bubble satisfies

$$
\dot{B}_t = r_t B_t - B_t \pi (Q_t - 1).
$$
(B.7)

The law of motion for aggregate capital still satisfies (20). The resource constraint is given by

$$
C_t + \pi (\xi Q_t K_t + B_t) = Y_t.
$$
(B.8)

The equilibrium system consists of five equations (20), (B.1), (B.5), (B.7), and (B.8) for five aggregate variables $(C_t, r_t, K_t, Q_t, B_t)$. The transversality condition also holds

$$
\lim_{T \to \infty} e^{-\int_0^T r_s ds} Q_T K_T = 0, \quad \lim_{T \to \infty} e^{-\int_0^T r_s ds} B_T = 0.
$$
(B.9)

Note that an equilibrium only determines the aggregate size $B_t$ of the bubble, but individual firm’s bubble size $B^j_t$ is indeterminate. So it is possible that some firms have no bubble and others have different sizes of bubbles.

We use a variable without the time subscript to denote its steady state value. Then (B.1) implies $r = \rho$ and hence the steady-state system is the same as that in the baseline model of Section 2. Our analysis of steady states in Sections 4 and 5 still applies to the case of risk averse households. We are unable to derive analytical results for local dynamics because the equilibrium system contains five equations. But it is straightforward to derive numerical solutions.
References


Miao, Jianjun and Pengfei Wang, 2013, Banking Bubbles and Financial Crises, working paper, Boston University.


Miao, Jianjun, Pengfei Wang, and Zhiwei Xu, 2013, A Bayesian DSGE Model of Stock Market Bubbles and Business Cycles, working paper, Boston University.


42