Chaotic Banking Crises and Banking Regulations

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Abstract

We study a model where limited enforcement permits bank owners to shift the risk of their asset portfolios to the depositors. Incentive compatible equilibria require the franchise value of the bank to exceed the value that the bank owners can obtain by undertaking excessively risky investments and defaulting on deposits when investment returns are low. Our model generates multiple stationary equilibria as well as chaotic equilibria that can lead to coordination failures, making bank runs, bank defaults, and banking crises more likely. We suggest that banking regulations, including leverage limits, restrictions on bank asset portfolios,central bank credit policies, as well as restrictions on bank size and deposit rate ceilings can be instituted not only to enhance stable franchise values and sound asset porfolios, but also to eliminate multiple and complex equilibria that may result in bank insolvency and moral hazard.

Keywords: Banking Crisis, Risk taking, Risk shifting, Chaos, Self-fulfilling Equilibria, Incentive constraints, Coordination failure.

JEL codes: E44, G01, G21

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1 Introduction

In response to the financial crisis of 2008 new legislation and regulations were introduced both in Europe and the US to assure the solvency of large banks and financial institutions, and to avoid the necessity of future government bailouts. In Europe Basel III Accords introduced regulatory standards on bank capital requirements by increasing bank liquidity and by establishing a minimum "leverage ratio."[1] In April 2014, the US Federal Reserve Bank, the FDIC, and the Office of the Comptroller of the Currency announced minimum leverage ratios of 5% for eight systemically important bank holding companies, and of 6% for their FDIC insured banks. Earlier in 2010 the US Congress passed the Dodd-Frank bill that increased the government oversight of large financial institutions. This bill included the Volcker rule, a modern version of the Glass-Steagall Act of 1932, attempting to restrict riskier investment bank activities and insulating such activities from commercial banking loans financed by bank deposits.

Such measures are designed to protect depositors by establishing capital requirements and minimum leverage ratios, and by limiting risky bank investments. They are instituted because a deregulated competitive environment may lead bank managers to acquire portfolios that expose depositors to excessive risk, and also create moral hazard problems under FDIC insurance. (See Keeley (1990), Gorton and Rosen (1995), Demsetz, Saidenberg, and Strahan (1997) and Hellmann, Murdock, and Stiglitz (2000), Matutes and Vives (2000), and Gorton (2010)). Excessive risk taking however may be avoided if banks operate in a regulated environment that assures sufficient ongoing profits, that is a sufficiently high franchise value (i.e., the present discounted value of payouts from operating the bank).[2] Bank owners and managers may then refrain from undertaking excessively risky investments for fear of bank runs and bankruptcy which results in their losing their equity, their firm-specific human capital, and the benefits of control (see Demsetz, Saidenberg, Strahan (1995)). The recent economics literature has modelled such risk-shifting to depositors and moral hazard problems by introducing borrowing or incentive compatibility constraints for banks. This literature is typically confined to static or three-period models as surveyed by Allen and Gale (2007).

We extend this literature by constructing an infinite-horizon model of banks. In the model

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[1] The generally applicable leverage ratio under the 2013 revised capital approaches is the ratio of a banking organization's tier 1 capital to its average total consolidated assets as reported on the banking organization's regulatory report minus amounts deducted from tier 1 capital. The banks were expected to maintain a leverage ratio in excess of 3% under Basel III.

[2] In addition to bank regulations and oversight, bank charters can also enhance the franchise value of banks with subsidies in the form of limited entry into banking, local deposit monopolies, interest-rate ceilings, and underpriced deposit insurance. See Gorton (2009)

households deposit their savings only in banks whose franchise value deters bank owners from investing in excessively risky assets. These constitute incentive compatibility constraints that arise under limited commitment, as in Kehoe and Levine (1993), Alvarez and Jermann (2000) and more recently, as in Gertler and Karadi (2011), Gertler and Kiyotaki (2013), Miao and Wang (2012b) for infinite-horizon models with banks. In the last three papers, bank managers can divert a fraction of bank assets for their own benefit and the franchise value of the bank acts as a deterrent to the diversion of assets by managers. Unlike these papers, we focus on banks' risk taking behavior.[3] Under incentive constraints bank owners and managers recognize the franchise value of their banks and they refrain from risks that may lead to the loss of their banking business. Competition among banks then drives profits not to zero, but to the point where the discounted value of franchise profits is at least as large as the value of undertaking risky investments that may be followed by deposit withdrawals, bank runs and the loss of the franchise value.

We show in section 3 that under the incentive constraints in our model we can obtain two stationary rational expectations equilibria, one with a high interest rate on deposits and a high level of deposits, the other with a low interest rate and a low level of deposits.[4] Furthermore, we show in section 4 that we can have a continuum of non-stationary chaotic equilibria, as well as equilibria with "branching" dynamics, that is, equilibrium dynamics described by difference correspondences with multiple values at each point in time, instead of the standard dynamics of difference equations. These complex equilibrium dynamics can occur even in the case of a unique steady state. Clearly, the plethora of such complicated equilibria creates problems of expectation coordination. The difficulty of rational expectations forecasting under such conditions may well lead to fragility, divergent expectations, speculative risk taking and bank runs. Therefore we can interpret bank regulations not only as mechanisms to assure that banks do not violate their incentive constraints, and do not invest with moral hazard, but also as tools to eliminate the multiplicity of complicated equilibria leading to coordination failure and costly bank runs.

In section 5 we discuss various banking regulations to eliminate such multiple and complex equilibria. The regulations and policies we study are capital or equity requirements, leverage

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[3] Miao and Wang (2012b) also consider other types of off-equilibrium behavior and outside options of banks. But they do not study banks' risk taking behavior.

[4] The literature on models of incentive compatibility or borrowing constraints due to limited enforcement that can generate stationary and non-stationary equilibria and bubbles in macroeconomic problems is too large for us to cite all relevant papers. Recent contributions include Benhabib and Wang (2013), Gu et al. (2014), Liu and Wang (2013), and Miao and Wang (2012a, b).

ratios, central bank credit policies to manage liquidity, limits on bank size, and restrictions on risky investments Such restrictions can come at a cost, in particular, at the cost of eliminating sustainable rational expectations equilibria with high lending and high interest on deposits.[5] The trade-off reflects the potential costs of financial fragility that results not only from moral hazard considerations, but also from the difficulty of coordinating expectations when multiple equilibria exist.

2   A Baseline Model

We consider an infinite-horizon environment. There is no aggregate uncertainty. Time is discrete and denoted by $t = 0, 1, 2, ...$ There are two types of identical agents: households and bankers. We will focus on frictions in the banking sector. We thus intentionally make the household problem as simple as possible.

2.1   Households

We model the supply of deposits to bank in a highly simplified fashion. There is a continuum of households of unit measure on $[0, 1]$. Each household $i \in [0, 1]$ receives constant income Y at the beginning of each period t. It has access to a storage technology that yields a return $r_t^i = a + x_t^i$ at the end of period t, where $a > 0$ is a constant and x_t^i is independently and identically drawn from a power function distribution with the cumulative distribution function $G(x) = (x/x_{\max})^\alpha$, $\alpha > 0$, on $[0, x_{\max}]$. The household can deposit $D_t^i \in [0, Y]$ at a bank and receive a return constant R_t. It maximizes expected utility

$$E \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right)$$

subject to the budget constraint

$$c_t^i = R_t D_t^i + \left(Y - D_t^i\right) r_t^i,$$

where u is strictly increasing and strictly concave.

Since the household's decision problem is essentially static due to our modeling, the household decision rule is simply given by

$$D_t^i = \begin{cases} Y & \text{if } r_t^i < R_t \\ 0 & \text{otherwise} \end{cases}. \tag{1}$$

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[5]See however section 5.3 below on central bank credit policy.

We can then compute total deposits as

$$D_t = \int D_t^i di = YG\left(R_t - a\right) = Y\left[\frac{R_t - a}{x_{\max}}\right]^\alpha.$$

It follows that the deposit supply function is given by

$$R_t = R\left(D_t\right) \equiv a + x_{\max}\left(\frac{D_t}{Y}\right)^{1/\alpha}.$$

Since $D_t \in [0, Y]$, $R_t \in [a, a + x_{\max}]$. Note that this supply function satisfies the property that $R\left(D\right)D$ is strictly convex in D. It turns out that this is the key property to generating multiple equilibria and chaotic dynamics. There are other approaches to the microfoundation of deposit supply with such a property.

If we let $\alpha = 1$,

$$R_t = a + \frac{x_{\max}}{Y}D_t = a + bD_t, \tag{2}$$

where $b = x_{\max}/Y$. We will use this simple deposit supply function in what follows and take a and b as parameters.

2.2 Banks

There is a continuum of competitive banks indexed by $j \in [0, 1]$. Each bank is managed by a risk-neutral banker. Each banker has access to two investment projects each period: a safe project and a risky project. Each unit of investment in the safe project yields a constant return A. Each unit of investment in the risky project generates a constant return A_r with probability $\pi \in (0, 1)$ and zero with probability $1 - \pi$. We assume that $A > \pi A_r$ so that it is socially optimal to invest in the safe project. This implies that the banker has no incentive to invest his own net worth in the risky project. However due to limited liability the banker will be tempted to engage in risk-shifting, and to invest in the risky project using funds from depositors if $A - R_t < \pi\left(A_r - R_t\right)$. In the event of failure, the banker is protected by limited liability and hence will not pay the depositors. Nevertheless the banker who gambles with depositors' money and invests in such a risky project would lose his reputation and the trust of depositors. He would not be able to attract any deposits in the future and would have to quit the banking business. We therefore introduce an incentive (compatibility) constraint associated with an optimal contracting problem under limited commitment, similar to those in Kehoe and Levine (1993) and Alvarez and Jermann (2000).

We consider the optimal implicit contract in which banker j will always invest in the safe project each period. If banker j deviates and invests in the risky project, he will receive profits

$AN_{jt} + \pi(A_r - R_t)D_{jt}$ in the current period but will be punished by being excluded from banking in the future. We can then write the incentive constraint as

$$V_t(N_{jt}) \geq AN_{jt} + \pi(A_r - R_t)D_{jt}, \tag{3}$$

where $V_t(N_{jt})$ denotes banker j's value function in the optimal contract, N_{jt} denotes the bank's net worth, and D_{jt} denotes deposits in the bank. This constraint ensures that banker j has no incentive to invest in the risky project in any period.[6]

Given that banker j always invests in the safe project in the optimal contract, we can write his flow-of-funds constraint as

$$X_{jt} + N_{jt+1} + \varphi(N_{jt+1}) \leq AN_{jt} + (A - R_t)D_{jt}, \tag{4}$$

where X_{jt} represents the banker's payoffs and the term $\varphi(N_{jt+1})$ is a strictly convex increasing function representing adjustment costs of bank capital. Capital adjustment costs can result from the moral hazard problem emphasized by Jensen (1986): the manager may divert some of the bank capital (net worth) for their own benefit and monitoring can become more difficult as the amount of asset in management increases. There can also be physical adjustment costs, for example disruption costs during installation of new software or IT systems with associated learning and maintenance costs required as banking capital grows. To derive a closed-form solution, we assume that $\varphi(N) = N^2/(2\lambda)$, $\lambda > 0$.

Assume that banker j may choose to default and exit after investing in the safe project in the end of period t. In this case he obtains profits $AN_{jt} + (A - R_t)D_{jt}$. If he chooses to stay in business, he obtains $X_{jt} - \phi + \beta V_{t+1}(N_{jt+1})$, where ϕ represents a fixed cost of managing the bank. We will show later that the fixed cost ϕ plays a crucial role in the subsequent analysis.[7] Assume that both bankers and depositors have the same discount factor β. We can then write banker j's decision problem by dynamic programming

$$V_t(N_{jt}) = \max_{N_{jt+1}, X_{jt}, D_{jt}} \{X_{jt} - \phi + \beta E_t V_{t+1}(N_{jt+1}), \ AN_{jt} + (A - R_t)D_{jt}\}, \tag{5}$$

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[6] For the role of reputation in a setting where unregulated shadow banks coexist with regulated banks under limited enforcement and incentive constraints, see Ordonez (2013).

[7] Fixed costs, for example administrative expenses are significant in the banking business. These costs can also be interpreted as an opportunity cost of the banker, who gives up an occupation in other industries. The bank also has to have equipment like computers and software and pay rent for its office building. In addition, the banker has to spend to spend costly effort to acquire information, conduct market research or manage and monitor its investment.All of these costs are fixed to some extent.

subject to (3) and (4). Using the flow-of-funds constraint (4), banker j's decision problem in (5) becomes

$$V_t(N_{jt}) = \max_{N_{jt+1}, D_{jt}} AN_{jt} + (A - R_t)D_{jt} + \{0, -N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1})\}, \quad (6)$$

subject to the incentive constraint (3). Banker j takes the market deposit rate R_t as given and chooses net worth N_{jt+1} and the amount of deposits D_{jt}.

The preceding problem is related to those in Gertler and Karadi (2011), Gertler and Kiyotaki (2013), and Miao and Wang (2012b). Gertler and Karadi (2011) and Gertler and Kiyotaki (2013) assume that a banker dies stochastically and consumes the bank net worth only when he dies. In addition, they assume that at the end of the period the banker can divert a fraction ζ of bank assets. If the banker diverts, depositors will force the bank into bankruptcy. In this case the incentive constraint that ensures the banker will never divert is given by

$$V_t(N_{jt}) \geq \zeta(N_{jt} + D_{jt}), \quad \zeta \in (0, 1].$$

Miao and Wang (2012b) generalize the approach of Gertler and Karadi (2011), Gertler and Kiyotaki (2013) by analyzing several other types of incentive constraints. However, they have not considered the incentive constraint as in (3). The key innovation of (3) is that the banker's deviation is to choose risky investments. We believe that this incentive problem is more prevalent than diverting funds in the banking sector.

Using (6), we can rewrite the incentive constraint (3) as

$$[\pi(A_r - R_t) - (A - R_t)] D_{jt} \tag{7}$$
$$\leq \max\{-N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1}), 0\}.$$

This becomes an endogenous borrowing constraint. The left-hand side of (7) represents additional gains from risk taking and the right-hand side represents the loss of profits from the future banking business or the opportunity cost of risk taking. Whenever $A > R_t$ and $\pi(A_r - R_t) > (A - R_t)$, the banker's optimization problem (6) implies that this constraint will bind. The intuition is as follows. When $A > R_t$, the banker would like to borrow as much as possible to invest in the profitable safe project. But when $\pi(A_r - R_t) > (A - R_t)$, risk taking is also profitable and the banker would like to gamble the depositors' money. The incentive constraint in (3) or (7) limits the banker's gambling behavior by restricting the amount of borrowings.

When $\pi(A_r - R_t) > (A - R_t)$, the incentive constraint in (3) ensures that staying in the banking business is always more profitable than quitting and obtaining $(A - R_t) D_t$ if $D_t > 0$ in any period. Thus the banker will never exit when $D_{jt} > 0$. Exit is possible only if $D_{jt} = 0$.

The first-order condition for N_{jt+1} is

$$1 + \varphi'(N_{jt+1}) = \beta E_t \frac{\partial V_{t+1}(N_{jt+1})}{\partial N_{jt+1}}, \tag{8}$$

where the left-hand side of this equation gives the marginal cost of increasing bank capital N_{jt+1} and the right-hand side represents the associated marginal benefit.

Conjecture that the value function takes the following form:

$$V_t (N_{jt}) = A N_{jt} + v_t, \tag{9}$$

where v_t is a term to be determined, which represents the future value of the bank. Following the finance literature, we may interpret $A N_{jt}$ as the bank's value of assets in place and v_t as the value of the growth opportunities. We may also interpret v_t as the franchise value of the bank.

Given the preceding conjectured value function, the envelope condition gives

$$\frac{\partial V_t(N_{jt})}{\partial N_{jt}} = A.$$

Substituting this equation into (8), we can derive that

$$N_{jt+1} = \lambda(\beta A - 1) \equiv \bar{N}. \tag{10}$$

Thus the optimal net worth is always constant. The following assumption ensures that $\bar{N} > 0$.

Assumption 1 $\beta A > 1$.

Since λ and $\bar{N}$ have a one-to-one relationship, we will treat $\bar{N}$ as an exogenous parameter instead of λ.

Using (9) and (10), we can derive that

$$-N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1}) = \frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta v_{t+1}.$$

It follows that the binding borrowing constraint (7) becomes

$$D_{jt} = \frac{\max\left\{\frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta v_{t+1}, 0\right\}}{\pi(A_r - R_t) - (A - R_t)}. \tag{11}$$

Substituting this expression into the Bellman equation (6) and matching coefficients, we obtain

$$v_t = \pi(A_r - R_t)D_{jt} = \pi(A_r - R_t)\frac{\max\left\{\frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta v_{t+1}, 0\right\}}{\pi(A_r - R_t) - (A - R_t)}. \tag{12}$$

Because the incentive constraint (3) binds, equation (9) implies that the future bank value v_t

is equal to the profits from investing in the risky project.

Now we summarize the solution to banker j's decision problem as follows:

Proposition 1 *Suppose that assumption 1 holds and*

$$A > R_t > \frac{A - \pi A_r}{1 - \pi} > 0. \tag{13}$$

Then banker j's value function $V_t(N_{jt})$ is given by (9), where v_t satisfies the difference equation (12) and the transversality condition $\lim_{t \to \infty} \beta^t v_t = 0$. The optimal net worth N_{jt+1} and deposit issuance D_{jt} are given by (10) and (11).

This proposition gives an individual bank's demanded deposits as a function of the market deposit rate R_t in (11). This demand is also affected by the future value of the growth opportunities v_t or the franchise value of the bank. As will be clear later, this is the key feature that generates multiple equilibria.

The transversality condition in the preceding proposition ensures that the value function satisfies $\lim_{t \to \infty} \beta^t V_t(N_{jt}) = 0$. Technically this condition ensures that the value function derived from the Bellman equation is equal to the optimal value in a sequence problem (Stokey, Lucas, and Prescott (1989, Theorem 4.3)). Intuitively this condition ensures that no asset is left with positive value in the infinite future.

The condition in (13) involves the deposit rate R_t, which is taken as given in the banker's decision problem. Since R_t is endogenous in general equilibrium, we need to check (13) ex post in equilibrium. By the microfoundation of the deposit supply in Section 2.1, the equilibrium deposit rate must satisfy $R_t \in [a, a + x_{\max}]$. Thus we can replace (13) by the following primitive assumption:

Assumption 2 *Let*

$$A > a + x_{\max} > a > \frac{A - \pi A_r}{1 - \pi} > 0. \tag{14}$$

2.3 Equilibrium

In equilibrium, we have $\int N_{jt+1} dj = N_{t+1} = \bar{N}$ and $\int D_{jt} dj = D_t$. In particular, total deposits demanded by banks are equal to total deposits supplied by depositors.

Using equation (12), we can derive that the aggregate deposit demand D_t satisfies the difference equation

$$D_t \left[\pi(A_r - R_t) - (A - R_t) \right] = \max \left\{ \frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta\pi(A_r - R_{t+1})D_{t+1}, 0 \right\}. \qquad (15)$$

Imposing the market-clearing condition and using the deposit supply function $R_t = R(D_t)$ in (15), we can derive that the equilibrium sequence of aggregate deposits $\{D_t\}$ satisfies the following difference equation:

$$\Phi(D_t) \equiv D_t \left[\pi(A_r - R(D_t)) - (A - R(D_t)) \right] = \max \{ \Psi(D_{t+1}), 0 \},$$

where

$$\Psi(D_{t+1}) \equiv \frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta\pi(A_r - R(D_{t+1}))D_{t+1}.$$

The expression $\Phi(D_t)$ represents the net benefit from investing in the risky project. The expression $\max \{ \Psi(D_{t+1}), 0 \}$ gives the loss of profits from the future banking business. This loss represents the opportunity cost of risk taking and reflects the bank franchise value. In equilibrium the cost must be equal to the benefit.

The transversality condition $\lim_{t\to\infty} \beta^t v_t = 0$ will be satisfied if $(A_r - R(D))D$ is bounded since $v_t = \pi(A_r - R(D_t))D_t \geq 0$. The boundedness property holds whenever $\{D_t\}$ takes values in a finite interval.

Under condition (13), Φ is a strictly increasing function. We can then write the equilibrium dynamic system as

$$D_t = \Phi^{-1}(\max \{ \Psi(D_{t+1}), 0 \}) \equiv f(D_{t+1}), \qquad (16)$$

for some function f. Note that this is a backward nonlinear dynamic system. The function f may not be invertible so that we may not be able to write forward dynamics explicitly. As a result, multiple equilibria can arise and equilibria can be quite complex, including cycles, chaos, branch switching, and sunspots.

A simple case for the existence of multiple equilibria is the following. When high future deposits D_{t+1} imply a high franchise value in the future, investing in risky assets using current deposits and endangering the future value of the franchise value becomes more costly. The banker does not have the incentive to invest in the risky project, and hence the incentive

constraint is relaxed. This in turn raises the amount D_t of deposits that the banker can borrow today. Thus the optimistic belief about future high deposits can be self-fulfilling. Similarly the pessimistic belief about low future deposits can also be self-fulfilling. This dynamic complementarity across time is different from the static complementarity in the Diamond and Dybvig (1983) model. In the Diamond-Dybvig model, a depositor will withdraw his deposit early if he thinks the others will do so as well, so a bank run will occur. As Peck and Shell (2003) show, an equilibrium with a bank run in the Diamond-Dybvig model always exists. They typically happen during economic downturns.

In our model we define a bank run as $D_t = D_{t+1} = 0$, when depositors no long lend to banks. We will show that the existence of a bank run equilibrium in our model depends on economic fundamentals so we can give more precise characterization of when bank runs are possible. Below we will illustrate the various equilibria of our model using a phase diagram.

Before analyzing equilibria with incentive problems, we briefly describe the first-best equilibrium. In the first best, the deposit rate is equal to the lending rate, $A = R_t$. The first-best deposit level D^{FB} satisfies $R\left(D^{FB}\right) = A$.

3   A First Look at Equilibria

In this section we analyze the steady states of the equilibrium system and local dynamics.

3.1   Steady States and Local Stability

The following proposition characterizes the steady states of the equilibrium system (16), which are fixed points of f.

Proposition 2 *Suppose that assumptions 1 and 2 hold. Suppose that*

$$Y\left[\pi\left(A_r - R\left(Y\right)\right) - \left(A - R\left(Y\right)\right)\right] > \max\left\{\frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta\pi\left[A_r - R\left(Y\right)\right]Y, 0\right\}. \quad (17)$$

(a) If

$$\frac{1}{2}(\beta A - 1)\bar{N} > \phi, \quad (18)$$

then there exists a unique steady state deposit level $D^* \in (0, Y)$. (b) If

$$\frac{1}{2}(\beta A - 1)\bar{N} = \phi, \quad (19)$$

and

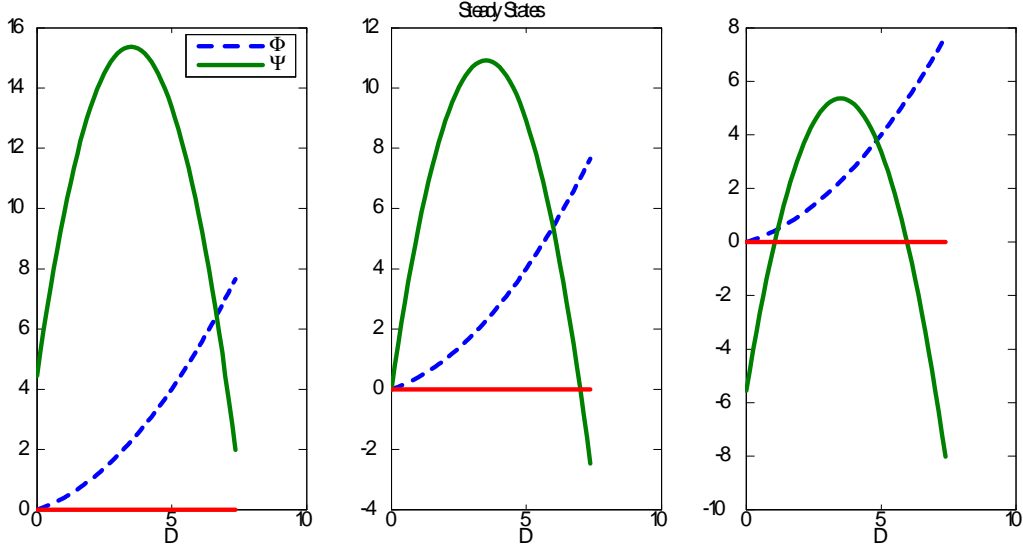$$A - R\left(0\right) > \pi\left(1 - \beta\right)\left(A_r - R\left(0\right)\right), \quad (20)$$

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Figure 1: Determination of steady states.

then there exist two steady-state deposit levels 0 and $D^{**} \in (0, Y)$. *(c) Suppose that (20) holds. If*

$$\frac{1}{2}(\beta A - 1)\bar{N} < \phi < \phi_{\max}, \tag{21}$$

where $\phi_{\max}$ is such that

$$\max_{D \in [0,Y]} \; [\Psi(D) - \Phi(D)] = 0, \tag{22}$$

then there exist three steady-state deposit levels $0 < D_L < D_H < Y$.

Assumptions 1 and 2 ensure that we can apply Proposition 1 and hence the dynamic system in (16) characterizes any equilibrium. Figure 1 illustrates Proposition 2. In each panel of this figure, the dashed and solid curves show the functions Ψ and Φ respectively. Given the conditions in Proposition 2, the function Φ is strictly convex, strictly increasing, and satisfies $\Phi(0) = 0$. The function Ψ is strictly concave and satisfies $\Psi(0) = (\beta A - 1)\bar{N}/2 - \phi$. Condition (17) implies that $\Psi(Y) < \Phi(Y)$. The intersection points of the curves for Φ and Ψ give the steady states. Figure 2 illustrates the corresponding phase diagrams in the backward dynamics. The intersection points of the phase curve $D_t = f(D_{t+1})$ and the 45-degree line give the steady states. The forward dynamics gives rise to a difference correspondence exhibiting "branching" with the possibility of two values of D_{t+1} given D_t, corresponding to the increasing and decreasing branches of f in Figure 2. (See section 4 below.)

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Under the condition in (18), $\Psi(0) > \Phi(0) = 0$. By (17), $\Psi(Y) < \Phi(Y)$. Thus the two curves for Φ and Ψ intersect only once as illustrated on the left panel of Figure 1. The intersection point gives the unique steady-state deposit level D^*. If the following condition

$$\left| f'(D^*) \right| = \left| \Psi'(D^*)/\Phi'(D^*) \right| < 1$$

is satisfied, then D^* is locally stable in the backward dynamics and hence is locally unstable in the forward dynamics: trajectories originating in the local neighborhood of D^* will diverge from D^*. Since $\{D_t\}$ is non-predetermined, $D_t = D^*$ for all t is the unique stable rational expectation solution. By contrast, if $|f'(D^*)| > 1$, then D^* is locally unstable in the backward dynamics and hence is locally stable in the forward dynamics[8]. This implies that, for any initial value of D_0 in the neighborhood of D^*, the solution D_t will converge to D^*. Thus equilibrium is indeterminate in the neighborhood of D^*.

Under the condition in (19), $\Psi(0) = 0$ and the two curves Φ and Ψ intersect twice as illustrated in the middle panel of Figure 1. The origin is an intersection point. The other intersection point gives the positive steady-state deposit D^{**}. The analysis of the stability of this steady state is the same as in the previous case. If condition (20) holds, then we can check that $f'(0) > 1$ so that 0 is an unstable steady state in the backward dynamics. This means that for any initial value $D_0 > 0$ in the neighborhood of 0, the equilibrium path D_t will converge to zero. Thus a bank run will eventually occur starting from any sufficiently small amount of deposits. The middle panels of Figures 1 and 2 illustrate this case. If condition (20) is violated, the bank run equilibrium is the unique steady state.

Under the condition in (21), $\Psi(0) < 0$ and the two curves Φ and Ψ also intersect twice as illustrated in the right panel of Figure 1. Since the right-hand side of the equilibrium system in (16) is $\max\{\Psi(D), 0\}$, zero deposit is also a steady state. This steady state is unstable in the forward dynamics since $f'(0) = 0$. The analysis of the stability of the two positive steady states D_L and D_H will be similar to the one given above. In particular, the low steady state D_L is unstable in the backward dynamics and stable in the forward dynamics. But the high steady state D_H can be either stable or unstable in the backward dynamics depending on parameter values.

Note that if $\phi > \phi_{\max}$, then $\Phi(D) > \Psi(D)$ for all D. The gain from risk taking is always larger than the loss. Depositors will never lend to banks $(D_t = 0)$ and hence bank run is the

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[8]Since the forward dynamics is given by a difference correspondence and exhibits branching, the local stability of D^* can be established only if we restrict attention to those local equlibrium trajectories originating in the neighborhood of D^* on the decreasing branch of f. See section 4 below.
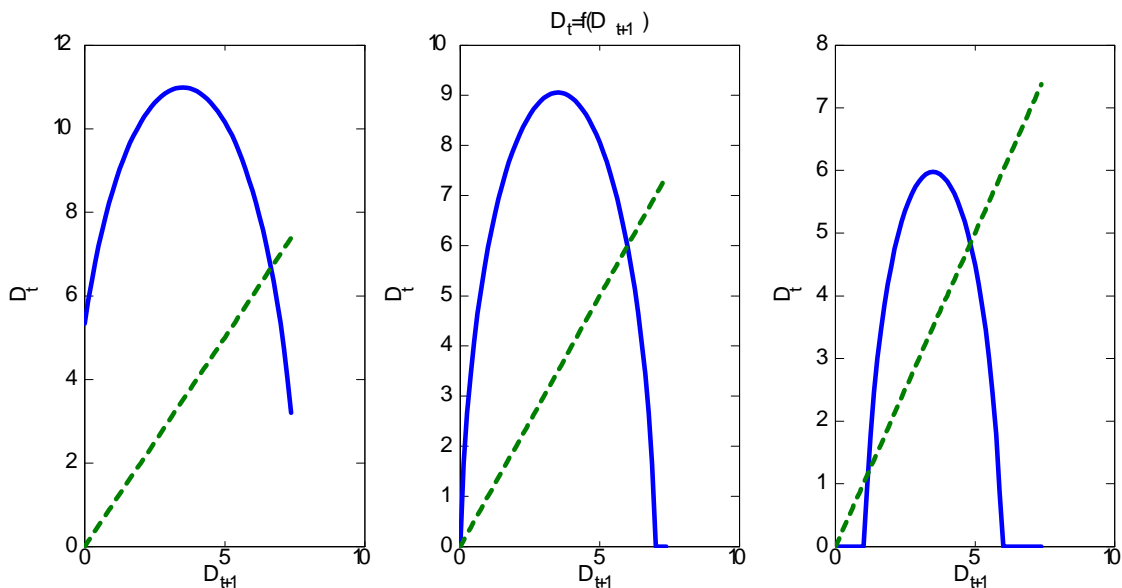
Figure 2: Phase diagram for the backward dynamics.

unique equilibrium.

From the analysis above, a crucial condition for a bank run equilibrium to exist is that $\phi \geq \phi_{\max}$. This condition depends on economic fundamentals. They are more likely to hold when ϕ is large, A is small, or $\bar{N}$ is small. This means that a bank run is more likely to happen if the bank is more cost inefficient, the investment is less profitable, or the bank net worth is lower.

An important feature shown in Figure 2 is that the phase curves are hill-shaped. This feature comes from the fact that the loss from risk taking $\Psi(D)$ or the franchise value of the bank is hill-shaped. The convexity of $R(D)D$ is crucial for this shape. We will show in the next section that the global dynamics can exhibit complex behavior such as chaos and cycles.

3.2 Incentive Constraints and Multiple Equilibria

Why can multiple steady states exist? We use the right panel of Figure 1 to illustrate the intuition. We ignore the trivial steady state at the zero deposit level. When the deposit level D is increased gradually from zero, the gain from risk taking $\Phi(D)$ rises. In the mean time the opportunity cost of risk taking or the continuation value from the future banking business, $\Psi(D)$, also rises from a value below $\Phi(D)$. At some high value of D, the cost exceeds the gain. Thus, by the intermediate value theorem, there is a value D_L such that $\Phi(D_L) = \Psi(D_L)$.

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This value gives a steady-state equilibrium. When the deposit level is increased further from D_L, the opportunity cost of risk taking $\Psi(D)$ will exceed the gain $\Phi(D)$. Such a deposit level cannot be an equilibrium. But when the deposit level is sufficiently high, the increased deposit rate $R(D)$ causes the opportunity cost of risk taking or the continuation value from the future banking business to decline. Since the gain from risk taking always increases with borrowings or deposits, $\Phi(D)$ will exceed $\Psi(D)$ eventually. By the intermediate value theorem, there exists another value $D_H > D_L$ such that $\Phi(D_H) = \Psi(D_H)$. This value D_H gives another steady state equilibrium.

So far we assumed that banks are competitive and showed how multiple equilibria and two steady states can exist. Along an equilibrium, say at the stationary equilibrium D_L, a unilateral increase in the deposit rate R_t that attracts deposits from other banks may seem profitable if the incentive constraint continues to hold as other banks follow suit within the period. For example, in a stationary equilibrium where $D_t = D_L$, raising the interest rate to $R(D_H)$ when all other banks follow suit can result in higher profits if the profits from investing in the risky asset $\pi(A_r - R_t(D_{jt}))D_{jt}$ are increasing in D_{jt}. Such a simultaneous move to D_H driven by seeking higher profits is an equilibrium because the incentive constraint also binds at D_H. However, if there is a lag in the response of other banks, then the bank initiating the response may attract a large amount of deposits away from other banks. For a large inflow ΔD_{jt}, since $A - R_t < \pi(A_r - R_t)$, the value of investing in the risky project will dominate the value of staying with the safe project. Therefore this bank will deviate by investing in the risky project. Even if the bank initiating the raise in R_t commits to limiting deposit intakes or staying with the safe project, this will not be credible. Therefore the depositors will not trust the lone bank offering the higher return. They will withhold their deposits, and competing banks will not follow suit in raising their rates. Therefore the initial equilibrium at D_L will remain. However if banks could coordinate on selecting R_t and $D(R_t)$ across equilibria at which (3) holds, they will coordinate on the most profitable equilibrium.

3.3 Example

We now give an explicitly solved example. Let the deposit supply function be $R_t = a + bD_t$, where $a > 0$ and $b > 0$ are some constants. Let

$$\eta \equiv \frac{1}{2}(\beta A - 1)\bar{N} - \phi.$$

Then

$$\Phi(D) = D[\pi A_r - A + (1 - \pi)(a + bD)],$$

$$\Psi(D) = \eta + \beta\pi(A_r - a - bD)D.$$

The equation $\Phi(D) = \Psi(D)$ becomes a quadratic equation

$$b((1 - \pi) + \beta\pi)D^2 + (\pi A_r(1 - \beta) - A + (1 - \pi + \beta\pi)a)D - \eta = 0.$$

We first solve the steady-state system $\Phi(D) = \max\{\Psi(D), 0\}$ explicitly. When $\eta > 0$, there is a unique positive steady state

$$D^* = \frac{-\Gamma_1 + \sqrt{\Gamma_1^2 + 4\Gamma_0\eta}}{2\Gamma_0} > 0,$$

where

$$\Gamma_0 \equiv b(1 - \pi + \beta\pi) > 0,$$

$$\Gamma_1 \equiv \pi(1 - \beta)A_r - A + (1 - \pi + \beta\pi)a.$$

When $\eta = 0$, there are two steady states 0 and

$$D^{**} = \frac{-\Gamma_1}{\Gamma_0} > 0,$$

where we assume $\Gamma_1 < 0$, which ensures that (20) is satisfied.

When $\eta < 0$, 0 is a steady state. We assume that $\Gamma_1^2 + 4\Gamma_0\eta > 0$, which ensures that (22) holds. Then there are two positive steady states

$$D_H = \frac{-\Gamma_1 + \sqrt{\Gamma_1^2 + 4\Gamma_0\eta}}{2\Gamma_0} > D_L = \frac{-\Gamma_1 - \sqrt{\Gamma_1^2 + 4\Gamma_0\eta}}{2\Gamma_0} > 0.$$

We can also solve for the backward dynamic system

$$D_t = \frac{-\xi + \sqrt{\xi^2 + 4(1 - \pi)b\max\{\Psi(D_{t+1}), 0\}}}{2(1 - \pi)b} \equiv f(D_{t+1}),$$

where $\xi \equiv \pi A_r - A + (1 - \pi)a$. Since $\Psi(D)$ is quadratic, f is not invertible and hill-shaped.

4 Complex Equilibrium Dynamics, Cycles, and Chaos

In addition to multiple steady states, our model can exhibit complicated equilibrium dynamics.[9] The trajectories of equilibrium deposits described by (16) exhibit "branching" since $f(D_{t+1})$ is

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[9]For an early application of chaotic dynamics in equilibrium OLG models, see Benhabib and Day (1982).

not invertible: given D_t, the value of $f(D_{t+1})$ is not uniquely pinned down, as shown in Figure 2. There can be two possible choices for D_{t+1} given D_t, one for each of the branches of $f(D_{t+1})$ to the right and left of its peak. One possibility is to introduce a selection mechanism for the branch at each t that is possibly a stochastic sunspot process. This is the approach followed by Christiano and Harrison (1999), and can obviously generate complicated equilibrium dynamics under the assumption that banks coordinate on such a sunspot. One can also construct a wide class of examples for the function $f(D_{t+1})$ whose dynamics exhibits additional complexity: the well-defined backward dynamics of the system can exhibit chaotic dynamics in the sense of Li and Yorke (1975). This implies that the dynamics will give rise to periodic cycles of every order, some of which will be stable,[10] as well as an infinite number of aperiodic or chaotic trajectories that are not asymptotic to cycles or fixed points. Furthermore the dynamics will exhibit sensitive dependence on initial conditions in that, for any $\delta > 0$, any $D > 0$ and a neighborhood of D, there exists y in the neighborhood and an integer n such that the distance $|f^n(D) - f^n(y)| > \delta$, where f^n denotes the n'th iterates of f.

This chaos however is in the backward dynamics of our model. We can follow Grandmont (1985) to construct a forecasting rule for D_{t+1} by the banks that will sustain any cycle of order k. For example the forecast function $D_{t+1} = f^e(D_{t-1})$ with $f^e(D_H) = D_H$ will not only convert the backward dynamics under f to forward "learning" dynamics, $D_t = f(f^e(D_{t-1}))$ but clearly sustain the steady state D_H. Similarly, a a period three cycle, (D^l, D^m, D^h) can be sustained by a forecast function $D_{t+1} = f^e(D_{t-1}, D_{t-2})$ for which $D^h = f^e(D^m, D^l)$, $D^m = f^e(D^l, D^h)$ and $D^l = f^e(D^h, D^m)$.[11] In the same way, forecast functions with $k-1$ lagged arguments can be constructed to sustain a cycle of any order k, although they get cumbersome for long cycles.

The above construction permits us to show that if we get chaos in backward dynamics, implying the existence of cycles of every order, that those cycles can be sustained under forward dynamics with an appropriate forecasting rule which is correct in the rational expectations sense. The question of establishing chaos for the original forward dynamics with branching when the backward map is chaotic has also been addressed by considering the full set of possible forward orbits consistent with the multi-valued forward map. Forward iterates of D_t

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[10] If he Schwartzian derivative of f,

$$S = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is negative at every D except at the point where $f'(D) = 0$ or is undefined, then there is a unique stable cycle.

[11] See Grandmont (1985) for further technical assumptions on the forecasting rule. As also shown in Grandmont (1985), if the cycle is locally stable under backward dynamics, it will also be locally stable in the foreward "learning" dynamics.

will now consist of sets rather than points. Applied to such sets it is possible to show that if the backward dynamics exhibit chaos, sensitive dependence on initial conditions in terms of sets defined by orbits will still obtain in the forward dynamics. Of course the backward and forward dynamics will also share the cyclic trajectories. Kennedy and Stockman (2008) show that under appropriate conditions forward dynamics are chaotic if and only if backward dynamics are chaotic.

It is of course easy, under conditions of multiple equilibria, to construct sunspot equilibria that coordinate expectations with the extrinsic uncertainty of a sunspot variable.[12] Since such constructions are by now quite standard, we leave this task to the reader.

We next use the linear deposit supply function studied in section 3.3 to construct numerical examples of cycles and chaos.

The Case of Two Positive Steady States   It is straightforward to construct examples of multiple equilibrium paths when there are two positive steady states. Consider first setting the parameters of our economy at the values give by $\beta = 0.99$, $A = 1.1$, $A_r = 1.2$, $\pi = 0.9$, $b = 1$, $a = 1$, $\phi = 0.447$, and $\bar{N} = 10$.[13] In this case, $\eta = (\beta A - 1)\bar{N}/2 - \phi = -0.002$. The left panel of Figure 3 illustrates the functions Φ and Ψ and the right panel illustrates the function f. We can compute two positive steady states, $D_L = 0.0287$ and $D_H = 0.0704$. The function f is monotonic in the interval (D_L, D_H). Trajectories originating in interval $(D_L, D_H]$ converge to D_H and diverge from D_L, implying that in forward dynamics they converge to D_L. We have therefore a continuum of equilibria already, even confining ourselves to one branch under forward dynamics and without resorting to jumping across branches.

Next we construct examples in our model of robust cycles of period two and three under backward dynamics. We reset the parameters of our model as $A = 1.2$, $A_r = 1.255$, $\pi = 0.95$, $b = 1$, $a = 1$, $\phi = 0.942$, and keep $\beta = 0.99$ and $\bar{N} = 10$. Figure 4 plots the functions f, f^2, and f^3. We find that the 45-degree line crosses f at the downward sloping branch. The domain for deposits is given by $[0, f(D_{\max})] = [0, 0.2440]$, where $D_{\max}$ achieves the maximum of $f(D)$. The function f now maps the invariant interval I into itself, where the interval $I = [f^2(D_{\max}), f(D_{\max})] = [0.0120, 0.2440]$.

There are now two positive steady states, $D_L = 0.0107$ and $D_H = 0.1888$, both of which are locally unstable in the backward dynamics. The high steady state D_H is surrounded by a

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[12] For example one can construct sunspot equilibria that simply randomizes across the two stationary equilibria.
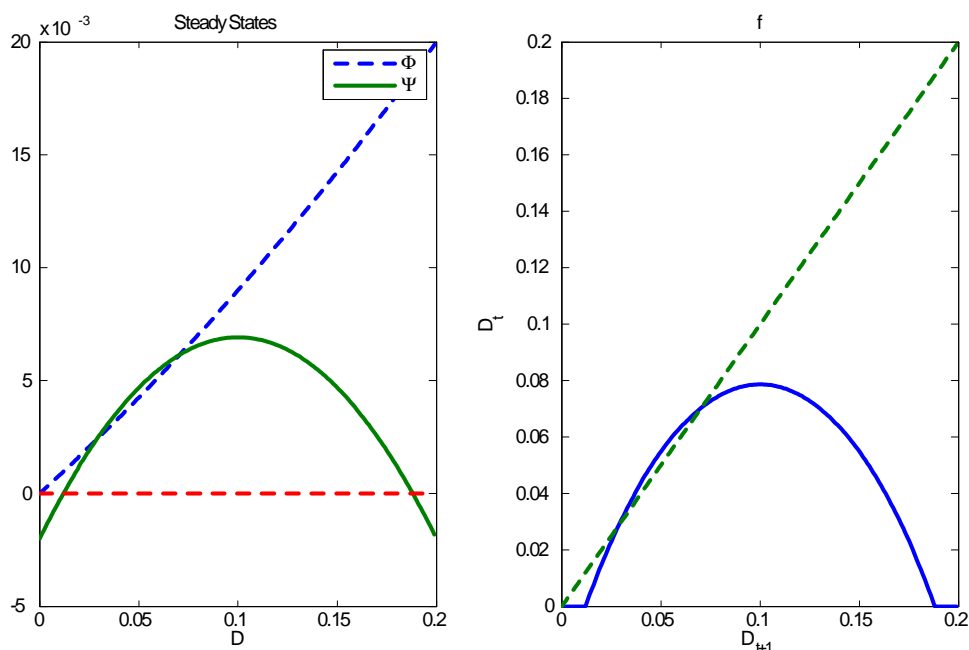[13] Picking $\bar{N}$ is equivalent to picking the parameter λ.

Figure 3: Multiple equilibria

two-period cycle with coordinates (0.0914, 0.2254) as illustrated on the left panel of Figure 5. The intuition is as follows. When bankers expect the future deposits to be at the high level 0.2254, the equilibrium deposit rate will be too high, causing the bank profits and franchise value to be low. This tightens the current incentive constraint, causing the bank to attract a low level of deposits at 0.0914. On the other hand, when the future deposits are expected to be at the low level 0.0914, the bank will have to pay a low deposit rate, causing the bank profits and franchise value to be high. This relaxes the incentive constraints and can support a high current level of deposits at 0.2254. These dynamics are self-fulfilling and constitute a period-two cycle equilibrium.

Furthermore in our example the backward dynamics under f produce two unstable period-three cycles with coordinates $(0.0348, 0.1092, 0.2393)$ and $(0.0493, 0.1512, 0.2361)$. We can use any one of these two cycles to discuss the intuition, say the first one (see the right panel of Figure 5). In period $t + 1$, when bankers expect deposits in period $t + 2$ to be at the high level $D^h \equiv 0.2393$, the deposit rate is too high, causing the bank profits and franchise value to be low. This tightens the incentive constraint and supports the low deposit level $D^l = f\left(D^h\right) = 0.0348$. In period t, when bankers expect the deposits at date $t + 1$ to be at the low level D^l, they
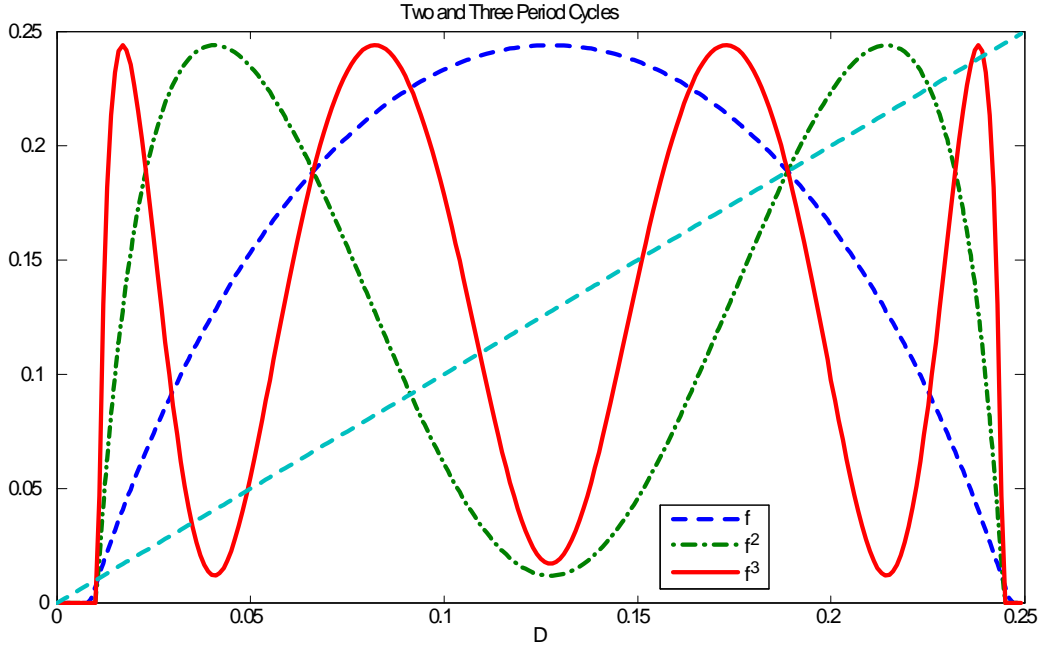
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Figure 4: Two and three period cycles. The backward map f has two positive steady states.

also expect the deposit rate at $t+1$ to be low and hence the franchise value to be high. This can support a period-t deposit level $D^m = f\left(D^l\right) = 0.1092$, which is higher than D^l but lower than D^h. Importantly, this deposit level is on the increasing branch of the backward map f. This implies that bankers believe that the period-t franchise value of the bank at D^m will be higher than at D^l. This allows the incentive constraint to be relaxed in period $t-1$ and can support a high deposit level $D^h = f\left(D^m\right)$. These dynamics are also self-fulfilling and constitute a period-three cycle equilibrium.

Note that the periodic cycles lie above D_L. Of course the cycles in backward dynamics are also cycles in forward dynamics. Having shown the existence of period-three cycles under the continuous function $f : I \to I$, we can immediately apply the theorem of Li and Yorke (1975) to establish the existence of an infinite number of distinct aperiodic trajectories as well as sensitive dependence on initial conditions. Formally, we can verify that the example satisfies the Li and Yorke condition for the existence of chaotic dynamics for the backward map f: there exists a D in the interval (D_L, D_H) such that $f^3\left(D\right) \leq D < f\left(D\right) < f^2\left(D\right)$.[14]

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[14]The emergence of chaos through succesive bifurcation of cycles has been characterized in the classic work of Sarkovskii (1995).
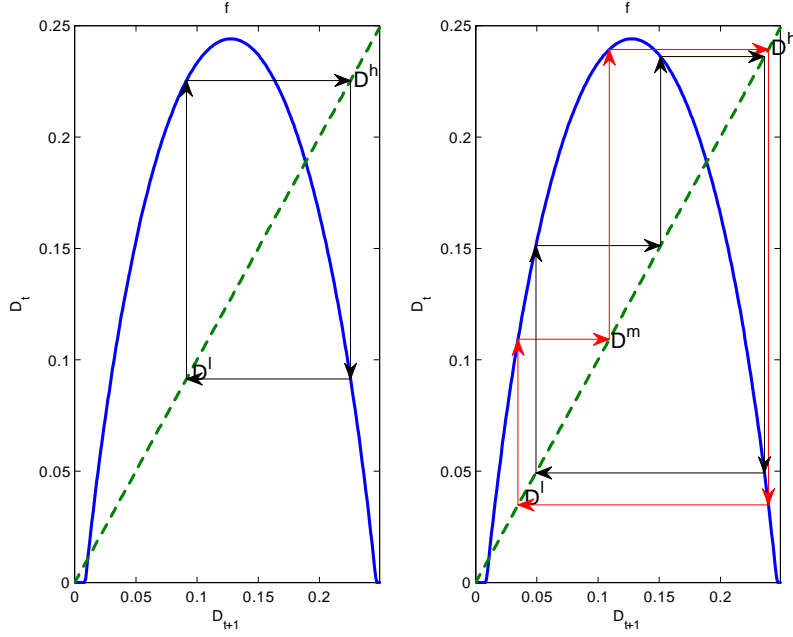
Figure 5: Two and three period cycles. The backward map f has two positive steady states.

The Case of a Unique Steady State Let $\beta = 0.99$, $A = 1.2$, $A_r = 1.27$, $\pi = 0.94$, $a = 0.99, b = 0.99$, $\bar{N} = 10$, and $\phi = 0.938$. In this case there is a unique steady state at 0.2207 as illustrated in Figure 6. The domain for deposits is given by $[0, 0.2900]$. The function f now maps the invariant interval I into itself, where the interval $I = [0.0015, 0.2900]$. There is a two-period unstable cycle with the periodic orbit $(0.0960, 0.2680)$. There are two unstable period-three cycles with coordinates $(0.0222, 0.1215, 0.2858)$ and $(0.0384, 0.1684, 0.2823)$. As discussed above, we can apply Li and Yorke (1975) to establish the existence of chaotic equilibria. Figure 7 illustrates how a two-period cycle and two three-period cycles can occur. The intuition is similar to that discussed earlier.

The above possibilities for the equilibrium trajectories of deposits, with sunspot selection under branching, cycles of every order under forward learning dynamics, and possibilities of chaos on dynamic sets induced by the multi-valued forward maps are extremely rich and very complex. It would be very hard for competitive banks and depositors to have the foresight to coordinate on one of the many complicated equilibria presented here. The inherent instabilities associated with the rich multiplicity of non-stationary equilibria is likely to lead to coordination failure and forecasting errors. The difficulty of coordinating on non-stationary equilibria may
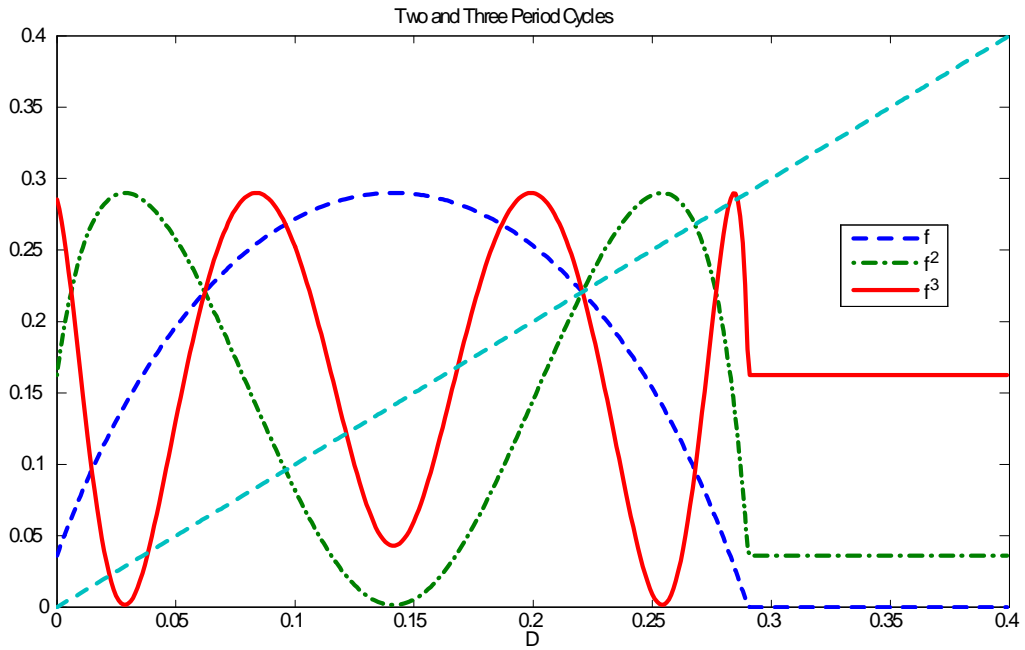
20

Figure 6: Two and three period cycles. The backward map f has a unique positive steady state.
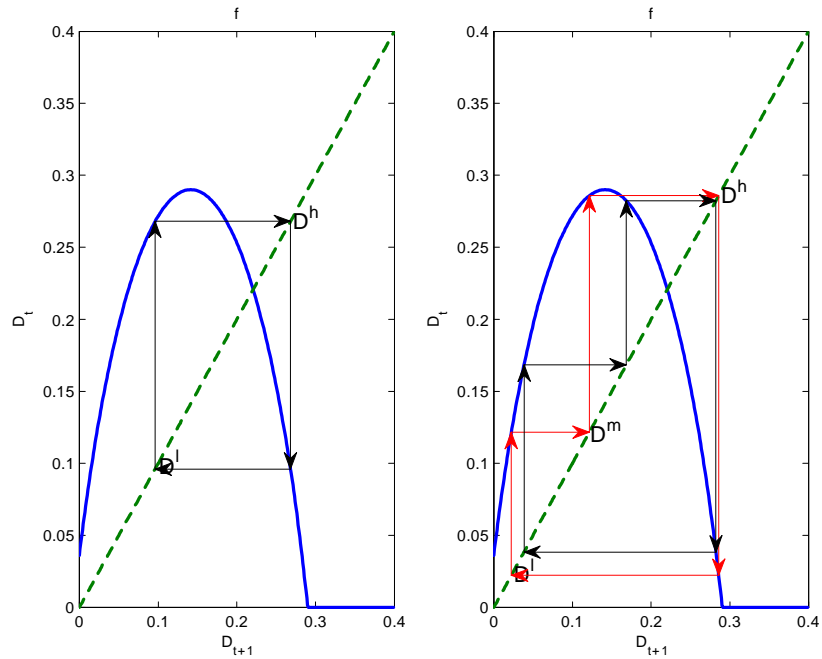

Figure 7: Two and three period cycles. The backward map f has a unique positive steady state.

lead banks to unexpected default or to encourage speculative investments in risky assets. Stationary steady state equilibria however, possibly one with high interest and high deposits, and another with low interest and low deposits, are much more likely to emerge as focal points for coordination, especially if guided by the appropriate regulation of leverage ratios. We therefore focus our attention on the stationary equilibria and consider the role of regulation and of minimum leverage ratios in order to minimize the possibilities multiple equilibria with consequent coordination failures, defaults, and the need for bailouts.

5 Banking Regulation and Leverage Restrictions

Our model suggests that under the existence multiple and possibly chaotic equilibria and "branching," the banking system may not function well, and the economy may face bank runs requiring costly bailouts. It is therefore natural to ask what type of policies can prevent such undesirable outcomes. In this section we use our baseline model to evaluate the consequences of different policies.

Proposition 2 establishes conditions for the existence of two stationary rational expectations equilibria with positive deposits, D_H and D_L. In addition we provided examples to show that there exist cyclic, chaotic, and branching non-stationary equilibria. Under these circumstances coordinating agent expectations on a particular equilibrium may prove to be very difficult. In uncertain times, and given the difficulty of coordinating expectations on a particular equilibrium, some pessimistic banks may choose to default, or may engage in Ponzi-like risky investments that divert equity returns to managers and can result in default. Such defaults, especially if they involve large "too big to fail" banks or occur in batches due to contagion, can be very costly in terms of freezing the financial system. They may require costly bailouts to curtail further runs, maybe even lead to the $\{D_t\} = \{0\}$ equilibrium. Banking regulators then may prefer to eliminate the high deposit and high leverage equilibrium $\{D_H\}$ to reduce the possibility of coordination failures and default whose social costs that can be particularly large under high leverage. Bankers on the other hand can argue that the cost of eliminating $\{D_H\}$ as an equilibrium is much too high in terms of forgone bank lending, bank profits, as well as forgone investment and output for the economy as a whole. They may also point out that constraining banks can lead to disintermediation, and to the emergence of large, less regulated, and still "too big to fail" shadow banks.

Eliminating the high deposit stationary equilibrium at $\{D_H\}$ may be achieved by specifying

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a maximum bank size, or alternatively by a minimum leverage ratio LR (defined as the ratio of bank capital N to total assets $D + N$) for banks.[15] Simply imposing a limit on bank size by specifying a maximum deposit amount, $\bar{D}$, may achieve this objective. Such a limit adds an additional constraint to the bank optimization problem and eliminates all equilibrium trajectories on which $D_t > \bar{D}$. The only ones that remain are those trajectories for which the size constraint remains slack: they are the ones that locally converge to D_L in the forward dynamics. In particular such remaining equilibrium trajectories remain on the increasing branch of f, and cyclic as well as chaotic equilibria are ruled out. We note that imposing a ceiling on the deposit rate R, as recently proposed by Hellmann, Murdock, and Stiglitz (2000), would also eliminate the high leverage equilibrium $\{D_H\}$ in the same fashion.

Recent policies however have been formulated in terms of leverage ratios or risk adjusted capital requirements. In addition, under the Volcker Rule, banks are prohibited from investing in some risky assets. In this section we will analyze the impact of these policies. We will also analyze the impact of the credit policy recently imposed in the Great Recession.

5.1 Leverage Ratio Restrictions

We first discuss the impact of the policy of leverage restrictions or capital requirements. We model this policy by the following constraint

$$D_{jt} \leq \varpi N_{jt}. \tag{23}$$

This constraint will alter the optimization problem of a bank j: the optimal bank equity is no longer $\bar{N}$. We therefore reconsider the bank optimization problem. We still assume that the deposit supply function is linear,

$$R_t = a + bD_t.$$

We focus on the case with $(\beta A - 1)\bar{N}/2 < \phi < \phi_{\max}$, so there are multiple steady states equilibrium (see Proposition 2). We want to show a leverage ratio or capital requirement policy can eliminate the possible multiple equilibria.

Bank j's optimization problem is described by the dynamic programming problem (6) subject to (3) and (23). The incentive constraint (3) can be written as (7).

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[15]This policy can be equivalently specified as a minimum capital requirement ratio. Note that Gertler and Karadi (2011) and Gertler and Kiyotaki (2013) define the leverage ratio as $(N + D)/N$. According to the Basel Accords, the tier 1 capital ratio is defined as the ratio of tier 1 capital to risk-adjusted assets and the leverage ratio is defined as the ratio of tier 1 capital to average total consolidated assets. In our model we do not distinguish between these two types of assets or different tiers of bank capital.

We will solve for an equilibrium in which $A > R_t$, $\pi(A_r - R_t) > A - R_t$, and the incentive constraint (7) will never bind. We can then ignore (7) and show that the constraint in (23) always binds, i.e., $D_{jt} = \varpi N_{jt}$.

Conjecture that the value function takes the following form:

$$V_t(N_{jt}) = [A + \varpi(A - R_t)]N_{jt} + v_t, \tag{24}$$

where v_t is a term independent of N_{jt}. Substituting this conjecture into (6) and using the first-order condition with respect to N_{jt+1}, we can show that

$$N_{jt+1} = \lambda\left[(\beta A - 1) + \beta\varpi(A - R_{t+1})\right].$$

Thus N_{jt+1} is independent of j and N_{jt}. We write this solution as

$$N_{t+1} = \bar{N} + \lambda\beta\varpi(A - R_{t+1}), \tag{25}$$

where we recall $\bar{N} = \lambda(\beta A - 1)$. Plugging this expression and (24) into (6), we can show that

$$
\begin{aligned}
v_t &= -N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1}) \\
&= \beta[A + \varpi(A - R_{t+1})]N_{t+1} - N_{t+1} - N_{t+1}^2/(2\lambda) - \phi + \beta v_{t+1} \\
&= \frac{1}{2}\left[(\beta A - 1) + \beta\varpi(A - R_{t+1})\right]N_{t+1} - \phi + \beta v_{t+1}.
\end{aligned}
\tag{26}
$$

In equilibrium,

$$R_{t+1} = a + bD_{t+1} = a + b\varpi N_{t+1}. \tag{27}$$

Plugging this expression into (25), we can derive

$$N_{t+1} = \frac{\bar{N} + \lambda\beta\varpi(A - a)}{1 + \lambda\beta b\varpi^2} \equiv N^*. \tag{28}$$

Substituting (27) and (28) into (26), we can derive that

$$v_t = v^* \equiv \frac{\frac{1}{2}\left[(\beta A - 1) + \beta\varpi(A - R(N^*))\right]N^* - \phi}{1 - \beta} \quad \text{for all } t,$$

where

$$R(N^*) \equiv a + b\varpi N^*.$$

For $A > R_t$ to hold in equilibrium, we need to assume

$$A > a + R(N^*). \tag{29}$$

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For $\pi(A_r - R_t) > A - R_t$ to hold in equilibrium, we need to assume

$$\pi(A_r - R(N^*)) > A - R(N^*).\tag{30}$$

Finally, for the incentive constraint (7) to never bind, we need to assume

$$[\pi(A_r - R(N^*)) - (A - R(N^*))]\,\varpi N^* < \frac{\frac{1}{2}\left[(\beta A - 1) + \beta\varpi(A - R(N^*))\right]N^* - \phi}{1 - \beta}.\tag{31}$$

We summarize the preceding analysis in the following result:

Proposition 3 *For parameter values β, a, b, A, A_r, π, ϕ, $\bar{N}$, and ϖ such that (29)-(31) hold, there exists a unique equilibrium in which $N_t = N^*$, $R_t = R(N^*)$, and $D_t = \varpi N^*$ for all t.*

We use a numerical example to illustrate this proposition. Set $\beta = 0.99$, $A = 1.2$, $A_r = 1.255$, $\pi = 0.95$, $b = 1$, $a = 1$, $\phi = 0.942$, and $\bar{N} = 10$. Without the capital requirement constraint (23), we have shown in section 4 that there are two steady-state equilibria $D_L = 0.0107$ and $D_H = 0.1888$, as well as other complex equilibria. If we impose the capital requirement constraint (23) and set $\varpi = 0.01886$, then we can show that conditions (29)-(31) hold and the unique equilibrium is given by $N_t = N^* = 10.0111$, $D_t = 0.18876$, and $R_t = 1.1888$ for all t. The complex equilibrium dynamics are eliminated. The intuition is that with the constraint (23), the deposit D_t is no longer linked to the expectations about future bank values. Thus self-fulfilling beliefs about future values cannot be initiated.

5.2 The Volcker Rule

Alternatively, we may also consider a type of Volcker Rule that directly prohibits investing in risky assets for which $A > \pi A_r$ (although bankers may object because they disagree about the direction of the inequality). We now study the consequence of such a rule in various economic environments. Without the risky asset, the bank's problem is given by the dynamic programming problem (6). But there is no incentive constraint (3). Since banks cannot engage in risk shifting, perfect competition will then eliminate bank profits from deposits. The interest rate on deposits will adjust so that $R_t = A$. Optimal bank equity will again be $N_{jt+1} = \lambda(\beta A - 1) \equiv \bar{N}$. To fully characterize the value function we again consider two cases. The first case is $\frac{1}{2}(\beta A - 1)\bar{N} - \phi < 0$. We have shown before that without a prohibition on investing in the risky asset, there are potential multiple steady states and complex dynamics. With a prohibition on the risky asset the results are different.

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It is straightforward to show that if $(\beta A - 1)\bar{N}/2 < \phi$, then all banks will exit in a competitive equilibrium under a rule that directly prohibits investing in risky assets. To see this, we only need to consider the problem of a single bank, since banks are identical. If a bank continues to operate, then

$$\max_{N_{jt}}\{-N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1})\} = \frac{\frac{1}{2}(\beta A - 1)\bar{N} - \phi}{1 - \beta} < 0.$$

When $R_t = A$, banks make no money on deposits. The condition $(\beta A - 1)\bar{N}/2 < \phi$ just means that fixed costs cannot be covered by profits on bank equity. Hence the bank will choose to exit. The prohibition on investing in the risky asset effectively removes the moral hazard problem and makes constraint (3) redundant. But the cost is that bank profits from deposits are driven to zero so that fixed costs cannot be covered.

In the case where $(\beta A - 1)\bar{N}/2 > \phi$, a rule prohibiting investments in risky assets is still be feasible and competition would raise the deposit rate $\{R_t\}$ until bank profits from deposits are driven to zero, a first-best equilibrium in our benchmark model. Nevertheless, it may not be easy to prevent banks from investing in off-balance-sheet risky assets[16]. This may lead to the type of instability that the central bank want to avoid. To make this point, we can extend our benchmark model to have three types of projects. We refer to the two types studied before as the safe project and the bad risky project. We introduce a third type: a good risky project that yields returns $A_r > 1$ with probability π and $a_r \in (0, 1)$ with probability $1 - \pi$. Good risky projects may be closely monitored by restricting them to collateralized loans to firms or households, with limitations and controls on the valuation of collateral assets like houses, or by risk weighing bank assets. Thus we suppose that good risky projects are not subject to the incentive problem.

Banks may still invest in a bad risky project off the balance sheet, which may be hard to monitor. Assume that the safe project yields return $A > 1$ and that $A_H \equiv A_r\pi + (1 - \pi)a_r > A > A_r\pi$. Here by construction the good risky project dominates the bad risky project so banks will never invest in the bad risky project. Competition among banks will then leads to $R_t = A_H$ and we then have $N_{t+1} = \lambda(\beta A_H - 1) > \bar{N}$. Under a stringent rule that prohibits investing in any risky asset, competition would result in $R_t = A$. Since $A_H > A$, there would be efficiency loss. If the prohibition rule succeeds preventing only the good risky project the

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[16] For example, a bank can sponsor a a mutual fund to purchase risky stocks. These off-balance-sheet assets can be quite significant. For example Citibank had an estimated $ 960 billion in off-balance-sheet assets in 2010, roughly equal to 6% of the GDP of the US.

situation may even be worse, as we have shown in section 4 that with only a bad risky asset there can be chaotic equilibria in the case with $\frac{1}{2}(\beta A - 1)\bar{N} - \phi > 0$.

The Volcker Rule, finally implemented in January 2014 in the United States tries, however imperfectly, to strike a balance. It attempts to exempt investments in good risky projects from its prohibitions but prohibits risky projects that are likely to be undertaken for risk-shifting purposes, and that may require costly bailouts.[17]

5.3 Credit Policy

We have focused on policies that achieve stability by removing equilibrium multiplicity. We now study the possibility that the central bank can support the equilibrium with high deposits using some unconventional monetary policies.[18]

Let L_{jt} be the discount window lending of the central bank to bank j. We first assume that the funds lent by the central bank are perfectly monitored, and hence are not subject to the risk-shifting problem studied before. Bank j's decision problem becomes

$$
\begin{aligned}
V_t(N_{jt}) &= \max_{N_{jt+1}, D_{jt}} \quad A(N_{jt} + D_{jt} + L_{jt}) - R_t(D_{jt} + L_{jt}) \\
&\quad + \{0, -N_{jt+1} - \varphi(N_{jt+1}) - \phi + \beta E_t V_{t+1}(N_{jt+1})\},
\end{aligned}
\tag{32}
$$

subject to the incentive constraint

$$
V_t(N_{jt}) \geq AN_{jt} + (A - R_t)L_{jt} + \pi(A_r - R_t)D_{jt}.
\tag{33}
$$

By (32), this incentive constraint is still equivalent to (7).

Next we assume that the funds lent by the central bank cannot be perfectly monitored and hence are subject to the risk-shifting problem studied above. In this case the incentive constraint becomes

$$
V_t(N_{jt}) \geq AN_{jt} + \pi(A_r - R_t)(D_{jt} + L_{jt}).
\tag{34}
$$

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[17]See for example, http://www.sec.gov/rules/final/2013/bhca-1.pdf

In particular the Volcker Rule "generally prohibits banking entities from engaging as principal in proprietary trading for the purpose of selling financial instruments in the near term or otherwise with the intent to resell in order to profit from short-term price movements" but specifically exempts from this prohibition the following categories : "Trading in U.S. government, agency and municipal obligations; Underwriting and market making-related activities; Risk-mitigating hedging activities; Trading on behalf of customers; Trading for the general account of insurance companies; and Foreign trading by non-U.S. banking entities."

[18]Cecchetti and Disyatat (2010) discuss cental bank credit and liquidity policies in times of financial market instability. In particular they note (see page 32) that the basic function of open market operations and liquidity management in such times is "... to regulate the level of aggregate reserves to ensure smooth functioning of the payments system and facilitate the attainment of the relevant policy interest rate target."

.

Let $\widetilde{D}_{jt} \equiv D_{jt} + L_{jt}$. The bank's problem is the same as the one studied in section 2 with the difference that D_{jt} is replaced by $\widetilde{D}_{jt}$.

Let $D_H > 0$ and R_H denote the larger steady-state deposit level and the associated interest rate characterized in Proposition 2. Suppose that the central bank establishes the following policy rule,

$$\frac{A - R_H}{A - R_t} D_H - D_t = L_t, \tag{35}$$

where D_t and L_t represent the aggregate deposit level and the aggregate discount loan volume. Assume that each bank takes the same level of discount loans so that $L_{jt} = L_t$. According to this rule, $D_t = D_{jt} = D_H$ and $L_{jt} = L_t = 0$ for all t and j constitute an equilibrium. We will show that this is the only equilibrium.

Proposition 4 *Suppose that the assumptions in part (c) of Proposition 2 hold so that there are two positive steady states $D_H > D_L > 0$. Let the central bank impose the policy rule in (35). Then there is a unique equilibrium in which $D_t = D_H$ for all t irrespective of whether there is a risk-shifting problem with discount loans.*

The preceding proposition shows that a credit policy rule in (35) can achieve stability at a more efficient steady state. The intuition for the stability is as follows. The feedback credit policy rule (35) sets a countercyclical discount window lending to banks. This makes the total one-period profits from deposits and government loans always equal to a constant: $(A - R_t)(L_{jt} + D_{jt}) = (A - R_H)D_H$. This in turn makes the franchise value of the bank v_t a constant. Since the amount of deposits that a bank can attract depends on its franchise value, this policy then automatically stabilizes households' expectations and hence the deposit level. It is worth pointing out that since the economy always stays in the steady state equilibrium, $L_t = 0$ for all t. By setting the policy rule given in (35), the central bank will never actually intervene along an equilibrium path. Of course we need the central bank to be credible and commit to the rule.

6   Conclusion

Banking regulations like leverage ratios, risk weighted capital requirements, restrictions on bank asset portfolios, central bank credit policies, and limits on bank size are imposed to contain risk-shifting to depositors, and to prevent moral hazard under government deposit insurance. Such regulations are aimed at minimizing the possibility bank insolvency, and to

prevent costly "too big to fail" government bailouts. In this paper we highlighted the possibility of multiple as well as complex dynamic equilibria that can generate financial fragility. Multiple and chaotic equilibria may lead to the failure of expectation coordination even when banks operate profitably under incentive constraints, and the franchise value of the bank exceeds the value of taking excessive risks. Appropriate bank regulations can then be designed to help avert the failure of expectation coordination that may lead to speculative and risky asset portfolios and bank runs. We showed that standard banking regulations including capital requirements, leverage ratios, restrictions on bank asset portfolios and credit policies can be targeted not only to ameliorate moral hazard problems, but also to minimize the dangers of the multiplicity of equilibria.

Appendix

A   Proofs

Proof of Proposition 1:   See the main text in Section 2.2.   Q.E.D.

Proof of Proposition 2:   Since $DR(D)$ is strictly convex, we can show that

$$\Phi'(D) = [\pi(A_r - R) - (A - R)] + (1 - \pi) DR'(D) > 0,$$

$$\Phi''(D) = 2(1 - \pi) R'(D) + D(1 - \pi) R''(D) > 0.$$

Thus Φ is an increasing and strictly convex function satisfying $\Phi(0) = 0$.

We can also show that Ψ is strictly concave since

$$\Psi''(D) = -\beta\pi \left[2R'(D) + DR''(D)\right] < 0.$$

Note that

$$\Psi(0) = \frac{1}{2}(\beta A - 1)\bar{N} - \phi.$$

If condition (18) holds, then $\Psi(0) > \Phi(0) = 0$. Condition (17) implies that $\Psi(Y) < \Phi(Y)$. By the intermediate value theorem, there is a positive solution $D^* \in (0, Y)$ to $\Phi(D) = \max\{\Psi(D), 0\}$. The positive solution is unique since $\Phi(D) - \Psi(D)$ is strictly concave.

If condition (19) holds, then $\Phi(0) = \Psi(0)$. When condition (20) holds, $\Psi'(0) > \Phi'(0)$. Thus the curve $\Psi(D) - \Phi(D)$ slopes upward at the origin and then eventually declines below the horizontal axis by (17). Thus there is a unique positive solution $D^{**} \in (0, Y)$ to the equation $\Phi(D) = \Psi(D) > 0$.

If condition (21) holds, $\Psi(0) < \Phi(0) = 0$. In addition,

$$\max_{D \in [0,Y]} \Psi(D) - \Phi(D) > 0 \text{ for } \phi < \phi_{\max}.$$

Thus the concave function $\Psi(D) - \Phi(D)$ starts at a negative function at $D = 0$, continuously increases to a positive maximum, and then decreases to a negative value at $D = Y$. It follows from the intermediate value theorem that there are two positive solutions $0 < D_L < D_H < Y$ to the equation $\Phi(D) = \max\{\Psi(D), 0\}$.   Q.E.D.

Proof of Proposition 3:   See the text in section 5.1.   Q.E.D.

Proof of Proposition 4: Conjecture that the value function still takes the form $V_t(N_{jt}) = AN_{jt} + v_t$, where v_t is a term independent of N_{jt}. First, suppose that there is no risk shifting problem for discount loans. Given the assumption that $A > R_t$ and $\pi(A_r - R_t) > A - R_t$, the incentive constraint (7) must bind. Using this equation and substituting the conjectured value function and (35) into (32), we can derive that

$$v_t = \frac{1}{2}(\beta A - 1)\bar{N} - \phi + \beta v_{t+1} + (A - R_H)D_H. \tag{A.1}$$

The only stable rational expectations solution is given by

$$v_t = \frac{\frac{1}{2}(\beta A - 1)\bar{N} - \phi + (A - R_H)D_H}{1 - \beta} \equiv v_H \tag{A.2}$$

for all t. Notice that Proposition 2 implies that we must have

$$\frac{1}{2}(\beta A - 1)\bar{N} - \phi + (A - R_H)D_H > 0$$

for a positive steady-state deposit level to exist.

Using the conjectured value function and the incentive constraint (33), we can derive that

$$\begin{aligned}
v_t &= (A - R_t)L_{jt} + \pi(A_r - R_t)D_{jt} \\
&= (A - R_H)D_H + D_t[\pi(A_r - R_t) - (A - R_t)], \tag{A.3}
\end{aligned}$$

where we have used (35) to derive the second equality.

Since $\pi(A_r - R_t) - (A - R_t) > 0$, the term $D_t[\pi(A_r - R_t) - (A - R_t)]$ is increasing in D_t. Thus there is a unique solution to (A.3) at D_H.

Next suppose that there is a risk shifting problem for discount loans. Then the incentive constraint becomes (34). Conjecture that the value function for the bank still takes the form $V_t(N_{jt}) = AN_{jt} + v_t$. By a similar method, we can show that the term v_t satisfies

$$\begin{aligned}
v_t &= \frac{1}{2}(\beta A - 1)\bar{N} - \phi + (A - R_t)(D_t + L_t) + \beta v_{t+1} \\
&= \frac{1}{2}(\beta A - 1)\bar{N} - \phi + (A - R_t)\left(\frac{A - R_H}{A - R_t}D_H\right) + \beta v_{t+1},
\end{aligned}$$

where we have used the policy rule to derive the second equation. Thus the unique rational expectations solution is also given by (A.2). The incentive constraint (34) becomes

$$v_t = \pi(A_r - R_t)(D_t + L_t) = \frac{\pi(A_r - R_t)}{A - R_t}(A - R_H)D_H, \tag{A.4}$$

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This equation determines a unique interest rate R_t. Notice that Proposition 2 implies that v_H and D_H satisfy

$$v_H = \pi(A_r - R(D_H))D_H. \tag{A.5}$$

Combining the two equations above, we deduce that there is a unique equilibrium in which $R_t = R(D_H)$ and hence $D_t = D_H$ for all t.   Q.E.D.

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