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MONEY AND MARKET INCOMPLETENESS IN OVERLAPPING GENERATIONS MODELS

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This paper investigates the extent to which the character of equilibria in stochastic overlapping generations models stems from the model's generic market incompleteness. In addition, it addresses the question of whether money serves to complete markets in these models. We show that money does not complete markets in the sense of expanding the set of state-contingent commodities that an individual may trade. The introduction of money effects state-contingent transfers of wealth between individuals; this is a generalization of a result obtained by Marshall, Sonstelie, and Gilles (1987). Numerical simulation of the model suggests that the risk-sharing induced in the monetary economy leads to substantial increases in welfare levels relative to the nonmonetary economy.

1. Introduction

There is a fundamental market incompleteness in the class of overlapping generations models which has developed out of Samuelson's classic (1956) analysis: agents born later in time cannot trade in markets with agents born at earlier dates. Put more formally, the natural market structure in an overlapping generations (OLG) economy is a sequential structure that limits individual trading opportunities relative to the standard construct of initial date markets which was used by Debreu (1959) to establish the welfare theorems for dynamic economies. This market incompleteness is of particular interest since it is now well understood that dynamic Pareto inefficiency can arise in a wide range of OLG models, including exchange and production economies as well as economies with and without uncertainty.

It is natural to conjecture that dynamic inefficiency in OLG models is attributable to market incompleteness and that monetary equilibria – when they exist – arise in OLG economies because money completes markets, i.e., money enriches the trading opportunities for economic agents at one or more

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dates. However, recent results by Marshall, Sonstelie, and Gilles (1987) indicate otherwise for a basic deterministic OLG model. First, Marshall, Sonstelie, and Gilles (henceforth MSG) show that the consumption outcomes of a sequential nonmonetary equilibrium of a basic OLG economy are identical to the consumption outcomes that would prevail in a full set of initial date (Arrow–Debreu) markets with transfers. Thus, although markets are incomplete in the OLG economy, these limitations are of no consequence for equilibrium outcomes. Second, MSG show that sequential monetary equilibria can also be represented as outcomes in Arrow–Debreu markets with an appropriate set of dated transfers to agents. MSG therefore argue that money does not complete markets; rather it acts to transfer wealth in ways that improve the efficiency of the economy.

The present paper studies the interactions of money and market incompleteness in a basic stochastic overlapping generations model. In contrast to the certainty analysis of MSG, it turns out that stochastic sequential OLG equilibria cannot be represented as outcomes that would prevail in a full set of initial date (Arrow–Debreu) markets. Rather, the sequential revelation of information in the OLG economy rules out a range of potential trades between agents born at different dates. Sequential market outcomes can be reproduced in complete initial date contingent claims markets only if certain trading restrictions are imposed. Therefore a natural conjecture is that the introduction of money in a stochastic setting works to complete markets in the sense that it increases the range of contingent transactions that an agent can undertake. Put more simply: to give money a chance to complete markets, it is important to start with an economy in which there are economically important departures from complete markets.

However, even under uncertainty, the introduction of money does not alter the number of state-contingent commodities that an individual agent can trade, so long as the sequential market structure implies that all gains from trade are exhausted between agents whose lifetimes overlap. In such equilibria money is a redundant asset, carrying a pattern of contingent payouts that can be provided in other ways (for example, by governmental tax/transfer policies). Thus, money does not complete markets in the basic stochastic OLG model. Rather, as under certainty, individual consumption allocations are different in a monetary OLG model because the introduction of money effects state-contingent wealth transfers between agents. Under uncertainty, money does not complete markets but it does alter patterns of intertemporal trade and interagent risk sharing.

To provide some indication of the quantitative importance of uncertainty on welfare and to indicate the nature of insurance provided by a monetary economy, we study several parametric versions of the basic OLG model. We find that the role of money is effectively to provide a great deal of insurance against the birth date risk faced by individuals in the sequential economy.

Although money does not complete markets, introducing money results in a substantial increase in expected utility for individuals born into the sequential economy. For the parameterized economies studied here, expected utility levels in the sequential monetary economies are very close to expected utility levels achieved in a benchmark Pareto-optimal allocation.

The organization of the paper is as follows. Section 2 describes the model and examines the equilibria that are obtained under timeless and sequential market structures without money. Section 3 characterizes sequential equilibria with valued money, and discusses the role of money in this setting. Section 4 provides a quantitative evaluation of several parameterized versions of the model economy, and investigates the nature and extent of trades facilitated by the introduction of money in these economies. Section 5 contains concluding comments.

2. A stochastic model without production

This section examines the question of market incompleteness in a stochastic OLG framework with two-period-lived agents and no storage or production opportunities. Agents' endowments are assumed to follow a stationary stochastic process. In this model it is generally not the case that equilibria in sequential markets can be replicated in complete Arrow–Debreu markets held before the economy begins. This is true unless one imposes trading restrictions in the Arrow–Debreu markets which prohibit agents from transferring resources across different states in their first period of life.[1] Thus, contrary to the economy studied by MSG, there is a nontrivial market incompleteness in the sequential economy which has important consequences for equilibrium consumption paths. The simple model discussed below illustrates this point; it will be easy to see that it must hold true for more complicated models.

2.1. The model

The stochastic process for endowments is assumed to be such that an agent's endowment at date t depends only on the state of nature that obtains at date t; let $w_t(s_t)$, $w_t(s_{t+1})$ denote the lifetime endowment profile for an agent born at time t in state s_t and where s_{t+1} is the state that obtains at time $t+1$ (i.e., when he is old). The state of nature, s, is assumed to follow a discrete stationary Markov process with state transition matrix Π. We use the notation $\pi(s_t = s)$ to denote the unconditional probability that state s occurs at time t, and the notation $\pi(s_t = s,\ s_{t+1} = s')$ to denote the probability that state s' occurs at time $t+1$ conditional on the event that state s occurs at time t.

[1] Wright (1987) studies a model which imposes these restrictions. We return to the discussion of Wright's model below.

Because of the stationarity of this stochastic process, these probabilities are independent of the date, and we shall make use of the notation $\pi(s_t = s) = \pi(s)$, $\pi(s_t = s,\ s_{t+1} = s') = \pi(s, s')$ whenever the dates are clear from the context.

To take a specific example, suppose that there are three possible states of nature, $s = 1, 2, 3$, and let s be i.i.d.: $\pi(s, s') = \pi(s')$. The (state-dependent) endowment profile for this economy is given by

	Endowment	
	Young	Old
State = 1	$w_y + \theta$	$w_o - \theta$
State = 2	w_y	w_o
State = 3	$w_y - \theta$	$w_o + \theta$

where $\theta \leq \min(w_y, w_o)$ so that endowments are always nonnegative.

This example has been constructed so that the economy-wide endowment is constant at the level $(w_y + w_o)$: there is no aggregate endowment risk even though there is idiosyncratic (individual) endowment risk. If individuals in this economy are risk-averse, they will want to diversify away this idiosyncratic risk. As is demonstrated below, the extent of risk reduction that agents can achieve depends on the market structure assumed for the economy. Two alternative market structures are examined. Finally, we assume throughout the existence of a government whose sole purpose is to effect state-dependent transfers or taxes between individuals.

2.2. *Timeless (Arrow–Debreu) markets*

Suppose that, in the economy described above, all agents who will eventually be born in the economy meet outside of time in complete Arrow–Debreu markets (markets in which all state- and date-contingent trades are permitted). In these markets, agents trade their endowments for consumption goods indexed by s, the state of nature in which consumption takes place, and by the index t, defined on the integers. In the sequential economy studied below, this index will be interpreted as the date of consumption. For simplicity, agents are assumed not to discount future consumptions.

Agents in this economy are distinguished by the index $j = 1, 2, 3, \ldots$ In the sequential economy, the index j will denote the agent's birth date. Since agents are 'alive' only in two periods, agent j values consumption only in periods j and $j+1$. Letting $c_j(s_t)$ denote agent j's state-contingent consumption at date t, agent j's expected utility is given by

$$\mathrm{E}U_j = \sum_{t=j}^{j+1} \sum_{s=1}^{3} \pi(s_t) u(c_j(s_t)), \tag{1}$$

where momentary utility, $u(c_j(s_t))$, is assumed to be of the constant elasticity form

$$u\left(c_j(s_t)\right) = \left(\frac{1}{1-\sigma}\right)\left\{c_j(s_t)^{1-\sigma} - 1\right\}.$$

The problem facing agent j in the Arrow–Debreu market structure is to maximize (1) subject to the budget constraint[2]

$$\sum_{t=1}^{\infty}\sum_{s=1}^{3} r(s_t)c_j(s_t) \leq \sum_{t=1}^{\infty}\sum_{s=1}^{3} r(s_t)\left[w_j(s_t) + T_j(s_t)\right], \tag{2}$$

where $r(s_t)$ is the price in the Arrow–Debreu markets of a unit of the consumption good in date t in state s, and where $T_j(s_t)$ is the government transfer to individual j in state s at date t. Since individual j has nonzero endowments and values consumption only in periods j and $j+1$, eq. (2) simplifies to

$$\sum_{s_j=1}^{3} r(s_j)c_j(s_j) + \sum_{s_{j+1}=1}^{3} r(s_{j+1})c_j(s_{j+1})$$
$$\leq \sum_{s_j=1}^{3} r(s_j)\left[w_j(s_j) + T_j(s_j)\right] + \sum_{s_{j+1}=1}^{3} r(s_{j+1})\left[w_j(s_{j+1}) + T_j(s_{j+1})\right]. \tag{3}$$

The market-clearing conditions for this economy are

$$\sum_j c_j(s_t) \leq \sum_j w_j(s_t) + \sum_j T_j(s_t) \quad \text{for all } t, s,$$

$$\sum_j T_j(s_t) = 0 \quad \text{for all } t, s.$$

The second of these constraints says that net transfers to individuals from the government must be zero in every state of the economy at each date. This reflects the assumption that the government does not produce or consume any output.

[2]The objective and constraints (1) and (2) would generally, in an Arrow–Debreu setting, be functions of histories, i.e., the realizations of all s_t up to and including date t. When preferences are separable and uncertainty takes the Markovian form assumed here, it is possible to simplify the objective and constraints in the manner presented above. See Baxter (1987) for more discussion of this point.

We examine first the 'no-intervention' equilibrium, defined as a situation in which transfers are always zero: $T_j(s_t) = 0$ for all j, s, t. Given the built-in stationarity of this economy, it is natural to search for a stationary equilibrium where the consumption levels depend on the state of nature, s, but not on the 'date' of consumption, t.

Because of the existence of the initial old generation which has endowment only in period $t = 1$, no intertemporal trade is possible. However, individuals still face uncertainty about the state of nature that will obtain at each date, including $t = 1$, so agents will pool the idiosyncratic risk associated with this uncertainty. The resulting equilibrium is

$$c_j(s_t) = w_y \quad \text{for} \quad t = j \quad \text{and for all } j, s,$$

$$c_j(s_t) = w_o \quad \text{for} \quad t = j + 1 \quad \text{and for all } j, s,$$

$$c_j(s_t) = 0 \quad \text{for all} \quad t \neq j, j + 1 \quad \text{and for all } j, s,$$

$$\begin{aligned} r(s_t)/r(s_{t+1}) &= \left[c_j(s_{t+1})/c_j(s_t)\right]^{\sigma} \\ &= \left[w_o/w_y\right]^{\sigma} \quad \text{for all } t, j, s. \end{aligned}$$

2.3. Sequential markets

This section studies the equilibrium obtained in a sequential market structure. The agent's problem is still to maximize expected utility [given by eq. (1)], subject to the constraints described below. In the sequential economy, agents are allowed to trade only after they are born. Further, it assumed that the timing of births is such that, by the time agent j is born in time period $t = j$, his endowment for his first period of life $[w_t(s_t)]$ has already been realized. The results of the paper do not depend on this assumption, as is discussed more fully in appendix A. This assumption is made because it enhances the heterogeneity that is the desirable characteristic of OLG models.[3] Further, with this assumption about the timing of information revelation, the consumption paths of agents of different ages are not perfectly correlated; individuals will bear idiosyncratic risk in their consumption profiles. From an empirical point of view this is a desirable characteristic.

[3] Most stochastic OLG models have the characteristic that young agents receive a deterministic first-period of life endowment either of goods or labor [see the references in Sargent (1987, p. 337)]. In these models, the timing of information revelation is not important. Huffman (1987) studies a model in which young agents face randomness in the endowment of the consumption good in the first period of their lives, but are not allowed to trade claims contingent on the realization of this process.

After the state of nature is revealed, young and old agents meet to trade in state-contingent consumption loans: $z(s_t = s,\ s_{t+1} = s')$ denotes a loan made in state s in period t paying off one unit in period $t+1$ if state s' is realized at that date. For simplicity, we write this as $z(s_t, s_{t+1})$. Similarly, let $q(s_t, s_{t+1})$ denote the price in period t of a unit of one unit of $z(s_t, s_{t+1})$. Thus an agent born in state s in period t faces the following pair of constraints:

$$c_t(s_t) \leq w_t(s_t) + T_t(s_t) - \sum_{s_{t+1}} q(s_t, s_{t+1}) z(s_t, s_{t+1}), \tag{4}$$

$$c_t(s_{t+1}) \leq w_t(s_{t+1}) + T_t(s_{t+1}) + z(s_t, s_{t+1}). \tag{5}$$

Eq. (4) is the constraint facing a young agent: his consumption is constrained by his endowment plus transfers while young, minus any state-contingent loans he wants to make. Eq. (5) is the constraint for an old agent: his consumption is constrained by his endowment plus transfers while old, plus receipts from state-contingent loans he made while young. Eqs. (4) and (5) may be consolidated into a single constraint, which says that an agent's lifetime consumption, conditional upon the state in which he is born, is equal to his conditional lifetime wealth:

$$\begin{aligned} &c_t(s_t) + \sum_{s_{t+1}} q(s_t, s_{t+1}) c_t(s_{t+1}) \\ &\leq \left[w_t(s_t) + T_t(s_t)\right] + \sum_{s_{t+1}} q(s_t, s_{t+1}) \left[w_t(s_{t+1}) + T_t(s_{t+1})\right] \qquad (6) \\ &\qquad \text{for} \quad s_t = 1, 2, 3. \end{aligned}$$

The individual faces three constraints of the form (6): one for each state of nature into which he may be born. After he is born, of course, the state has been realized and only one of these constraints is relevant. But the fact that these constraints are separate means that the constraints on an individual in the sequential markets structure are stronger than the single constraint facing an agent in the timeless markets environment [eq. (3)]. In the sequential environment, constraint (6) requires agents' consumption to equal wealth conditional upon the state realized at birth. In the complete (Arrow–Debreu) markets environment, the constraint (3) permits the individual to allocate consumption across states in any way he chooses. The complete markets environment thus allows agents to transfer wealth from good states on his date of birth to bad states; the sequential environment does not.

Because each agent lives only two periods, in equilibrium there will be no trade in contingent consumption loans: $z(s_t, s_{t+1}) = 0$ for all t, s. The sequential equilibrium is characterized by autarky. All agents consume their endowments plus transfers in each period: $c_t(s_t) = w_t(s_t) + T_t(s_t)$ and $c_t(s_{t+1}) =$

$w_t(s_{t+1}) + T_t(s_{t+1})$. In equilibrium, agents are unable to diversify away any of the risk associated with variation in state-dependent endowments and transfers. But in the example economy studied here, *all* of this risk is diversifiable from the point of view of the economy as a whole.

It is clear from this simple example that the incompleteness of markets associated with the sequential OLG model has important consequences for equilibrium consumption patterns. In expected utility terms, risk-averse agents are worse off under the sequential structure since they bear idiosyncratic risk that is diversifiable in the Arrow–Debreu timeless markets setting.[4] Thus, comparison of the real equilibria obtained in the timeless and the sequential market structures suggests that the market incompleteness inherent in the sequential structure is important for the difference in the real allocations achieved by these two structures.

2.4. Trading restrictions in timeless markets

In a recent paper, Wright (1987) has suggested the following interpretation of a stochastic OLG economy in which the timeless markets equilibrium and the sequential markets equilibrium are equivalent. Suppose that agents are distinguished both by the date at which they are born *and* by the state of nature in which they are born. Then in the timeless markets environment, each agent is (by definition) trading contingent on the state of nature that will be realized on his birth date. That is, an individual born at date j in state $s = 1$ is viewed as a different individual from someone born at date j in state $s = 2$. Let $\mathrm{E}(U_j|s_j)$ denote the expected utility of an individual who is born in period j in state s_j. In this environment, the objective maximized by the individual becomes

$$\mathrm{E}(U_j|s_j) = u\big(c_j(s_j)\big) + \sum_{s_{j+1}=1}^{3} \pi(s_j, s_{j+1}) u\big(c_j(s_{j+1})\big), \tag{7}$$

subject to the constraint

$$\begin{aligned} & r(s_j)c_j(s_j) + \sum_{s_{j+1}=1}^{3} r(s_{j+1})c_j(s_{j+1}) \\ & \leq r(s_j)\big[w_j(s_j) + T_j(s_j)\big] + \sum_{s_{j+1}=1}^{3} r(s_{j+1})\big[w_j(s_{j+1}) + T_j(s_{j+1})\big]. \end{aligned} \tag{8}$$

[4] If one were to ask agents with utility function (1), before the economy begins, whether they would prefer the Arrow–Debreu complete markets structure or the sequential structure, agents would choose the complete markets structure.

The constraint for this problem is of the same form as the constraints faced by an agent in the sequential economy studied above. The objective maximized by Wright's individual [eq. (7)] differs from the objective [eq. (1)] maximized by the individual living in the sequential economy described above. However, the separation of trading opportunities imposed by the constraints (8) means that equilibria in Wright's timeless market structure are identical to equilibria with sequential markets.

Thus with the trading restrictions implied by Wright's definition of an 'agent', there is an equivalence between Arrow–Debreu markets and sequential markets. Agents in the timeless markets environment have no incentive to smooth consumption across different states of nature on their birth date, since this would involve transferring wealth to someone else! The idea of treating agents born in different states of nature as different agents is a potentially useful device for computing equilibria in OLG economies, since it allows one to exploit the fact that the equilibria in Wright's economy are constrained Pareto-optimal.[5]

However, it is important to understand the economic interpretation of the device employed by Wright in his demonstration of the equivalence between regimes of timeless and sequential markets. Relative to the initial date market structure studied in this paper, Wright effectively imposes a set of trading restrictions on individuals. These restrictions force individuals in the timeless markets environment to operate subject to a set of distinct budget constraints in which no trade across birth date states of nature is possible.

3. Monetary equilibrium in the sequential economy

This section characterizes the properties of monetary equilibrium in the sequential stochastic OLG economy and investigates the role of money in this environment. To insure that money is valued in equilibrium, the endowment profile of the economy has been chosen to satisfy the sufficient conditions for an equilibrium with valued fiat money derived by Peled (1982). In the context of this economy, the sufficient conditions say that the young agent's endowment is greater than the old agent's endowment in every state of nature. There is a constant amount, M, of fiat money initially held by the old. In order to focus on the effects of money alone, transfers are set to zero [$T_j(s_t) = 0$ for all j, s, t].

In this economy, only the young agents have a nontrivial decision problem. The notation below has been simplified to reflect the conjecture that a stationary equilibrium exists: $c_y(s)$ denotes the young agent's consumption in state s; $c_o(s')$ denotes the consumption of an agent for whom state s' obtains

[5]See Baxter (1987, 1988) for discussion of methods and problems in computing such equilibria.

when he is old; $w_y(s)$, $w_o(s)$ denotes young and old endowments in state s; $p(s)$ is the price of money (the inverse of the price level).

The decision problem of an agent born in state s is to choose $c_y(s)$, $c_o(s, s')$, state-dependent consumption loans, $z(s, s')$, and money holdings M to maximize lifetime expected utility (1), subject to the constraints

$$c_y(s) \leq w_y(s) - \sum_{s'} q(s, s') z(s, s') - p(s) M, \tag{9}$$

$$c_o(s') \leq w_o(s') + z(s, s') + p(s') M. \tag{10}$$

Eq. (9) is the budget constraint for a young agent born in state s. His consumption when young must be less than or equal to his young-period endowment, minus the present value of contingent consumption loans he makes and his accumulation of real cash balances. Eq. (10) is the budget constraint for this agent when he is old, given that state s' obtains in his old age. His old-age consumption is constrained by his old-age endowment, plus the value of consumption loans made while young which pay off in state s', plus the current real value of his cash balances. The agent's consolidated or 'present value' budget constraint is

$$c_y(s) + \sum_{s'} q(s, s') c_o(s') \leq w_y(s) + \sum_{s'} q(s, s') w_o(s') + \left(\sum_{s'} q(s, s') p(s') - p(s) \right) M. \tag{11}$$

Eq. (11) shows the necessary condition for no arbitrage opportunities in money; the condition is

$$\sum_{s'} q(s, s') p(s') - p(s) \leq 0, \tag{12}$$

with this condition holding with equality if money is valued in equilibrium. If this condition did not hold, an agent could make an infinite amount of money by buying money today and selling claims to money tomorrow.[6]

[6]Suppose $\sum_{s'} q(s, s') p(s') > p(s)$. Then an agent would wish to sell claims to one unit of money (say, \$1) tomorrow in each state of nature, i.e., claims to \$1 for sure. This would yield $\sum q(s, s') p(s')$ goods units in receipts. This amount is enough to buy the \$1 needed to back the claims, $p(s)$ goods units, and still have some goods left over.

Equilibrium consumptions as functions of contingent prices $q(s,s')$ are found by solving the first-order conditions:

$$c_y(s) = \left[\frac{\Omega(s)}{1 + \sum_{s'} q(s,s') \left(\frac{q(s,s')}{\pi(s)} \right)^{-1/\sigma}} \right], \tag{13}$$

$$c_o(s') = \left[\frac{q(s,s')}{\pi(s')} \right]^{-1/\sigma} \left[\frac{\Omega(s)}{1 + \sum_{s'} q(s,s') \left(\frac{q(s,s')}{\pi(s)} \right)^{-1/\sigma}} \right], \tag{14}$$

where

$$\Omega(s) \equiv \left\{ w_y(s) + \sum_{s'} q(s,s') w_o(s,s') \right\} \tag{15}$$

is the wealth of an agent born in state s. Finally, equilibrium consumption must satisfy the economy's resource constraint in each period:

$$c_y(s) + c_o(s) \leq w_y(s) + w_o(s) \quad \text{for all } s, t. \tag{16}$$

3.1. Does money complete markets?

The analysis of section 2 demonstrated that the economy's market structure has important implications for equilibrium consumption and asset prices. Further, whether money is allowed to have value in the sequential OLG economy affects equilibrium allocations and asset prices. But it does not follow immediately that money 'completes markets' in OLG models. In fact, Marshall, Sonstelie, and Gilles (1986) have convincingly argued that it does not, at least in deterministic environments. They established that, for any monetary equilibrium in the sequential economy, it is possible to redistribute wealth in such a way that the timeless Arrow–Debreu market structure replicates the consumption patterns of the monetary equilibrium.

The question therefore remains whether this correspondence holds in stochastic models. Specifically: can we effect transfers of state-dependent endowments in such a way that the equilibrium in the timeless Arrow–Debreu market replicates the equilibrium of the monetary economy? It is easy to see that this cannot be the case. The reason is that in the timeless, Arrow–Debreu environment agents bear no diversifiable risk, but they do (in general) bear some diversifiable risk in the sequential monetary environment. This can be seen from the form of the budget constraint (11) and from the fact that equilibrium consumptions are state-dependent [eqs. (13) and (14)]. In the sequential environment, equilibrium consumption paths display idiosyncratic randomness that is due to the presence of idiosyncratic 'birth date risk' which is diversifiable at the social level but not the individual level. A characteristic of the timeless markets equilibrium, on the other hand, is that all diversifiable risk has been eliminated, and tampering with agents' wealths can do nothing to change this.

Thus there is an important sense in which markets are incomplete in stochastic OLG frameworks. Under complete markets, agents face a single budget constraint [eq. (3)] which reflects their ability to transform resources across all dates and states (i.e., all histories). The form of eq. (11) suggests that money does not complete markets since, even with the introduction of money, the individual still faces a set of distinct budget constraints – one for each state of the world into which he is born. That is: the budget constraints of the form (11) are distinct in the sense that the budget constraints of the form (6) were distinct, and differ from the complete markets consolidated budget constraint (3) which allows the individual to effect trades across different histories preceding his birth. Money therefore does not complete markets because it does not enlarge the set of contingent quantities that an individual may trade. Comparison of the budget constraint (11) with the budget constraint (3) demonstrates that the role of money in the sequential stochastic OLG model is to effect state-contingent transfers of wealth between individuals. As we shall see from the examples in the next section, the introduction of money will result in equilibria characterized by a nonzero amount of intergenerational risk sharing – risk sharing that was absent in the nonmonetary sequential equilibrium.

Thus we have uncovered a generalization of the MSG result: in the stochastic OLG model, money may be viewed as carrying out Pareto-improving *contingent* wealth transfers. With these contingent wealth transfers there is an equivalence between sequential nonmonetary equilibria and equilibria achieved in the monetary economy. Since the sequential nonmonetary equilibrium is characterized by autarky, these transfers are easily computed as $T_j(s_t) = c_j^{\mathrm{m}}(s_t) - w_j(s_t)$ where $c_j^{\mathrm{m}}(s_t)$ denotes equilibrium consumption in the monetary economy. Thus the equilibrium of the monetary economy could also be achieved by an appropriate choice of tax/transfer policies by the government.

4. Quantitative evaluation

In this section we study several parameterized versions of the model in order to investigate the nature of the transfers effected in the monetary economy. In

Table 1

A simple example.

Logarithmic utility: $\sigma = 1$ Money stock: $M = 1$
Three i.i.d. states, equal state probabilities: $\pi(s) = \frac{1}{3}$, $s = 1, 2, 3$

State	**Young**	**Old**
	Endowments	
1	6	4
2	7	3
3	8	2
	Equilibrium consumption levels in monetary equilibrium	
1	4.2293	5.7707
2	4.9342	5.0658
3	5.6391	4.3609
	Transfers which render nonmonetary sequential equilibrium (autarky) equivalent to monetary equilibrium	
1	−1.7707	1.7707
2	−2.0658	2.0658
3	−2.3609	2.3609

	Equilibrium contingent claim prices $q(s,s')$			**Equilibrium price of money $p(s)$**
s	$s' = 1$	$s' = 2$	$s' = 3$	
1	0.24430	0.27829	0.32327	1.7707
2	0.28501	0.32467	0.37715	2.0658
3	0.32573	0.32105	0.43103	2.3609

	Benchmark optimal equilibrium	**Nonmonetary equilibrium**	**Monetary equilibrium**
Certainty equivalent (C.E.) consumption level	5	4.4781	4.9662
Difference in C.E. consumption as percentage of benchmark	—	10.4%	0.7%
Percentage of expected utility in benchmark case	100%	93.1%	99.6%

addition, we undertake welfare comparisons of the different market structures. As is often the case in fully specified equilibrium models, there do not exist analytic expressions for equilibrium quantities and prices. Since neither the monetary nor the nonmonetary sequential equilibrium is Pareto-optimal, equilibrium cannot be found as a solution to a social planner's problem. Baxter (1987, 1988) discusses a method for computing suboptimal dynamic equilibria that is potentially applicable to overlapping generations economies with long-lived agents. Because individuals live only two periods in this model, a simple iterative scheme (outlined in appendix B) can be used to compute the model's equilibrium. These calculations yield equilibrium values of prices $p(s)$, $q(s, s')$

Table 2

A decrease in relative risk aversion.

Endowments and parameters as in example 1, except $\sigma = 0.33$ instead of $\sigma = 1$

State	Young	Old
	Endowments	
1	6	4
2	7	3
3	8	2
	Equilibrium consumption levels in monetary equilibrium	
1	4.0889	5.9111
2	4.9616	5.0384
3	5.8469	4.1531
	Transfers which render nonmonetary sequential equilibrium (autarky) equivalent to monetary equilibrium	
1	−1.9111	1.9111
2	−2.0384	2.0384
3	−2.1531	2.1531

	Equilibrium contingent claim prices $q(s,s')$			Equilibrium price of money $p(s)$
s	$s'=1$	$s'=2$	$s'=3$	
1	0.39480	0.31092	0.33161	1.9111
2	0.31443	0.33163	0.35369	2.0384
3	0.33212	0.35029	0.37359	2.1531

	Benchmark optimal equilibrium	Nonmonetary equilibrium	Monetary equilibrium
Certainty equivalent (C.E.) consumption level	5	4.8319	4.9827
Difference in C.E. consumption as percentage of benchmark	—	3.4%	0.3%
Expected utility as percentage of expected utility in benchmark case	100%	96.6%	99.6%

and consumptions $c_y(s), c_o(s)$ as functions of σ, $w_y(s)$, $w_o(s)$, $\pi(s)$, and M. Tables 1–4 exhibit equilibria for several parameterized versions of this model.[7]

Several characteristics of the monetary equilibria exhibited in tables 1–4 are worth discussing. First, compared to the autarky equilibria obtained in the sequential economy without money, the sequential monetary equilibria exhibit smoother consumption patterns and higher expected utility. The equilibrium variability of consumption decreases as the coefficient of relative risk aversion,

[7]Manuelli (1987) provides a proof of the uniqueness of stationary equilibrium under conditions satisfied by this economy.

Table 3

An increase in relative risk aversion.

Endowments and parameters as in example 1, except $\sigma = 3$ instead of $\sigma = 1.0$

State	Young	Old
	Endowments	
1	6	4
2	7	3
3	8	2
	Equilibrium consumption levels in monetary equilibrium	
1	4.4726	5.5274
2	4.9382	5.0618
3	5.3587	4.6413
	Transfers which render nonmonetary sequential equilibrium (autarky) equivalent to monetary equilibrium	
1	−1.5274	1.5274
2	−2.0618	2.0618
3	−2.6413	2.6413

	Equilibrium contingent claim prices $q(s, s')$			Equilibrium price of money $p(s)$
s	$s' = 1$	$s' = 2$	$s' = 3$	
1	0.17548	0.22909	0.29796	1.5274
2	0.23688	0.30926	0.40223	2.0618
3	0.30345	0.39617	0.51527	2.6413

	Benchmark optimal equilibrium	Nonmonetary equilibrium	Monetary equilibrium
Certainty equivalent (C.E.) consumption level	5	3.4968	4.9584
Difference in C.E. consumption as percentage of benchmark	—	30.0%	0.8%
Expected utility as percentage of expected utility in benchmark case	100%	95.6%	99.9%

σ, rises. Equilibrium asset prices exhibit correspondingly decreased variability as σ rises.

If market structures are viewed as representing institutional or legal constraints which could potentially be modified, it makes sense to compare the welfare levels achieved under alternative market structures. We take as our benchmark for comparison the Pareto-optimal allocation $c_y(s) = c_o(s) = 5$. This allocation Pareto-dominates the timeless markets equilibrium $c_y(s) = w_y$, $c_o(s) = w_o$. Further, it is the equilibrium yielding the highest expected utility in the set of equilibria that Pareto-dominates the complete markets equilibrium.

Table 4

A mean-preserving spread relative to table 3.

State	Young	Old
	Endowments	
1	6	4
2	7	3
3	8	2
	Equilibrium consumption levels in monetary equilibrium	
1	4.4590	5.5410
2	4.9217	5.0783
3	5.3395	4.6605
	Transfers which render nonmonetary sequential equilibrium (autarky) equivalent to monetary equilibrium	
1	−1.5410	1.5410
2	−2.0783	2.0783
3	−2.6605	2.6605

	Equilibrium contingent claim prices $q(s,s')$			Equilibrium price of money $p(s)$
s	$s'=1$	$s'=2$	$s'=3$	
1	0.22435	0.08991	0.37903	1.5410
2	0.30258	0.12136	0.51120	2.0783
3	0.38734	0.15522	0.65439	2.6605

	Benchmark optimal equilibrium	Nonmonetary equilibrium	Monetary equilibrium
Certainty equivalent (C.E.) consumption level	5	3.4026	4.9548
Difference in C.E. consumption as percentage of benchmark	—	32.0%	0.9%
Expected utility as percentage of expected utility in benchmark case	100%	95.2%	99.9%

Each table also exhibits the transfers necessary in the nonmonetary sequential economy to make equilibrium consumption the same as in the monetary economy. By examining these transfers we can learn about the effects of introducing money into the sequential economy. The pattern of transfers shows that there are two effects induced by the alteration in asset prices which occur in the monetary economy. First, a nonzero amount of intergenerational (intertemporal) trade takes place; this is reflected in the fact that transfers to the old (young) are always positive (negative). Second, a nonzero amount of intratemporal risk sharing takes place; this is reflected in the fact that the transfer to an old person is always highest in the state in which his endowment is lowest (state 3). The converse is of course true for young agents: the negative transfer (or tax) on the young person is highest in the state in which his endowment is highest (state 3).

We are interested in comparing welfare levels among the various market structures (complete markets and sequential markets with and without money). One way to make this comparison is to compare certainty-equivalent consumption levels under the alternative market structures. Another way is to compare expected utility. The tables provide information on both of these measures.

In order to compare certainty-equivalent consumption levels, we first compute the constant consumption level yielding the same expected utility as the nonmonetary sequential markets equilibrium. The difference between this level and the (constant) level of consumption attained in the benchmark Pareto-optimal equilibrium, measured as a percentage of consumption in the Pareto-optimal equilibrium, is interpreted as a measure of the welfare cost of the sequential market structure without intervention. In the examples above, this measure ranges from 3% (in table 2, where agents have a low level of risk aversion) to 32% (in table 4, where agents have a high level of risk aversion, and where a mean-preserving spread has increased the probability weights on the extreme outcomes). Proceeding in the same way, it was found that this measure of the welfare cost of sequential markets in the economy with money is much smaller, ranging from 0.3% (in table 3) to 0.8% (in table 3). These results suggest that, in a welfare sense, the monetary equilibrium is much closer to the Pareto-optimal benchmark equilibrium than to the autarky sequential markets equilibrium.

Comparing expected utility levels across the different market structures yields similar conclusions. For each market structure, expected utility was computed as a percentage of expected utility achieved in the benchmark Pareto-optimal equilibrium. For the nonmonetary sequential equilibrium, this measure ranged from 95.2% (table 4) to 97% (table 2). The monetary equilibrium yielded expected utility levels very close to the benchmark Pareto-optimal equilibrium: the relative expected utility measure ranged from 99.6% (tables 1 and 2) to 99.9% (tables 3 and 4). In these examples, therefore, we find that introducing money into the sequential economy results in equilibria that are very close – in an expected utility sense – to the benchmark Pareto-optimal equilibrium. This is true even though there is still a nontrivial market incompleteness in the equilibrium with valued money. Apparently the idiosyncratic risk associated with consumption allocations in the particular monetary equilibria we have studied has inconsequential effects on welfare.

5. Conclusions

This paper has investigated the extent to which the character of equilibria in stochastic sequential overlapping generations models stems from the model's generic market incompleteness. This incompleteness arises naturally as a result of the model's overlapping demographic structure combined with sequential resolution of uncertainty as the economy moves through time. In addition, this paper has addressed the question of whether money serves to complete markets in these models.

First, we have shown that the market incompleteness inherent in the sequential structure represents a real constraint on individuals' trading opportunities. This is shown by demonstrating that sequential equilibria in a stochastic OLG model cannot generally be represented as equilibria in Arrow–Debreu markets held outside of time. This result stands in contradistinction to the result of Marshall, Sonstelie, and Gilles (1987) that, in a deterministic environment, sequential equilibria can be represented as equilibria timeless markets equilibria with transfers.

Second, we have shown that money does not complete markets in the sense of expanding the set of state-contingent commodities that an individual may trade. Money does, however, influence equilibrium consumption patterns. Comparison of the budget constraints for individuals in the monetary and nonmonetary sequential economies shows that the introduction of money may be viewed as effecting state-contingent transfers of wealth between individuals. These state-contingent transfers could alternatively have been carried out by the government.

Finally, the quantitative analysis of section 4 shows that the introduction of money into the sequential economy induces intertemporal trade and intratemporal risk sharing through its effect on state-contingent asset prices. The introduction of money therefore has important welfare implications. In the particular parameterized economies that we study, expected utility levels achieved in monetary equilibrium are very close to expected utility achieved in a benchmark Pareto-optimal equilibrium.

Appendix A

Sequential markets with trade before information revelation

This appendix examines the properties of equilibrium in a sequential markets structure in which agents are born *before* the state of nature is revealed for that period.[8] Consider the model developed in section 2, except that there are only two states of nature, with state-dependent endowments for young and

[8] I wish to thank the referee for encouraging me to develop the ideas presented in this appendix.

old agents given by

$$w_y(s) = \begin{cases} w_y + \varepsilon & \text{if} \quad s = 1, \\ w_y - \varepsilon & \text{if} \quad s = 2, \end{cases}$$

$$w_o(s) = \begin{cases} w_o - \varepsilon & \text{if} \quad s = 1, \\ w_o + \varepsilon & \text{if} \quad s = 2, \end{cases}$$

where w_y, w_o, ε are positive constants, where it is assumed that $w_y > w_o + 2\varepsilon$, and where ε is small enough so that $w_y(s)$, $w_o(s)$ are always strictly positive.

It is easy to verify that the complete markets equilibrium is as before: young agents always consume w_y and old agents always consume w_o, no matter what state of nature actually occurs.

Now, suppose that the economy is operating in real time, i.e., is a sequential economy. Suppose that the old and young agents meet each period *before* the current state of nature is revealed, and are allowed to make trades contingent on the state of nature that will be revealed after trade is concluded. The state transition matrix is assumed to be as follows:

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} \rho & 1-\rho \\ 1-\rho & \rho \end{bmatrix}, \qquad 0 < \rho < 1,$$

which has the stationary probability distribution $(\frac{1}{2}, \frac{1}{2})$. We assume that the probability distribution at the beginning of time is $(\frac{1}{2}, \frac{1}{2})$ to avoid time dependence in transition probabilities.

Some additional notation is necessary at this point. Let s_{-1} denote the state of nature that was realized in the previous period (call it period $t-1$), let s denote the state of nature that will be realized after the close of trading in the current period (period t), and let s' denote the state of nature that will be realized in the next period (period $t+1$). Since the state of nature follows a stationary Markov process, s_{-1} is sufficient to determine the conditional probability of realizing particular states in the current and subsequent periods. Let $\pi_1(s_{-1}, s)$ denote the probability that state s occurs in the current period (period t), given that s_{-1} occurred in period $t-1$, and let $\pi_2(s_{-1}, s')$ denote the probability that state s' occurs in period $t+1$ given that state s_{-1} occurred in period $t-1$.

The young agent's decision problem is now to maximize expected utility conditional on s_{-1} (the state of nature realized in the period before his birth), which is given by

$$\mathrm{E}(U) = \sum_s \pi_1(s_{-1}, s) u\big(c_y(s)\big) + \sum_{s'} \pi_2(s_{-1}, s') u(c_o(s')), \tag{A.1}$$

where momentary utility, u, has the constant elasticity form given in eq. (1) above. Letting $f_1(s_{-1}, s)$ denote the price of a unit of the consumption good in state s in period t given that state s_{-1} occurred in period $t-1$, and letting $f_2(s_{-1}, s')$ denote the price of a unit of the consumption good in state s' in period $t+1$ given that state s_{-1} occurred in period $t-1$, the lifetime budget constraint for this agent is given by

$$\begin{aligned}&\sum_s f_1(s_{-1}, s)c_y(s) + \sum_{s'} f_2(s_{-1}, s')c_o(s')\\&\leq \sum_s f_1(s_{-1}, s)w_y(s) + \sum_{s'} f_2(s_{-1}, s')w_o(s').\end{aligned} \tag{A.2}$$

Transfers have been set to zero for simplicity. Letting $\lambda(s_{-1})$ be the multiplier on the wealth constraint (A.2) and letting $\mathrm{D}u(.)$ denote the first derivative of u, the first-order conditions for this individual's problem are

$$\begin{aligned}&\pi_1(s_{-1}, s)\mathrm{D}u(c_y(s)) = \lambda(s_{-1})f_1(s_{-1}, s) \quad \text{for all } s,\\&\pi_2(s_{-1}, s')\mathrm{D}u(c_o(s')) = \lambda(s_{-1})f_2(s_{-1}, s') \quad \text{for all } s',\end{aligned} \tag{A.3}$$

together with the budget constraint (A.2). To compute the equilibrium, we make use of the following:

(i) In equilibrium, there will be no intertemporal trade, but agents alive at the same dates will exhaust all gains from trade.
(ii) Budget constraints will be satisfied with equality.
(iii) Resource constraints must be satisfied.

Eq. (A.3) implies that

$$\frac{\mathrm{D}u(c_y(s|s=1))}{\mathrm{D}u(c_y(s|s=2))} = \frac{\mathrm{D}u(c_o(s|s=1))}{\mathrm{D}u(c_o(s|s=2))}, \tag{A.4}$$

which is the condition guaranteeing efficient risk sharing; ratios of marginal utilities across states are equated across individuals. With a utility function of the assumed constant elasticity form, this condition simplifies to

$$\frac{c_y(s|s=1)}{c_y(s|s=2)} = \frac{c_o(s|s=1)}{c_o(s|s=2)}. \tag{A.5}$$

The resource constraints are

$$c_y(s) + c_o(s) = w_y + w_o, \qquad s' = 1, 2. \tag{A.6}$$

These conditions imply that

$$\begin{aligned} c_y(s) &= \theta(s_{-1})[w_y + w_o] = \theta(s_{-1})w, \\ c_o(s) &= (1-\theta(s_{-1}))[w_y + w_o] = (1-\theta(s_{-1}))w, \end{aligned} \tag{A.7}$$

where $\theta(s_{-1})$ is a scaling function to be determined and where w denotes the aggregate endowment, $w_y + w_o$. Recall that in the complete markets setting the young consumed w_y and the old consumed w_o, no matter what state occurred. That is, the scaling function $\theta(s_{-1})$ was the constant function:

$$\theta(s_{-1}) = w_y/(w_y + w_o) = w_y/w. \tag{A.8}$$

We conjecture that contingent prices have the following form:

$$f_1(s_{-1}, s) = \alpha(s_{-1})\pi_1(s_{-1}, s),$$

$$f_2(s_{-1}, s') = \alpha(s_{-1})\pi_2(s_{-1}, s'),$$

for some function $\alpha(s_{-1})$. The idea is to use these conjectured prices to determine wealth levels in order to pin down $\theta(s_{-1})$. Absence of intertemporal trade in equilibrium means that individuals' consumptions must equal their endowments on a period-by-period basis. Thus the young agent's budget constraint is given by

$$\sum_s [\alpha(s_{-1})\pi_1(s_{-1}, s)][\theta(s_{-1})w] = \sum_s \alpha(s_{-1})\pi_1(s_{-1}, s)w_y(s),$$

implying that

$$\theta(s_{-1})w = \sum_s \pi_1(s_{-1}, s)w_y(s). \tag{A.9}$$

Analogously for the old agent, we have

$$(1-\theta(s_{-1}))w = \sum_s \pi_1(s_{-1}, s)w_o(s). \tag{A.10}$$

Thus we find that the proportion of the total endowment consumed by the young agent, $\theta(s_{-1})$, is given by

$$\theta(s_{-1}) = \sum_s \pi_1(s_{-1}, s)\frac{w_y(s)}{w}. \tag{A.11}$$

The question we want to answer is the following: does allowing trade between the old and young prior to the resolution of uncertainty result in an equivalence between the sequential equilibrium and the equilibrium obtained in complete markets (Arrow–Debreu markets before the beginning of time)? Answering this question involves comparing the $\theta(s_{-1})$ functions. In the complete markets setting, $\theta(s_{-1})$ is given by eq. (A.8). In the sequential markets setting, $\theta(s_{-1})$ is given by eq. (A.11). These two are equivalent only if $\sum_s \pi_1(s_{-1}, s) w_y(s) = w_y$; i.e., only if the expected endowment conditional on s_{-1} is equal to the unconditional expected endowment. This will not generally be the case, except when the states of nature are serially independent. Therefore equilibrium in the sequential setting will generally differ from equilibrium in the complete markets setting, whether or not trade between current young and old is allowed prior to the resolution of uncertainty in the current period.

The assumption that agents are born after the realization of uncertainty in their first period of life is nevertheless important for the result that agents have idiosyncratic randomness in their equilibrium consumption patterns. But this assumption is not important for the result that equilibria in sequential markets generally differ from equilibria in complete markets.

Appendix B

Algorithm to compute the monetary equilibria of the examples of section 4

Step 1. Read in values of σ, M, $\pi(s, s')$, $w_y(s)$, $w_o(s)$.

Step 2. Read in initial values for $q(s, s')$ – any positive numbers.

Step 3. Compute $\Omega(s)$ (wealth of an agent born in state s) for $s = 1, 2, 3$, using eq. (15).

Step 4. Compute $c_y(s)$, $c_o(s)$ for $s = 1, 2, 3$, using eqs. (13) and (14).

Step 5. Compute $p(s)$ for $s = 1, 2, 3$, using the young individual's budget constraint, eq. (9), and the equilibrium condition that $z(s, s') = 0$.

Step 6. Compute new values for $q(s, s')$ for $s, s' = 1, 2, 3$ from the following equation, which is eq. (13) divided by eq. (14):

$$q(s, s') = \pi(s') * \left[c_y(s)/c_o(s')\right]^{\sigma}.$$

Step 7. Check the no-arbitrage condition, eq. (12).

Step 8. If money is valued (which is the case in all the examples) and if the right-hand side of (12) is negative and within a specified tolerance band, cease iteration. If not, go to step 3 and iterate until this condition is satisfied.

Step 9. Print out equilibrium consumption plans and asset prices.

References

Baxter, Marianne, 1987, Evaluating overlapping generations models, Manuscript (University of Rochester, Rochester, NY).

Baxter, Marianne, 1988, Approximating suboptimal dynamic equilibria: An Euler equation approach, Rochester Center for Economic Research working paper no. 139 (University of Rochester, Rochester, NY).

Cooley, Thomas and J. Sonstelie, 1984, Equilibrium asset prices under non-neutral taxation, Manuscript (University of California, Santa Barbara, CA).

Debreu, Gerard, 1959, Theory of value, Cowles Foundation monograph no. 17 (Yale University Press, New Haven, CT).

Huffman, Gregory, 1987, A dynamic equilibrium model of asset prices and transactions volume, Journal of Political Economy 95, 138–159.

Manuelli, Rodolfo, 1987, Existence and optimality of currency equilibrium in stochastic overlapping generations models: The pure endowment case, MEDS discussion paper (Northwestern University, Evanston, IL).

Marshall, John, J. Sonstelie, and C. Gilles, 1987, Money and redistribution: Revisionist notes on a problem of Samuelson, Journal of Monetary Economics 20, 3–24.

Peled, Dan, 1982, Informational diversity over time and the optimality of monetary equilibria, Journal of Economic Theory 28, 255–274.

Samuelson, Paul, 1958, An exact consumption loan model of interest with or without the social contrivance of money, Journal of Political Economy 66, 467–482.

Sargent, Thomas, 1987, Dynamic macroeconomic theory (Harvard University Press, Cambridge, MA).

Wright, Randall, 1987, Market structure and competitive equilibrium in dynamic economic models, Journal of Economic Theory 41, 189–201.