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Approximating suboptimal dynamic equilibria

An Euler equation approach

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This paper develops a method for numerical approximation of dynamic competitive equilibria which can be applied quite generally. This method produces approximations to the equilibrium decision rules via a direct attack on the stochastic Euler equations which define competitive equilibrium. This approach does not require the Pareto optimality of competitive equilibrium. We provide a detailed discussion of the implementation of this computational method, and discuss considerations important for the choice of one computational technique versus another. For illustrative purposes, the paper presents several examples based on the analysis of taxation in the one-sector neoclassical model.

1. Introduction

Many research questions necessarily involve the study of suboptimal dynamic equilibria. For example, one might be interested in studying the effect of a change in the income tax laws on the joint time series behavior of investment, production, and asset returns under rational expectations. Since taxes induce a wedge between private and social returns, the resulting dynamic equilibrium is suboptimal and cannot generally be studied with

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methods which rely on the Pareto optimality of competitive equilibrium. Since dynamic equilibrium problems do not generally have analytic solutions, one must choose between (i) studying special cases of these problems in which closed forms exist or (ii) using numerical methods to compute approximate equilibria.

This paper develops a method for numerical approximation of dynamic competitive equilibria which can be applied quite generally.[1] In a dynamic competitive equilibrium problem, individual agents are assumed to make their decisions in a privately rational manner. The result of this maximization is a set of first-order necessary conditions or 'stochastic Euler equations' for the individual's problem; these conditions restrict the dynamic evolution of the individual's choice variables. When combined with aggregate consistency conditions, the stochastic Euler equations restrict the dynamic behavior of the entire economic system.

On a theoretical level, we are familiar with viewing dynamic competitive equilibrium in an economy as a set of functions which satisfy the economy's stochastic Euler equations and aggregate consistency conditions. The properties of equilibrium can therefore be explored by finding numerical approximations to these equilibrium functions. The method described in this paper finds discrete approximations to the equilibrium functions that solve the stochastic Euler equations. That is, approximate equilibria are computed using an algorithm which involves discretization of the state space as in Bertsekas (1976) and Sargent (1979). This paper provides a detailed discussion of this method, and demonstrates its use by application to the stochastic one-sector neoclassical growth model of Brock and Mirman (1972), modified to allow distortionary taxation.

While this paper is organized around the problem of capital accumulation in the presence of distortions, the basic computational approach is applicable to a wide variety of problems in which competitive equilibrium can be characterized as a system of Euler equations. For example, this method can be used to study overlapping generations (OLG) economies with long-lived agents, as discussed in Baxter (1987). Because equilibria in OLG economies are generally suboptimal, these equilibria cannot be studied using methods which rely on the optimality of competitive equilibrium. The method can also be used to study monetary economies in which the introduction of money means that competitive equilibrium is not socially optimal, as in Cooley and Hansen (1989), Baxter (1991), and Hodrick, Kocherlakota, and Lucas (1991). Other potential applications of the methodology are to economies in which

[1]This paper develops in more detail the computational strategy outlined in Baxter (1987), and which is briefly discussed in Baxter, Crucini, and Rouwenhorst (1990). Working independently, Danthine, Donaldson, and Smith (1987), Bizer and Judd (1989), and Coleman (1990, 1991) have developed methods that are similar to the one described here, in that the central insight is to work with the system's first-order conditions.

suboptimality of competitive equilibrium is due to (i) the existence of monopoly power at the firm level, as in dynamic versions of the model of Blanchard and Kiyotaki (1987); (ii) productive externalities of the types studied by Romer (1986), Lucas (1988), and Baxter and King (1990); or (iii) incompleteness in asset markets, as in Persson and Svensson (1987) and Svensson (1988).

The paper proceeds as follows. Section 2 describes the method of obtaining approximate equilibrium policy functions via iteration on Euler equations. The presentation is organized around the stochastic one-sector growth model with distortionary taxation, and highlights the conceptual similarity between this method and the more familiar method of value function iteration. Section 3 applies the Euler equation approach to computing equilibrium policy functions for several variants of this problem, and evaluates the computational accuracy of the approach. This is done by comparing approximate policy functions to exact policy functions in the context of an example economy possessing a closed-form solution for the policy function. Section 4 addresses the problem of the choice of computational method. Section 5 contains concluding remarks and discusses directions for future research.

2. The Euler equation approach

The Euler equation approach is illustrated within the framework of the neoclassical model of capital accumulation under uncertainty. In this model, individuals maximize expected utility:

$$\mathrm{E}\sum_{t=0}^{\infty}\beta^t u(c_t), \tag{1}$$

where β is a discount factor between zero and one, the utility function $u(.)$ is assumed to be twice continuously differentiable, and where the expectation taken at time zero is conditioned on the initial capital stock, k_0, and the initial value of the technology shock, A_0. Agents face a sequence of resource constraints of the form

$$A_t f(k_t) + (1-\delta)k_t \leq c_t + k_{t+1}, \tag{2}$$

where A_t is a technology shock; k_t is the capital stock; $f(k)$ is the production function, assumed to be twice continuously differentiable with $f'(k) > 0$ and $f''(k) < 0$; and δ is the rate of depreciation of capital. The technology shock, A_t, follows a discrete Markov process with state transition matrix Π. Agents in this model own the capital stock and directly operate the technology. In period t, they receive output from production, $A_t f(k_t)$, and there is undepreciated capital left over after production in the amount $(1-\delta)k_t$. They allocate this gross output between current consumption, c_t, and capital to be used in production in the subsequent period, k_{t+1}. Thus the period $t+1$ capital stock is determined at the end of period t, and cannot be adjusted after the period technology shock A_{t+1} is realized at the beginning of period $t+1$.

Using the notation Dx to denote the derivative of the function x, the first-order necessary conditions for the consumer's problem are

$$\mathrm{D}u(c_t)=\beta \mathrm{E}\{[A_{t+1}\mathrm{D}f(k_{t+1})+(1-\delta)]\,\mathrm{D}u(c_{t+1})\},$$

$$\mathrm{E}\left\{\lim_{t\to\infty}\beta^t\,\mathrm{D}u(c_t)k_{t+1}\right\}=0, \tag{3}$$

and the budget constraint (2). Since this problem is recursive we let unprimed variables denote period t, single primes denote period $t+1$, and double primes denote period $t+2$. Making these substitutions and using the resource constraint to substitute for c, eq. (3) becomes

$$\begin{aligned}&\mathrm{D}u(Af(k)+(1-\delta)k-k')\\&=\beta\mathrm{E}\{[A'\mathrm{D}f(k')+(1-\delta)]\,\mathrm{D}u(A'f(k')+(1-\delta)k'-k'')\}.\end{aligned} \tag{4}$$

Under the assumptions imposed on this problem, there is unique function relating the optimal choice of k' to the current level of k and the current technology shock A; call this function h:

$$k'=h(k,A). \tag{5}$$

To take a specific example, suppose that there are only two possible realizations of the technology shock, $A_t\in\{\bar{A},\underline{A}\}$, and that A_t follows a Markov process with transition matrix Π. Fig. 1 plots the policy functions for k' as a function of k, conditional on the technology shock. One fixed point is at $k=\bar{k}$; this is the level of capital that would obtain if the economy turned out always to have the high realization of the technology shock, $A_t=\bar{A}$ for all t even though in each period there is positive probability that $A=\underline{A}$ in some future period.[2] There is a second fixed point $k=\underline{k}$, which is the level of

[2]In general, the function $h(k,\bar{A})$ is not the function that would obtain in a deterministic version of the model in which the technology variable always took on the value $\bar{A}$ [similarly for $h(k,\underline{A})$]. In the stochastic model, there is always a positive probability associated with each of the two shocks, and the equilibrium policy functions incorporate individuals' response to the risk associated with the randomness in A.

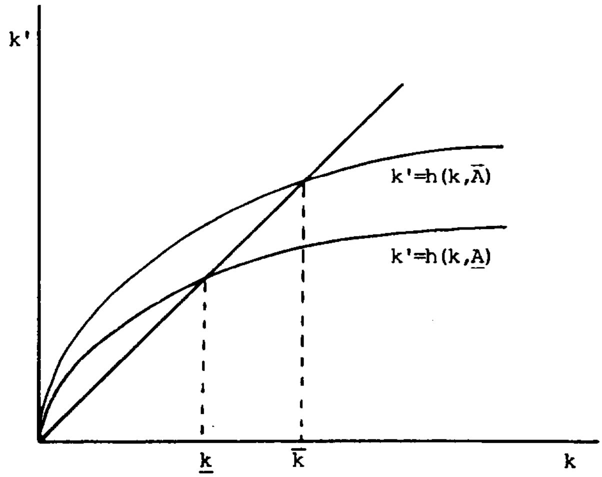

Fig. 1

capital that would obtain if the economy always had the low realization of the technology shock, $A_t = \underline{A}$ for all t. In addition, all the points in the interval $(\bar{k}, \underline{k})$ generally have positive mass in the stationary distribution of k.

Except for very special choices of the parameters of preferences and technologies, it is not possible to solve (4) to obtain a closed-form solution for the function $h(k, A)$. We turn now to a discussion of two approaches to computing approximations to the equilibrium policy function. The first is the approach of stochastic dynamic programming and value function iteration. This method relies on the equivalence between competitive equilibrium and Pareto optimum in the economy under consideration.[3] The second approach is new, and involves iteration on a stochastic Euler equation. This approach does not rely on the Pareto optimality of competitive equilibrium.

2.1. Stochastic dynamic programming and value function iteration

Since the problem described above has a recursive structure, it can be studied using the methods of stochastic dynamic programming. Thus, the problem can be rewritten as

$$v(k_t, A_t) = \max_{c_t, k_{t+1}} u(c_t) + \beta \mathrm{E}\{v(k_{t+1}, A_{t+1})\}, \tag{6}$$

[3]Two recent papers which use this computational approach are Sargent (1979) and Greenwood, Hercowitz, and Huffman (1988).

subject to the constraint (2). The function v gives the value, in utility terms, of entering a period with capital equal to k_t and encountering the technology shock A_t, assuming that the agent makes individually optimal decisions. Eq. (6) is a functional equation in the unknown function v. Using (2) to substitute for c_t in eq. (4), we obtain

$$v(k_t, A_t) = \max_{k_{t+1}} u(A_t f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta \mathrm{E}\{v(k_{t+1}, A_{t+1})\}. \tag{7}$$

Define the operator T by

$$Tv = \max_{k_{t+1}} u(A_t f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta \mathrm{E}\{v(k_{t+1}, A_{t+1})\}. \tag{8}$$

Since the form of (8) does not depend on the time period, t, time subscripts can be suppressed and (8) can be written as

$$Tv = \max_{k'} \{u(Af(k) + (1-\delta)k - k') + \beta \mathrm{E} v(k', A')\}, \tag{9}$$

where, as above, variables without superscripts refer to the current period (t) and primed variables refer to the subsequent period ($t+1$). Solving for the unknown function v involves finding a fixed point of the mapping T, i.e., finding the function v for which $Tv = v$.

Because the mapping T defined by eq. (8) is a contraction mapping,[4] iteration on T produces a sequence of functions which converge to the unique fixed point of the mapping. This suggests that iteration on the mapping can be used as a computational approach to finding an approximation to the optimal value function. The approximate nature of the solution is due to the computational necessity of 'discretizing the state space', i.e., choosing a discrete grid for k and A over which the value function will be defined. Having done this, the computational problem involves finding the approximate value function, v^*, which is defined on the (k, A) grid and which approximately solves the equation $Tv^* = v^*$.

The iterative procedure begins by choosing an initial v^* function, defined on the (k, A) grid. Call this function v_0^*. Given v_0^*, application of the

[4] T can be shown to be a contraction by verifying that it satisfies Blackwell's (1965) conditions. For a more detailed discussion of this problem, see Stokey and Lucas (1989).

operator T yields a new v^* function; call this new function v_1^*:

$$v_1^* = Tv_0^* = \max_{k'} \{u(Af(k) + (1-\delta)k - k') + \beta \mathrm{E} v_0^*(k', A') \mid k', A\},$$

where the maximization is over values of k in the chosen grid and is conditional on the current value of A. Subsequent iterations proceed in the same way, generating a sequence of v^* functions, $\{v_j^*\}$. The iterative process continues until the sequence of v^* functions converges according to a criterion selected by the researcher.

Often the value function chosen as the starting point for the iterative procedure is the zero function, $v_0^* = 0$. This choice of v_0^* means that the sequence of functions produced by application of the operator T has an economic interpretation as the sequence of value functions for finite economies, with the horizon lengthening one period at each iteration. Thus, v_1^* is the approximate value function for an economy with one period left to go, v_2^* is the approximate value function for an economy with two periods left to go, and so forth. In the one-sector growth model, the limit of the value functions for the finite horizon economies is the value function for the infinite horizon economy.

If the economy under study satisfies the conditions of the second welfare theorem, the optimal solution obtained by value function iteration may be interpreted as a competitive equilibrium. In cases where competitive equilibrium is not optimal, the approach outlined above is generally invalid. It can be used only if there is a way to rewrite the competitive problem as an optimum problem which properly reflects the constraints of the competitive problem. In general, studying suboptimal equilibria requires a direct attack on the first-order necessary conditions of the individual's problem. The Euler equation approach proceeds in exactly this way.

2.2. *The Euler equation approach*

Unlike value function iteration, the method described here does not rely on the second welfare theorem. For illustrative purposes, we consider the neoclassical model of capital accumulation described above, modified to allow distortionary taxation in the form of an income tax with lump-sum rebates of the proceeds of the tax. This problem cannot be recast as a fictitious planner's problem, so the method of value function iteration cannot be used.[5] In this example, taxes can be functions of the Markovian technology shock and the level of the aggregate capital stock. The point of this is to

[5]If taxes are not rebated in a lump-sum manner, the tax shocks may be reinterpreted as technology shocks, as discussed in Abel and Blanchard (1983), and value function iteration could then be used to compute the equilibrium. With lump-sum rebates the problem cannot be recast in this way as a fictional 'planner's problem', and an alternative to value function iteration must be found. This problem, therefore, is a natural one for illustrating the Euler equation method.

demonstrate that state dependence of the tax function does not add to the complexity of the problem to be solved. The economy's capital stock (per capita) is denoted by K, and the individual agent's choice of capital is denoted by k. This is not a very restrictive specification of taxes since, in equilibrium, all endogenous variables (output, consumption, investment, etc.) are functions of the state vector (K, A). Let $\tau(K_t, A_t)$ denote the tax rate in period t – note that taxes depend on the aggregate capital stock, not on individual's capital accumulation decisions. The behavior of the government is very simple: it levies taxes and returns the proceeds in a lump-sum manner. Letting TR_t denote period t transfer payments, the government's budget constraint is just $TR_t = \tau(K_t, A_t)A_t f(K_t)$, for all t.

The problem facing the representative agent in this economy is

$$\max_{\{c_t, k_{t+1}\}} \mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \mid A_0, k_0 \right\}, \tag{10}$$

subject to

$$c_t + k_{t+1} \leq (1 - \tau(K_t, A_t))A_t f(k_t) + (1 - \delta)k_t + TR_t, \tag{11}$$

$$K_{t+1} = H(K_t, A_t). \tag{12}$$

Eq. (11) is the individual's resource constraint; the first two terms on the right-hand side are after tax gross output, and the last term is the lump-sum rebate of the government's tax revenues. Eq. (12) is private agents' perceived law of motion for aggregate capital, K_t. As before, it is convenient to suppress time subscripts, and the arguments of the tax function are suppressed as well: τ should be read as $\tau(K, A)$. The first-order necessary condition for maximization with respect to choice of capital is $\mathrm{D}u(c) = \beta\mathrm{E}\{[(1 - \tau')A'\mathrm{D}f(k') + (1 - \delta)]\mathrm{D}u(c')\}$. Using (11) to substitute for c_t yields

$$\begin{aligned} &\mathrm{D}u((1-\tau)Af(k) + (1-\delta)k + \tau f(K) - k') \\ &= \beta\mathrm{E}\{[(1-\tau')A'\,\mathrm{D}f(k') + (1-\delta)]\,\mathrm{D}u((1-\tau')A'f(k') \\ &\quad + (1-\delta)k' + \tau' A' f(K') - k'')\}. \end{aligned} \tag{13}$$

Individual maximization yields equilibrium decision rules of the form:

$$k' = h(k, A; K, H). \tag{14}$$

In equilibrium, the capital agents choose to carry out of the period is a function of capital brought into the period, k, and the current technology shock, A. Individuals take as given the tax function $\tau(K, A)$ and the current level of the aggregate capital stock, K. They condition on their beliefs about the law of motion for aggregate capital as summarized by the function $K' = H(K, A)$.

A rational expectations equilibrium requires that the law of motion for k_{t+1} coincides the perceived law of motion for K_{t+1}:

$$h(k, A; K, H) = H(K, A). \tag{15}$$

This condition is sometimes referred to as a 'consistency condition', meaning that individual's beliefs are consistent with the outcomes of the economy's equilibrium: in equilibrium, k (capital chosen by the representative agent) must equal aggregate capital, K. Imposing this consistency condition on the first-order condition yields

$$\begin{aligned} &\mathrm{D}u(Af(k) + (1-\delta)k - k') \\ &= \beta \mathrm{E}\{[(1-\tau')A'\mathrm{D}f(k') + (1-\delta)]\,\mathrm{D}u(A'f(k') \\ &\quad + (1-\delta)k' - k'')\}. \end{aligned} \tag{16}$$

Finding the competitive equilibrium means finding the function h of the form given by (15) which solves (16) and for which the implied function h is such that $h(k, A; K, H) = H(K, A)$. Below, we use the notation $h(k, A)$ when referring to equilibrium policy functions: functions for which $h(k, A; K, H) = H(K, A)$.

2.3. An iterative approach to approximating stochastic Euler equations

This subsection provides a detailed description of the Euler equation approach to computing approximate equilibrium policy rules. This procedure is similar in spirit to the method of iterating on the value function described earlier. As with the value function approach, the first step is to discretize the state space by choosing a grid for k and A.[6] And, as with value function iteration, the method of iterating on Euler equations can be viewed as

[6]The grid chosen will depend on the specific problem under study. If, for example, the exogenous state variable, A, is not discrete in the original problem but instead is normally distributed, Tauchen's (1985) method can be used to provide a discrete Markov-chain approximation to the normal distribution. The grid for the endogenous state variable will depend on whether the researcher is interested only in the steady state properties of the economy, in which case the grid should be concentrated in this area. If, on the other hand, the researcher wishes to study transitional dynamics, the grid must cover a larger area containing the initial value of the endogenous variable.

generating a sequence of optimal policy rules for finite economies with the horizon lengthening one period at each iteration. This perspective will be used in the following discussion of the computational algorithm. Under this perspective, we view ourselves as working backward from the end of the economy, in a manner similar to stochastic dynamic programming.

We therefore begin by considering an economy that will terminate at the end of period N. In an N period economy agents will plan to consume all of their capital by the end of period N, setting $k_{N+1}=0$ regardless of the levels of k_N and A_N. Thus, the equilibrium policy function relating k' to (k, A) for an economy with zero periods to go is the zero function: $k_{N+1} = h_0(k_N, A_N)=0$. Now, step back one period and consider the problem of the optimal choice of capital in period $N-1$. This involves solving the period $N-1$ version of eq. (16), using the fact that $k_{N+1}=h_0(k_N, A_N)=0$. Thus, the period $N-1$ version of (16) is

$$\begin{aligned}&\mathrm{D}u\left[A_{N-1}f(k_{N-1})+(1-\delta)k_{N-1}-k_N\right]\\&=\beta\mathrm{E}\{[(1-\tau_N)A_N\,\mathrm{D}f(k_N)+(1-\delta)]\,\mathrm{D}u(A_Nf(k_N)\\&\quad+(1-\delta)k_N)\}.\end{aligned}\tag{17}$$

Eq. (17) defines a first-order stochastic difference equation in k. Solving the equation involves finding, for each (k_{N-1}, A_{N-1}) pair, the equilibrium amount of capital to take out of the period, k_N. Solving (17) involves using the initial policy function, h_0, together with the stochastic Euler equation to generate a new policy function, h_1, i.e., the solution to (17) is the function $k_N = h_1(k_{N-1}, A_{N-1})$.

It is good computing practice to use known properties of the problem to simplify the computations. The one-sector growth model is very well-behaved in that it has smooth, concave preferences and technology. We can use these properties to improve the efficiency of our computational algorithm. In other applications, the restrictions placed on the model's 'building blocks' by the economic phenomenon being modeled should analogously be used to streamline the computations. In this spirit, we rewrite eq. (16) as follows, denoting the right-hand side of (16) as $MV(k'\mid A)$ (the expected marginal value of additional unit of capital invested for use in the subsequent period, conditional on the current value of the technology shock, A):

$$Af(k)+(1-\delta)k=k'+\mathrm{D}u^{-1}(MV(k'\mid A)).\tag{18}$$

We have isolated all terms in k' on the right-hand side of (18) and all terms in k on the left-hand side; this will turn out to be very useful.

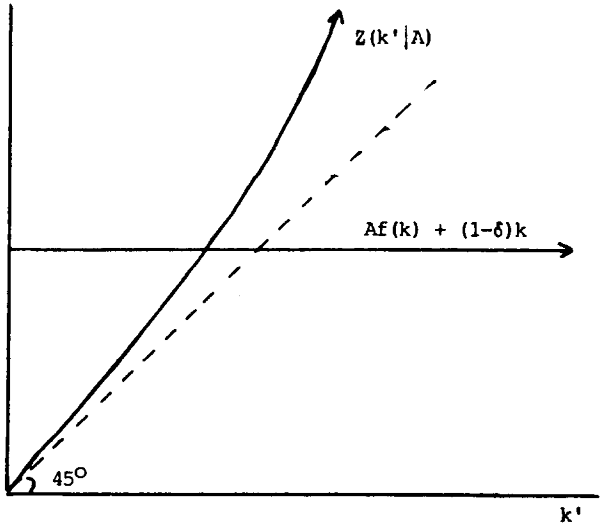

Fig. 2

A two-step computational procedure is used to trace out the new policy function, h_1. The first step of the computational procedure is to compute the right-hand side of (18) for each (k', A) pair in the grid. Call this function Z:

$$Z(k' \mid A) \equiv k' + \mathrm{D}u^{-1}(MV(k' \mid A)). \tag{19}$$

In this discrete environment Z is a two-dimensional matrix, with each column corresponding to a particular value of A and rows corresponding to different values of the capital stock.

The second step is to find, for each (k, A) pair, the value of k' in the matrix which comes closest to solving (18). This is the point at which it is very useful to have segregated terms in k from terms in k'. For each point (k, A), we simply compute the left-hand side of (18), $Af(k) + (1-\delta)k$, and then search the appropriate column of the Z matrix for the value of k' that solves (18). Further, we know from the structure of the model that, conditional on A, Z is nondecreasing in k', as illustrated in fig. 2. This makes the search procedure very straightforward. Given (k, A), if we choose a candidate value for k' such that $Af(k) + (1-\delta)k > Z(k' \mid A)$, we know that we must try a larger value for k'. Further, the problem is simple in that, if we have found one value of k' that solves (18) for given k and A, we need not look for another. This need not, of course, be true for other problems. But the virtue of a grid method is that it can locate all the zeros of a function by a simple-minded search over the whole grid.

Applied to eq. (17), this two-stage procedure yields a function giving k_N as a function of A_{N-1} and k_{N-1}, call this function h_1: $k_N = h_1(k_{N-1}, A_{N-1})$. This is the equilibrium policy function for k' as a function of (k, A) for an economy with one period left to go. Now, step back again and consider the version of eq. (16) that applies to the economy in period $N-2$. Using (19) and the function $k_N = h_1(k_{N-1}, A_{N-1})$ obtained from the first iteration, the period $N-2$ Euler equation can be written as

$$A_{N-2}f(k_{N-2}) + (1-\delta)k_{N-2} = Z(k_{N-1}|A_{N-2}).$$

Proceeding as in the first iteration, compute $Z(k_{N-1}, A_{N-2})$ for each (k_{N-1}, A_{N-2}) pair. Now, for each pair (k_{N-2}, A_{N-2}) search the matrix Z for the value of k_{N-1} that comes closest to solving (19). Call this function h_2: $k_{N-1} = h_2(k_{N-2}, A_{N-2})$.

The way to proceed in the third and subsequent steps should, by now, be clear. At step j, the policy function h_{j-1} is used to evaluate k'', and the resulting equation is solved to obtain a new policy function h_j as described above. Iteration continues until the sequence of functions $\{h_j\}$ converges, i.e., when h_j changes only a small amount between iterations. One natural norm is the sup norm with tolerance level $d \geq 0$; iteration stops when

$$\max_{A,k} |h_j(k, A) - h_{j-1}(k, A)| \leq d.$$

In the course of implementing this algorithm, the initial function $h_0 = 0$ has been found to work well in the sense that convergence is reasonably rapid. With $h_0 = 0$, the sequence of functions $\{h_j\}$ generated by the iterative procedure has a natural economic interpretation in much the same way as in value function iteration discussed earlier. The sequence $\{h_j\}$ can be viewed as an approximation to the sequence of equilibrium policy functions for finite economies with j periods left to go. However, convergence will be more rapid if the initial function h_0 is closer to the equilibrium policy function under the norm specified by the researcher. A good starting point for many applications is the function given by log-linear approximations to the true decision rules, as described for example in King, Plosser, and Rebelo (1989). These log-linear approximations are easy to compute, and in problems with large state spaces a good starting point is likely to be essential to keeping computation time to a manageable level.

There are no generally applicable theoretical results giving precise necessary and sufficient conditions under which the Euler equation approach will yield the competitive equilibrium for a problem.[7] But if the algorithm converges, if it satisfies the stochastic Euler equations, and if a check of the second-order conditions guarantees that a maximum has been found, then one can be sure that the procedure has found an equilibrium function in the space of policy functions. One cannot conclude, however, that this is the unique equilibrium function. Nevertheless, if multiple equilibria were suspected in a particular problem, one way to search for these would be to apply the algorithm repeatedly, starting with a different initial policy function each time.

3. Computation of equilibrium in the one-sector model

This section presents the results of applying the Euler equation approach to several variations of the distorted one-sector model described above. We begin by examining a special case of an undistorted economy for which a closed form solution exists. The approximate policy rules are compared to those computed from the closed form, in order to assess the accuracy of the approximation. Subsequently, several examples of economies with distortionary taxation are presented.

3.1. A closed form example

As an initial application of the approximation methodology, we study an economy possessing a closed form solution for the policy function $k' = h(k, A)$. In this economy, individuals maximize an objective function of the following form:

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \mid A_0, k_0.$$

The production function is Cobb–Douglas and is subject to technology shocks, A, which are identically and independently distributed:

$$y = Af(k) = Ak^{\alpha}, \tag{20}$$

and there is 100% depreciation of capital in each period, $\delta = 1$. The government taxes output at the rate τ, where τ is given by the function

$$\tau(k, K) = 1 - M(k/K)^{-\phi}, \tag{21}$$

[7] Coleman (1991) provides sufficient conditions for convergence of an algorithm related to this one, for an economy similar to the one studied here. However, interesting economic examples – including the one studied in section 3.3 below – violate these sufficient conditions.

where M and ϕ are constants. According to this specification, the tax rate levied on an individual firm (with capital equal to k) depends on the size of the firm, in terms of capital input, relative to the average firm size, K. If $\phi = 0$ the tax rate does not depend on relative firm size; the tax rate is constant at the level $1 - M$. If $\phi > 0$ the tax rate increases with firm size, and if $\phi < 0$ the converse is true. The revenues of the firm (after taxes) are $(1 - \tau)y = AMK^{\phi}k^{\alpha-\phi}$, so that $\alpha > \phi$ is required for a positive marginal revenue product of private capital.

In equilibrium, the consistency condition guarantees that the average tax rate is $1 - M$, whatever the value of ϕ. But individuals considering their own capital accumulation choices will take into account the effect of changing their relative capital stocks on their average tax rate.

The Euler equation for this economy is given by

$$(c_t)^{-1} = \beta \mathrm{E}_t\{(c_{t+1})^{-1}(\alpha - \phi)(1 - \tau_{t+1})A_{t+1}k_{t+1}^{\alpha-1}\}. \tag{22}$$

To derive the closed form solution for this economy, we conjecture that the marginal propensity to consume is constant at the level z. Thus $c_t/y_t = z$, and $k_{t+1}/y_t = 1 - z$ for all t. Substituting this conjecture into (22) yields, after some manipulation, $z = 1 - \beta M(\alpha - \phi)$. In this economy, the solution for the equilibrium path of capital is given by

$$k' = 1 - zy = \beta M(\alpha - \phi)Ak^{\alpha}. \tag{23}$$

Fig. 3 plots the exact equilibrium policy functions for this economy with a grid containing 500 points for k and two points for A, with the following parameter values: $\beta = 0.95$, $\alpha = 0.33$, $M = 1$, $\phi = 0.0$, $\underline{A} = 1.0$, $\bar{A} = 1.2$. This is the simplest possible example: an economy in which the tax rate is zero ($M = 1$) and does not depend on relative firm size ($\phi = 0$). On the graph there is one policy function conditional on each value of the technology shock, and one fixed point corresponding to each value of the technology shock. Fig. 3 also plots the approximate equilibrium policy functions for this economy, computed with the same capital grid of 500 points, and using the norm defined in eq. (22) with $d = 0.0$. As seen from fig. 3, the approximate function is indistinguishable from the exact function. To get a closer look at the approximation error, fig. 4 graphs the approximation error (exact minus approximate) against the capital stock for each of the two policy functions. The maximum approximation error is less than 3×10^{-4} in absolute value. Convergence of the policy functions is quite rapid, requiring 10 iterations beginning with an initial function h_0 which was the constant function $k' = 0.002$.[8]

[8]The grid for capital contains 500 points, and each iteration requires about five seconds on an IBM-compatible 386 computer running at 16 MHz.

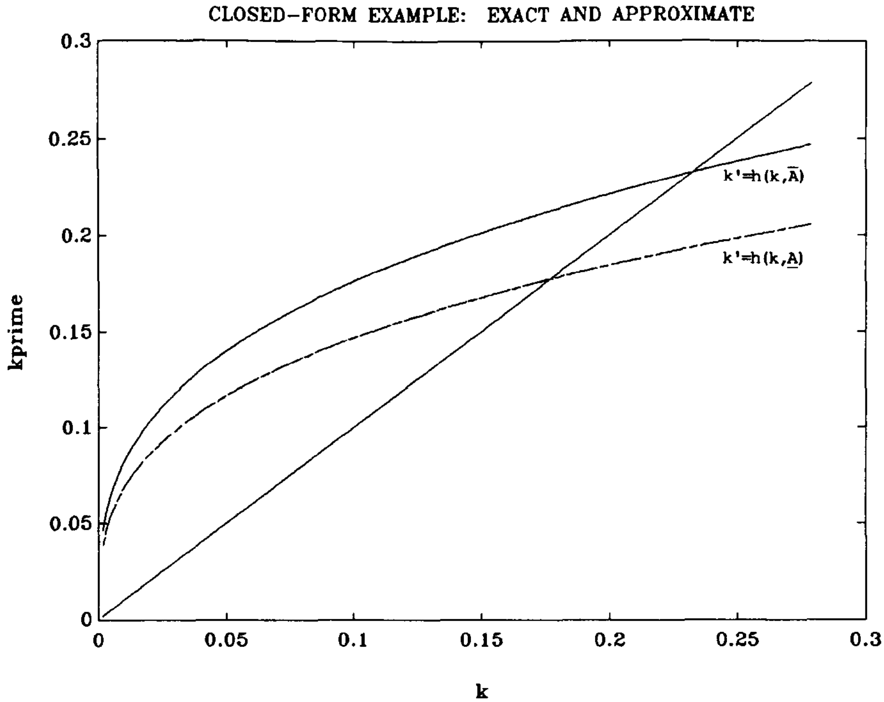

Fig. 3

In fig. 5 we set the average tax rate to a more realistic rate of 30% ($M = 0.7$), and explore the implications of variation in ϕ. To focus on ϕ, we set $\underline{A} = \bar{A} = 1.0$. Two cases are considered: $\phi = 0.10$ (tax rates increase with relative firm size) and $\phi = -0.10$ (tax rates decrease with relative firm size). Recall that an economically sensible solution requires $\phi < \alpha$.

The first panel of fig. 5 plots output and the investment functions for these two cases; panel 2 plots output and consumption. In both panels the dotted line is output, the solid line corresponds to the case $\phi = 0.10$, and the dashed line corresponds to the case $\phi = -0.10$. Steady states are indicated by small dots. Panel 3 plots the tax rate as a function of relative firm size. For $\phi = 0.10$ (the solid line) the tax rate increases with firm size, and for $\phi = -0.10$ (the dashed line) the tax rate decreases with firm size. In both cases, however, the tax rate is 30% when $k = K$.

Panel 4 plots the marginal propensity to consume (MPC) against the level of the capital stock. In each case, the MPC is computed from the approximate solution as $MPC = (y - k')/y$. The exact MPC, computed as $MPC = 1 - \beta M(\alpha - \phi)$, is also plotted. In each case, the approximation error is so

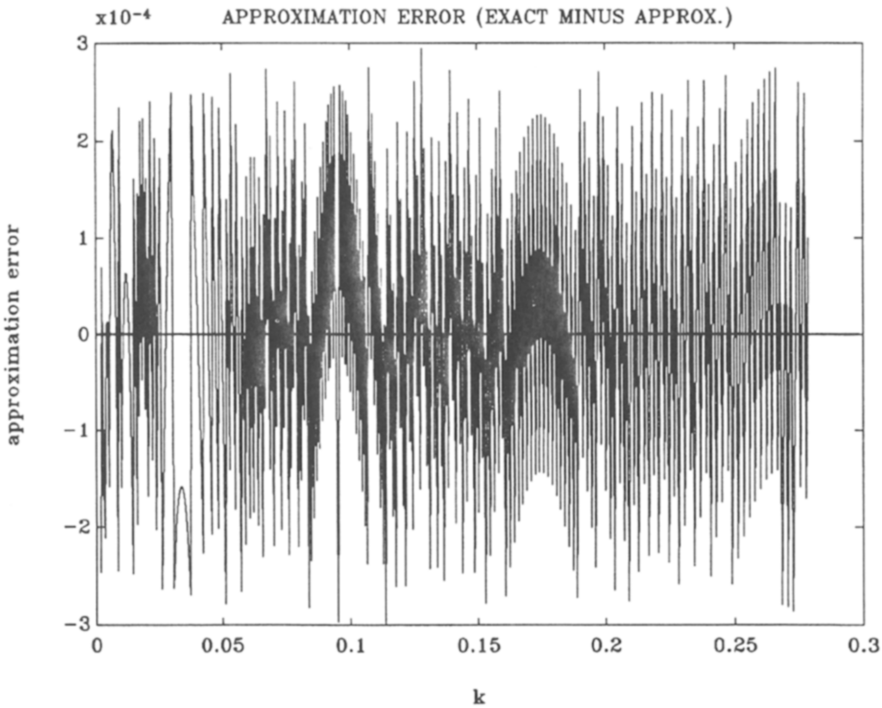

Fig. 4

small as to be hardly noticeable on the graph. To give an idea of the size of the approximation error, the true *MPC* for the case $\phi = -0.10$ is 0.8471, which is also the mean of the approximate *MPC*. The standard deviation of the approximate *MPC* in this case is about 4×10^{-4}.

Looking at fig. 5, we see that investment as a function of the capital stock is lower for the $\phi = 0.10$ case, in which taxes increase with relative firm size. The equilibrium level of output is correspondingly lower, and the *MPC* is correspondingly higher. The intuition behind this is easily understood: the higher the level of ϕ, the more sharply taxes increase with relative firm size. This leads individuals to cut back the size of their firms and, since in equilibrium everyone is the same, the aggregate capital stock will be lower. Further, the allocation of output between consumption and investment will be tilted toward higher consumption. Panel 2 shows that consumption in the two cases appears roughly equal; in fact consumption is slightly higher in the $\phi = -0.10$ case. Despite the fact that the *MPC* is lower in this case, equilibrium output is higher by a sufficient amount to support a higher level of consumption. This can be seen in panel 2 (which plots the level of

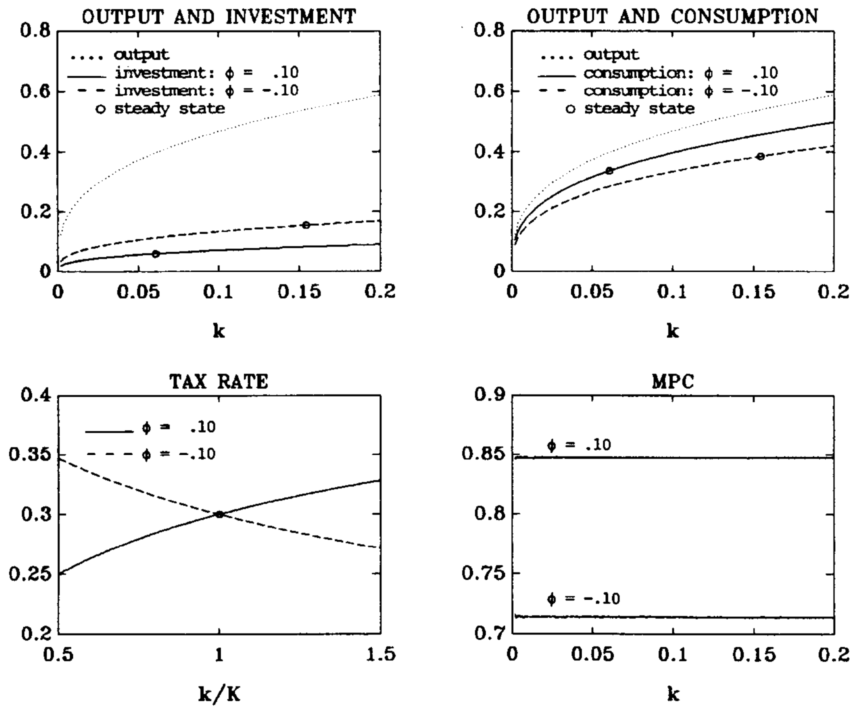

Fig. 5

consumption) and in panel 4 (which plots the marginal propensity to consume out of output).

3.2. Realistic taxation and realistic depreciation

Having verified that the Euler equation approach yields highly accurate solutions in cases where the model possesses a closed form solution, we turn now to investigate of the effects of taxation in the one-sector model with realistic depreciation. Fig. 6 plots equilibrium policy functions for the economy described above with decpreciation of capital at the rate of 10% per year and with a zero tax rate. There are two steady states, one for each value of the technology shock; the upper one is at a level of capital of 4.56 and the lower one is at a level of 2.93. The stationary distribution of capital is contained in this interval. Fig. 7 plots equilibrium policy functions for the economy with 10% depreciation, and in which output is taxed at a 30% rate, corresponding roughly to the average level of U.S. government tax revenues as a fraction of GDP. In this economy, the steady states are at 2.84 and 1.63.

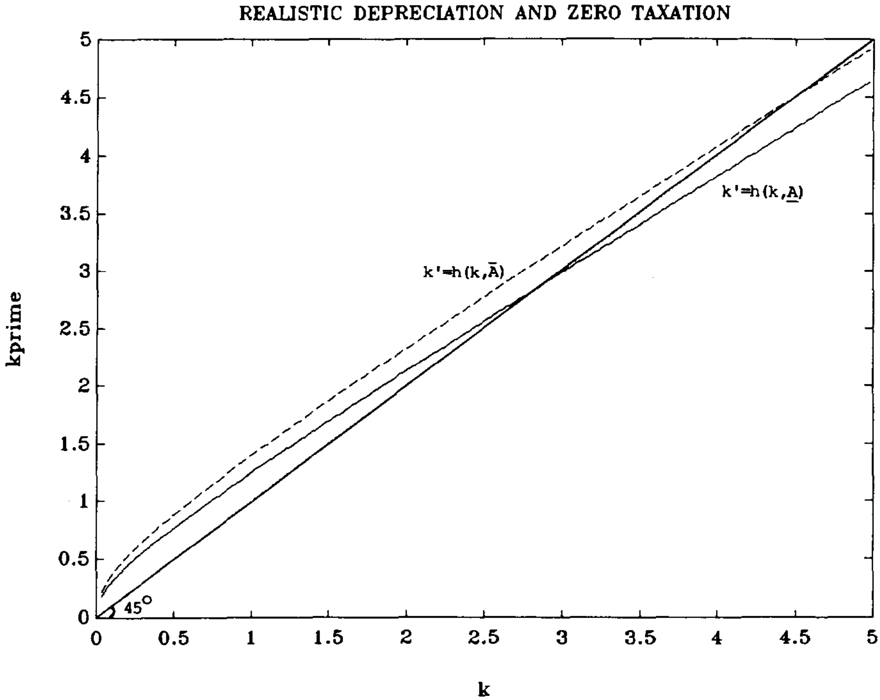

Fig. 6

The upper steady state in the taxed economy is 60% below that of the untaxed economy, and the lower steady state is 80% below the corresponding steady state in the untaxed economy. Thus a tax rate of 30% on output leads to a greater than proportional decline in the steady state distribution of the capital stock in this economy. Computation of the stationary distribution of capital is straightforward, and would be necessary for answering questions about relative welfare in the taxed and untaxed economies.

3.3. A last example

As a last example, consider an economy with 10% depreciation and a tax policy which implies that the after-tax gross and net marginal products of capital do not decline monotonically with the tax rate. This invalidates standard theorems that could otherwise be used to prove that this algorithm will produce the competitive equilibrium for this economy.[9] Fig. 8 plots the

[9]See Stokey and Lucas (1989, ch. 18) and Coleman (1991).

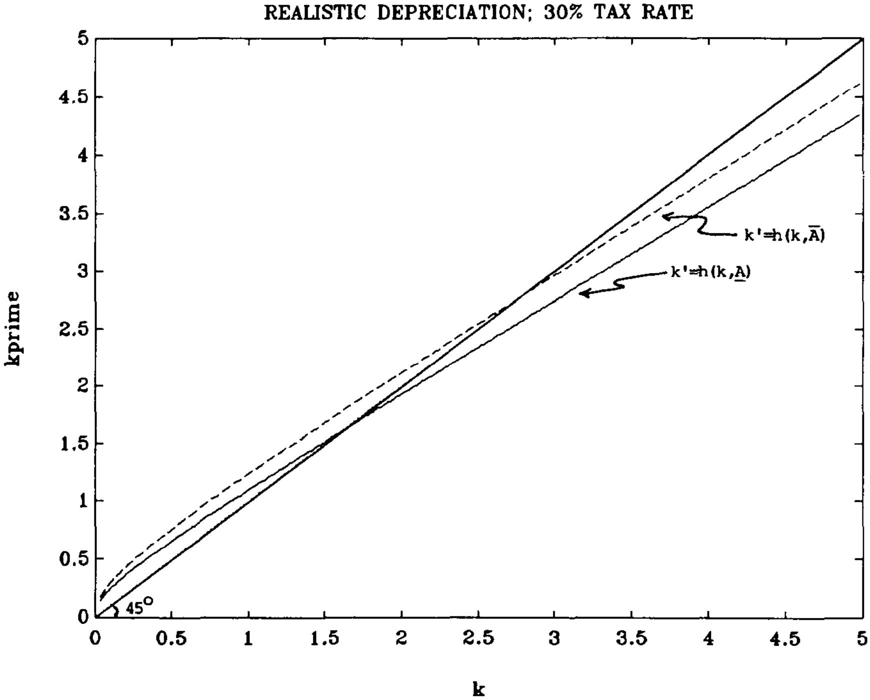

Fig. 7

policy functions for this example. Although the policy functions increase monotonically in k, the slope of the functions changes sign. In particular, the function $k' = h(k, \bar{A})$ crosses the 45-degree line more than once. Fig. 9 plots the tax rates as a function of the capital stock, the after-tax net marginal product of capital, consumption, and the error in the Euler equation. The first panel shows that the tax rates needed to generate this example are extreme – subsidies of 200% are needed to generate a rising after-tax net marginal product of capital (panel 2), and a corresponding region of the policy function in which the policy function $k' = h(k, A)$ is convex. The second panel shows that the after-tax marginal product of capital initially falls but then rises. This effect is mirrored in the policy functions in fig. 8. The third panel shows the effect of the rising after-tax marginal products on consumption. Since it becomes increasingly profitable to invest at the higher levels of capital stocks, individuals choose in state $\bar{A}$ to have consumption be a declining function of capital when the level of capital is high. Panel 4 verifies that we have found an equilibrium, since the error in the Euler equation is small [the error is defined as the difference between the

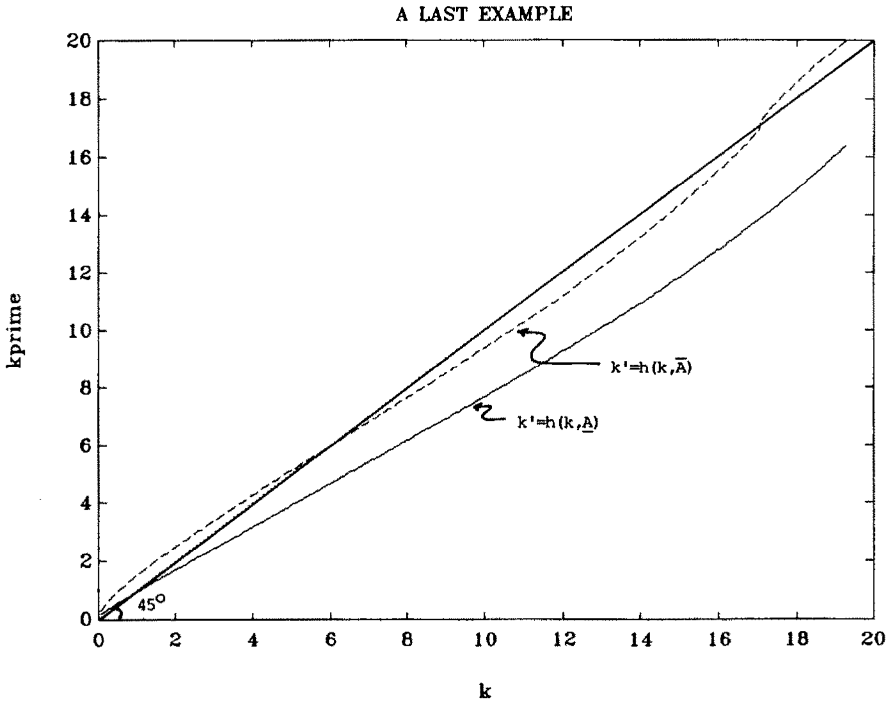

Fig. 8

left- and right-hand sides of eq. (16) where the approximate solution, h, is used to evaluate k' and k''].

The fact that such an extreme tax policy was needed to generate this example suggests that most examples of empirical relevance will be much more well-behaved, and will satisfy conditions necessary to prove that the Euler equation approach delivers the right answer. Nevertheless, it is useful to know that this algorithm still converges smoothly and rapidly, and yields a function which satisfies the Euler equation even for cases in which proofs cannot be supplied.

4. Uses of the Euler equation methodology

This section discusses contexts in which the Euler equation methodology may be particularly useful. An alternative computational approach currently in general use is log-linear approximation of the Euler equations around the model's steady state; this approach which yields log-linear approximations to the decision rules. King, Plosser, and Rebelo (1989) provide a detailed

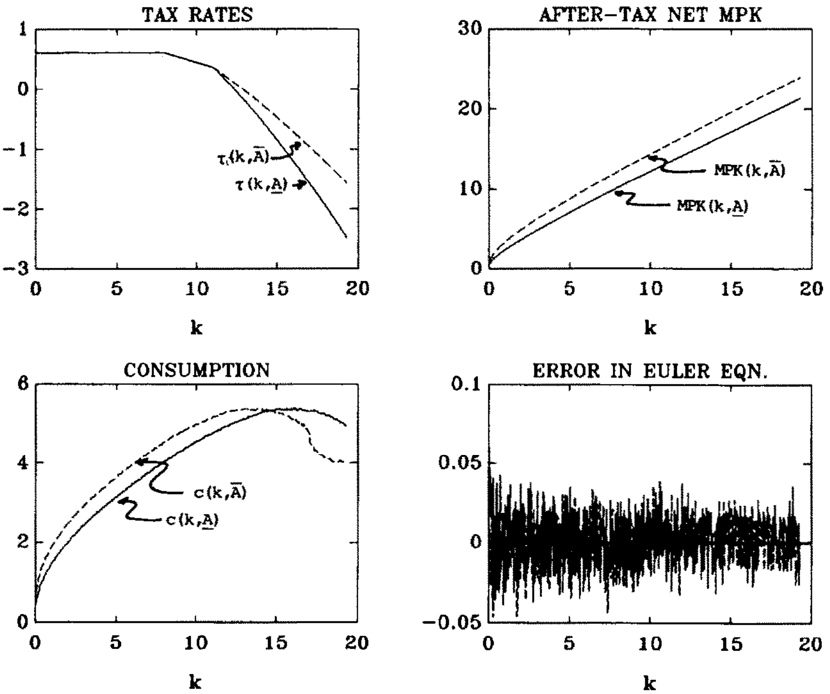

Fig. 9

discussion of one way to implement this approach. Other approximation methods have recently been developed: see the survey by Taylor and Uhlig (1990) for an overview.[10] We focus on comparing the Euler equation approach with log-linear approximation because it is currently the most widely-used alternative approach for computing suboptimal dynamic equilibria. As other methods become more widely used, the considerations discussed in this section will still continue to be relevant.

Two considerations are paramount in the choice of a computational technique: speed and accuracy. Log-linear approximation is very fast: in the examples studied in section 3, it runs several times faster than the Euler equation approach.[11] The Euler equation approach is, however, more accu-

[10]More recently, Judd (1990) has developed a method which produces polynomial approximations to the equilibrium policy functions.

[11]Log-linear approximations to the decision rules for the economies studied in this paper can be computed in about fifteen to thirty seconds on an IBM-compatible 286 or 386 machine running at 16 MHz. The Euler equation approach requires about two to four minutes to produce the decision rules for the economies studied in this paper, depending on the parameterization (especially of the discount factor and the depreciation rate) and on the initial functions chosen.

rate. It provides an exact solution to a discrete version of the original problem, and converges to the exact solution to the continuous problem as the grid becomes arbitrarily fine. These features suggest the following considerations as important for the choice of one technique versus the other.

A unique steady state: The log-linear approximation method computes linearly approximate decision rules around the fixed point of the deterministic analog to the original economy. If the economy under study possesses more than one fixed point, this methodology is generally inappropriate except for studying local dynamics near a particular steady state. The Euler equation approach, on the other hand, produces the entire policy function and therefore will uncover multiple steady states if these exist.

Further, many problems that one might wish to study will involve permanent policy shifts that will permanently change the steady state or fixed point of the certainty-equivalent economy. Because the log-linear approximation methods are designed to study local dynamics around one steady state, they cannot generally be used to study movement from one steady state to another. The Euler equation methodology can easily handle shifts in steady states.

Transitional dynamics: In the one-sector growth model studied above, the decision rules for output, investment, and consumption were very close to log-linear in the range containing the stationary distribution, if the technology shocks A are not unrealistically volatile. But these decision rules are very nonlinear in the region characterized by low levels of the capital stock. To see this, fig. 10 plots the logarithm of the policy functions plotted in fig. 7: the case of realistic depreciation and a 30% tax rate. In the vicinity of the steady state, the logs of the policy functions look reasonably linear. The log-linearly approximate decision rules (conditional on the value of the technology shock) are roughly the straight lines drawn tangent to the true decision rule at the steady state point. The log-linearly approximate decision rules have a higher slope than the true decision rules for low levels of the capital stock, and the difference in slopes is greater, the lower the initial capital stock. Thus the log-linear approximation methodology will underpredict the speed at which the economy transits to the steady state. If we wished to use this model to characterize the dynamics of a poor economy accumulating capital and growing toward its stochastic steady state, the linear approximation method would involve a nontrivial amount of error. Suppose, for example, that we imagine a developing economy beginning with a low level of capital, say, $k_0 = 0.05$ [$\log(k_0) = -3.0$]. All other parameters are as specified in the construction of figs. 7 and 10. Let us compare the speed of adjustment under log-linear approximation with the speed of adjustment under Euler equation approximation. We will say that the economy has reached the steady state

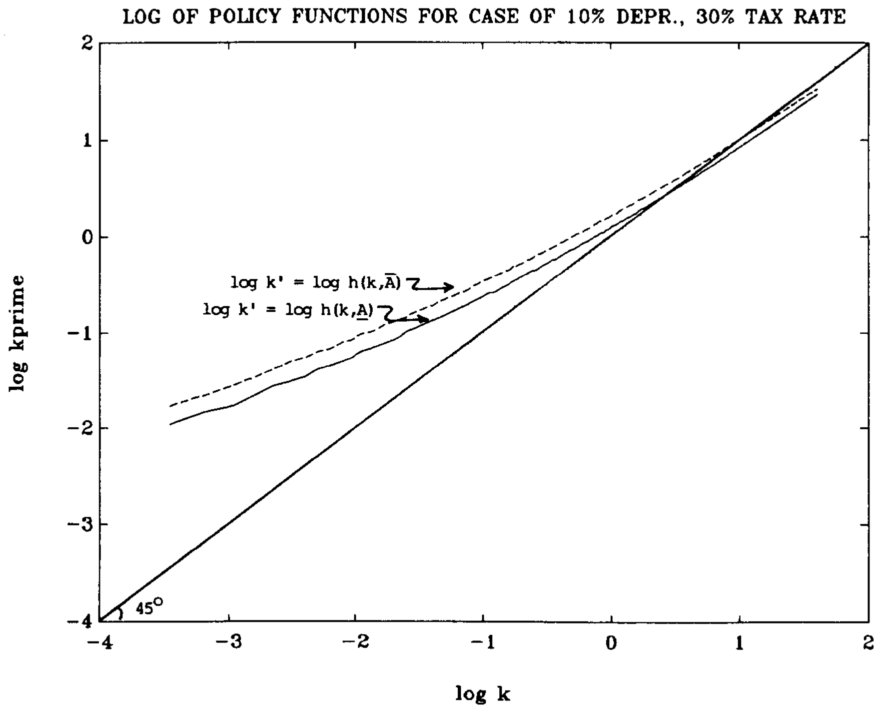

Fig. 10

when the economy reaches the lower bound of the stationary distribution, $k \cong 1.5$. For this economy to move halfway to the steady state, the log-linear approximation method predicts that this economy will take about 8.5 years, compared with the correct answer of only about 4.5 years as computed by Euler equation approximation. To reach the steady state, about 16 years are required (as computed under the Euler equation approach) – a much shorter time than the 22 years predicted by log-linear approximation.

Asset pricing: Although the decision rules for quantities in the one-sector growth model are approximately linear in the stochastic steady state, there is no reason to expect that equilibrium prices will also be linear. In particular, asset prices depend on individuals' attitudes toward risk. Since linear approximation methods produce decision rules characterized by certainty equivalence, asset prices computed under this approach can differ markedly from asset prices computed using the Euler equation approach which correctly accounts for the effects of risk aversion on stock prices. Although the decision rule for consumption is very close to linear inside the stationary distribution, variations in the marginal utility of consumption may not be small. Since it is the marginal utility schedule that is critical for asset pricing, it does not necessarily follow from the linearity of the consumption schedule that linear approximations provide accurate approximations for the purpose of asset pricing.

Large shocks: The one-sector stochastic growth model has been shown to be remarkably well-approximated by log-linear approximation methods when the underlying shocks are the small productivity disturbances typical of real business cycle models. When the underlying shocks to the model are large, however, this approximation method may not deliver accurate solutions. Dotsey and Mao (1990) study a one-sector economy subject to tax shocks, in which the tax shocks are roughly consistent with movements in U.S. tax rates since WWII. They use the methods of this paper to compute 'exact' solutions for their model, and they find that this method is quite accurate [using the evaluation criteria discussed in Taylor and Uhlig (1990)]. For this model, however, log-linear approximation involved a substantial degree of approximation error.

'Corners' in the policy function: Many economic problems involve inequality restrictions that show up as 'corners' in the policy functions. For example, one sensible restriction to place on an economy is that investment is irreversible as, for example, in Sargent (1979). The time-to-build structures of Kydland and Prescott (1982) and Backus, Kehoe, and Kydland (1990) also assume that investments ('starts') are irreversible. In a single-good, one-country setting, subject to productivity shocks of empirically reasonable volatility, these nonnegativity constraints are unlikely to bind very frequently in the steady state. In a multi-country setting with the possibility of international capital flows, however, the nonnegativity constraint on net investment is very likely to bind in the absence of costs of adjustment or other barriers to capital flows. This arises because the neoclassical model possesses a strong 'accelerator' mechanism, under which there is a strong incentive to send capital to the most productive location.

In a two-country version of the one-sector growth model, Baxter and Crucini (1990) found that some impediment to international capital flows was necessary to maintain realistic investment–output relationships in each country. They solved this problem by introducing small convex costs of adjustment in capital. Introducing adjustment costs makes it unlikely that the economy will frequently hit the corners associated with nonnegativity of gross investment. However, since the adjustment cost technology penalizes large changes in the capital stock, the resulting investment function will display greater curvature than the model without adjustment costs. This suggests that log-linear approximation may become less accurate as frictions are introduced.

Other examples of 'corners' induced by the basic economic structure of a model include short-sales constraints on asset holdings in multi-agent economies, and nonnegativity of cash holdings in cash-in-advance economies, as in Hodrick, Kocherlakota, and Lucas (1991) and Baxter, Fisher, King, and Rouwenhorst (1990). These corners are only important if they bind within the stationary distributions of the endogenous variables. But linear approximation methods implicitly assume that these constraints do not bind, and may therefore involve large approximation errors in situations where the constraints actually do bind. Because the Euler equation approach is 'exact', it uncovers corners in the policy function as a matter of routine, so long as the relevant first-order conditions are written in the manner of the Kuhn–Tucker theorem.

Larger state spaces: As the dimension of the state space grows, or as one adds decision variables to the problem (such as adding the labor–leisure choice to the problem studied in this paper), the computational complexity of the problem necessarily grows as well. Under the linear approximation method, the computational time increases approximately linearly with the number of state variables. Under the Euler equation approach, the grid for the state variables grows roughly exponentially in the number of state variables. Letting N_A denote the number of grid points for the technology shock in the one-sector model and letting N_K denote the number of grid points for the capital stock, there were $N_A \times N_K$ points in the state space. If we contemplate adding variable labor supply to this model, with N_N points for labor effort, there are now $N_A \times N_K \times N_N$ points in the state space. Each of these points need not be evaluated at each iteration and for each current state vector: we discussed in section 2.3 ways in which the basic economic structure of the problem can be used to improve on a naive search procedure. Nevertheless, the Euler equation approach shares with discrete dynamic programming the 'curse of dimensionality'.

As the size of the state space grows, it is not efficient to write one's own equation-solving routine, which has implicitly been done in our solution method for the one-sector model, as discussed in section 3. Recall that we computed the matrix $Z(k' \mid A)$ defined by eq. (19) and for each value of (k, A) located the value of k' such that $Af(k) + (1-\delta)k = Z(k' \mid A)$. Because of the structure of the problem, there was a simple and very efficient method for searching Z for the equilibrium value of k'. In problems with more than one endogenous state variable, this search procedure will be more difficult. It is sensible, therefore, to use packaged routines for solving sets of nonlinear equations to perform these calculations. Because these equation-solving routines will return values which do not lie on the grid for the state variables, an interpolation scheme must be used to evaluate the policy function for points which do not lie on the grid. In the context of the one-sector model, for example, suppose that we are at iteration j of the solution procedure, and are considering a value for k' which does not lie on the grid. We must evaluate $k'' = h_{j-1}(k', A)$. Since k' does not lie on the grid, we compute a value for k' by interpolating between neighboring points which are on the grid, using the function h_j. Coleman's (1990, 1991) implementation of the Euler equation approach proceeds in exactly this way.

Combining the Euler equation method and log-linear approximation: While the Euler equation method is 'exact,' the linear approximation method is nevertheless much faster than the Euler equation method. This suggests that the two methods are most profitably used together. For example, we suggested above (i) that using linearly-approximate decision rules as the starting function for the Euler equation method could provide important time savings and (ii) that an important diagnostic use of the Euler equation approach is to check on the extent of nonlinearity in the equilibrium decision rules. Thus one might want to begin a computational attack on a problem by using the Euler equation method over a fairly small grid, and using the log-linear approximate decision rules as starting points. This will help determine whether there is important curvature in the decision rules, and whether there are 'corners' in the decision rules inside the stationary distribution. Based on these diagnostics, the researcher would then employ log-linear approximation methods if the nonlinearities and/or corners are unimportant, and would employ the Euler equation method if this has been shown to be necessary.

5. Conclusions

This paper has developed a new method for obtaining equilibrium policy functions by means of iteration on stochastic Euler equations. The chief advantage of this method is that it can be used to compute accurate numerical solutions for problems in which competitive equilibrium is not Pareto optimal. The method is computationally fast for low-dimensional problems. With this new technology in hand, we can quantitatively evaluate a much wider range of theoretical economies; this class includes any model whose equilibrium is characterized by a set of stochastic Euler equations and which possesses a kind of turnpike property. Thus, the stochastic neoclassical growth model with distortionary taxation which was the focal point of this paper could be generalized to allow variable labor, other kinds of tax policies, productive externalities, and additional sources of randomness such as preference shocks or labor-augmenting technical change.

But the applications are not limited to neoclassical capital theory. Other fruitful applications include the study of economies whose equilibria are suboptimal because of (i) monopolistic market structure, (ii) absence of complete markets, perhaps due to private information, (iii) money introduced in a way that leads to socially suboptimal decisions, or (iv) an overlapping-generations demographic structure. The development of numerical methods for studying suboptimal dynamic economies represents a first step toward quantitative analysis of policy in an equilibrium setting.

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