4. Duality in Consumer Theory

Definition 4.1 *For any utility function $U(x)$, the corresponding indirect utility function is given by:*

$$V(p,w) \equiv \max_x \{U(x) \mid x \geq 0,\ px \leq w\}$$
$$\equiv \max_x \{U(x) \mid x \in B_{p,w}\},$$

so that if x^* is the solution to the UMP, then $V(p,w) = U(x^*)$.

Note that

$$V(p,w) \equiv \max_x \{U(x) \mid x \geq 0,\ px \leq w\}$$

and

$$x(p,w) \equiv \operatorname*{argmax}_x \{U(x) \mid x \geq 0,\ px \leq w\},$$

so that

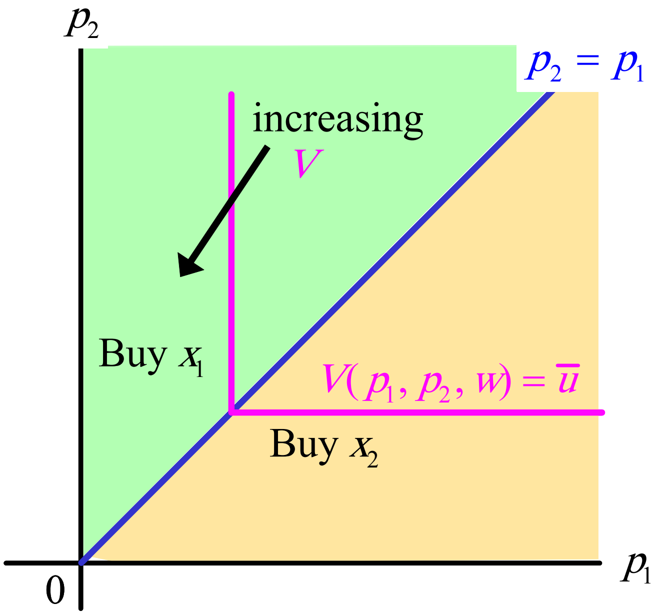
$$V(p,w) \equiv U(x(p,w)).$$

Example 4.1 *Find the demand correspondence and the indirect utility function for the linear utility function $U = x + y$.*

- **With the given utility function, x and y are perfect substitutes and the MU s are both 1 so the consumer will buy only the cheaper good.**
- **Let $p_m = \min\{p_x, p_y\}$. Demand for the cheaper good will be w/p_m and demand for the more expensive good will be 0.**
- **If $p_x = p_y$ then demand for the goods can be any combination such that expenditures add up to w.**

- The consumer will always buy w/p_m units of the goods, so his utility must also be w/p_m. Therefore, the indirect utility function is

$$v(p_x, p_y, w) = \frac{w}{\min\{p_x, p_y\}}$$

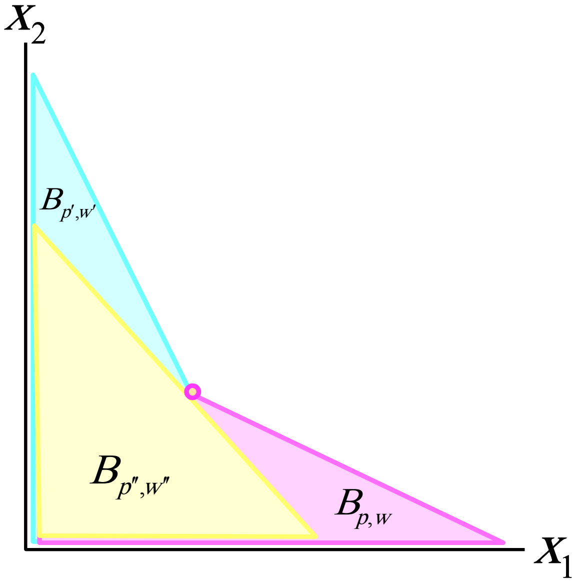


- ¿¿Quasiconcave??

Proposition 4.1 *Let $p'' = \alpha p + (1-\alpha)p'$ and $w'' = \alpha w + (1-\alpha)w'$ for $\alpha \in [0, 1]$. Then*

$$B_{p'',w''} \subset B_{p,w} \cup B_{p',w'}.$$

(If a new price and wealth vector is a convex combination of two price and wealth vectors, then the new budget set will be contained with the union of the two original budget sets.)



PROOF. **We prove the contrapositive:**

- **If** $x \notin B_{p,w}$ **and** $x \notin B_{p',w'}$, **then** $x \notin B_{p'',w''}$.
- **But this must be true, because:**
  - **if** $px > w$ **and** $p'x > w'$
  - **then** $[\alpha p + (1-\alpha)p']x > \alpha w + (1-\alpha)w'$.

■

Proposition 4.2 *If U is continuous and locally nonsatiated (lns), then V is:*

i). Homogeneous of degree 0.

ii). Strictly increasing in w and monotonically decreasing in p.

iii). Quasiconvex (no-better-than sets, $B(p,w)$, are convex).

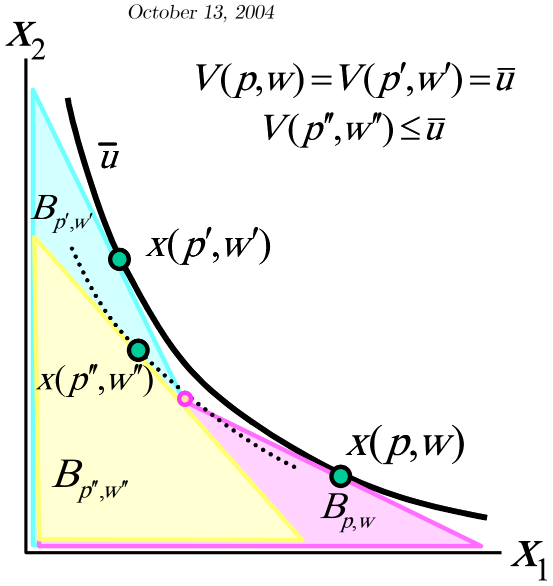
iv). Continuous in p and w.

INFORMAL PROOF.

i). Homogeneity: V doesn't change if the budget set doesn't change.

ii). Strictly increasing in w; decreasing in p:

- **nonsatiated preferences $\Longrightarrow$ strictly increasing in w.**
- **decreasing in p, because**
  - **increases in p make the budget set smaller**
  - **new budget set is inside the old one.**



iii). Quasiconvex: suppose $V(p,w) = V(p',w') = \bar{u}$.

- **Let p'', w'' is a convex combination of p, w and p', w'.**
- **From previous proposition, we know:**
  - **if $x \in B_{p'',w''}$ then it must be in either $B_{p,w}$ or $B_{p',w'}$**
  - **since $\bar{u}$ is the maximum utility available in those sets we have $V(p'',w'') \leq V(p,w) = V(p',w')$.**

iv). Continuity:

- $B_{p,w}$ is “continuous” in p and w
    - for small changes in p and w, additional and excluded commodity bundles are very close to the ones already there.
    - The continuity of U does the rest.
    - Yes, this is not really a proof, but the idea is the right one.

∎

Definition 4.2 *Given $U(x)$, the expenditure minimization problem (EMP) is*

$$\min_x px$$

$$s.t.\ \ U(x) \geq u$$

Definition 4.3 *Given p, u, the expenditure function e is defined by*

$$e(p,u) = px^*,$$

where x^* solves EMP.

- The expenditure function yields the minimum expenditure required to reach utility u at prices p.
- More formally:

$$e(p,u) = \min_x \{px \mid U(x) \geq u\}$$

Example 4.2 *Find the expenditure function for the linear utility function $U = x + y$. How much do we have to spend to get 100 units of utility if $p_x = 5$ and $p_y = 7$?*

- We already know that the indirect utility function is

$$v(p_x, p_y, w) = \frac{w}{\min\{p_x, p_y\}}.$$

- To find his expenditure function we set

$$u = \frac{w}{\min\{p_x, p_y\}}$$

  and solve for w. We have

$$e(p_x, p_y, u) \equiv w = u \min\{p_x, p_y\}\,.$$

- Expenditure to get $u = 100$ when $p_x = 5$ and $p_y = 7$.

$$e(5, 7, 100) = 100 \min\{5, 7\} = 500.$$

Proposition 4.3 `(Duality)` *Given $U(x)$, continuous and lns and a constant vector of prices $p \gg 0$, we have*

i). If x^* solves the UMP for $w > 0$,

- *then x^* solves the EMP when u is set to $U(x^*)$*
- *and $px^* = w$*
- *therefore $e(p, v(p, w)) = w$*

ii). If y^* solves the EMP for $u > U(0)$,

- *then y^* solves the UMP when w is set to py^**
- *and $U(y^*) = u$*
- *therefore $v(p, e(p, u)) = u$.*

INFORMAL PROOF.

i). Given that x^* solves UMP.

- **Suppose that x^* does not solve the EMP.**
- **Then there is an x'such that $U(x') \geq U(x^*)$ but costs less ($px' < px^*$).**
- **so that, in the UMP, we can spend a little more than px' without violating the budget constraint.**
- **so by lns, we can find x'' with $px'' < w$ and $U(x'') > U(x') \geq U(x^*)$, a contradiction.**
- **Nonsatiation implies $px^* = w$.**

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ii). Given that y^* solves EMP.

- **Suppose that y^* does not solve the UMP.**
- **Then there is a y' such that $U(y') > U(y^*) \geq u$ with $py' \leq w = py^*$.**
- **then because U is continuous, we can choose $y'' < y'$ with $U(y'') > U(y^*)$ but $py'' < py' \leq py^*$, a contradiction.**
- **The continuity of U implies that $U(y^*) = u$, for otherwise money could be saved by allowing $U(y^*)$ to fall without violating the utility constraint of EMP.**

■

Proposition 4.4 *For U continuous and nonsatiated, $e(p,u)$ is*

i). Homogeneous of degree 1 in p.

ii). Strictly increasing in u; increasing in p.

iii). Concave in p.

iv). Continuous in p, u.

INFORMAL PROOF.

i). Homogeneous in p:

$$\begin{aligned} e(\alpha p, u) &= \min_x \{\alpha p x \mid U(x) \geq u\} \\ &= \alpha \min_x \{p x \mid U(x) \geq u\} \\ &= \alpha e(p,u). \end{aligned}$$

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ii). Strictly increasing in u and increasing in p:

- For utility:
  - By definition $e(p,u)$ is the required expenditure to obtain u.
  - Suppose $u' > u$ could be obtained by consuming x' without increasing expenditures.
  - By continuity of u we could obtain $u'' > u$ even if we consume a little less than x', that is at a lower expenditure than $e(p,u)$, a contradiction.
- For prices:
  - Suppose that $p' > p$.
  - Then if x' solves the EMP at prices p' with $U(x') \geq u$, we have
  $$e(p', u) = p'x' \geq px' \geq e(p,u)$$
  - [Why not $p'x' > px'$?]

iii). Concave in p. Think of a consumer who normally consumes x'' at prices p''

- Suppose she spends a day at prices p and another day at prices p', where

$$p'' = \frac{1}{2}(p + p')\,.$$

- Could reach same utility at same average expense by consuming x'' both days.
- But can save money by adapting her choice of goods to the current prices.
- By substituting cheap for expensive goods, you can get same utility for less money at more extreme prices than at average prices.

- More formally:
  - Suppose that for $\alpha \in (0, 1)$, $p'' = \alpha p + (1 - \alpha)p'$
  - and suppose that x'' solves the EMP with utility u, so that $U(x'') \geq u$.
  - Then $px'' \geq e(p, u)$ and $p'x'' \geq e(p', u)$ [why?]
  - Therefore

$$\begin{aligned} e(p'', u) &= p''x'' = (\alpha p + (1 - \alpha)p')x'' \\ &= \alpha px'' + (1 - \alpha)p'x'' \\ &\geq \alpha e(p, u) + (1 - \alpha)e(p', u). \end{aligned}$$

iv). Continuous in p, u. Follows from:

- continuity of the constraint set $\{x \mid U(x) \geq u\}$ as a function of u
- and continuity of the objective function px in x and p.

■

Definition 4.4 *Hicksian demand $h(p, u)$ is a consumption vector x^* that solves the EMP.*

- We have

$$e(p, u) = \min_x \{px \mid U(x) \geq u\}$$

and

$$h(p, u) = \operatorname*{argmin}_x \{px \mid U(x) \geq u\}$$

- Also, $e(p, u) = ph(p, u)$.

Because $x(p, w)$ solves the UMP and $h(p, u)$ solves the EMP, the proposition on utility duality tells us:

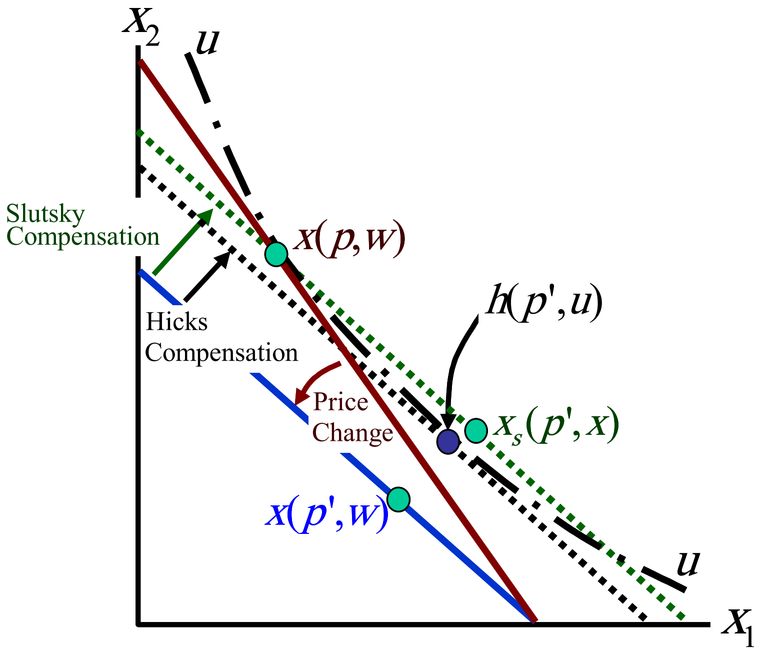
Proposition 4.5 *Given that U is continuous, $u > U(0)$ and $w > 0$, we have*

i). $x(p, w) = h(p, v(p, w))$,

- *Walrasian demand at wealth w = Hicksian demand at utility level produced by w.*

ii). $h(p, u) = x(p, e(p, u))$,

- *Hicksian demand at utility u = Walrasian demand with wealth required to reach u.*



- Graph above shows the difference between
  - Slutsky compensated demand $x_s(p', x)$
  - and Hicksian demand $h(p', u)$.

Suppose a consumer has consumption vector $x(p, w)$ and utility $u = U(x(p, w))$,

- and then prices change from p to p'.
- We have

$$\begin{aligned} x_s(p', x) &= x(p',\ p'x(p, w)) \\ h(p', u) &= x(p',\ e(p', u)) \end{aligned}$$

- Slutsky compensated demand = Walrasian demand when the consumer is given sufficient wealth to buy his original consumption vector $x(p, w)$.
- Hicksian demand = Walrasian demand when the consumer is given sufficient wealth to reach his original utility level, $u = U(x(p, w))$.

- We know $U(x_s(p',x)) \ge U(h(p',u))$. [Why?]
- As the price change $p' - p$ gets small, difference between Hicksian demand and Slutsky demand becomes second-order small.
- We will show that

$$S(p,w) \equiv \left.\frac{\partial x_s(p',x)}{\partial p'}\right]_{p'=p} \equiv \left.\frac{\partial h(p',u)}{\partial p'}\right]_{p'=p}$$

- Both have the same derivatives at $p' = p$.
- Therefore, the Slutsky Equation is true for Hicksian compensated demand.
- "Compensated demand" usually refers to Hicksian demand
- Slutsky demand is rarely used.

Proposition 4.6 *(M-C 3.E.4; Law of Demand) On average, when prices rise, the substitution effect is negative. More formally:*

- *If $U(x)$ is continuous and lns, and*
- *$h(p,u)$ is a function,*
- *then for all p'' and p'*

$$(p'' - p')[h(p'',u) - h(p',u)] \le 0.$$

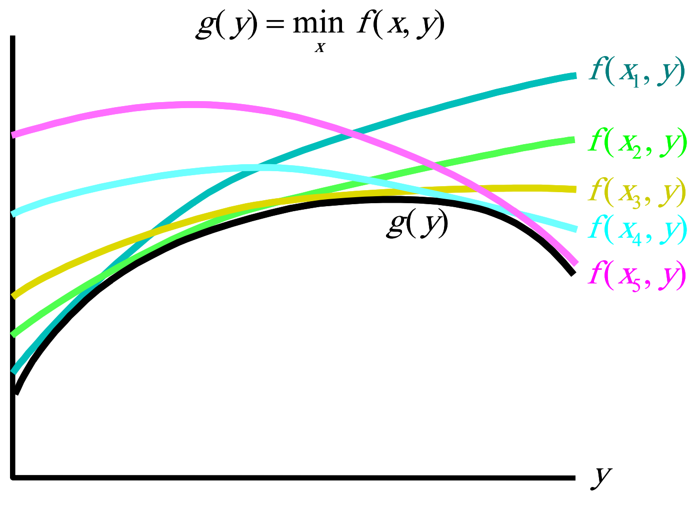
PROOF. We have

- (1) $p''h(p'',u) \le p''h(p',u)$ [why?]
  - $\Longrightarrow$(2) $p''h(p'',u) - p''h(p',u) \le 0$
- (3) $p'h(p',u) \le p'h(p'',u)$. [why?]
  - $\Longrightarrow$(4) $p'h(p',u) - p'h(p'',u) \le 0$
- Add (2) and (4) and factor the results.

■

4.1 The Envelope Theorem

- **Suppose that a family of functions is described by $f(x, y)$ for different fixed parameters x and a variable y.**
- **At each point, we compare the values of all functions in the family, and choose the minimum value.**
- **This creates a new function $g(y) \equiv \min_x f(x, y)$. The function $g(y)$ is called the lower envelope of $f(x, y)$.**
- **In the figure, the family members and the lower envelope are plotted as functions of y.**



- **The theorem says that the slope of the envelope at any point is the same as the slope of the member of the family that it touches.**
- **M-C has a more general version of the theorem: don't worry about it, because it is quite messy.**

Proposition 4.7 (Envelope Theorem) *Let* $g(y) = \min_x f(x, y)$, *where* $f(x, y)$ *is differentiable. Then*

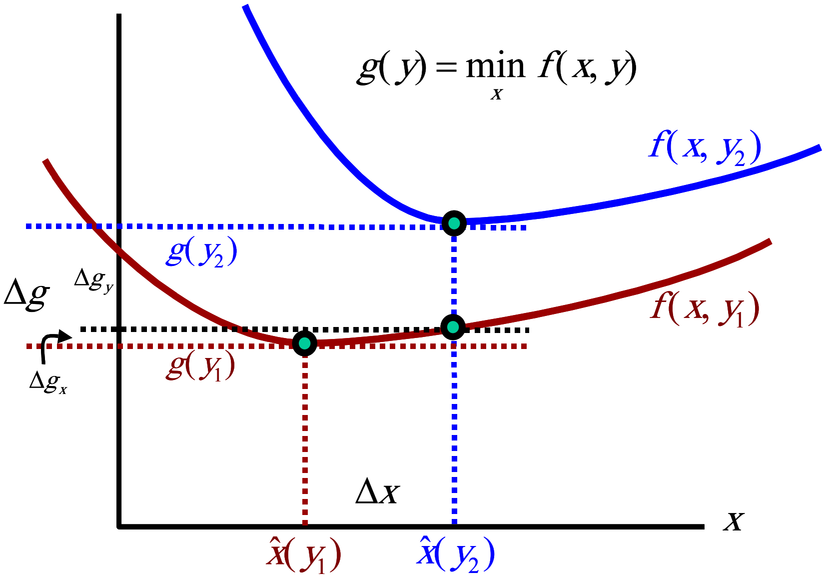
$$g'(y) = \left.\frac{\partial f(x, y)}{\partial y}\right]_{x=\hat{x}(y)}$$

where $\hat{x}(y)$ *is the value of* x *that minimizes* $f(x, y)$.

The intuition

- As y changes x also must change because x must always minimizes $f(x, y)$.
- If y changes by Δy, the change $\Delta g(y)$ comes from two sources
  - directly from Δy
  - and from Δx (which is caused by Δy).

- The envelope theorem says that if Δy is small, the part of Δg that comes from Δx (labeled Δg_x on the graph) is near 0, because...
  - The curves are flat at $\hat{x}(y)$, because $\hat{x}(y)$ minimizes $f(x, y)$.
  - So at $x = \hat{x}(y)$, if y is held constant, Δx produces a small change in $f(x, y)$.
  - Almost all of the change in $f(x, y)$ comes directly from Δy.



PROOF. Let $\hat{x}(y)$ be the solution of $\min_x f(x, y)$. The f.o.c for $\hat{x}(y)$ is

$$\left.\frac{\partial f}{dx}\right]_{x=\hat{x}(y)} = 0.$$

We can now write:

$$g(y) = f(\hat{x}(y), y)),$$

so, by the chain rule,

$$g'(y) = \left.\frac{\partial f}{dx}\right]_{x=\hat{x}(y)} \hat{x}'(y) + \frac{\partial f}{\partial y}.$$

The first term is 0. ■

Proposition 4.8 *If $U(x)$ is continuous and lns, and $h(p, u)$ is a function, then*

$$h(p, u) = \nabla_p e(p, u).$$

PROOF.

- We know that

$$e(p, u) = \min_x \{px \mid U(x) = u\}$$

- Notice the equality constraint [why equality?]
- We can write this as a saddle-point problem:

$$e(p, u) = \max_\lambda \min_x \{px - \lambda[u - U(x)]\}$$

- Envelope theorem says: in calculating $\partial e/\partial p$, λ and x can be treated as constants at their optimal values.
- The only term that contains p explicitly is px.
- Thus $\nabla_p e(p, u) \equiv \partial e/\partial p = x^* \equiv h(p, u)$.

■

Proposition 4.9 *(M-C 3.G.2) For the Jacobian matrix* $\partial h(p,u)/\partial p$ *we have:*

i). $\partial h(p,u)/\partial p = \partial^2 e(p,u)/\partial p^2$

ii). $\partial h(p,u)/\partial p$ *is negative semidefinite,*

iii). $\partial h(p,u)/\partial p$ *is symmetric, and*

iv). $[\partial h(p,u)/\partial p] \cdot p = 0$

PROOF. We have:

i). 2nd derivative of $e(p,u)$: Immediate from $h(p,u) = \partial e(p,u)/\partial p$

ii). Negative semidefinite: From concavity of expenditure function.

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iii). Symmetric:

- The off-diagonal elements of $\partial h(p,u)/\partial p$ are the cross-partial derivatives of $e(p,u)$.
- But well-behaved functions have symmetric cross-partial derivatives (i.e. $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$).

iv). $[\partial h(p,u)/\partial p] \cdot p = 0$

- $h(p,u)$ is homogeneous of degree 0 in p.
- Result follows from Euler's formula.

■

Proposition 4.10 *(Slutsky equation for Hicksian demand.) Given $U(x)$ strictly quasiconcave and well-behaved and the corresponding indirect utility function $V(p,w)$, we have*

$$\frac{\partial x_i(p,w)}{\partial p_j} = \frac{\partial h_i(p,u)}{\partial p_j} - \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$$

where $u = V(p,w)$.

PROOF. The proof depends on the previously-established identity $h(p,u) \equiv x(p, e(p,u))$.

- By chain rule:

$$\frac{\partial h_i(p,u)}{\partial p_j} \equiv \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w}\frac{\partial e(p,u)}{\partial p_j}$$

- But

$$\frac{\partial e(p,u)}{\partial p_j} \equiv h_j(p,u) \equiv h_j(p, V(p,w)) \equiv x_j(p,w)$$

- Substitution completes the proof.

■

Proposition 4.11 *(Roy's Identity) Given $U(x)$ strictly quasiconcave and well-behaved and the corresponding indirect utility function $V(p,w)$, we have*

$$x_j(p,w) = -\frac{\partial V(p,w)}{\partial p_j} \Big/ \frac{\partial V(p,w)}{\partial w}.$$

PROOF.

- First, the intuition:
  - $$\frac{\partial V(p,w)}{\partial p_j} \Big/ \frac{\partial V(p,w)}{\partial w} \overset{\circ}{=} \frac{\Delta u}{\Delta p_j} \Big/ \frac{\Delta u}{\Delta w} = \frac{\Delta w}{\Delta p_j} = -x_j(p,w)$$
  - We overlooked some little details:
  - for example, $x_j(p,w)$ changes when p changes,
  - but $x_j(p,w)$ is a utility maximizer, so the envelope theorem tells us that we can ignore this change.

- **Formal proof:**
  - Let $u = V(p, w)$, so that $w = e(p, u)$
  - We have $V(p, e(p, u)) \equiv u$
  - Hold u constant. By the implicit-function theorem, we have:

$$\frac{\partial e(p,u)}{\partial p_j} \equiv -\frac{\partial V(p,w)}{\partial p_j} \Big/ \frac{\partial V(p,w)}{\partial w}$$

  - but

$$\begin{aligned}\frac{\partial e(p,u)}{\partial p_j} &\equiv h_j(p,u) = h_j(p, V(p,w)) \\ &\equiv x_j(p,w).\end{aligned}$$

■

- The chart below summarizes the duality between the UMP and the EMP.
- It is taken (with editorial errors corrected) from M-C, p. 75.