Manove's Microeconomic Theory Slides

[A substantial portion of these notes are based on materials from the textbook for this course: Mas-Colell, Winston and Green, Microeconomic Theory, Oxford University Press, 1995]

-2. Administrative Matters

- Michael Manove / Jonathan Treussard
- My webpage

  http://web.bu.edu/econ/faculty/manove/

- Times:
  - Classes: MW 5:00-6:30 room CAS 326
  - Problem Sections: F 9:00-10:30 SMG 302
  - My Office Hours: T 5:00-6:30 and F 11:40-1:10
  - Jonathan's Office Hours: M 12:30-2 and W 9-10:30

- **Textbook: Andreu Mas-Colell, Michael Winston, Jerry Green. Microeconomics.**
- **Please buy it. Excellent reference book.**
- **I will try to follow the book as much as possible.**
- **Please read the book and take your own class notes.**
- **Your problem sets are the most important part of the course.**

-2.1 How To Get an A in EC 701

- **Attend class and discussion section.**
- **Go to bed by 11pm; get up by 7am.**
- **Do not smoke (at all) or drink (too much).**
- **Do not memorize material.**
- **Do not copy the solutions to the problem sets.**
- **Do not solve problems in a mechanical way.**
- **Explain the material and problem solutions to your friends, your brothers, your dogs, and your aunts.**
- **Do economics, not mathematics.**

-1. Do Economics, not Mathematics

- **This is an example of an economics problem. We will review the material in detail during the next two weeks.**

Example -1.1 *Suppose utility is given by*

$$U(x_1, x_2) = \min\{x_1, x_2\} + a(\max\{x_1, x_2\} - \min\{x_1, x_2\})$$

where

$$0 < a < 1.$$

Find the demand function for x_1 *and* x_2*.*

- **How should we think about this problem?**
- **Can you think of some goods that might fit this example?**

- **Suppose x_1 represents spoons and x_2 represents forks.**
  - $\min\{x_1, x_2\}$ **provides one unit of utility for each pair of one spoon and one fork.**
  - $a(\max\{x_1, x_2\} - \min\{x_1, x_2\})$ **provides a units of utility for the remaining forks (if there are more forks) or the remaining spoons (if there are more spoons).**

- **Which is the better purchase, singles (only spoons or only forks) or pairs?**
  - **The consumer gets 1 unit of utility for each pair (one spoon and one fork) plus a units of utility for each single (additional spoons or forks).**
  - **Suppose p_1 is the price of spoons and p_2, the price of forks and w, wealth.**
  - **Then price of pairs is $p_1 + p_2$, so the utility per dollar from buying pairs is**

$$\frac{1}{p_1 + p_2}.$$

  - **If the consumer buys singles, she will buy only the cheaper single. So assume that $p_1 \leq p_2$ so that the consumer will buy spoons (she will be indifferent if prices are the same).**
  - **Then for singles, the utility per dollar is a/p_1.**
  - **Therefore, the consumer will buy only singles if**

$$\frac{a}{p_1} > \frac{1}{p_1 + p_2}$$

which is equivalent to

$$p_1 < \frac{a}{1-a} p_2.$$

The consumer will buy only pairs if the reverse is true.

  - **Does this make sense if $a = 0$? if $a = .99$?**

- Demand:
  - If the consumer buys only singles, she will buy w/p_1 spoons,
  - but if she buys pairs, she will buy $w/(p_1+p_2)$ pairs of spoons and forks.
- Therefore, assuming that $p_1 < p_2$, the demand funtion is given by

$$x_1 = \begin{cases} \frac{w}{p_1} & \text{for} \quad p_1 < \frac{a}{1-a}p_2 \\ \frac{w}{p_1+p_2} & \text{for} \ p_2 > p_1 > \frac{a}{1-a}p_2 \end{cases}$$

and

$$x_2 = \begin{cases} 0 & \text{for} \quad p_1 < \frac{a}{1-a}p_2 \\ \frac{w}{p_1+p_2} & \text{for} \ p_2 > p_1 > \frac{a}{1-a}p_2 \end{cases}$$

- Challenge: solve this problem using formal mathematical tools!

0. Why Study Micro Theory?

- Basic tools for all of economics, including empirical economics
  - Microeconomic theory itself doesn't say very much. You use theory as a tool so that you can say things about economy.
  - Tools are basic and general.
  - Tools are useful for building models
  - All of my own research uses the tools I will teach you.

0.1 The principal topics in EC701

- **Neoclassical Preference and Choice: how individuals make decisions.**
  - **Probably the weakest part of microeconomics. Complete rationality assumed.**
  - **Little psychology, little empirical evidence.**
  - **Source of preferences or reasons for choices are not explained.**
  - **Consistency in preference or choice is required.**
  - **Theory of demand.**
  - **Modern decision theory is more complicated and interesting.**

- **Production: decision-making by owners of firms.**
  - **Strong rationality more reasonable, especially for large firms: built into firm structure, entrepreneurs, managers, accountants, outside directors, etc.**
  - **Revenues, costs, profit maximization**
  - **Theory of supply**

- Theory of Strategic Interaction (Game Theory)
  - When all agents are small (competitive markets) or when only one agent is large, price mediates all interactions between agents.
    - Little interesting strategic interaction.
    - If you know the price, you don't care what the other agents are doing.
  - More than one large agent $\Longrightarrow$ strategic interaction usually important
    - You want to respond to the actions of other agents
  - Game theory is important tool for models.

- Competitive equilibria
  - Prices summarize strategies of all agents.
  - Equilibrium exists when prices and strategies are consistent.
- Other types of equilibria are defined by game theoretical solution concepts.

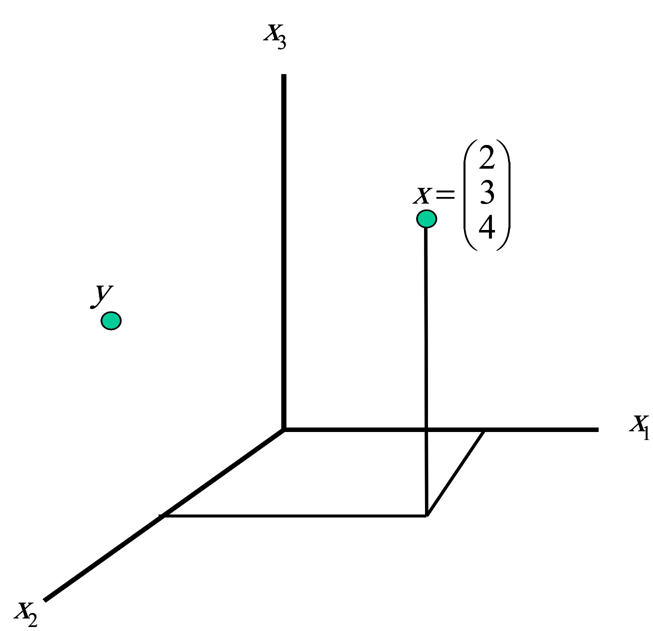
1. Preference and Choice

1.1 Preferences

- Preferences are psychological entities.
- Most aspects of preferences are usually ignored by economists.
  - Origin ignored
  - Causes ignored
  - Intensity ignored
  - Dynamics ignored

General representation of preferences in economics:

- Commodity space: each point is a bundle of goods.

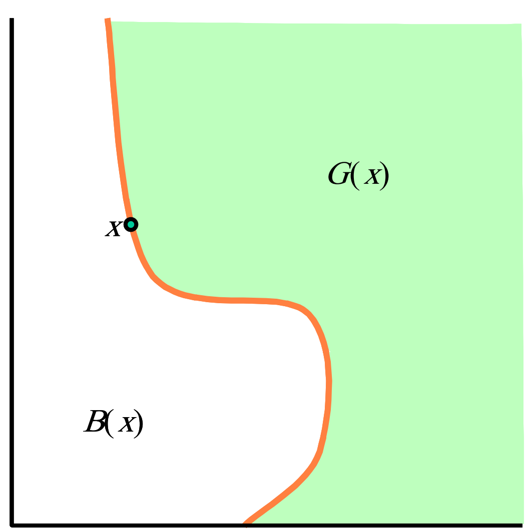


- Binary relation $\succsim$ defined on commodity space X
- $x \in X$ is a vector of quantities of each commodity in existence.
- For $x, y \in X$, $y \succsim x$ means that y is at least as preferred (desired) as x.
- Rationality:
  - Completeness: Either $y \succsim x$ or $x \succsim y$ or both.
    - Obviously false for real people:
    - People don't know characteristics of most goods.
    - People don't know how characteristics will affect them.
    - Worse: people make decisions without knowing their preferences.
  - Transitivity (consistency): If $y \succsim x$ and $x \succsim v$ then $y \succsim v$.

- Failure of rationality:
  - (read pp 7-8 Mas-Colell small print); Matthew Rabin, JEL
  - Framing, and other heuristic biases
  - Taste a function of state (dynamics)
- Unrealistic models may be useful. Why?

Contour Sets

- **As-good-as sets (upper contour sets): for each $x \in X$, define $G(x) \equiv \{y \mid y \succsim x\}$**
  - **$G(x)$ is the set of all bundles such that are as good (preferred or desired) as x.**
  - **$G(x) \subset X$**
  - **G is a function that maps points in X into subsets of X.**
    - **$\mathcal{P}(X)$ denotes the power set of X, which is the set of all subsets of X**
    - **$G : X \to \mathcal{P}(X)$**
  - **The function G is a completely general, precise mathematical description of $\succsim$.**

- **For well behaved G, boundaries of the sets $G(x)$ are the indifference curves. Why?**
  - 

- **We also define no-better-than sets (lower contour sets), $B(x) \equiv \{y \mid x \in G(y)\}$.**
- **Indifference curves are given by $I(x) \equiv B(x) \cap G(x)$.**

Utility Functions

- If $U : X \to \mathbb{R}$ has the property:

$$U(x) \geq U(y) \iff x \succsim y.$$

  then U is a utility function that describes the preferences $\succsim$.

  - For example, if $U(x) = -3.7$ and $U(y) = -4$, then $x \succsim y$.
  - The utility values provide an order for the commodity bundles but have no other meaning.
  - In classical theory, utility was intended to be a measure of consumer satisfaction (a psychological construct) ...
  - ...in the same way that IQ is intended to be a measure of intelligence.

- Not all preferences can be represented by a utility function, but:

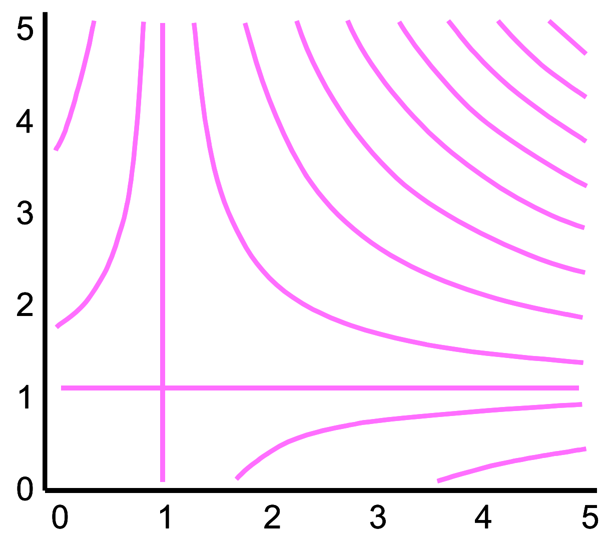
Proposition 1.1 *If the preference relation $\succsim$ can be described by a utility function U, then $\succsim$ is rational.*

Prove this as an exercise.

Example 1.2 *Inma likes wine, and Inma likes beer. The more the better. But Inma doesn't like them together. Construct a continuous utility function and as-good-as sets for this example.*

- Beer (c) and wine (v) are both good, but when both are present, there is a negative effect.
- How can we represent this?
  - One possibility is a negative product term, $-cv$, in the utility function.
    - If $c, v > 0$, some utility is lost.
    - But if $c = 0$ and $v > 0$ (or $c > 0$ and $v = 0$), then no utility is lost.

- For example $U(c, v) = c + v - cv$



- What are the utility values of the various indifference curves?

◦ **Another example:** $U(c, v) = |c - v|$.

2. Consumer Choice (Neoclassical Model)

- **Notation**

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_L \end{bmatrix}, \text{ commodity vector (consumption bundle)}$$

- **Commodity space:** $X = \{x \mid x \in R^L_+\}$, **positive orthant of** L-**dimensional real Euclidean space.**

- We assume linear prices: price per unit not a function of how much you buy.
- Nonlinear prices?
  - Price of leisure?
  - Mobile phone calls? (not truly competitive)

- $$p = \begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ p_L \end{bmatrix} \text{, price vector}$$
- Price space, $p \in R^L_+$, also.
- px = expenditure required to buy x [$px \equiv p \cdot x \equiv$ dot product].
  - Example:

$$p = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 4 \end{bmatrix}, x = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 8 \end{bmatrix}$$

then

$$px = (2 \times 3) + (1 \times 5) + (3 \times 2) + (4 \times 8) = 49.$$

- **Budget set:** $B_{p,w} = \{x \mid px \leq w\}$, **where** $w =$ **income or wealth.**
- **Budget frontier (line or hyperplane)** $\bar{B} = \{x \mid px = w\}$

Proposition 2.2 *Consider three points in space,* v, a *and* b,*and consider the lines* $\overline{va}$ *and* $\overline{vb}$. *Then* $\overline{va}$ *and* $\overline{vb}$ *are orthogonal (perpendicular)* $\iff$ *the inner product* $(a - v)(b - v) = 0$.

PROOF. **(informal; for 2-dim vectors) Suppose the lines are orthogonal, and suppose that the vector** $a - v$ **makes an angle** θ **with the** x**-axis. Then**

$$a - v = \lambda_1 \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

and

$$b - v = \lambda_2 \begin{bmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{bmatrix} = \lambda_2 \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix},$$

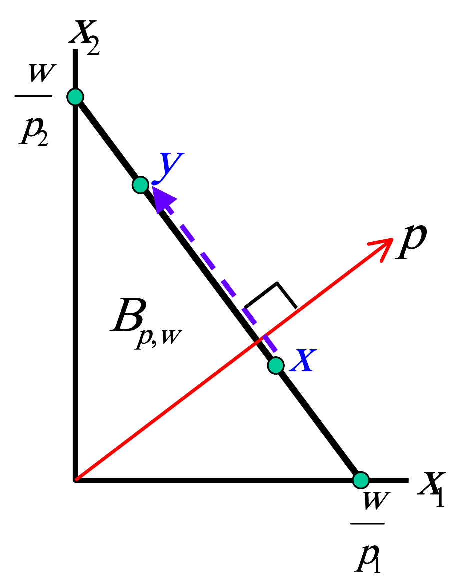
and these have zero inner products. ■

Proposition 2.3 p *is orthogonal (perpendicular) to the budget frontier* $\bar{B}$ *(that is, to any line in* $\bar{B}$ *)*

PROOF. **If** $x, y \in \bar{B}$, **then the inner product is** $p(y - x)$.

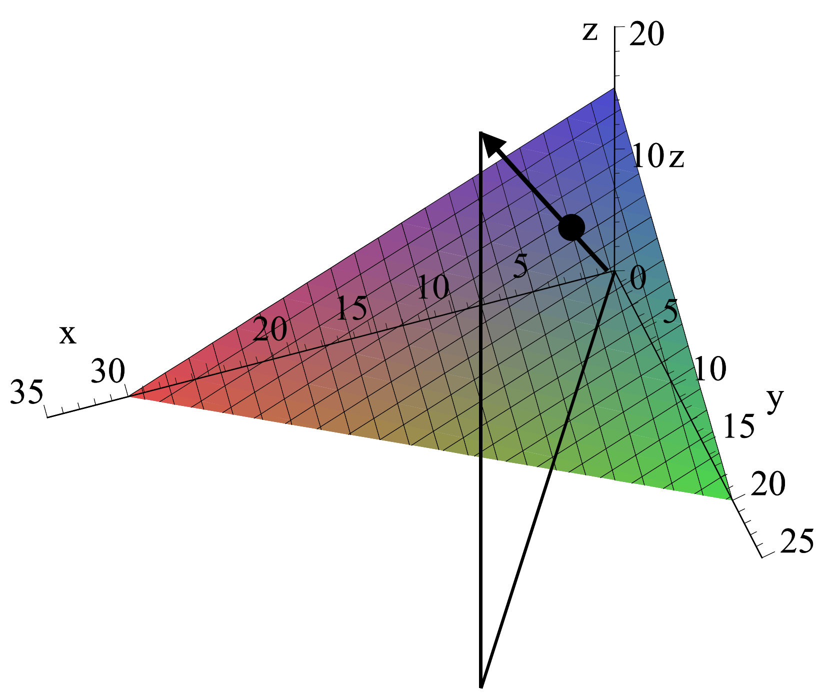
$$p(y - x) = py - px = w - w = 0.$$

■



Example 2.3 *Suppose we have commodities in amounts x, y and z, with $p = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $w = 60$. Draw a graph of the price vector and the budget set.*

- **The intercepts of the budget set are they amounts of each good the consumer could buy if he bought nothing else: $60/2$, $60/3$ and $60/4$.**
- **The budget set is bounded by the two-dimensional plane with those intercepts.**
- **The price vector must be orthogonal to the buget set.**



2.1 Choice and Revealed Preference

- Let $P \equiv \mathbb{R}_+^L$ denote the space of prices, $W \equiv \mathbb{R}_+$, the space of income values (or wealth) and $X \equiv \mathbb{R}_+^L$, commodity space.
- Let $\succsim$ define preferences on X, and for each $x \in X$, and let $G(x)$ and $B(x)$ represent the corresponding upper and lower contour sets of x.
- Let $\mathcal{B}(X)$ denote a set of subsets of commodity space X.

Definition 2.1 *The function C is a* choice function *on the sets $\mathcal{B}(X)$ if for any set $S \in \mathcal{B}(X)$, $C(S)$ is a nonempty subset of S. Think of C as a function that identifies the commodity bundles a person would choose within a set of commodity bundles.*

Definition 2.2 *The choice function C is said to* correspond to preferences *$\succsim$ if for any $S \in \mathcal{B}(X)$, $C(S)$ is given by*

$$C(S) \equiv \{x \in S \mid S \subset B(x)\},$$

where $B(x)$ is the lower contour set of x.

- This definition says that no points in S are better than the points in $C(S)$.
- When C corresponds to preferences, you can think of C as a function that chooses the best commodity bundles in S.

- Note that $C(S) \subset S$.
- If $\bar{x} \in C(S)$ and $x \in S$, then $\bar{x} \succsim x$.
- How would you illustrate $C(S)$ by using indifference curves?

Definition 2.3 *If for some $S \subset X$, we have $x, y \in S$ and $x \in C(S)$, we say "x is revealed as good as y."*

Definition 2.4 *If for some $S \subset X$, we have $x, y \in S$ and $x \in C(S)$ and $y \notin C(S)$, we say "x is revealed preferred to y."*

Definition 2.5 *The choice function C satisfies the weak axiom of revealed preference if whenever x is revealed as good as y, then y cannot be revealed preferred to x.*

- The weak axiom of revealed preference assures consistency in choice.

Proposition 2.4 *If $\succsim$ is rational, then C satisfies the weak axiom of revealed preference.*

PROOF. Suppose x is revealed as good as y.

- Then for some S, we have $x, y \in S$ and $x \in C(S)$.
- Therefore $x \succsim y$.
- Let $\hat{S}$ be a different set, and suppose $x, y \in \hat{S}$ and $y \in C(\hat{S})$.
- We show that $x \in C(\hat{S})$ [otherwise y would be preferred to x]
  - Let $z \in \hat{S}$.
  - We know that $y \succsim z$.
  - By transitivity $x \succsim z$.
  - But this is true for all $z \in \hat{S}$, which implies that $x \in C(\hat{S})$.

Therefore y is not revealed preferred to x, and the weak axiom is satisfied. ■

2.2 Demand Functions and Comparative Statics

Definition 2.6 *A demand function $x : P \times W \to X$, maps prices and income (or wealth) into vectors of consumption choices as follows:* $x(p, w) = C(B_{p,w})$.

- In general, demand is a correspondence; $x(p, w) \subset X$, but we usually assume that $C(B_{p,w})$ contains only one point so that demand is a function.
- Definition of demand implies that its value is completely determined by the budget set $B_{p,w}$ and the choice function C.
  - No money in this model.
  - Price and wealth determine budget set, nothing more.

- What kind of information about the consumer does the demand function contain?
  - How much a consumer will buy?
  - How much a consumer will buy when prices are at the market equilibrium?
  - How much a consumer would want to buy at every reasonable combination of income and prices?

Definition 2.7 *Let X and Y be vector spaces. A function $f : X \to Y$ is homogeneous of degree n if for all $x \in X$ and $\alpha > 0$, $f(\alpha x) = \alpha^n f(x)$.*

- The function behaves like polynomial of the form as^n along any ray coming out from the origin. (If $n = 0$, it is constant along any ray.)
- Example: Is $f(x, y) \equiv xy$ a homogeneous function? Of what degree?

Proposition 2.5 *If f homogeneous of degree n, then for $n > 0$*

$$f(x) \equiv \frac{1}{n}\frac{\partial f(x)}{\partial x}x,$$

and for $n = 0$,

$$\frac{\partial f(x)}{\partial x}x \equiv 0.$$

Example 2.4 *$f(x) = x^n$ is a scalar function homogeneous of degree n, because $f(ax) \equiv a^n f(x)$. Proposition implies:*

$$f(x) = \frac{1}{n}f'(x)x$$

or

$$x^n = \frac{1}{n}nx^{n-1}x,$$

which is clearly correct.

PROOF. Start with $y = f(x)$. Let $x = \alpha z$. For $n > 0$,

$$\frac{\partial y}{\partial \alpha} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} = \frac{\partial f}{\partial x}z \text{ [chain rule]}$$

By homogeneity, we also have $y = f(\alpha z) = \alpha^n f(z)$, so

$$\frac{\partial y}{\partial a} = n\alpha^{n-1}f(z),$$

and we can write:

$$n\alpha^{n-1}f(z) \equiv \frac{\partial f(\alpha z)}{\partial x}z$$

True for all α, so set $\alpha = 1$. Then $x = z$, and we have,

$$f(x) \equiv \frac{1}{n}\frac{\partial f}{\partial x}x.$$

For $n = 0$, $f(\alpha z) \equiv f(z)$, so differentiating both sides with respect to α gives

$$\frac{\partial f(\alpha z)}{\partial x} z = 0 \text{ [chain rule]}$$

and setting $\alpha = 1$

$$\frac{\partial f(x)}{\partial x} x \equiv 0.$$

■

- Results for homogeneous functions expressed in scalar notation:
  - For $n > 0$,

$$f_i(x_1, ..., x_n) = \frac{1}{n} \sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} x_j.$$

  - For $n = 0$,

$$\sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} x_j \equiv 0.$$

    - If f is constant on αx, then

$$\sum_j \frac{\partial f_i(x_1, ..., x_n)}{\partial x_j} \Delta x_j \equiv 0.$$

    - But moving along the ray αx, Δx_j is porportional to x_j, so we can substitute x_j for Δx_j.
    - Weighted average of derivatives along path must be 0 for function constant along the path.

- We apply this to demand functions:

Proposition 2.6 *A demand function is homogeneous of degree 0 (in p and w).*

PROOF. For any p, w and $\alpha > 0$,

$$B_{p,w} = \{x \mid px \leq w\} = \{x \mid \alpha px \leq \alpha w\} = B_{\alpha p, \alpha w}.$$

Therefore,

$$x(p,w) \equiv C(B_{p,w}) = C(B_{\alpha p,\alpha w}) \equiv x(\alpha p, \alpha w).$$

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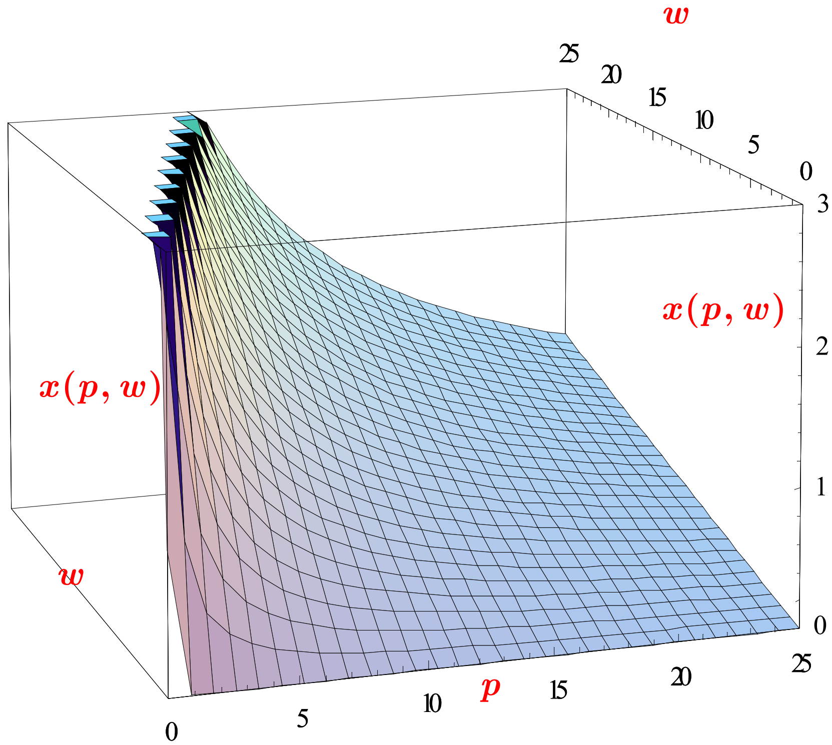
- 3 dimensional plot of $x(p,w) = w/p$
- Discontinuous at origin
- Not graphed near origin.
- Constant along every ray.



- **Homogeneity of degree zero means that the absolute level of prices and wealth doesn't matter. No money illusion.**

- **Only the relative values have an effect.**

Definition 2.8 *The price-elasticity of demand ε_{ij} of commodity i to the price of commodity j is given by*

$$\varepsilon_{ij} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}.$$

- **Therefore,**

$$\varepsilon_{ij} = \text{approximately } \frac{\Delta x_i}{x_i} \Big/ \frac{\Delta p_j}{p_j} = \frac{\%\Delta x_i}{\%\Delta p_j},$$

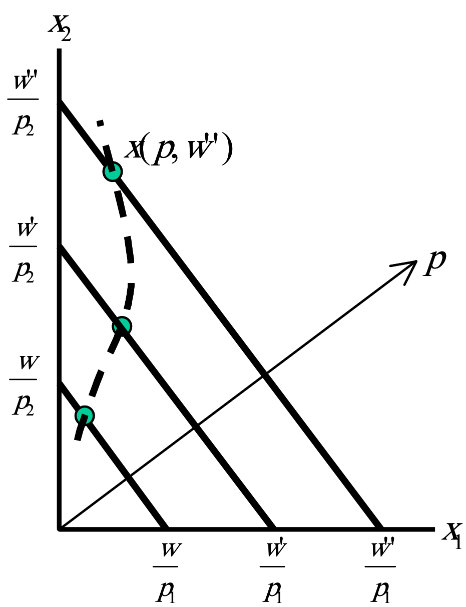
the ratio of the percentage change in quantity to the percentage change in price.

Definition 2.9 *The income-elasticity of demand ε_{iw} is given by*

$$\varepsilon_{iw} = \frac{\partial x_i}{\partial w}\frac{w}{x_i}$$

- **Why useful?**
- **Homogeneity $\Longrightarrow$ for each commodity i, $\varepsilon_{i1} + \varepsilon_{i2} + \ldots + \varepsilon_{iL} + \varepsilon_{iw} = 0$.**
- **Why?**

- **Income (Wealth) -Consumption Curve**
- **Expansion path.**



Assumption 2.1 *Walras Law:* $px = w$ *(all wealth is spent).*

Proposition 2.7 *Walras Law implies*

$$-\sum_i p_i \frac{\partial x_i}{\partial p_j} = x_j.$$

INFORMAL PROOF. **Suppose price of j increases by Δp_j.**

- **Then your demand will change by** $\Delta x_i = \dfrac{\partial x_i}{\partial p_j} \Delta p_j$
- **This creates an expenditure reduction on commodity i of** $-p_i \Delta x_i = -p_i \dfrac{\partial x_i}{\partial p_j} \Delta p_j$.
- **Total expenditure reduction for all commodities is** $-\sum_i p_i \dfrac{\partial x_i}{\partial p_j} \Delta p_j$
- **But the price increase created an added expense for j equal to** $x_j \Delta p_j$.
- **By Walras Law the expenditure reduction must equal the added expense:**

$$-\sum_i p_i \frac{\partial x_i}{\partial p_j} \Delta p_j = x_j \Delta p_j.$$

- **Cancel the** Δp_j.

■

Proposition 2.8 *Walras law implies*

$$\sum_i p_i \frac{\partial x_i}{\partial w} = 1.$$

- **Intuition: If wealth increases by $1, your expenditures must increase to absorb that dollar.**

Definition 2.10 *A price change Δp (a vector) is "Slutsky compensated" if income (or wealth) is adjusted by the amount $\Delta w_s = x(p, w)\Delta p$ (so that the consumer can still afford to buy the quantity demanded before the price change).*

Proposition 2.9 *Suppose*

- *a demand* `function`, $x(p, w)$, *satisfies Walras Law,*
- *price changes from p to p'*
- *and Slutsky compensation changes w to w'*
- *Let $x \equiv x(p, w)$ and $x' \equiv x(p', w')$.*
- *Assume $x \neq x'$.*
- *Then*

$$(p' - p)(x' - x) < 0$$

$$\Updownarrow$$

$$\left\{ \begin{array}{l} x \text{ satisfies the weak axiom} \\ \text{of revealed preference} \end{array} \right.$$

PROOF.

Compensation + Demand Function + Walras Law

$$\Updownarrow$$

$$p'x = p'x'$$

which means that x' is revealed preferred to x

Revealed preference axiom

$$\Updownarrow$$

$$px' > px$$

(otherwise x would be revealed preferred to x')

Therefore:

Compensated price change + Walras Law + Revealed Preference

$$\Updownarrow$$

$$p'x = p'x'$$

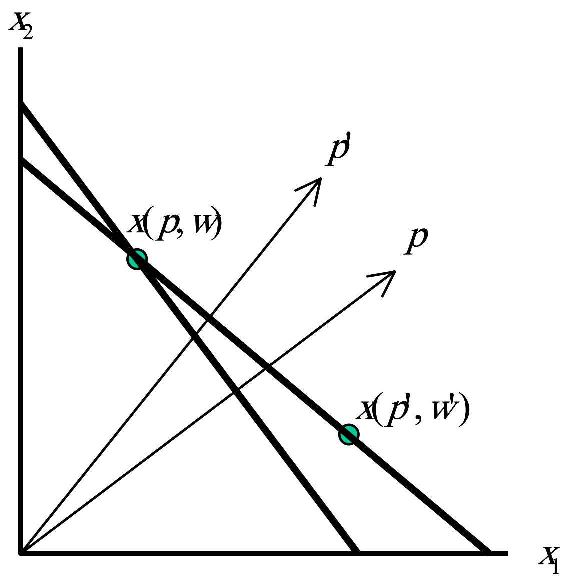
and $px' > px$

$$\Updownarrow$$

$$(p' - p)(x' - x) < 0$$

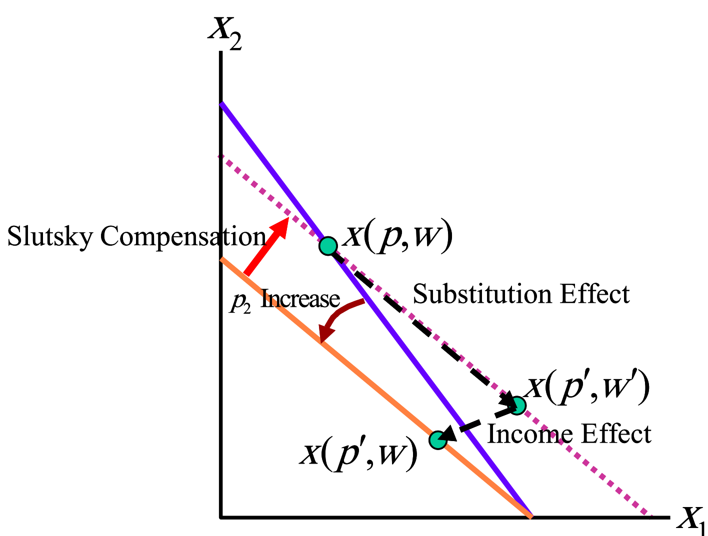
■

- **Drawing satisfies week axiom of revealed preference, because new demand was not in budget set at the old prices.**



2.3 Income and Substitution Effect of Uncompensated Price Change.

- **For simplicity assume** $x(p, w) \neq x(p', w')$.



Definition 2.11 *Suppose price changes from p to p'. The* substitution effect *is*

$$\Delta x_s = x(p', w') - x(p, w),$$

where w' reflects Slutsky compensation. The substitution effect allows prices to change while holding buying power constant.

- From previous proposition:

Weak axiom of revealed preference $\Longleftrightarrow$ $\Delta x_s \Delta p < 0$ for all Δp.

Definition 2.12 *The* income effect *is given by*

$$\Delta x_w = x(p', w) - x(p', w'),$$

the difference between demand at the new prices without and with Slutsky compensation.

- Net (uncompensated) change in demand is

$$\Delta x = \Delta x_s + \Delta x_w = x(p', w) - x(p, w)$$

Example 2.5 *Suppose that Pierre's utility is given by*

$$U(x, y) = x + y,$$

and suppose prices and wealth are $p_x = 4$, $p_y = 5$ and $w = 60$. If p_x changes to $p'_x = 6$, find the change in the quantities demanded and decompose them into the substitution and income effects.

Solution:

- Before the price change, Pierre buys only good x. Why?
- Therefore $x = 15$ and $y = 0$.
- After the price change, Pierre buys only good y.
- Therefore $x = 0$ and $y = 12$.
- Slutsky compensation for the price change would enable Pierre to buy $x = 15$ at the price $p'_x = 6$, so the compensation must be $\Delta w = 30$.
- Pierre would then purchase $y = 18$ and $x = 0$. Why?
- This means that the substitution effect is $\Delta x_s = -15$ and $\Delta y_s = 18$.
- Therefore the income effect is $\Delta x_w = 0$ and $\Delta y_w = -6$.

We define substitution and income effects in terms of derivatives.

Definition 2.13 *The* `Slutsky compensated demand function` $x_s(p', z)$ *is the demand at prices* p' *when when the consumer is always provided with the wealth needed to buy the consumption vector* z*; that is:*

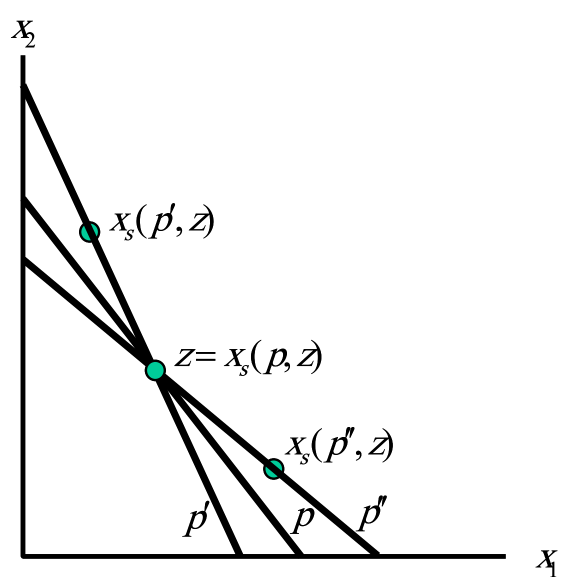
$$x_s(p', z) \equiv x(p', p'z)$$

Proposition 2.10 *Given* p *and* w*, we have*

$$x_s(p, x(p, w)) = x(p, w).$$

- **Why?**

- **Illustration of a Slutsky demand function.**
- **The consumer can always buy** z**, if she wants to.**



- Let $\bar{w}$ denote the level of buying power (real income).

Definition 2.14 *The* `Slutsky derivative`

$$\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}}$$

is the derivative of compensated demand $x_s(p', z)$ with respect to prices p' evaluated at $p' = p$ while holding buying power constant at $z = x(p,w)$; that is:

$$\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}} = \left.\frac{\partial x_s(p',z)}{\partial p'}\right]_{p'=p,\ z=x(p,w)}.$$

Proposition 2.11 *The* `Slutsky Equation` *(in scalar notation):*

$$\frac{\partial x_i(p,w)}{\partial p_j} \equiv \left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}} - \frac{\partial x_i(p,w)}{\partial w} x_j(p,w)$$

- The first right-hand term describes the substitution effect, and the second, the income effect (in the limit as price changes become small).

PROOF. Let $z = x(p,w)$ and $w' = p'z$. Then,

$$\frac{\partial x_{si}(p',z)}{\partial p'_j} = \left.\frac{\partial x_i(p',w')}{\partial p'_j}\right]_{p'_j, w' \text{ varying}}$$

$$= \frac{\partial x_i(p',w')}{\partial p'_j} + \frac{\partial x_i(p',w')}{\partial w'}\frac{\partial w'}{\partial p'_j} \quad \text{[chain rule]}$$

$$= \frac{\partial x_i(p',w')}{\partial p'_j} + \frac{\partial x_i(p',w')}{\partial w'} z_j$$

Replacing z_j by $x_j(p,w)$, setting $p' = p$ (and $w' = w$) and denoting

$\left.\frac{\partial x_{si}(p',z)}{\partial p'_j}\right]_{p'=p}$ by $\left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}}$ yields

$$\left.\frac{\partial x_i(p,w)}{\partial p_j}\right]_{\bar{w}} = \frac{\partial x_i(p,w)}{\partial p_j} + \frac{\partial x_i(p,w)}{\partial w}\frac{\partial w}{\partial p_j}.$$

We obtain the Slutsky equation by rearranging terms. ■

Definition 2.15 *A quadratic expression, in which every term is quadratic, is* positive definite *if its value is positive for any nonzero values of the variables. The expression is* negative definite *if its value is negative for any nozero values of the variables.*

Example 2.6 $-x^2 + xy - y^2$ *is negative definite. Why?*

Hint: Find the maximum value the expression can take for a given x*.*

- Note: a quadratic expression can be written by use of a symmetrix matrix.

$$\begin{pmatrix} x & y \end{pmatrix} \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -x^2 + xy - y^2$$

- The matrix is defined to be positive (negative) definite when the expression is.

Proposition 2.12 *The Slutsky matrix* $\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}}$ *is negative definite.*

INFORMAL PROOF.

- $\Delta x_s \mathrel{\hat{=}} \left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}} \Delta p$
- The weak axiom of revealed preference implies $\Delta p \Delta x_s < 0$.
- Therefore, for all sufficiently small Δp,

$$\Delta p \left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}} \Delta p < 0.$$

- But the sign of the left-hand side stays the same if Δp is multiplied by any scalar. Why?
- Therefore, the inequality is true for any Δp, and the matrix must be negative definite.

■

- Note that

$\left.\frac{\partial x(p,w)}{\partial p}\right]_{\bar{w}}$ is negative definite

$\Longrightarrow$ all diagonal terms are negative (among other things)

$\Longrightarrow$ "own" substitution effect is always negative.