4-6 <u>Nonhomogeneous Recurrences</u>: They are of the form

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n)$$, b constant $p(n)$: polynomial

<u>Example</u> ① $t_n - 2t_{n-1} = 3^n$ (1)

Reduce to homogeneous - First multiply by 3.

$$3t_n - 6t_{n-1} = 3^{n+1} \quad (2)$$

substitute n by $n-1$

$$3t_{n-1} - 6t_{n-2} = 3^n \quad (2)$$

Now (2) − (1):

$$t_n - 5t_{n-1} + 6t_{n-2} = 0 \quad (3)$$

This can be solved by using previous method:
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$$x^2 - 5x + 6 = (x-2)(x-3)$$

and solutions are of the form:

$$t_n = c_1 2^n + c_2 3^n \quad (4)$$

It is no longer true that arbitrary choice of constants c_1 and c_2 in (4) produces a solution to the recurrence even when initial conditions are not taken into accounts. Even the basic solutions $t_n = 2^n$ and $t_n = 3^n$, that are solutions to (3) are not solutions to (1). What is this? Well simply (1) & (3) are not equivalent. (3) can be solved given arbitrary values for t_0 and t_1 (the initial conditions), whereas equation (1) implies that: $t_1 = 2t_0 + 3$. So, the general solutions to the original recurrence can be determined as a function of t_0 by solving 2 linear equations in the unknowns c_1 and c_2

$$c_1 + c_2 = t_0 \qquad n=0$$

$$2c_1 + 3c_2 = 2t_0 + 3 \qquad n=1$$

Solving these, we obtain $c_1 = t_0 - 3$, $c_2 = 3$. Thus the general sol.:

$$t_n = (t_0 - 3)2^n + 3^{n+1}$$

and thus $t_n \in \Theta(3^n)$ regardless of initial conditions.

we could've concluded that $t_n \in \Theta(3^n)$ from equation (4). But the equation alone is not sufficient. It could be that $c_2 = 0$. Anyway c_2 could've been obtained directly. Substitute Eq. (4) into the original recurrence.

$$\begin{aligned} 3^n &= t_n - 2t_{n-1} \\ &= (c_1 2^n + c_2 3^n) - 2(c_1 2^{n-1} + c_2 3^{n-1}) \\ &= c_1 2^n + c_2 3^n - c_1 2^n - 2c_2 3^{n-1} \end{aligned}$$

finally:

$$3^n = c_2 3^n - 2 c_2 3^{n-1}$$

$$3 \times 3^{n-1} = 3 c_2 3^{n-1} - 2 c_2 3^{n-1} \Rightarrow \boxed{3 = c_2}$$

<u>Example</u> 2 Find the general solution of the following recurrence:

$$t_n - 2t_{n-1} = (n+5)3^n \qquad n \geq 1 \qquad (1)$$

To transform this equation into a homogeneous recurrence is slightly more complicated than in the previous example:

a. Write down the recurrence

b. In the recurrence, replace n by $n-1$ and multiply by -6

c. replace n in the recurrence by $n-2$ and then multiply by 9.

$$\begin{array}{lllll} t_n & - 2t_{n-1} & & & = (n+5)3^n \\ & - 6t_{n-1} & + 12 t_{n-2} & & = -6(n+4)3^{n-1} \\ & & 9t_{n-2} & - 18 t_{n-3} & = 9(n+3)3^{n-2} \end{array}$$

Adding these 3 equations we obtain a homogeneous recurrence:

$$t_n - 8t_{n-1} + 21 t_{n-2} - 18 t_{n-3} = 0$$

characteristic polynomial:

$$x^3 - 8x^2 + 21x - 18 = (x-2)(x-3)^2$$

all solutions are of the form:

$$t_n = c_1 2^n + c_2 3^n + c_3 n 3^n \qquad (2)$$

Once again, any choice of values for the constants c_1, c_2 and c_3 in equation ② provides a solution to the homogeneous recurrence, but the original one imposes restrictions on these constants because it requires that:

$$t_1 = 2t_0 + 18 \quad , \quad t_2 = 2t_1 + 63 = 4t_0 + 99 .$$

Thus, the general solution is found by solving the following system of linear equations.

$$\begin{aligned} c_1 + c_2 &= t_0 & n=0 \\ 2c_1 + 3c_2 + 3c_3 &= 2t_0 + 18 & n=1 \\ 4c_1 + 9c_2 + 18c_3 &= 4t_0 + 99 & n=2 \end{aligned}$$

This implies that:

$$c_1 = t_0 - 9 \quad , \quad c_2 = 9 \quad \text{and} \quad c_3 = 3$$

Therefore the general solution is:

$$t_n = (t_0 - 9)\, 2^n + (n+3)\, 3^{n+1}$$

and thus $t_n \in \Theta(n 3^n)$, regardless of the initial condition.

<u>Examples ③</u> The number of movements of a ring required in the Towers of Hanoi problem is:

$$t(m) = \begin{cases} 0 & \text{if } m = 0 \\ 2t(m-1) + 1 & \text{otherwise} \end{cases}$$

This can be written as:

$$t(m) - 2t(m-1) = 1$$

RHS characteristic equation : $(x-2)$

LHS " " : $(x-1)$

General characteristic equation is : $(x-2)(x-1)$

so : $\quad t(m) = c_1 1^m + c_2 2^m$

we need 2 initial conditions to solve: we know $t(0) = 0$ and for the second, we use the original recurrence:

$t(1) = 2t(0) + 1 = 1$

this gives us 2 linear equations:

$$c_1 + c_2 = 0 \qquad m = 0$$

$$c_1 + 2c_2 = 1 \qquad m = 1$$

From this we obtain the solution: $c_1 = -1$, $c_2 = 1$

and therefore:

$$t(m) = 2^m - 1$$

If we were to determine the order only we would'nt need to calculate the constants. Only the fact that:

$t(m) = c_1 + c_2 2^m$ is sufficient to conclude that $c_2 > 0$ and thus $t(m) \in \Theta(2^m)$

The general behind nonhomogeneous is as follows:

given a recurrence of the form

$$a_0 t_n + a t_{n-1} + \cdots + a_k t_{n-k} = b^n p(n) \quad \text{where}$$

b is a constant and $p(n)$ a polynomial of degree d.

It is sufficient to use the following characteristic polynomial

$$(a_0 x^k + a_1 x^{k-1} + \cdots + a_k)(x-b)^{d+1}$$

(again d: degree of polynomial $p(n)$)

Once this polynomial is obtained, proceed as in the homogeneous case, except that some of the equations needed to determine the constants are obtained not from the initial conditions, but from the recurrence itself.

Example 4: consider the recurrence $t_n = 2t_{n-1} + n$

this can be rewritten:

$$t_n - 2t_{n-1} = n \qquad (1)$$

$b = 1$, $p(n) = n$, $d = 1$

The characteristic polynomial is thus:

$$(x-2)(x-1)^2$$

where 1 is a root of multiplicity 2.

All solutions are therefore of the form

$$t_n = c_1 2^n + c_2 1^n + c_3 n 1^n \qquad ②$$

Provided that $t_0 \geqslant 0$ and therefore $t_n \geqslant 0, \forall n$, we conclude immediately that $t_n \in O(2^n)$

If we substitute ② in ①, we obtain,

$$\begin{aligned} n &= t_n - 2t_{n-1} \\ &= (c_1 2^n + c_2 + c_3 n) - 2(c_1 2^{n-1} + c_2 + c_3(n-1)) \\ &= (2c_3 - c_2) - c_3 n \end{aligned}$$

from which we read:

$$2c_3 - c_2 = 0 \quad , \quad c_3 = -1 \rightarrow c_2 = -2$$

regardless of the inial conditions

substituting in ②:

$$t_n = c_1 2^n - n - 2 \qquad ③$$

Provided that $t_0 \geqslant 0$ and thus $t_n \geqslant 0 \; \forall n$, equation ③ implies that c_1 must be strictly positive, and we can conclude that:

$$t_n \in \Theta(2^n)$$

without no need to solve explicitly for c_1.

By now you may be convinced that, for all practical purposes, there is no need to worry about the constants: the exact order of t_n can always be read off directly from the general solution. This is WRONG! Perhaps you think that the constants obtained by the simpler technique of substituting the general solution into the original recurrence are always sufficient to determine its exact order. WRONG again!

Example ⑤: consider $t_n = \begin{cases} 1 & \text{if } n=0 \\ 4t_{n-1} - 2^n & \text{otherwise} \end{cases}$

rewrite the recurrence $\quad t_n - 4t_{n-1} = -2^n \qquad ①$

$b = 2$, $p(n) = -1$, $d = 0$

characteristic polynomial is thus: $(x-4)(x-2)$

All solutions are of the form:

$$t_n = c_1 4^n + c_2 2^n \qquad ②$$

We may be tempted to assert that $t_n \in \Theta(4^n)$ since, it is clearly the dominant term in ②.

Since we are not in any hurry, let's substitute ② in ① and check:

$$\begin{aligned} -2^n &= t_n - 4t_{n-1} \\ &= c_1 4^n + c_2 2^n - 4(c_1 4^{n-1} + c_2 2^{n-1}) \\ &= -c_2 2^n \qquad \text{thus } c_2 = 1 \end{aligned}$$

regardless of initial conditions. Knowing c_2 is not of any relevance to determining the exact order of t_n as given by ②.

Let's use t_0 to determine c_1:

$$t_n = c_1 4^n + 2^n \rightarrow \quad t_0 = 1, \quad 1 = c_1 + 1$$

which means $c_1 = 0$. and thus $t_n = 2^n \rightarrow \Theta(2^n)$

Our previous assertion that $t_n \in \Theta(4^n)$ was WRONG!

This example illustrates the importance of the initial condition for some recurrences, whereas previous examples had shown that the asymptotic behaviour of many recurrences is not affected by the initial condition, at least when $t_0 \geq 0$.

A further generalization of the same type of argument allows us finally to solve recurrences of the form

$$a_0 t_n + a_1 t_{n-1} + \cdots + a_k t_{n-k} = b_1^n p_1(n) + b_2^n p_2(n) + \cdots$$

where the b_i's are distinct constants and the $p_i(n)$ are polynomials in n respectively of order d_i. Such recurrences are solved using the following characteristic polynomial

$$(a_0 x^k + a_1 x^{k-1} + \dots + a_k).(x - b_1)^{d_1+1}.(x-b_2)^{d_2+1} \dots ,$$

once this is obtained the rest is as before.

Example ⑥: consider the recurrence $t_n = \begin{cases} 0 & n=0 \\ 2t_{n-1} + n + 2^n & \text{other} \end{cases}$

rewrite the recurrence:

$$t_n - 2t_{n-1} = n + 2^n \qquad (1)$$

$b_1 = 1$, $p_1(n) = n$, $b_2 = 2$, $p_2(n) = 1$, $d_1 = 1$, $d_2 = 0$

The characteristic polynomial is:

$$(x-2)(x-1)^2(x-2)$$

1 and 2 are roots of multiplicity 2 each. All solutions to the recurrence therefore have the form:

$$t_n = c_1 1^n + c_2 n 1^n + c_3 2^n + c_4 n 2^n \qquad (2)$$

We can conclude that $t_n \in O(n 2^n)$ without calculating the constants, but we need to know whether $c_4 > 0$ or not. To know this substitute ② in ①:

$$n + 2^n = (2c_2 - c_1) - c_2 n + c_4 2^n$$

Equating the coefficients of 2^n, we obtain $c_4 = 1$ and therefore $t_n \in \Theta(n 2^n)$. Other coefficients can be derived easily, but they would be irrelevant to decide the order only. If you're curious about the constant values here they are: where $t_1 = 3$, $t_2 = 12$, $t_3 = 35$

$$\begin{array}{lllllr} c_1 & & + c_3 & & = 0 & n=0 \\ c_1 & + c_2 & + 2c_3 & + 2c_4 & = 3 & n=1 \\ c_1 & + 2c_2 & + 4c_3 & + 8c_4 & = 12 & n=2 \\ c_1 & + 3c_2 & + 8c_3 & + 24c_4 & = 35 & n=3 \end{array}$$

and solutions are: $c_1 = -2$, $c_2 = -1$, $c_3 = 2$, $c_4 = 1$

and

$$t_n = n2^n + 2^{n+1} - n - 2$$

4-7 Change of Variable:

Sometimes it helps to make a change of variable to solve some complicated recurrences.

Example ①: Consider the previous recurrence

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ 3T(n/2) + n & \text{if } n \text{ is a power of } 2, n>1 \end{cases}$$

in order to transform this into a form that we know how to solve, we replace n by 2^i. The new recurrence is $t_i = T(2^i)$. This transformation is useful because $n/2$ becomes $2^i/2 = 2^{i-1}$.

$$t_i = T(2^i) = 3\,T(2^{i-1}) + 2^i = 3t_{i-1} + 2^i$$

or simply: $\quad t_i - 3t_{i-1} = 2^i$

This is of a form we know how to handle. The characteristic polynomial is: $\quad (x-3)(x-2)$

and hence all solutions for t_i are of the form

$$t_i = c_1 3^i + c_2 2^i$$

We use the fact that $T(2^i) = t_i$ and thus $T(n) = t_{\log n}$ when $n = 2^i$ to obtain:

$$T(n) = c_1 3^{\log n} + c_2 2^{\log n}$$
$$= c_1 n^{\log 3} + c_2 n$$

when n is a power of 2, which is sufficient to conclude that

$$T(n) \in O(n^{\log 3} \mid n \text{ is a power of } 2)$$

However, we need to show that c_1 is strictly positive before we can assert something about the exact order of $T(n)$.

We know that $\quad T(1) = 1 \quad , \quad T(2) = 3T(1) + 2 = 5$

Thus: $\quad c_1 + c_2 = 1 \qquad n=1$

$\qquad 3c_1 + 2c_2 = 5 \qquad n=2$

solving these equations, we obtain $c_1 = 3$ and $c_2 = -2$.

Therefore: $\quad T(n) = 3n^{\log 3} - 2n \quad$ when $n = 2^i$

Example ②: Consider the recurrence

$$T(n) = 4\,T\left(\tfrac{n}{2}\right) + n^2 \qquad ①$$

when n is a power of 2, $n \geq 2$.

Like the previous example: $t_i = T(2^i) = 4\,T(2^{i-1}) + (2^i)^2$

$$= 4\,t_{i-1} + 4^i$$

rewrite, $t_i - 4t_{i-1} = 4^i$

charac. Polyn. is $(x-4)^2$ and all solutions are of the form: $t_i = c_1 4^i + c_2\, i\, 4^i$ in terms of $T(n)$ this is

$$T(n) = c_1 n^2 + c_2 n^2 \log n \qquad ②$$

Substituting ② in ① yields

$$n^2 = T(n) - 4\,T(n/2) = c_2 n^2$$

and thus $c_2 = 1$. Therefore

$$T(n) \in \Theta(n^2 \log n \mid n \text{ is a power of } 2)$$

Example ③: Consider the recurrence $T(n) = 2\,T(n/2) + n \log n$ ①

when n is a power of 2, $n \geq 2$. Like before

$$t_i = T(2^i) = 2\,T(2^{i-1}) + i\,2^i$$

$$= 2\,t_{i-1} + i\,2^i$$

Thus:

$$t_i - 2t_{i-1} = i\,2^i$$

charac. Polyn. is: $(x-2)(x-2)^2 = (x-2)^3$ and all solutions are of the form: $t_i = c_1 2^i + c_2\, i\, 2^i + c_3\, i^2\, 2^i$

in terms of $T(n)$, this is

$$T(n) = c_1 n + c_2 n \log n + c_3 n \log^2 n \qquad ②$$

Substituting ② in ① we get:

$$n \log n = T(n) - 2\,T(n/2) = (c_2 - c_3)\,n + 2c_3\, n \log n,$$

which implies that $c_2 = c_3$ and $2c_3 = 1$, and thus $c_2 = c_3 = \frac{1}{2}$. Therefore:

$$T(n) \in \Theta(n \log^2 n \mid n \text{ is a power of } 2)$$

regardless of initial conditions.

Remark: in the preceding examples, the recurrences $T(n)$ only applies when n is a power of 2. It is therefore inevitable that the solution obtained should be in conditional asymptotic notation. In each case, it is sufficient to add the condition that $T(n)$ is eventually nondecreasing to be able to conclude that the asymptotic results obtained apply unconditionally for all values of n. This follows from the smoothness rule, since the functions $n^{\log 3}$, $n^2 \log n$ and $n \log^2 n$ are smooth.

Example ④: This example is one of the most important recurrences for algorithmic purposes. It is particularly useful, for the analysis of Divide and Conquer algorithms.

$$T(n) = l\,T(n/b) + cn^k \qquad n > n_0 \qquad (1)$$

$T: \mathbb{N} \to \mathbb{R}^+ - \{0\}$ $\quad n_0 \geq 1$, $l \geq 1$, $b \geq 2$ and $k \geq 0$ are integers, c is a strictly positive real number.

T: is a nondecreasing function.

When $\frac{n}{n_0}$ is an exact power of b, that is when

$$n \in \{ bn_0, b^2n_0, b^3n_0, \dots, \dots \}$$

An appropriate change of variable would be:

$$n = b^i n_0.$$

$$t_i = T(b^i n_0) = l\,T(b^{i-1} n_0) + c(b^i n_0)^k$$
$$= l\,t_{i-1} + c\,n_0^k\, b^{ik}$$

we rewrite this in the form:

$$t_i - l\,t_{i-1} = (cn_0^k)(b^k)^i$$

The right hand side is of the form $a^i p(i)$, where $p(i) = cn_0^k$ a constant polynomial of degree 0, and $a = b^k$.

Thus the characteristic polynomial is:

$(x-l)(x-b^k)$ whose roots are l and b^k.

Of course, it is tempting to say that all solutions are of the form
$t_i = c_1 l^i + c_2 (b^k)^i$ ② _ but this is incorrect.

To write this in terms of $T(n)$, note that $i = \log_b (n/n_0)$ when n is of the proper form, and thus

$$d^i = (n/n_0)^{\log_b d} \quad \text{for arbitrary positive values of } d.$$

Therefore,

$$T(n) = \left(c_1 / n_0^{\log_b l}\right) n^{\log_b l} + \left(\frac{c_2}{n_0^k}\right) n^k$$

$$= c_3 n^{\log_b l} + c_4 n^k \qquad ③$$

for appropriate new constants c_3 and c_4.

Substitute ③ in ① to find c_3 and c_4.

$$c n^k = T(n) - l\, T(n/b)$$

$$= c_3 n^{\log_b l} + c_4 n^k - l\left(c_3 (n/b)^{\log_b l} + c_4 (n/b)^k\right)$$

$$= \left(1 - \frac{l}{b^k}\right) c_4 n^k$$

Therefore, $c_4 = \dfrac{c}{1 - l/b^k}$. To express $T(n)$ in asymptotic notation, we need only to keep the dominant term in ③.

There are three cases.

- If $l < b^k$ then $c_4 > 0$ and $k > \log_b l$. $c_4 n^k$ dominates equation ③. We conclude $T(n) \in \Theta(n^k \mid (\frac{n}{n_0})$ is a power of $b)$. Since n^k is a smooth function and $T(n)$ is nondecreasing by assumption, therefore $T(n) \in \Theta(n^k)$

- If $l > b^k$ then $c_4 < 0$ and $\log_b l > k$. c_3 is therefore positive (since c_4 is negative). The term $c_3 n^{\log_b l}$ dominates equation ③. Furthermore, $n^{\log_b l}$ is a smooth function and $T(n)$

is eventually nondecreasing. Therefore $T(n) \in \theta(n^{\log_b l})$.

- If $l = b^k$, we have a problem, c_4 formula involves a division by zero! The reason for this is: In this case the characteristic polynomial has a single root of multiplicity 2, rather than 2 distinct roots. The general solution in this case is:

$$t_i = c_5 (b^k)^i + c_6 \, i \, (b^k)^i$$

In terms of $T(n)$, this is

$$T(n) = c_7 n^k + c_8 n^k \log_b (n/n_0) \qquad (4)$$

substituting (4) in (1), our usual manipulation yields $c_8 = c$. Therefore, the dominant factor is $c n^k \log_b n$ in (4) (remember c was assumed to be strictly positive.) Since $n^k \log n$ is smooth and $T(n)$ is non decreasing, we conclude that $T(n) \in \theta(n^k \log n)$.

Putting all together,

$$T(n) \in \begin{cases} \theta(n^k) & \text{if } l < b^k \\ \theta(n^k \log n) & \text{if } l = b^k \\ \theta(n^{\log_b l}) & \text{if } l > b^k \end{cases}$$

4-8 Range Tranformations:

When we make a change of variable, we transform the domain of the recurrence. Instead, it may be useful to transform the range to obtain a recurrence in a form that we know how to solve. Both transformations can sometimes be used together. Let's give an example.

Example (1): Consider the following recurrence which defines $T(n)$ when n is a power of 2.

$$T(n) = \begin{cases} \frac{1}{3} & n=1 \\ nT^2(\frac{n}{2}) & \text{otherwise} \end{cases} \quad (1)$$

First do a change of variable: let t_i denote $T(2^i)$

$$t_i = T(2^i) = 2^i\, T^2(2^{i-1})$$
$$= 2^i\, t_{i-1}^2$$

At first, it seems none of the previous techniques applies since it is not linear; and non constant coeff. 2^i.
To transform the range, we create another recurrence by using u_i to denote $\log t_i$

$$u = \log t_i = i + 2 \log t_{i-1}$$
$$= i + 2u_{i-1} \quad (2)$$

This time once rewritten as: $u_i - 2u_{i-1} = i$
charac. polyn. $(x-2)(x-1)^2$
and all solutions are of the form:

$$u_i = c_1 2^i + c_2 1^i + c_3 i 1^i$$

substituting in (2) we get

$$i = u_i - 2u_{i-1}$$
$$= c_1 2^i + c_2 + c_3 i - 2(c_1 2^{i-1} + c_2 + c_3(i-1))$$
$$= (2c_3 - c_2) - c_3 i$$

and thus: $c_3 = -1$, $c_2 = 2$, $c_3 = -2$
and the general solution for $u_i = c_1 2^i - i - 2$
The general solution for t_i

$$t_i = 2^{u_i} = 2^{c_1 2^i - i - 2}$$

$$T(n) = t_{\log n} = 2^{c_1 n - \log n - 2} = \frac{2^{c_1 n}}{4n}$$

using initial conditions: $T(1) = 1/3$ this implies, $T(1) = \frac{2^{c_1}}{4} = \frac{1}{3}$
$c = \log(\frac{4}{3}) = 2 - \log 3$

$$T(n) = \frac{2^{2n}}{4n3^n}$$