THREE PARADOXES FOR THE 'SMOOTH AMBIGUITY' MODEL OF PREFERENCES

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P. Klibanoff, M. Marinacci and S. Mukerji, A smooth model of decision making under ambiguity, *Econometrica* (2005) KMM

- presented in unqualified terms as a general model (as alternative to multiple-priors): no suggestion of limited scope or explanation of where model does NOT make sense

- described as "offering flexibility in modeling ambiguity" and permitting "a wide variety of patterns of ambiguity"

I will dispute these claims, using Ellsberg-style thought experiments

KMM MODEL

- Anscombe-Aumann acts $f : \Omega \to \Delta(C)$, $C \subset \mathbb{R}$
- $\mathcal{F}$: set of all AA acts
- $U(f) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))\, dp(\omega)\right) d\mu(p), \;\; f \in \mathcal{F}$
- Nice story
- KMM also make claims re interpretation

- $(u, \mu, \phi) =$ (risk attitude, ambiguity, ambiguity attitude)

- "ambiguity aversion" modeled by concave ϕ

- Can mimic risk analysis!!
  Arrow-Pratt measure of ambiguity aversion $-\phi''/\phi'$

- Has been applied in macro-finance: Chen, Ju and Miao (2009), Ju and Miao (2009), Hansen (2007)

But what is being assumed via U??
What is justification for above interpretations?

KMM FOUNDATIONS (slightly restated)

- Expand domain of preference to $\mathcal{F} \cup \mathcal{F}^2$
- $\mathcal{F}^2$: "2nd-order acts", $f^2 : \Delta(\Omega) \to C$
  bets on the true probability law
- observability? in laboratory

AXIOM 1: vNM on lotteries (constant acts in $\mathcal{F}$)
with vNM index u, strictly increasing, continuous

AXIOM 2: SEU on $\mathcal{F}^2$, with vNM index v strictly increasing, continuous
v and μ "unique"

Say that $f \in \mathcal{F}$ *induces* $f^2 \in \mathcal{F}^2$ if

$$\int_\Omega f(\omega)\, dp(\omega) \sim f^2(p) \text{ for every } p$$

AXIOM 3: For all $f, g \in \mathcal{F}^2$, $f \succeq g$ iff $f^2 \succeq g^2$, where f^2 and g^2 are induced by f and g

PROOF of sufficiency for U on $\mathcal{F}$: let $v = \phi(u)$, $f \in \mathcal{F}$ and f^2 induced by f. Then

$$U(f) = \int_{\Delta(\Omega)} \phi \circ u \left(f^2(p)\right) d\mu = \int_{\Delta(\Omega)} \phi \circ u \left(\int_\Omega f(\omega)\, dp(\omega)\right) d\mu$$

$$= \int_{\Delta(\Omega)} \phi \left(\int_\Omega u(f(\omega)) dp(\omega)\right) d\mu \quad \blacksquare$$

TRANSLATION OF FUNCTIONAL FORM

I will argue that

1. Translation misses problematic features of behavior on $\mathcal{F}$
2. Translation is "poor" - introduction of $\mathcal{F}^2$ generates problems
3. $\mathcal{F}^2$ plays critical role not only in axioms, but also in interpretation, and interpretation matters

Alternative foundations for the model on $\mathcal{F}$: Seo (2009)

THOUGHT EXPERIMENT 1

Ellsberg: $R = 30$, $B, G = 0$ or 60

Bets on the color			
	R	B	G
f_1	100	0	0
f_2	0	100	0
f_3	100	0	100
f_4	0	100	100

Ellsbergian choices: $f_1 \succ f_2$ and $f_3 \prec f_4$

Intuition: uncertainty about the composition

A 'Second-Order' Urn: $r = 30$, $b, g = 0$ or 60

Draw a ball, and outcome i implies the composition p_i of a 'normal' urn, $i = r, b, g$

$$p_r = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right),\ p_b = \left(\frac{1}{3}, \frac{2}{3}, 0\right),\ p_g = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$$

Bets on the composition			
	p_r	p_b	p_g
F_1	100	0	0
F_2	0	100	0
F_3	100	0	100
F_4	0	100	100

INTUITIVE HYPOTHESIS

"Ellsbergian choices in betting on color $\Longrightarrow$ Ellsbergian choices in betting on composition"

- *necessary condition for a model that: (i) claims that uncertainty about the urn's composition is the reason for the classic Ellsbergian choices, AND (ii) makes predictions about choices between bets on the true composition of the urn*

KMM model satisfies the hypothesis only if ϕ is linear and ambiguity does not matter.

Experiment has nothing to say about any other model of ambiguity averse preferences!

ARE WE USING THE WRONG STATE SPACE Ω?

Take $\Omega = \{R, B, G\} \times \boldsymbol{\Delta}(\{R, B, G\})$, or $\Omega = \{R, B, G\} \times \{r, b, g\}$

This does NOT solve the problem

IS THE PROBLEM ONLY WITH "STRUCTURED UNCERTAINTY"?

"Second-order urn" $\{r, b, g\}$: you are told that one color is twice as likely as each of the others. AND the outcome will determine the composition of the 'normal' urn by:

$$p_r = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right),\ p_b = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right),\ p_g = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

Bet on both urns (stakes $u = 1$ and 0)

Suppose ϕ is concave and nonlinear (ambiguity averse), μ uniform

Then certainty equivalents for bets on R and r are different even though the urns are isomorphic!

$\phi(u(c_r)) = \frac{1}{3}\phi(1) + \frac{2}{3}\phi(0)$, and
$\phi(u(c_R)) = \int \phi(p(R))\, d\mu = \frac{1}{3}\phi\left(\frac{1}{2}\right) + \frac{2}{3}\phi\left(\frac{1}{4}\right) < \phi(u(c_r))$

NOTE: SEU is plausible on second-order bets ("uniform urn")
- there is no Ellsberg Paradox within either urn

Difficulty arises because of *asymmetry* in the modeling of preference on bets on the two kinds of urns

KMM Reply (2009): "... implications that undoubtedly conflict with some plausible behavior, but are the result of exactly the kind of pragmatic modeling choices that are often useful in building tractable and insightful models"

".... such a limitation should be viewed as a pragmatic modeling choice and is of little importance in many applications"

WHY IMPORTANT?

We are mostly interested in choice within $\mathcal{F}$. But:

1. shaky foundations: what are axioms for? doesn't it matter if they are violated even if it is on a "secondary domain"?

2. Interpretation of μ and ϕ presumes UNIQUENESS, but they are NOT pinned down uniquely by the ranking on $\mathcal{F}$ alone

3. Interpretation is vital for evaluating applications. In a quantitative empirical exercise one needs to judge whether parameter values make sense and this requires that they have an interpretation

Remaining thought experiments concern (primarily) the ranking on $\mathcal{F}$ alone, ignoring foundations ....

They are based on variants of Ellsberg's "2-urn experiment," and rely on the Anscombe-Aumann domain of acts, and the value of randomization

ELLSBERG'S "TWO-URN" PARADOX FOR SEU

URN:	$R + B = 100$	
	R	B
f_R	100	0
f_B	0	100
$\frac{1}{2}f_R + \frac{1}{2}f_B$	$\left(100, \frac{1}{2}\right)$	$\left(100, \frac{1}{2}\right)$

$$\frac{1}{2}f_R + \frac{1}{2}f_B \succ f_R \sim f_B$$

- Randomization smooths out ("hedges") ambiguity.
  Violates IA - ambiguity aversion

THOUGHT EXPERIMENT 2

Builds on the observation that (for strictly concave ϕ), KMM utility function U on $\mathcal{F}$ satisfies: For all AA acts f_1 and f_2,

$$[f_1 \sim f_2 \sim \tfrac{1}{2}f_1 + \tfrac{1}{2}f_2] \implies \tfrac{1}{2}f_1 + \tfrac{1}{2}h \sim \tfrac{1}{2}f_2 + \tfrac{1}{2}h \text{ for all } h \in \mathcal{F}$$

False for multiple-priors or CEU, ... - the behavior to be described is easily accommodated by other models

PROOF: $[f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2] \Longrightarrow$

$$
\begin{aligned}
\int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2\right) dp\right) d\mu &= \int \phi\left(\int_\Omega u\left(f_i\right) dp\right) d\mu \Longrightarrow \\
\int_\Omega u\left(f_1\right) dp &= \int_\Omega u\left(f_2\right) dp \quad \mu\text{-a.s.} \Longrightarrow \\
\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) dp &= \int_\Omega u\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right) dp \quad \mu\text{-a.s.} \Longrightarrow \\
\int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) dp\right) d\mu &= \int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right) dp\right) d\mu \quad \Longrightarrow \\
U\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) &= U\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right)
\end{aligned}
$$

$$[f_1 \sim f_2 \sim \tfrac{1}{2}f_1 + \tfrac{1}{2}f_2] \implies \tfrac{1}{2}f_1 + \tfrac{1}{2}h \sim \tfrac{1}{2}f_2 + \tfrac{1}{2}h \text{ for all } h \in \mathcal{F}$$

Roughly, randomization to reduce ambiguity is valuable either everywhere or (almost) nowhere - “uniformity”

KMM Reply (2009) argues for the former

EXPERIMENT 2 is intended to illustrate where $f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2$ is intuitive (at least plausible), but not RHS

Only weak concavity of ϕ is assumed

TWO (ambiguous) urns $R_i + B_i = 50$ - urns are “independent”.
Draw one ball from each

	Bets for Experiment 2			
	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	100	100	0	0
f_2	100	0	100	0
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	100	$\left(100, \frac{1}{2}\right)$	$\left(100, \frac{1}{2}\right)$	0
h	0	0	100	100
$\frac{1}{2}f_1 + \frac{1}{2}h$	$\left(100, \frac{1}{2}\right)$	$\left(100, \frac{1}{2}\right)$	$\left(100, \frac{1}{2}\right)$	$\left(100, \frac{1}{2}\right)$
$\frac{1}{2}f_2 + \frac{1}{2}h$	$\left(100, \frac{1}{2}\right)$	0	100	$\left(100, \frac{1}{2}\right)$

Uncontentious that $f_1 \sim f_2$, and $\frac{1}{2}f_1 + \frac{1}{2}h \succ \frac{1}{2}f_2 + \frac{1}{2}h$
Lack of hedging suggests also $\frac{1}{2}f_1 + \frac{1}{2}f_2 \sim f_1$
Similarly for any stakes $c^* > c$ in place of 100 and 0

Contradicts smooth model (with ϕ weakly concave)

KMM REPLY: $\Omega = S_1 \times S_2$, let $P = \{p, p'\} \subset \Delta(S)$,

$$p = \left(\frac{1}{5}, \frac{4}{5}\right), \; p' = \left(\frac{4}{5}, \frac{1}{5}\right)$$

Let μ be uniform on $P \otimes P = \{p_1 \otimes p_2 : p_1, p_2 \in P\}$

	Expected Utility Payoffs $\int u(g)\, dm$			
	$p \otimes p$	$p \otimes p'$	$p' \otimes p$	$p' \otimes p'$
f_1	4	16	16	64
f_2	16	64	4	16
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	10	40	10	40

NEW THOUGHT EXPERIMENT (in progress)

Preceding begs question: what is the meaning of "stochastic independence"?

I claim that KMM model does not capture independence

Two *independent* "urns": $\Omega = S_1 \times S_2$, $S_1 = S_2 = S$

Draw from both urns and bet on outcomes

Stakes for all bets are 1 and $b > 0$ (in utils)

$A_i \subset S_i$ denotes both event and bet; similarly $A_1 \times A_2$

INTUITIVE HYPOTHESIS: Let bet on A_i be indifferent to betting on Heads with objective probability π_i, $i = 1, 2$. Then the bet on $A_1 \times A_2$ is indifferent to betting on Heads with objective probability $\pi_1 \pi_2$

Easily accommodated by multiple-priors: let $P \subset \Delta(S)$ and

$$P \otimes P = \{p_1 \otimes p_2 : p_1, p_2 \in P\}$$

Take $P \otimes P$ to be the Gilboa-Schmeidler set of priors
(There are other specifications that work)

But the KMM model "has difficulty"

CLAIM: The Intuitive Hypothesis (for all $0 \leq b < 1$) is inconsistent with the following application of the KMM model:

(i) State space $\Omega = S_1 \times S_2$

(ii) ϕ concave but not linear

(iii) μ has support on $P \otimes P$ for some binary nonsingleton set $P \subset \Delta(S)$, where each measure in P is countably additive and nonatomic [extends to finite P under suitable conditions]

WHY? Let $P = \{p_1, ..., p_n\}$

If $b \equiv 0$, Hypothesis is equivalent to:

$$
\begin{aligned}
& \phi^{-1}\left(\int_{P \otimes P} \phi\left(p\left(A_1\right) q\left(A_2\right)\right) d\mu\left(p, q\right)\right) \\
= \ & \phi^{-1}\left(\int_P \phi\left(p\left(A_1\right)\right) d\mu\right) \cdot \phi^{-1}\left(\int_P \phi\left(q\left(A_2\right)\right) d\mu\right)
\end{aligned}
$$

Apply Lyapunov

Implies: for all $t > 0$ and $A \subset S$,

$$\begin{aligned} & \phi^{-1}\left(\int_P \phi(tp(A))d\mu\right) \\ = \ & t \cdot \phi^{-1}\left(\int_P \phi(p(A))\, d\mu\right) \end{aligned}$$

Or: for all $x \in D = \{(p_1(A), ...p_n(A)) : A \subset S\} \subset [0, 1]^n$,

$$\phi^{-1}\left(\Sigma_{i=1}^n \mu_i \phi(tx_i)\right) = t \cdot \phi^{-1}\left(\Sigma_{i=1}^n \mu_i \phi(x_i)\right)$$

Suppose D is 'large' - contains diagonal in its interior; automatic if $n = 2$ by Lyapunov

Then ϕ is α-homogeneous

Admitting also $b > 0$ implies linearity

CALIBRATION?

- Quantitative discipline or can explain anything

- Equity premium puzzle: can one match moments of returns using a "reasonable" degree of relative risk aversion ($RRA \leq 10$)?
  Essence is comparison of behavior *across settings*

- KMM Reply (2009): reject the above interpretation of the equity premium puzzle and insist it is about 'internal' consistency only -
  calibration across settings is impossible

- Lucas (2003): "No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high levels of risk aversion to do it."

- The smooth model has been interpreted as justifying "calibration" of ambiguity aversion - ϕ is not tied math'ly to the state space
  Chen, Ju and Miao (2009), Ju and Miao (2009) use thought experiments to judge the plausibility of ambiguity aversion parameters estimated from asset market data

- I construct an example with two Ellsberg-style settings where "intuitive" behavior is *incompatible* with the "same KMM agent" in the sense of (u, ϕ, μ_1) and (u, ϕ, μ_2)

POSITIVE LESSONS?