Supplementary Appendix to

LEARNING UNDER AMBIGUITY*

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Abstract

This appendix provides a detailed treatment of the portfolio choice problem studied in our paper “Learning under Ambiguity”, filling in details of calculations that were omitted in the text of the paper.

1 Investor Problem

Section 1 of this appendix restates the portfolio problem studied in Section 5 of “Learning under Ambiguity”. Section 2 defines beliefs and derives the dynamics of beliefs under ambiguity. Section 3 computes optimal portfolio weights for a benchmark Bayesian investor and a myopic ambiguity-averse investor.

Time is measured in months; there are \( k \) trading dates per month. The state space is \( S = \{1, 0\} \).\(^1\) The return on stocks \( R(s_t) = e^{r(s_t)} \) realized in period \( t \) is either high or low: we fix (log) return realizations \( r(1) = \sigma / \sqrt{k} \) and \( r(0) = -\sigma / \sqrt{k} \). In addition to stocks, the investor also has access to a riskless asset with constant per period interest rate \( R^f = e^{r^f/k} \), where \( r^f < \sigma \). We consider investors who plan for \( T \) months starting in month \( t \) and who care about terminal wealth according to the utility function \( V_T(W_{t+T}) = \log W_{t+T} \). Investors may rebalance their portfolio at all \( k (T - t) \) trading dates between \( t \) and \( T \). Let \( \omega^\star_{t,T,k} \) denote the optimal fraction of wealth invested in stocks at date \( t \) for an investor who plans for \( T \) months, when there are \( k \) trading dates per month.

Consider the investor’s problem when beliefs are given by a general process of one-step-ahead conditionals \( \{P_\tau(s^\tau)\} \). The history \( s^\tau \) of state realizations up to trading date

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\(^1\)In this appendix, we call the binary states 1 and 0, rather than \( hi \) and \( lo \) as in the text of “Learning under Ambiguity”, because this allows simpler notation below.
\[ \tau = t + j/k \] where \( j \) is the \( j \)th trading date in month \( t + 1 \). The sample of binary returns up to some integer date \( \tau \) follows from solving (1) when \( \Theta \) is a parameter space. Let \( \Phi(\cdot; s^t) \) denote the monthly standard deviation of log returns.

\[
\phi_\tau = \phi_k(\bar{r}_\tau) := \frac{1}{2} + \frac{1}{2\sqrt{\frac{1}{\bar{r}_\tau}}}
\]

where \( \bar{r}_\tau := \frac{1}{\tau} \sum_{j=1}^{\tau} r_j \) is the mean of the monthly return sample. For given \( k \), the sequence \( \{\phi_\tau\} \) pins down a sequence of monthly log returns \( \{\sum_{j=1}^{k} \log R(s_{\tau+j/k})\} \) that is identical to the data sample \( \{r_\tau\} \).

## 2 Beliefs

As a Bayesian benchmark, we assume that the investor has an improper beta prior over the probability \( p \) of the high state, so that the posterior mean of \( p \) after \( t \) months (or \( tk \) state realizations) is equal to \( \phi_\tau \), the maximum likelihood estimator of \( p \). The Bayesian’s probability of a high state next period is then also given by \( \phi_\tau \). The optimal portfolio follows from solving (1) when \( \mathcal{P}_\tau(s^\tau) \) is a singleton set containing only the measure that puts probability \( \phi_\tau \) on the high state.

For an ambiguity-averse investor, beliefs are defined as in Section 3 of “Learning under Ambiguity”. We briefly review the general model here. Beliefs are represented by

\[
(\Theta, \mathcal{M}_0, \mathcal{L}_k, \alpha),
\]

where \( \Theta \) is a parameter space, \( \mathcal{M}_0 \) is a set or priors on \( \Theta \), \( \mathcal{L}_k \) is a set of likelihoods and \( \alpha \) is a number between zero and one. A theory \( \ell^t \) is a sequence of likelihoods. Let \( \mu_t(\cdot; s^t, \mu_0, \ell^t) \) denote the posterior derived from the theory \( (\mu_0, \ell^t) \) by Bayes’ Rule, given the data \( s^t \).

The set of posteriors contains posteriors that are based on theories not rejected by a
likelihood ratio test:

\[ \mathcal{M}_{t,k}^\alpha(s^t) = \{ \mu_t(s^t; \mu_0, \ell^t) : \mu_0 \in \mathcal{M}, \ell^t \in \mathcal{L}_k^t, \int \Pi_{j=1}^t \ell_j(s_j|\theta) d\mu_0(\theta) \geq \alpha \max_{\tilde{\mu} \in \mathcal{M}_0} \int \Pi_{j=1}^t \tilde{\ell}_j(s_j|\theta) d\tilde{\mu}_0 \}. \] (3)

The set of one-step-ahead conditional belief is defined by

\[ \mathcal{P}_t(s^t) = \left\{ p_t(\cdot) = \int_\Theta \ell(\cdot | \theta) d\mu_t(\theta) : \mu_t \in \mathcal{M}_t^\alpha(s^t), \ell \in \mathcal{L}_k \right\}, \] (4)

This process enters the specification of recursive multiple priors preferences in (1).

The particular representation \((\Theta, \mathcal{M}_0, \mathcal{L}_k, \alpha)\) assumed for the portfolio choice problem is defined as follows. The ambiguity averse investor perceives the mean monthly log return as \(\theta + \lambda_t\), where \(\theta \in \Theta := \mathbb{R}\) is fixed and can be learned, while \(\lambda_t\) is driven by many poorly understood factors affecting returns and can never be learned. The set \(\mathcal{L}_k\) consists of all \(\ell(\cdot | \theta)\) such that

\[ \ell^k(hi|\theta) = \frac{1}{2} + \frac{1}{2} \frac{\theta + \lambda}{\sigma \sqrt{k}}, \text{ for some } \lambda \text{ with } |\lambda| < \tilde{\lambda}. \] (5)

The set of priors \(\mathcal{M}_0\) on \(\Theta\) consists of Dirac measures. For simplicity, we write \(\theta \in \mathcal{M}_0\) if the Dirac measure on \(\theta\) is included in the set of priors, and we define

\[ \mathcal{M}_0 = \{ \theta : |\theta| \leq \tilde{\lambda} + 1/\sigma \}. \]

The condition ensures that the probability (5) remains between zero and one for all \(k \geq 1\).

Beliefs depend on history only via the fraction of high returns \(\phi_t = \hat{\phi}_k(\bar{r}_t)\). We write \(\theta \in \mathcal{M}_{t,k}^\alpha(\bar{r}_t)\) if the Dirac measure on \(\theta\) is included in the posterior set at the end of month \(t\) after history \(\hat{\phi}_k(\bar{r}_t)\). The posterior set for fixed \(k\) can be characterized as follows:

**Proposition S1.** The posterior set is a subinterval \([\theta_k(\bar{r}_t), \bar{\theta}_k(\bar{r}_t)]\) of \(\mathcal{M}_0\) with both bounds strictly increasing in \(\bar{r}_t\).

**Proof.** The history \(s^t\) consists of \(tk\) realizations of the state. Write the likelihood of a sample \(s^t\) under some theory, here identified with a pair \((\theta, \lambda^t)\), as

\[ L_k(s^t, \theta, \lambda^t) = \prod_{\tau=0}^{t-1} \prod_{j=1}^k \left( \frac{1}{2} + \frac{1}{2} \frac{1 + \lambda_{\tau+j/k}}{\sigma \sqrt{k}} \right)^{s_{\tau+j/k}} \left( \frac{1}{2} - \frac{1}{2} \frac{1 + \lambda_{\tau+j/k}}{\sigma \sqrt{k}} \right)^{1-s_{\tau+j/k}}. \] (6)

Let \(\tilde{\lambda}^t\) denote the sequence that maximizes (6) for fixed \(\theta\). This sequence is independent of \(\theta\) and has \(\tilde{\lambda}_i = \tilde{\lambda}\) if \(s_i = 1\) and \(\tilde{\lambda}_i = -\tilde{\lambda}\) if \(s_i = 0\), for all \(i \leq t\). It follows that
$L_k \left( s^t, \theta, \tilde{\lambda}^t \right)$ depends on the sample only through the fraction $\phi_t$ of high returns observed. The posterior set can be written

$$
\mathcal{M}_{t,k}^{\alpha} (s^t) = \left\{ \theta : \frac{1}{tk} \log L \left( s^t, \theta, \tilde{\lambda}^t \right) \geq \max_{\tilde{\theta}} \frac{1}{tk} \log L \left( s^t, \tilde{\theta}, \tilde{\lambda}^t \right) - \frac{1}{tk} \log (\alpha) \right\} \quad (7)
$$

Indeed, if $\theta \in \mathcal{M}_{t,k}^{\alpha}$, then there exists some $\lambda^t$ such that the theory $(\theta, \lambda^t)$ passes the criterion for an admissible theory put forward in (3). Thus the theory $(\theta, \tilde{\lambda}^t)$ must also pass that criterion, since its likelihood is at least as high. In contrast, if $\theta \notin \mathcal{M}_{t,k}^{\alpha}$, then there is no $\lambda^t$ such that the theory $(\theta, \lambda^t)$ passes the criterion.

Using the fraction of high states $\phi_t = \frac{1}{tk} \sum_i s_i$, rewrite the log data density as

$$
\log L \left( s^t, \theta, \tilde{\lambda}^t \right) = f (\theta; \phi_t) := \phi_t \log \left( \frac{1}{2} + \frac{1}{2} \frac{\theta + \tilde{\lambda}}{\sigma \sqrt{k}} \right) + (1 - \phi_t) \log \left( \frac{1}{2} - \frac{1}{2} \frac{\theta - \tilde{\lambda}}{\sigma \sqrt{k}} \right).
$$

The likelihood ratio criterion becomes

$$
f (\theta; \phi_t) \geq \max_{\tilde{\theta}} \{ f (\theta; \phi_t) \} - \frac{1}{tk} \log (\alpha) \quad (8).
$$

For $\phi_t \in (0, 1)$, the function $f$ is strictly concave and achieves a unique maximum at the MLE

$$
\theta^\ast (\phi_t) = (2\phi_t - 1) \left( \sigma \sqrt{k} + \tilde{\lambda} \right) = (2\phi_k \langle \tilde{r}_i \rangle - 1) \left( \sigma \sqrt{k} + \tilde{\lambda} \right) = \tilde{r}_t \left( 1 + \tilde{\lambda} / \sigma \sqrt{k} \right).
$$

Since $f$ is strictly concave and $\lim_{\theta \to \lambda + \frac{1}{\sigma}} f (\theta; \phi_t) = \lim_{\theta \to \lambda - \frac{1}{\sigma}} f (\theta; \phi_t) = -\infty$, the set of $\theta$s that pass the likelihood ratio test is a subinterval of $\mathcal{M}_0$, with bounds that satisfy (8) with equality. Using $\phi_t = \phi_k \langle \tilde{r}_i \rangle$, the posterior set can be written as an interval $[\tilde{\theta}_k (\tilde{r}_t), \tilde{\theta}_k (\tilde{r}_t)]$.

To see why both bounds are strictly increasing in $\tilde{r}_t$, suppose that $\tilde{\theta}$ satisfies (8) with equality. Apply the implicit function theorem to obtain

$$
\frac{d\tilde{\theta}}{d\phi}_{\tilde{\theta} = \tilde{\phi}} = \frac{-f_2 \left( \tilde{\theta}; \phi_t \right) + f_2 \left( \theta^\ast (\phi_t); \phi_t \right)}{f_1 \left( \tilde{\theta}; \phi_t \right)},
$$

where $f_i$ is the derivative of $f$ with respect to its $i$th argument. Since $f$ is strictly concave, $f_1 \left( \tilde{\theta}; \phi_t \right) > 0$ if $\tilde{\theta} < \theta^\ast (\phi_t)$ and $f_1 \left( \tilde{\theta}; \phi_t \right) < 0$ if $\tilde{\theta} > \theta^\ast (\phi_t)$. In addition, it can be verified that $f_{21} \left( \tilde{\theta}; \phi_t \right) > 0$, so that $f_2 \left( \tilde{\theta}; \phi_t \right) - f_2 \left( \theta^\ast (\phi_t); \phi_t \right) > 0$ if $\tilde{\theta} > \theta^\ast (\phi_t)$, but $f_2 \left( \tilde{\theta}; \phi_t \right) - f_2 \left( \theta^\ast (\phi_t); \phi_t \right) > 0$ if $\tilde{\theta} > \theta^\ast$. Taken together, these facts imply that $d\tilde{\theta} / d\phi > 0$. Since $\phi_k \langle \tilde{r}_t \rangle > 0$, it follows that the bounds are strictly increasing in $\tilde{r}_t$.

A simple formula for the posterior set is obtained by taking the limit as $k \to \infty$: 


Proposition S2. The limit of the posterior set is given by

\[
\lim_{k \to \infty} \theta_k(\bar{r}_t), \lim_{k \to \infty} \bar{\theta}_k(\bar{r}_t) = \left[ \bar{r}_t - t^{-\frac{1}{2}}\sigma b, \bar{r}_t + t^{-\frac{1}{2}}\sigma b \right],
\]

where \( b_\alpha = \sqrt{-2\log \alpha} \).

Proof. Substitute for \( \theta^* \) and \( \phi_t = \phi_k(\bar{r}_t) \) defined in (2) to obtain

\[
(1 + \frac{\bar{r}_t}{\sigma \sqrt{k}}) \log \left( 1 + \frac{\theta + \bar{\lambda}}{\sigma \sqrt{k}} \right) + \left( 1 - \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \log \left( 1 - \frac{\theta - \bar{\lambda}}{\sigma \sqrt{k}} \right)
\geq \left( 1 + \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \log \left( 1 + \frac{\bar{\lambda} + \bar{\lambda}}{\sigma \sqrt{k}} \right) + \left( 1 - \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \log \left( 1 - \frac{\bar{\lambda} - \bar{\lambda}}{\sigma \sqrt{k}} \right) - 2 \frac{\log(\alpha)}{tk}
\]

For each term of the form \( \log(1 + x) \), perform a Taylor expansion around \( x = 0 \):

\[
\left( 1 + \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \left( \frac{\theta + \bar{\lambda}}{\sigma \sqrt{k}} - \frac{1}{2} \frac{(\theta + \bar{\lambda})^2}{\sigma^2 k} + O \left( k^{-\frac{3}{2}} \right) \right) + \left( 1 - \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \left( -\frac{\theta - \bar{\lambda}}{\sigma \sqrt{k}} - \frac{1}{2} \frac{(\theta - \bar{\lambda})^2}{\sigma^2 k} + O \left( k^{-\frac{3}{2}} \right) \right)
\geq \left( 1 + \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \left( \bar{r}_t + \bar{\lambda} - \frac{1}{2} \frac{(\bar{r}_t + \bar{\lambda})^2}{\sigma^2 k} + \bar{r}_t \bar{\lambda} + O \left( k^{-\frac{3}{2}} \right) \right) + \left( 1 - \frac{\bar{r}_t}{\sigma \sqrt{k}} \right) \log \left( -\frac{\bar{r}_t - \bar{\lambda}}{\sigma \sqrt{k}} - \frac{1}{2} \frac{(\bar{r}_t - \bar{\lambda})^2}{\sigma^2 k} - \bar{r}_t \bar{\lambda} + O \left( k^{-\frac{3}{2}} \right) \right) - 2 \frac{\log(\alpha)}{tk}
\]

Multiply out to obtain

\[
\frac{\bar{\lambda}}{\sigma \sqrt{k}} + 2\bar{r}_t \frac{\theta}{\sigma^2 k} - \frac{1}{2} \frac{(\theta + \bar{\lambda})^2}{\sigma^2 k} - \frac{1}{2} \frac{(\theta - \bar{\lambda})^2}{\sigma^2 k} + O \left( k^{-\frac{3}{2}} \right)
\geq \frac{\bar{\lambda}}{\sigma \sqrt{k}} + 2\bar{r}_t^2 \frac{1}{\sigma^2 k} - \frac{1}{2} \frac{(\bar{r}_t + \bar{\lambda})^2}{\sigma^2 k} - \frac{1}{2} \frac{(\bar{r}_t - \bar{\lambda})^2}{\sigma^2 k} + O \left( k^{-\frac{3}{2}} \right) - 2 \frac{\log(\alpha)}{tk}
\]

All the terms involving \( \bar{\lambda} \) cancel from this inequality. Multiplying by \( \sigma^2 k \) thus yields

\[
(\bar{r}_t - \theta)^2 \leq \sigma^2 \frac{-2\log \alpha}{t} + O \left( k^{-\frac{1}{2}} \right),
\]

which implies (9).

Finally, consider the set of one-step-ahead beliefs \( \mathcal{P}_{t,k}(\bar{r}_t) \). Following (4), it contains all likelihoods of the type (5) for some \( \theta \in \mathcal{M}_{t,k}(s^t) \) and \( |\lambda| < \bar{\lambda} \). It can thus
be summarized by an interval of probabilities for the high state next period, denoted \( [p_k(\bar{r}_t), \bar{p}_k(\bar{r}_t)] \), with both bounds strictly increasing in the sample mean \( \bar{r}_t \):

\[
\begin{align*}
\bar{p}_k(\bar{r}_t) & = \frac{1}{2} + \frac{1}{2} \frac{\theta_k(\bar{r}_t) - \bar{\lambda}}{\sigma/\sqrt{k}} \\
\bar{p}_k(\bar{r}_t) & = \frac{1}{2} + \frac{1}{2} \frac{\hat{\theta}_k(\bar{r}_t) + \lambda}{\sigma/\sqrt{k}}.
\end{align*}
\]

To get an idea about the shrinkage of the interval of possible equity premia, consider the lowest and highest mean log returns per month. In the limit as \( k \to \infty \),

\[
\begin{align*}
\lim_{k \to \infty} (\theta_k(\bar{r}_t) - \bar{\lambda}), \lim_{k \to \infty} (\hat{\theta}_k(\bar{r}_t) + \lambda) &= [\bar{r}_t - \bar{\lambda} - t^{\frac{1}{2}} \sigma b, \bar{r}_t + \bar{\lambda} + t^{\frac{1}{2}} \sigma b].
\end{align*}
\]

### 3 Optimal portfolio weights

It is helpful to begin with the maximization step in (1), given some arbitrary probability \( 0 < p < 1 \) for the high state. The optimal weight on stocks is

\[
\begin{align*}
w(p) &= e^{r_f/k} p \left( e^{\sigma/\sqrt{k}} - e^{r_f/k} \right) + (1 - p) \left( e^{-\sigma/\sqrt{k}} - e^{r_f/k} \right) \\
&= \frac{e^{r_f/k} e^{-\sigma/\sqrt{k}} - e^{r_f/k} e^{-\sigma/\sqrt{k}}}{e^{r_f/k} - e^{-\sigma/\sqrt{k}}}
\end{align*}
\]

Taylor expansions of the exponential terms lead to

\[
\begin{align*}
w(p) &= \left( 1 + \frac{r_f}{k} + O \left( \frac{1}{k^2} \right) \right) \times \\
&\quad \frac{p \left( \sigma/\sqrt{k} + \frac{1}{2} \frac{\sigma^2}{k} - \frac{r_f}{k} + O \left( \frac{1}{k^2} \right) \right) + (1 - p) \left( -\frac{\sigma}{\sqrt{k}} + \frac{1}{2} \frac{\sigma^2}{k} - \frac{r_f}{k} + O \left( \frac{1}{k^2} \right) \right)}{\left( \sigma/\sqrt{k} + \frac{1}{2} \frac{\sigma^2}{k} - \frac{r_f}{k} + O \left( \frac{1}{k^2} \right) \right) \left( \frac{r_f}{k} + \frac{\sigma}{\sqrt{k}} - \frac{1}{2} \frac{\sigma^2}{k} + O \left( \frac{1}{k^2} \right) \right)}
\end{align*}
\]

\[
\begin{align*}
&= \frac{(2p - 1) \sigma/\sqrt{k} + \frac{1}{2} \frac{\sigma^2}{k} - r_f \frac{r_f}{k} + O \left( k^{-\frac{3}{2}} \right)}{\frac{\sigma^2}{k} + O \left( k^{-\frac{3}{2}} \right)}.
\end{align*}
\]

We are now ready to derive optimal portfolio weights. Begin with the Bayesian case:

**Proposition S3.** The optimal Bayesian portfolio weight for horizon \( T > 0 \) is

\[
\lim_{k \to \infty} \omega_{t,k,T}^* (\bar{r}_t) = \frac{\bar{r}_t + \frac{1}{2} \frac{\sigma^2}{k} - r_f \frac{r_f}{k}}{\sigma^2} =: \omega_t^{bay}.
\]

**Proof.** Consider the optimal portfolio weight for the Bayesian investor with horizon \( T > 0 \). There is no minimization step in (1); as a result, the objective function on the
left hand side is the sum of (i) the objective for a myopic investor and (ii) continuation utility, which depends on the investment horizon $T$, but is independent of the optimal portfolio weight. The optimal weight is therefore independent of the horizon $T$. The optimal weight is now obtained by evaluating $w(p)$ at $p = \phi_k (\bar{r}_t)$:

$$w^*_{t,k,T}(\bar{r}_t) = w(\phi_k (\bar{r}_t))$$

$$= \frac{(2\phi_k (\bar{r}_t) - 1) \sigma / \sqrt{k} + \frac{1}{2} \sigma^2 / k - r^f / k + O \left( k^{-\frac{3}{2}} \right)}{\sigma^2 / k + O \left( k^{-\frac{3}{2}} \right)}$$

$$= \frac{\bar{r}_t / k + \frac{1}{2} \sigma^2 / k - r^f / k + O \left( k^{-\frac{3}{2}} \right)}{\sigma^2 / k + O \left( k^{-\frac{3}{2}} \right)},$$

where the third equality uses the definition of $\phi_k$ (recall (2)). Taking the limit as $k \to \infty$ delivers the Bayesian solution.

Consider next the problem with a nondegenerate set of one-step-ahead conditionals, but assume $T = 1/k$. In (12) above, the belief set $P_t,k (\bar{r}_t)$ was defined in terms of an interval $[\underline{p}_k (\bar{r}_t), \bar{p}_k (\bar{r}_t)]$ for the probability of the high state, with bounds strictly increasing in $\bar{r}_t$.

**Proposition S4.** The optimal portfolio weight of the myopic ambiguity averse investor ($T = 1/k$) is

$$\lim_{k \to \infty} \omega^*_{t,k,1/k}(\bar{r}_t) = \sigma^{-2} \max \left\{ \bar{r}_t + \frac{1}{2} \sigma^2 - r^f - \left( \bar{\lambda} + t^{-\frac{1}{2}} \sigma b_\alpha \right), 0 \right\}$$

$$+ \sigma^{-2} \min \left\{ \bar{r}_t + \frac{1}{2} \sigma^2 - r^f + \bar{\lambda} + t^{-\frac{1}{2}} \sigma b_\alpha, 0 \right\}$$

$$= \max \left\{ \omega^*_{t} - \sigma^{-2} \left( \bar{\lambda} + t^{-\frac{1}{2}} \sigma b_\alpha \right), 0 \right\}$$

$$+ \min \left\{ \omega^*_{t} + \sigma^{-2} \left( \bar{\lambda} + t^{-\frac{1}{2}} \sigma b_\alpha \right), 0 \right\}.$$ 

**Proof.** For given $p$, the optimal weight is $w(p)$. To solve the minimization step, substituting $w(p)$ back into the objective (1). We now need to find $p \in [\underline{p}_k (\bar{r}_t), \bar{p}_k (\bar{r}_t)]$ to minimize

$$g(p) = p \log \left( e^{r^f / k} + w(p) \left( e^{\sigma / \sqrt{k}} - e^{r^f / k} \right) \right)$$

$$+ (1 - p) \log \left( e^{r^f / k} + w(p) \left( e^{-\sigma / \sqrt{k}} - e^{r^f / k} \right) \right)$$

$$= r^f / k + p \log p + (1 - p) \log (1 - p) + \log \left( e^{\sigma / \sqrt{k}} - e^{-\sigma / \sqrt{k}} \right)$$

$$- p \log \left( e^{r^f / k} - e^{-\sigma / \sqrt{k}} \right) - (1 - p) \log \left( e^{r^f / k} - e^{r^f / k} \right).$$
The function \( g \) is strictly convex on \((0, 1)\) and achieves a minimum at

\[
\hat{p}_k = \frac{e^{r^t/k} - e^{-\sigma/\sqrt{k}}}{e^{\sigma/\sqrt{k}} - e^{-\sigma/\sqrt{k}}}. \tag{16}
\]

The minimizer \( \hat{p}_k \) is in the unit interval because we have assumed that \( r^t < \sigma \). It is precisely the probability at which the one-step-ahead conditional equity premium \( \mathbb{E} [R(s_{t+1})] - R^t \) is equal to zero. The conditional premium appears also as the bracketed term in the numerator of (13), so that \( w(\hat{p}_k) = 0 \). It follows that the solution to the minimization step is

\[
p_k^*(\bar{r}_t) = \begin{cases} 
\bar{p}_k(\bar{r}_t) & \text{if } \hat{p}_k > \bar{p}_k(\bar{r}_t) \\
\bar{p}_k & \text{if } \hat{p}_k \in [\bar{p}_k(\bar{r}_t), \hat{p}_k(\bar{r}_t)] \\
\bar{p}_k(\bar{r}_t) & \text{if } \hat{p}_k > \hat{p}_k(\bar{r}_t).
\end{cases}
\]

Substituting into (13) and using \( w(\hat{p}_k) = 0 \), we can express the optimal portfolio weight as a function of the sample mean:

\[
\omega_{t,k,1/k}^*(\bar{r}_t) = \begin{cases} 
\frac{w(\bar{p}_k(\bar{r}_t))}{\hat{p}_k(\bar{r}_t)} & \text{if } \hat{p}_k > \bar{p}_k(\bar{r}_t) \\
0 & \text{if } \hat{p}_k \in [\bar{p}_k(\bar{r}_t), \hat{p}_k(\bar{r}_t)] \\
\frac{w(\bar{p}_k(\bar{r}_t))}{\hat{p}_k(\bar{r}_t)} & \text{if } \hat{p}_k < \bar{p}_k(\bar{r}_t).
\end{cases}
\]

We now compute \( \lim_{k \to \infty} \omega_{t,k,1/k}^*(\bar{r}) \). Since the functions \( p_k \) and \( \bar{p}_k \) are strictly increasing, the nonparticipation region of the state space can be represented by an interval of sample means \([\bar{r}_{lo}(k), \bar{r}_{up}(k)]\). If \( \bar{r}_t > \bar{r}_{up}(k) \) the evidence about the equity premium is so positive that investment in stocks is positive even under the lowest probability for the high state. The upper bound \( \bar{r}_{up}(k) \) is the unique solution to \( p_k(\bar{r}_t) = \hat{p}_k \) or, equivalently,

\[
\left(2p_k(\bar{r}_t) - 1\right) \sigma/\sqrt{k} = (2\hat{p}_k - 1) \sigma/\sqrt{k} \tag{17}
\]

Use (12) and (11) above to rewrite the left hand side as:

\[
\left(2p_k(\bar{r}_t) - 1\right) \sigma/\sqrt{k} = \frac{1}{k} \left(\bar{r}_t - \bar{\lambda} - \sqrt{\sigma^2 - 2\log\alpha} + O\left(k^{-\frac{1}{2}}\right)\right) = \frac{1}{k} \left(\bar{r}_t - \sigma b_\alpha/\sqrt{t} - \bar{\lambda}\right) + O\left(k^{-\frac{1}{2}}\right). \tag{18}
\]

Substitute (18) and (16) into (17) to obtain

\[
\frac{1}{k} \left(\bar{r}_t - \sigma b_\alpha/\sqrt{t} - \bar{\lambda}\right) + O\left(k^{-\frac{1}{2}}\right) = \frac{2e^{r^t/k} - e^{\sigma/\sqrt{k}} - e^{-\sigma/\sqrt{k}}}{e^{\sigma/\sqrt{k}} - e^{-\sigma/\sqrt{k}}} \frac{\sigma}{\sqrt{k}}.
\]

Taylor expansions of the exponential terms on the right side around 1 lead to
\[
\frac{1}{k} \left( \bar{r}_t - \sigma b_\alpha / \sqrt{t} - \bar{\lambda} \right) + O \left( k^{-\frac{3}{4}} \right) = \frac{2r^f/k - \sigma^2/k \sigma}{2\sigma/\sqrt{k}} \sqrt{k},
\]
so that the upper bound of the nonparticipation region can be written as
\[
\bar{r}_{up} (k) = \sigma b_\alpha / \sqrt{t} + \bar{\lambda} + r^f - \frac{1}{2} \sigma^2 + O \left( k^{-\frac{1}{4}} \right).
\]
(19)

The lower bound \( \bar{r}_{lo} (k) \) is the unique solution to \( \bar{p}_k (\bar{r}_t) = \hat{p}_k \). By an argument similar to the one above, it can be written as
\[
\bar{r}_{lo} (k) = -\sigma b_\alpha / \sqrt{t} - \bar{\lambda} + r^f - \frac{1}{2} \sigma^2 + O \left( k^{-\frac{1}{4}} \right).
\]
(20)

Consider next the optimal weight in the case \( \bar{p}_k (\bar{r}_t) > \hat{p}_k \). Evaluating \( w (p) \) from (15) at \( p = \bar{p}_k (\bar{r}_t) \) and replacing the first term in the numerator using (18) yields
\[
w (\bar{p}_k (\bar{r}_t)) = \frac{\bar{r}_t - \sigma b_\alpha / \sqrt{t} - \bar{\lambda} + \frac{1}{2} \sigma^2 - r^f + O \left( k^{-\frac{1}{4}} \right)}{\sigma^2 + O \left( k^{-\frac{1}{4}} \right)} = \frac{\bar{r}_t - \bar{r}_{up} (k) + O \left( k^{-\frac{1}{4}} \right)}{\sigma^2 + O \left( k^{-\frac{1}{4}} \right)}.
\]
(21)

The case \( \hat{p}_k > \bar{p}_k (\bar{r}_t) \) is analogous, but with \( \bar{r}_{up} (k) \) replaced by \( \bar{r}_{lo} (k) \).

Combine the bounds (19)-(20) and the formulas for the optimal weight to deduce that the sequence of functions \( \left\{ \omega^*_{t, k, 1/k} (\bar{r}_t) \right\} \) converges pointwise to
\[
\lim_{k \to \infty} \omega^*_{t, k, 1/k} (\bar{r}_t) = \begin{cases} 
\frac{\bar{r}_t + \sigma b_\alpha / \sqrt{t} + \frac{1}{2} \sigma^2 - r}{\sigma^2} & \text{if } \bar{r}_t < r^f - \frac{1}{2} \sigma^2 - \sigma b_\alpha / \sqrt{t} - \bar{\lambda} \\
0 & \text{if } |\bar{r}_t + \frac{1}{2} \sigma^2 - r^f| < \sigma b_\alpha / \sqrt{t} + \bar{\lambda} \\
\frac{\bar{r}_t - \sigma b_\alpha / \sqrt{t} - \bar{\lambda} + \frac{1}{2} \sigma^2 - r}{\sigma^2} & \text{if } \bar{r}_t > r^f - \frac{1}{2} \sigma^2 + \sigma b_\alpha / \sqrt{t} + \bar{\lambda}.
\end{cases}
\]