A Central Limit Theorem, Loss Aversion and Multi-Armed Bandits*

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Abstract

This paper establishes a central limit theorem under the assumption that conditional variances can vary in a largely unstructured history-dependent way across experiments subject only to the restriction that they lie in a fixed interval. Limits take a novel and tractable form, and are expressed in terms of oscillating Brownian motion. A second contribution is application of this result to a class of multi-armed bandit problems where the decision-maker is loss averse.

Keywords: Central limit theorem, oscillating Brownian motion, multi-armed bandit, loss aversion, sequential sampling, rectangular sets of measures, robustness

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1 Introduction

1.0.1 Outline

The martingale version of the central limit theorem (CLT) considers a sequence \((X_i)\) of random variables having zero conditional mean and constant conditional variance \(\sigma^2\), and shows that (under suitable additional conditions) the distribution of \(\sum_{i=1}^n X_i / \sqrt{n}\) converges to the normal \(N(0, \sigma^2)\) as \(n \to \infty\). (The classical result for identically and independently distributed random variables is an immediate special case). This paper establishes a CLT under the relaxed assumption on variance according to which conditional variances can vary in a largely unstructured history-dependent way across experiments subject only to the restriction that they lie in a fixed interval \([\sigma^2, \bar{\sigma}^2]\), in which case limits take a novel and tractable form. This CLT is the main technical contribution of the paper. One well-known motivation for generalizing from a single probability distribution (hence single variance) to a set of probability distributions (hence set of variances) is robustness to model uncertainty or ambiguity. Our second contribution is to highlight an alternative application of our CLT in which model uncertainty plays no role, specifically to a class of multi-armed bandit (MAB) problems.

The decision-maker in our model maximizes expected utility relative to a single probability measure, but, unlike in the rest of the MAB literature, she is loss averse. Our CLT is used to derive large horizon approximations to the value function and corresponding strategies for particular specifications of the MAB problem.

The following special case illustrates our CLT. Let \((\Pi_1^n \Omega_i, \{\mathcal{G}_n\}_{n=1}^\infty)\) be a filtered space modeling a sequence of experiments. The set of possible outcomes for the \(i\)th experiment is \(\Omega_i\). For each \(n\), \(\mathcal{G}_n\) is a \(\sigma\)-algebra on \(\Pi_1^n \Omega_i\) representing the observable events regarding experiments \(1, \ldots, n\), and \(\mathcal{G} = \sigma(\bigcup_{n=1}^\infty \mathcal{G}_n)\). Let \(\mathcal{P}\) be a set of probability measures on \((\Pi_1^n \Omega_i, \mathcal{G})\), where all measures in \(\mathcal{P}\) are equivalent on each \(\mathcal{G}_n\). For each \(i \geq 1\), \(X_i : \Pi_1^n \Omega_j \to \mathbb{R}\) is \(\mathcal{G}_i\)-measurable. Assume that conditional means of the \(X_i\)s satisfy

\[
E_Q[X_i|\mathcal{G}_{i-1}] = 0 \text{ for all } Q \in \mathcal{P} \text{ and all } i \geq 1, \tag{1.1}
\]

and that their conditional variances satisfy

\[
\text{ess sup}_{Q \in \mathcal{P}} E_Q[X_i^2|\mathcal{G}_{i-1}] = \sigma^2 \text{ and ess inf}_{Q \in \mathcal{P}} E_Q[X_i^2|\mathcal{G}_{i-1}] = \sigma^2 \text{ for all } i \geq 1. \tag{1.2}
\]

Let \(\varphi : \mathbb{R} \to \mathbb{R}\) be given by

\[
\varphi(x) = \begin{cases} 
1 - \exp(-(x - c)) & x \geq c \\
\theta^{-1}(\exp(\theta(x - c)) - 1) & x < c
\end{cases} \tag{1.3}
\]

1 See Bergemann and Valimaki (2008) for references to a range of applications of the bandit framework in economics.
where $c \in \mathbb{R}$ and $\theta = \sigma / \sigma$. Then, under suitable additional assumptions, we obtain the following:

$$
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} E_Q \left[ \varphi \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] = E_{P^*}[\varphi(W^c_1)], \quad (1.4)
$$

Here the right side denotes the expectation of ($\varphi$ composed with) the time 1 value of an oscillating Brownian motion (Keilson and Wellner 1978; Lejay and Pigato 2018), defined as follows: Given a standard Brownian motion $(B_t)$ on a probability space $(\Omega^*, \mathcal{F}^*, P^*)$, with the induced natural filtration $(\mathcal{F}_t)_{t \geq 0}$, then $(W_t)$ denotes the unique strong solution of the stochastic differential equation (SDE)

$$
Y_t = \int_0^t \sigma(Y_s) dB_s, \quad t \geq 0,
$$

where

$$
\sigma(y) = \begin{cases} 
\sigma & \text{if } y \geq c \\
\sigma / \sigma & \text{if } y < c.
\end{cases}
$$

If $\sigma = \sigma = \sigma$, then the oscillating Brownian motion reduces to a Brownian motion and the right side of (1.4) reduces to the expected value of $\varphi$ under $\mathcal{N}(0, \sigma^2)$, as in the classical (martingale) CLT. More generally, the limit admits the following closed-form expression (see Appendix A.4):

$$
E_{P^*}[\varphi(W^c_1)] = \begin{cases} 
\Phi(-\frac{c}{\sigma}) - \Phi(\frac{c}{\sigma}) + e^{\frac{c^2}{2\sigma}} \left( e^{-c} \Phi(-\frac{c}{\sigma} + \frac{\sigma}{2}) - e^{c} \Phi(-\frac{c}{\sigma} - \frac{\sigma}{2}) \right) & c \leq 0 \\
\frac{\sigma}{\sigma} \left[ \Phi(-\frac{c}{\sigma}) - \Phi(\frac{c}{\sigma}) + e^{\frac{c^2}{2\sigma}} \left( e^{-\frac{c^2}{2\sigma}} \Phi(-\frac{c}{\sigma} + \frac{\sigma}{2}) - e^{\frac{c^2}{2\sigma}} \Phi(-\frac{c}{\sigma} - \frac{\sigma}{2}) \right) \right] & c > 0
\end{cases}, \quad (1.5)
$$

where $\Phi$ is the standard normal cdf. The closed-form for the limit in (1.4) is noteworthy given the complexity of the left side and the fact that the same limit applies to a large class of sets $\mathcal{P}$. Though the availability of a closed-form for the limit is special, the relevance of an oscillating Brownian motion is valid more generally, for many functions $\varphi$ and for a large class of sets $\mathcal{P}$, thus establishing a form of universality for this class of processes (Theorem 3.1).

The key assumption underlying our theorem, in addition to (1.1) and (1.2), is that the set $\mathcal{P}$ is "rectangular", which means that it is closed with respect to the pasting of alien marginals and conditionals. Rectangularity was introduced in Epstein and Schneider (2003) in the context of recursive utility theory, where an axiomatic analysis demonstrated its role in modeling dynamic behavior for an ambiguity-averse decision-maker. Here we demonstrate its relevance also to central limit theorems. Rectangularity endows $\mathcal{P}$ with a recursive structure that yields

\footnote{It has been studied and applied also in robust stochastic dynamic optimization (Iyengar, 2005; Shapiro, 2016), in the literature on dynamic risk measures (Riedel, 2004; Cheridito, Delbaen and Kupper, 2006; Acciaio and Penner, 2011), and in continuous-time modeling in finance (Chen and Epstein, 2002).}
a form of the law of iterated expectations; in particular, it is vacuously satisfied when $P$ is a singleton and conditionals are derived by applying Bayes’ rule. (Section 2 provides a precise definition of rectangularity and further interpretation.)

Above we indicated two possible interpretations of the (nonsingleton) set $P$. The first, whereby the multiplicity of measures reflects ambiguity and hence the lack of confidence in any single probability law, is well-known (in decision theory, Gilboa and Schmeidler (1989) is a seminal article). Accordingly, we focus below on the second interpretation, which is original to this paper. It assumes a Bayesian decision-maker, perfectly confident in her understanding of the environment, who must choose between lotteries (probability distributions over outcomes). The clearest example is a multi-armed bandit problem, where each arm delivers a random reward (a lottery) and the choice of which arm to pull can be identified as the choice of a lottery. Such an identification extends to a situation where arms are chosen sequentially and where there is learning about unknown parameters. The bottom line is that the set $P$ can be identified with the feasible set of strategies regarding which arm to choose at each history. Consequently, for each horizon $n$, we interpret the supremum in (1.4) as modeling the choice of an optimal strategy and the limiting result as a large-horizon approximation to the maximum attainable expected utility. It merits emphasis that the latter depends only on the variance bounds $\sigma^2$ and $\sigma^2$ but not on other features of $P$ (or on the very specific functional form in (1.3)).

The form of the utility or payoff function in (1.4) is noteworthy and constitutes another novel feature of the analysis. The specification does not embody a role for ambiguity, but deviates from the standard specification of global risk aversion. Rather, we assume loss aversion, which was introduced via cumulative prospect theory by Tversky and Kahneman (1992), and has since been well-established empirically and widely applied in economics and finance (see for example, Kahneman and Tversky 2000, Kobberling and Wakker 2005, Barberis 2013, and the references therein). Its essential elements are (i) a reference point; (ii) utility depends only on gains and losses relative to that reference point rather than on the total payoff (or total wealth); (iii) risk aversion (concavity) for gains and risk loving (convexity) for losses; and (iv) greater sensitivity to losses than to gains. In our sequential setting, gains/losses are incurred at each stage and we posit that utility depends on their $n^{1/2}$-weighted average where $n$ is the number of stages. This "explains" (1.3)-(1.4), where the reference point is $c$. To our knowledge, this is the first study of loss aversion in MAB problems.\(^3\)

\(^3\)Xu and Zhou (2013) and Ebert and Strack (2015) study optimal stopping problems assuming prospect theory. Their focus is on the probability-weighting aspect of prospect theory and loss aversion plays no role in their analyses. Two theoretical studies of loss aversion in a sequential context are Easley and Yang (2015) and Shi et al (2015). Guasoni et al (2020) study shortfall aversion, which shares the spirit of loss aversion but which is more directly relevant to preference
Other features of the model - payoff streams entering via their sums and the focus on large horizon approximations - are motivated partly by tractability. Our assumptions seem well-suited to contexts where many trials can be conducted within a short period of time, which would also support the absence of discounting. The reliance on the \( n^{1/2} \)-weighted average has the sensible implication that a given arithmetic average is more impactful if it applies over a longer horizon. Finally, we suggest that "tractability" is a concern not only for the modeler but also for the decision-maker within the model. In that light, view the decision-maker, who is trying to comprehend a much more complicated and intractable optimization problem, as using our simplifying assumptions to determine a "sensible" strategy. From this perspective, and given also loss aversion, some may view our MAB application as more descriptive than prescriptive.

We proceed as follows. The next section describes rectangularity formally. The CLT is presented in section 3 and section 4 describes the application to a multi-armed bandit problem. Related literature is discussed in section 5. Proofs of the CLT and related results are presented in Appendix A. Proofs for the bandit application are in Appendix B.

2 Rectangularity

Adopt the primitives \((\Pi_1^n \Omega_i, \{\mathcal{G}_n\}_{n=1}^\infty), \mathcal{G}, (X_i)\) and \(\mathcal{P}\) introduced in the introduction. They satisfy the assumptions noted above, namely that all measures in \(\mathcal{P}\) are equivalent on each \(\mathcal{G}_n\) and (1.1)-(1.2).

Denote by \(\mathcal{H}\) the set of all r.v. \(X\) on \((\Pi_1^n \Omega_i, \mathcal{G})\) satisfying \(\sup_{Q \in \mathcal{P}} E_Q[|X|] < \infty\). For any \(X\) in \(\mathcal{H}\), its upper and lower expectations are defined respectively by

\[
E[X] \equiv \sup_{Q \in \mathcal{P}} E_Q[X], \quad \mathcal{E}[X] \equiv \inf_{Q \in \mathcal{P}} E_Q[X] = -E[-X],
\]

and its conditional upper and lower expectations are defined respectively by

\[
E[X|\mathcal{G}_n] \equiv ess \sup_{Q \in \mathcal{P}} E_Q[X|\mathcal{G}_n], \quad \mathcal{E}[X|\mathcal{G}_n] \equiv ess \inf_{Q \in \mathcal{P}} E_Q[X|\mathcal{G}_n].
\]

Say that \((X_i)\) satisfies the Lindeberg condition if

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E \left[ |X_i|^2 I_{(|X_i| > \sqrt{n} \epsilon)} \right] = 0, \quad \forall \epsilon > 0. \tag{2.1}
\]

over deterministic consumption streams rather than over lotteries. In the MAB literature, the majority of papers assume risk neutrality.
To formulate the remaining assumption, rectangularity, requires some additional notation. Write \( \omega_{(n)} = (\omega_1, \ldots) \), \( \omega^{(n)} = (\omega_1, \ldots, \omega_n) \),

\[
\mathcal{P}_{0,n} = \{ P_{\mathcal{G}_n} : P \in \mathcal{P} \} \text{ and } \\
\mathcal{G}_{(n+1)} = \{ A \subset \Pi^\infty_{n+1} \Omega_i : \Pi^n_1 \Omega_i \times A \in \mathcal{G} \}.
\]

A probability kernel from \((\Pi^n_1 \Omega_i, \mathcal{G}_n)\) to \((\Pi^\infty_{n+1} \Omega_i, \mathcal{G}_{(n+1)})\) is a function \( \lambda : \Pi^n_1 \Omega_i \times \mathcal{G}_{(n+1)} \rightarrow [0, 1] \) satisfying:

**Kernel 1:** \( \forall \omega^{(n)} \in \Pi^n_1 \Omega_i, \lambda (\omega^{(n)}, \cdot) \) is a probability measure on \((\Pi^\infty_{n+1} \Omega_i, \mathcal{G}_{(n+1)})\),

**Kernel 2:** \( \forall A \in \mathcal{G}_{(n+1)}, \lambda (\cdot, A) \) is a \( \mathcal{G}_n \)-measurable function on \( \Pi^n_1 \Omega_i \).

Any pair \((p_n, \lambda)\) consisting of a probability measure \( p_n \) on \((\Pi^n_1 \Omega_i, \mathcal{G}_n)\) and a probability kernel \( \lambda \) as above, induces a unique probability measure \( P \) on \((\Pi^\infty_{n+1} \Omega_i, \mathcal{G})\) that coincides with \( p_n \) on \( \mathcal{G}_n \). It is given by, \( \forall A \in \mathcal{G} \),

\[
P(A) = \int_{\Pi^n_1 \Omega_i} \int_{\Pi^\infty_{n+1} \Omega_i} 1_A(\omega^{(n)}, \omega_{(n+1)}) \lambda (\omega^{(n)}, d\omega_{(n+1)}) p_n (d\omega^{(n)}). \tag{2.2}
\]

For \( Q \in \mathcal{P} \), let \( Q(\cdot \mid \mathcal{G}_n) \), denote its induced (regular) conditional. Then it defines a probability kernel \( \lambda \) by: \( \forall \omega^{(n)} \in \Pi^n_1 \Omega_i, \)

\[
\lambda (\omega^{(n)}, A) = Q(\Pi^n_1 \Omega_i \times A \mid \Pi^n_1 \mathcal{F}_i) (\omega^{(n)}), \forall A \in \mathcal{G}_{(n+1)}. \tag{2.3}
\]

A feature of such a kernel is that the single measure \( Q \) is used to define the conditional at every \( \omega^{(n)} \). We are interested in kernels for which the measure to be conditioned can vary with \( \omega^{(n)} \). Thus say that the probability kernel \( \lambda \) is a \( \mathcal{P} \)-kernel if: \( \forall \omega^{(n)} \in \Pi^n_1 \Omega_i \exists Q \in \mathcal{P} \) such that (2.3) is satisfied.

Finally, say that \( \mathcal{P} \) is rectangular (with respect to the filtration \( \{ \mathcal{G}_n \} \)) if: \( \forall n \forall p_n \in \mathcal{P}_{0,n} \) and for every \( \mathcal{P} \)-kernel \( \lambda \), if \( P \) is defined as in (2.2), then \( P \in \mathcal{P} \).

The significance of rectangularity is illuminated by the following lemma. (Its proof can be found in [Chen and Epstein (2020)].)

**Lemma 2.1.** \( \mathcal{P} \) rectangular implies the following (for any \( 0 \leq m \leq n \leq N \)).

(i) **Stability by composition:** For any \( Q, R \in \mathcal{P} \), \( \exists P \in \mathcal{P} \) such that, for any \( X \in \mathcal{H} \),

\[
E_P[X|\mathcal{G}_m] = E_Q[E_R[X|\mathcal{G}_n]|\mathcal{G}_m].
\]

(ii) **Stability by bifurcation:** For any \( Q, R \in \mathcal{P} \), and any \( A_n \in \mathcal{G}_n \), \( \exists P \in \mathcal{P} \) such that, for any \( X \in \mathcal{H} \),

\[
E_P[X|\mathcal{G}_n] = I_{A_n}E_Q[X|\mathcal{G}_n] + I_{A_n^c}E_R[X|\mathcal{G}_n].
\]
Law of iterated upper expectations: For any $X \in \mathcal{H}$,
\[
\mathbb{E}[\mathbb{E}[X|\mathcal{G}_n]|\mathcal{G}_m] = \mathbb{E}[X|\mathcal{G}_m]. \tag{2.4}
\]

Let $\{X_i\}$ be a sequence in $\mathcal{H}$. Then, for any continuous bounded functions $f, h$
\[
\mathbb{E}\left[f\left(\sum_{i}^{n-1} X_i\right) + h\left(\sum_{i}^{n-1} X_i\right) X_n^2\right] = \mathbb{E}\left[f\left(\sum_{i}^{n-1} X_i\right) + h\left(\sum_{i}^{n-1} X_i\right) X_n^2|\mathcal{G}_{n-1}\right].
\]
Particularly, if the upper and lower conditional variances of $X_n$ satisfy
\[
\mathbb{E}[X_n^2|\mathcal{G}_{n-1}] = \sigma^2, \mathbb{E}[X_n^2|\mathcal{G}_{n-1}] = \sigma^2,
\]
then
\[
\mathbb{E}\left[h\left(\sum_{i}^{n-1} X_i\right) X_n^2|\mathcal{G}_{n-1}\right] = \sigma^2\left[h\left(\sum_{i}^{n-1} X_i\right)\right]^+ - \sigma^2\left[h\left(\sum_{i}^{n-1} X_i\right)\right]^-. \tag{Superscripts $+$ and $-$ denote the positive and negative parts respectively.}
\]

Part (iii) gives the law of iterated expectations for upper expectations (a similar condition for lower expectations is implied). (iv) is an extension that is used in the proofs of our CLTs. Parts (i) and (ii) of the lemma describe direct implications of $\mathcal{P}$ being "closed with respect to the pasting of alien marginals and conditionals," which property we view as the semi-formal definition of rectangularity. The following example will be used in the bandit application below. (See Chen and Epstein (2020) and Epstein and Schneider (2003) for further discussion of rectangularity under the interpretation of $\mathcal{P}$ as modeling ambiguity.)

Example (Trinomial): This is a dynamic extension of the trinomial model in Levy et al (1994). For each $i = 1, 2, \ldots$, let $\Omega_i = \Omega = \{1, -1, 0\}$ and let $X_i$ be the co-ordinate r.v.,
\[
X_i(\omega) = X(\omega) = \omega_i, \quad \omega = (\omega_1, \ldots, \omega_i, \ldots) \in \prod_{i=1}^{\infty} \Omega_i = \Omega^\infty.
\]
Define the set $\mathcal{L}$ of probability measures on $\{1, -1, 0\}$
\[
\mathcal{L} = \left\{ \left( \frac{p}{2}, \frac{p}{2}, 1 - p \right) : p \in L \subset (0, 1) \right\}. \tag{2.5}
\]
Finally, let $\mathcal{P}$ consist of all probability measures on $\prod_{i=1}^{\infty} \Omega_i$ such that, for each $i$ and each history $\omega^{(i)} = (\omega_1, \ldots, \omega_i)$, the induced conditional probability measure on $\Omega_{i+1}$ lies in $\mathcal{L}$ (that is, $\mathcal{P}$ is the set of all measures whose one-step-ahead conditionals, at every history, lie in $\mathcal{L}$).

\footnote{See also Avellaneda et al (1995).}
Then conditions (1.1) and (1.2) are evident, with
\[ \sigma^2 = \inf_{p \in L} p \quad \text{and} \quad \bar{\sigma}^2 = \sup_{p \in \bar{L}} p. \]

(The Lindeberg condition (2.1) and equivalence on each \( G_n \) are also satisfied.) In addition, \( \mathcal{P} \) is rectangular (Chen and Epstein (2020, Lemma 3.1) prove a more general result). Here is a way to think about how rectangularity is built-in and what it means. While \( \mathcal{L} \) is the set of possible probability laws for each \( X_i \), independent of history, there remains the question of how experiments may differ. In spite of \( \mathcal{L} \) being common to all \( i \), experiments as modeled by \( \mathcal{P} \) are not necessarily identical. Indeed, any measure in \( \mathcal{L} \) is possible as the law describing the \( i^{th} \) experiment at history \( \omega^{(i-1)} \) in conjunction with any possibly different measure in \( \mathcal{L} \) being the law describing the \( j^{th} \) experiment at history \( \tilde{\omega}^{(j-1)} \). As a result, besides the restriction imposed by \( \mathcal{L} \), the set \( \mathcal{P} \) imposes no restrictions on the pattern of heterogeneity across experiments.\(^5\)

### 3 A Central Limit Theorem

We extend (a version of) the classical CLT to admit a set \([\sigma^2, \bar{\sigma}^2]\) of variances while maintaining the assumption of a fixed zero mean. Throughout \((B_t)\) denotes a standard Brownian motion under a probability space \((\Omega^*, \mathcal{F}^*, P^*)\) and \((\mathcal{F}_t)_{t \geq 0}\) is the natural filtration generated by \((B_t)\).

In the classical case, the limiting distribution is normal, which is the distribution of \( B_1 \). In the more general case, the corresponding (upper and lower) limits are not given by the normal distribution, but are described instead using the following generalization of Brownian motion. For any \( \alpha, \beta > 0 \), the oscillating Brownian motion with parameter \((\alpha, \beta)\) and threshold \( c \in \mathbb{R} \) is the unique strong solution \((W^{\alpha,\beta,c}_t)\) of the SDE (see Le Gall 1984)

\[ Y_t = \int_0^t \sigma (Y_s) dB_s, \quad t \geq 0, \quad (3.1) \]

where the diffusion coefficient \( \sigma \) is the positive two-valued function, discontinuous at the threshold \( c \),

\[ \sigma(y) = \alpha I_{[c,\infty)}(y) + \beta I_{(-\infty,c)}(y), \quad \forall y \in \mathbb{R}. \quad (3.2) \]

Two oscillating Brownian motions to be considered are \((W^{\alpha,\beta,c}_t)\) and \((\tilde{W}^{\alpha,\beta,c}_t)\), which we denote more simply by \((W^c_t)\) and \((\tilde{W}^c_t)\) respectively.

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\(^5\)In contrast, the set \( \mathcal{P}^{\text{prod}} \) consisting of all (nonidentical) product measures that can be constructed from \( \mathcal{L} \) is not rectangular. It violates (2.4) as well.
Theorem 3.1. Let the sequence $(X_i)$ be such that $X_i \in \mathcal{H}$ for each $i$, and where $(X_i)$ satisfies (1.1) and (1.2), with conditional upper and lower variances $\bar{\sigma}^2 \geq \underline{\sigma}^2 > 0$. Assume also the Lindeberg condition (2.1), that measures in $\mathcal{P}$ are equivalent on each $\mathcal{G}_i$, and that $\mathcal{P}$ is rectangular. Set $\theta = \bar{\sigma}/\underline{\sigma}$. For any $c \in \mathbb{R}$ and $\varphi_1 \in C_0^3(\mathbb{R})$, with $\varphi_1(0) = 0$, define functions

$$
\varphi(x) = \begin{cases}
\varphi_1(x - c) & x \geq c \\
-\frac{1}{\bar{\sigma}} \varphi_1(-\theta(x - c)) & x < c
\end{cases}
$$

(3.3)

$$
\overline{\varphi}(x) = \begin{cases}
\varphi_1(x - c) & x \geq c \\
-\theta \varphi_1(-\frac{1}{\bar{\sigma}}(x - c)) & x < c
\end{cases}
$$

(3.4)

(1) If $\varphi_1''(x) \leq 0$ for $x \geq 0$, then

$$
\lim_{n \to \infty} \mathbb{E} \left[ \varphi \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] = \mathbb{E}_{\mathcal{P}}[\varphi(W_1)],
$$

(3.5)

$$
\lim_{n \to \infty} \mathbb{E} \left[ \overline{\varphi} \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] = \mathbb{E}_{\mathcal{P}}[\overline{\varphi}(W_1)].
$$

(3.6)

(2) If $\varphi_1''(x) \geq 0$ for $x \geq 0$, then

$$
\lim_{n \to \infty} \mathbb{E} \left[ \overline{\varphi} \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] = \mathbb{E}_{\mathcal{P}}[\overline{\varphi}(W_1)],
$$

(3.7)

$$
\lim_{n \to \infty} \mathbb{E} \left[ \varphi \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] = \mathbb{E}_{\mathcal{P}}[\varphi(W_1)].
$$

(3.8)

Note that part (2) follows immediately from (1) by applying (1) to $-\varphi_1$.

Remark 3.2. The condition $\varphi_1(0) = 0$ is a simplifying normalization. If it is violated, then the asserted limits follow by applying the theorem to the translated function $\varphi_1 - \varphi_1(0)$. If conditional means do not satisfy (1.1), then the arguments in the proof can be applied to $S^Q_n$, $S^Q_n \equiv \sum_{i=1}^{n} (X_i - E_Q[X_i|\mathcal{G}_{i-1}])$ for each $Q \in \mathcal{P}$, to show that

$$
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} E_Q \left[ \varphi \left( \frac{S^Q_n}{\sqrt{n}} \right) \right] = \mathbb{E}_{\mathcal{P}}[\varphi(W_1)].
$$

Results (3.6)-(3.8) can be rewritten similarly.

As noted previously, conditions (1.1) and (1.2) reduce to the assumptions used in the standard form of the classical martingale CLT, and rectangularity is satisfied vacuously in the classical context. The third main assumption, or restriction, is
the form of functions $\varphi$ and $\varphi'$. In the classical case, because of the (countable) additivity of probability measures, limit results for the expected value of indicator functions for intervals imply those for all (suitably integrable) integrands. Here we can also derive limiting results for more general integrands, but their expressions are more complex and hence less transparent and tractable, and consequently are excluded. Moreover, the stated theorem has at least four strong points. First, in (3.5), for instance, the limit is defined by the distribution of $W_1^c$, for which the pdf can be described in closed-form (see Theorem 1 in Keilson and Wellner (1978), and also Appendix A.4 below). When $c = 0$, it is given by the particularly simple expression

$$q(y) = \begin{cases} q^*(y; \sigma) \left[ \frac{2\sigma}{\sigma + \sigma^2} \right] & y \geq 0 \\ q^*(y; \sigma) \left[ \frac{2\sigma}{\sigma + \sigma^2} \right] & y < 0 \end{cases}$$

(3.9)

where $q^*(y; \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left(-\frac{(y/\sigma)^2}{2}\right)$ is the pdf for $N(0, \sigma^2)$. Availability of the pdf enhances tractability, for example, it leads to the closed-form (1.5) in the introduction.

Second, the theorem yields limit results also for indicators corresponding to all one-sided intervals (Corollary 3.4), and, moreover, these limits have closed-forms. Third, though the theorem restricts the integrand relative to what is familiar from the classical context, it relaxes the classical assumption of a fixed variance to permit a set of variances, accommodating thereby a relatively unstructured form of heteroscedasticity.

Finally, section 4 shows that part (1), specifically (3.5), is useful for analysing (some) bandit problems where the decision-maker is loss averse, in particular where her utility index is concave for "gains" and convex for "losses". This pattern for risk attitudes is at the heart of the result (3.5); in particular, it is exhibited by the function $\varphi$ if gains/losses are defined relative to the reference point $c$. For perspective, if instead of defining $\varphi$ by (3.3), we took $\varphi$ to be any (suitably bounded, smooth and) globally concave function, then the limit in (3.5) would equal the expected value of $\varphi$ under $N(0, \sigma^2)$, as in the classical case with fixed variance $\sigma^2$ (Peng (2019), Proposition 2.2.15). In other words, oscillating Brownian motion with fluctuating volatility would be replaced by a Brownian motion with constant volatility.

Remark 3.3. It is noteworthy that the limit, in (3.5), for example, is valid for many different $\varphi_1$ and $\varphi'$. In particular, given rectangularity and the other nonparametric assumptions stated for $\varphi$, only the implied variance bounds matter for

$^6$Note, however, that $\varphi_1$, which is the building block for both $\varphi$ and $\varphi'$, is restricted only by nonparametric smoothness and curvature conditions.
the limit (paralleling the familiar robustness of the classical CLT). For example, neither other moments of the \(X_i\)s nor the presence or absence of measures yielding variances strictly between \(\sigma^2\) and \(\sigma^2\) matter for the limit. In fact, the following stronger statement can be proven: For all functions \(\varphi \in C([-\infty, \infty])\), where the functional form in (3.3) is NOT assumed, \(\lim_{n \to \infty} \mathbb{E}[\varphi(\Sigma_{i=1}^n X_i/\sqrt{n})]\) exists and it depends on the set \(\mathcal{P}\) only through the implied variance bounds \(\sigma^2\) and \(\sigma^2\). (Details are beyond the scope of this paper.)

The example in the introduction is the special case of (3.5) where \(\varphi_1(x) = 1 - e^{-x}\) in (3.3). As another example, let \(\varphi_1(x) = x^2\) in (3.4), which yields

\[
\varphi(x) = \begin{cases} 
(x - c)^2 & x \geq c \\
-\frac{1}{\beta}(x - c)^2 & x < c.
\end{cases}
\]

Then, part (2) of the theorem applies and the limit in (3.7) equals

\[
E_{P^*}[\varphi(W_1^d)] = \begin{cases} 
-\frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2\sigma^2}} \left( 2c + \frac{\pi^2 + c^2}{\sigma^2} e^{\frac{c^2}{2\sigma^2}} \sqrt{2\pi} \text{erf}\left(\frac{c}{\sqrt{2\sigma}}\right) \right) & c \leq 0 \\
-\frac{1}{\sqrt{2\pi}} e^{-\frac{c^2}{2\sigma^2}} \left( 2c + \frac{\pi^2 + c^2}{\sigma^2} e^{\frac{c^2}{2\sigma^2}} \sqrt{2\pi} \text{erf}\left(\frac{c}{\sqrt{2\sigma}}\right) \right) & c > 0
\end{cases}
\] (3.10)

where \(\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy\) is the Gauss error function (see Appendix A.4).

Indicator functions for the one-sided intervals \((-\infty, d]\) and \([d, \infty)\) provide another example where limits can be expressed in closed-form. Such indicators can be approximated by the functions defined in Theorem 3.1, which leads to the following corollary (see Appendix A).

**Corollary 3.4.** Adopt the assumptions in Theorem 3.1. Then, for any \(d \in \mathbb{R}\), we have

\[
\begin{align*}
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( \frac{\Sigma_{i=1}^n X_i}{\sqrt{n}} \geq d \right) &= P^* \left( W_1^d \geq d \right) \\
\lim_{n \to \infty} \inf_{Q \in \mathcal{P}} Q \left( \frac{\Sigma_{i=1}^n X_i}{\sqrt{n}} \geq d \right) &= P^* \left( W_1^d \geq d \right)
\end{align*}
\] (3.11)

and

\[
\begin{align*}
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( \frac{\Sigma_{i=1}^n X_i}{\sqrt{n}} \leq d \right) &= P^* \left( W_1^d \leq d \right) \\
\lim_{n \to \infty} \inf_{Q \in \mathcal{P}} Q \left( \frac{\Sigma_{i=1}^n X_i}{\sqrt{n}} \leq d \right) &= P^* \left( W_1^d \leq d \right)
\end{align*}
\] (3.12)

Furthermore, the right sides of (3.11) and (3.12) are given respectively by

\[
P^* \left( W_1^d \geq d \right) = \frac{2\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right) I_{\{d > 0\}} + \left( 1 - \frac{2\sigma}{\sigma + \sigma} \Phi \left( \frac{d}{\sigma} \right) \right) I_{\{d \leq 0\}}
\]

and

\[
P^* \left( W_1^d \leq d \right) = \left( 1 - \frac{2\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right) \right) I_{\{d > 0\}} + \frac{2\sigma}{\sigma + \sigma} \Phi \left( \frac{d}{\sigma} \right) I_{\{d \leq 0\}}
\]
To illustrate, let $d > 0$, and compute that
\[
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( \frac{\Sigma^n X_i}{\sqrt{n}} \geq d \right) = \frac{2\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right) > \Phi \left( -\frac{d}{\sigma} \right) = \max_{\sigma} \Phi \left( -\frac{d}{\sigma} \right).
\]

(3.13)

Adopt the robustness-to-ambiguity interpretation. In the classical approach, with variance $\sigma^2$ assumed constant, robustness is often addressed via "sensitivity analysis." Here that might take the form of considering all $\sigma \in [\sigma, \bar{\sigma}]$, and computing the maximum probability (according to $\mathbb{N}(0, \sigma^2)$) of the event $\{\Sigma^n X_i/\sqrt{n} \geq d\}$, which occurs at $\sigma = \sigma$. Thus (3.13) shows that our model attaches more weight to the right tail than does traditional sensitivity analysis. The intuition is clear. In sensitivity analysis, scenarios with different values for $\sigma$ are considered, but it is assumed that the variance is constant within each scenario. In contrast, we allow also for variance to vary in many ways across histories and experiments, some of which would make it more likely that $(\Sigma^n X_i/\sqrt{n}) \geq d$ for large $n$; for example, if the variance for $X_{n+1}$ is large (say equal to $\sigma^2$) when $(\Sigma^n X_i/\sqrt{n}) < d$ and small (say equal to $\sigma^2$) when $(\Sigma^n X_i/\sqrt{n}) > d$.

The same is true for left tails. In fact, for $d > 0$,
\[
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( \frac{\Sigma^n X_i}{\sqrt{n}} \leq -d \right) = \lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( \frac{\Sigma^n X_i}{\sqrt{n}} \geq d \right).
\]

It is noteworthy that this asymptotic symmetry is valid even though we have not assumed any symmetry properties for $\mathcal{P}$.

Finally, we note the following straightforward implication for two-sided intervals: For any $d > 0$, $\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} Q \left( -d \leq \frac{\Sigma^n X_i}{\sqrt{n}} \leq d \right) \in \mathcal{I}$, where
\[
\mathcal{I} = \left[ 1 - \frac{2\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right) - \frac{2\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right), 1 - \frac{4\sigma}{\sigma + \sigma} \Phi \left( -\frac{d}{\sigma} \right) \right].
\]

4 Multi-Armed Bandits

Our objective here is to demonstrate how Theorem 3.1, specifically (3.5), can be applied to a multi-armed bandit problem where the arms share a common mean return but differ in riskiness as measured by variances. The model can admit also alternative interpretations and thus we refer to actions rather than arms. Crucial is that beliefs vary with the strategy used to choose actions.
4.1 Strategy-dependent beliefs

Uncertainty is represented by the state space \((\Omega, \mathcal{G})\), where: \(\Omega = \prod_{i=1}^{n} \Omega_i\), \(\Omega_i \subseteq \mathbb{R}\) is finite, \(\mathcal{G}_i\) is induced by histories \(\omega^{(i)} = (\omega_1, \ldots, \omega_i)\), \((\mathcal{G}_0 = \{\Omega, \emptyset\})\), and \(\mathcal{G} = \sigma(\bigcup_{i} \mathcal{G}_i)\). Let \(A\) be a finite set of actions. The timing is as follows: For each \(i \geq 1\), \(\omega^{(i-1)}_{i-1}\) is observed, \((\omega_0 = \emptyset)\), \(a_i \in A\) is chosen and then the resulting outcome \(\omega_i\) is realized. A strategy is given by \(s = (s_i)^{\infty}_{i=1}\), where \(s_i : (\Omega, \mathcal{G}_{i-1}) \rightarrow A\), and \(s_i(\omega) \in \Gamma_i(\omega)\) for all \(\omega\). The corresponding set of strategies is \(S\).

Define \(X_i(\omega) = \omega_i\).

Action choice is determined by expected utility maximization. The remaining primitives - beliefs and the vNM utility index - are described next.

Beliefs are strategy-dependent. Fix a strategy \(s\). At each history \(\omega^{(i-1)}\), \(P^s_i = P^s_i(\cdot | \omega^{(i-1)}) \in \Delta(\Omega_i)\) gives the conditional belief about \(X_i\). The set \(\{P^s_i : i \geq 1, s \in S\}\) is a primitive that represents beliefs (which may be taken to be subjective or objective). The dependence of conditional beliefs on the strategy \(s\) arises for two reasons. First, beliefs about the next outcome depend on the chosen action \(a_i = s_i(\omega^{(i-1)}\). Second, past outcomes may have resulted from different actions, and the inference to be drawn from history requires that one apply the correct likelihood function corresponding to the actions \(a_j = s_j(\omega^{(j-1)}\), \(j \leq i\).

By the Ionescu-Tulcea extension theorem, the 1-step-ahead conditionals \(P^s_i\) can be pasted together to obtain a measure \(P^s\),

\[P^s \in \Delta(\prod_{i=1}^{n} \Omega_i, \mathcal{G})\]

such that its 1-step-ahead conditionals (obtained by Bayesian updating) are the \(P^s_i, i \geq 1\).

Remark 4.1. To clarify notation, \(P^s_i\) depends on \(s\) only via \((s_j)^{i}_{j=1}\) and \(s_1 \in A\) is a constant. Therefore, \(P^s_i \in \Delta(\Omega_i)\) depends on actions (rather than strategies) and \(\{P^s_a : a \in A\}\) can be thought of as representing prior beliefs.

The specification of the utility index is nonstandard (at least in models of sequential choice) because we assume loss aversion. In particular, we assume that outcomes for each action are evaluated according to whether they produce gains or losses relative to a reference point, which we take to be their common mean. More precisely, assume that, for every \(s \in S\),

\[E_{P^s_i} [X_i | \mathcal{G}_{i-1}] = m\]

for all \(i \geq 1\) and all \(s \in S\).

\(^7\)For any given \(n\), \(s \in S\) induces the contingent plan \((s_i)^{n}_{i=1}\), which is adequate if one is interested only in the \(n\)-horizon case. Because we will be interested in varying horizons, it is convenient to define a strategy to apply to all finite horizons.
Then we view $X_i - m$ as the gain/loss at stage $i$. With the change of notation whereby $X_i - m$ is denoted $X_i$, then $X_i$ describes the gain/loss at stage $i$ and,

$$E_{P_i^s} [X_i \mid G_{i-1}] = 0 \text{ for all } i \geq 1 \text{ and all } s \in S. \quad (4.1)$$

Since gains/losses are incurred at each stage, we posit that, for any horizon length $n$, utility depends on their $n^{1/2}$-weighted average. Therefore, a given arithmetic average gain/loss is more impactful if it applies over a longer horizon.

The utility index is $\varphi$ defined as in (3.3), with $\theta = \sigma / \tilde{\sigma}$, yielding

$$\varphi(x) = \begin{cases} \varphi_1(x - c) & x \geq c \\ -\frac{\sigma}{\tilde{\sigma}} \varphi_1 \left(-\frac{\sigma}{\tilde{\sigma}}(x - c)\right) & x < c \end{cases} \quad (4.2)$$

Assume that $\varphi_1$ is (strictly) increasing and (strictly) concave for $x > c$. Then, $\varphi$ is increasing globally, concave for $x > c$ (corresponding to gains) and convex for $x < c$ (corresponding to losses), implying risk aversion for gains and risk seeking for losses. In addition,

$$\varphi'(c - x) > \varphi'(c + x), \text{ for } x > 0, \quad (4.3)$$

modeling a greater sensitivity to losses than to gains. An implication is that

$$-\varphi(c - x) > \varphi(c + x), \text{ for } x > 0; \quad (4.4)$$

hence the lottery $(c - x; \frac{1}{2}; c + x; \frac{1}{2})$ is strictly inferior to $0$ for sure $(x > 0)$.

**Remark 4.2.** There are different definitions of "loss aversion" in the literature. Our use of the above properties of utility as defining the term is consistent with terminology in Kahneman and Tversky (1979) and Wakker and Tversky (1993), for example. Another view relates loss aversion to the local behavior of utility near $c$, and to nondifferentiability at $c$ (Kobberling and Wakker (2005), for example).

The function $\varphi$ defined in (4.2) is continuously differentiable and accordingly, kinks play no role in our analysis. Neither does probability weighting, which is often tied to loss aversion because of the latter’s origins in prospect theory but is ruled out here.

The preceding leads finally to the optimization problem (for each $n$)

$$V_n = \sup_{s \in S} E_{P_i^s} \varphi \left(\Sigma_1^n X_i / \sqrt{n}\right). \quad (4.5)$$

The finite horizon problem is not tractable (for us). However, by application of our CLT (Theorem 3.1) we can describe the limit of $V_n$ as $n \to \infty$, and (in special cases) also supporting strategies. Moreover, we do this even though $\varphi_1$ in (4.2) is restricted only by nonparametric monotonicity and concavity assumptions and the other primitive, $\{P_i^s : i \geq 1, s \in S\}$, is largely unrestricted. Moreover, the large horizon approximations that we derive are invariant to the many admissible changes in these primitives.
Remark 4.3. Before proceeding, we elaborate on why tractability for (4.5) is problematic. Adopt the bandit interpretation with actions corresponding to arms. A common method is to establish the optimality of index-based strategies, most commonly using the Gittins index (Gittins and Jones 1974). When arms can be valued separately, then at each stage and history an index summarizes each arm and comparison of these indices determines which arm to pull. This approach does not work in our model because arms cannot be delinked for at least two reasons: (i) outcomes from one arm may be informative about the distribution describing other arms because of common unknown parameters (see section 4.3); (ii) because of loss aversion risk attitude depends on the sign of the sum of past payoffs from all arms.

The following lemma provides the first step in justifying application of our CLT.

Lemma 4.4. Under the conditions stated above, there exists a rectangular set 
\( \mathcal{P} \subset \Delta (\Pi_1^\infty \Omega_1, \mathcal{G}) \) such that, for each \( n \),

\[
V_n = \sup_{s \in \mathcal{S}} E_{P_s} \left[ \varphi \left( \sum_1^n X_i / \sqrt{n} \right) \right] = \sup_{Q \in \mathcal{P}} E_{Q} \left[ \varphi \left( \sum_1^n X_i / \sqrt{n} \right) \right]. \tag{4.6}
\]

Proof: Define \( \mathcal{P} = \{ P^s : s \in \mathcal{S} \} \subset \Delta (\Pi_1^\infty \Omega_1, \mathcal{G}) \) to be the set of all measures constructed as above. Then \( \mathcal{P} \) is rectangular, that is, pasting alien marginals and conditionals leaves one within \( \mathcal{P} \). This stems from the fact that distinct strategies can be pasted together in the sense that: For all \( n \), \( A_n \in \mathcal{G}_n \), and \( s, s' \in \mathcal{S} \), then

\[
\widehat{s} = (s_1, ..., s_n, \hat{s}_{n+1}, \hat{s}_{n+2}, ...) \in \mathcal{S}, \text{ where} \]

\[
\widehat{s}_i (\omega^{(n)}, \cdot) = I_{A_n} (\omega^{(n)}) s_i (\omega^{(n)}, \cdot) + I_{A_n^c} (\omega^{(n)}) s'_i (\omega^{(n)}, \cdot), \forall i > n.
\]

The point is simply that any action can be chosen at any stage and at any history.

Turn to (4.6). It is well-known that, due to the Law of Iterated Expectations, an expected utility maximization problem such as (4.5) can be analysed by backward induction. It follows that, for each history, the choice between actions is equivalent to the choice between the 1-step-ahead conditionals they induce. In other words, strategies matter only through the 1-step-ahead conditionals they induce. But the sets of 1-step-ahead conditionals induced by \( \mathcal{S} \) and by \( \mathcal{P} \) are identical. This implies (4.6).

Remark 4.5. Reformulation of a sequential optimization problem in terms of a rectangular set of measures is not limited to bandit problems. The essential features leading to an equivalence such as (4.6) are that the optimization problem (here (4.5)) can be solved by backward induction, and that the choice of action at \( i \) does not affect what is feasible at any later stage. The result does not require the functional form restriction (4.2) for \( \varphi \), the assumptions on (1.1) and (1.2) on
means and variances, or aggregation of payoffs via the $\sqrt{n}$-weighted average (for example, it is valid also if the latter is replaced by the sum or by the arithmetic average of payoffs).

Our CLT requires more than rectangularity. To guarantee (1.1) and (1.2), assume (4.1) and

$$
\text{ess sup}_{s \in \mathcal{S}} E_{P^*_s}[X_i^2 | \mathcal{G}_{i-1}] = \sigma^2 \quad \text{and} \quad \text{ess inf}_{s \in \mathcal{S}} E_{P^*_s}[X_i^2 | \mathcal{G}_{i-1}] = \sigma^2, \; i \geq 1. \quad (4.7)
$$

The required equivalence of measures in $\mathcal{P}$ is satisfied if, for each $i$, $P^*_s$ has full support on $\Omega$ for every $s$. The Lindeberg condition (2.1) is immediate given that $\bar{\Omega}$ is finite.

**Corollary 4.6.** Under the conditions stated above, (in particular, (4.1), (4.7), full support, $\varphi$ defined by (4.2)),

$$
\lim_{n \to \infty} V_n \begin{cases} = \varphi(0) & c = 0 \\ > \varphi(0) & c > 0 \\ < \varphi(0) & c < 0 \end{cases} \quad (4.8)
$$

To interpret (4.8), consider first the case $c = 0$. Thus, for large $n$, maximum expected utility is approximately equal to that achievable when the payoff to each action is riskless, hence identically equal to the common mean denoted $m$ above, implying zero gains and losses for sure. In other words, risk is a matter of indifference in the limit. The freedom to switch between arms in response to experience is critical. If one arm must be chosen ex ante for all trials, then maximum expected utility is negative, hence less than $\varphi(0) = 0$. (The classical CLT applies to each arm separately and, by (4.4), $\varphi(\cdot)$ has negative expected value under the normal $N(0, \sigma^2)$ for any positive variance.)

Further perspective (for the $c = 0$ result) follows from two additional comparisons. First, consider the following lottery: Toss a fair coin. If Heads, then receive a positive prize according to $N(0, \sigma^2)$ conditioned on $\mathbb{R}_+$ and if Tails receive a negative prize according to $N(0, \sigma^2)$ conditioned on $\mathbb{R}_-$. This lottery has negative expected utility using $\varphi$. It is less attractive because the ability to choose actions sequentially affords some influence over positive versus negative outcomes, while in the lottery that influence belongs to nature alone. The second comparison is with the implication of the law of large numbers (LLN), where payoffs are unweighted averages $\frac{\sum \mathcal{P} X_i}{n}$. The LLN also yields the conclusion that risk does not matter asymptotically, but it differs from (4.8) because it implies indifference for all bounded and continuous $\varphi$, not only for functions $\varphi$ having the form in (4.2).
Condition (4.6), with $\sqrt{n}$ replaced by $n$, is valid, and by Peng (2019, Theorem 2.4.1),
\[
\lim_{n \to \infty} \sup_{Q \in \mathcal{P}} E_Q[\varphi(\Sigma^n_i X_i/n)] = \varphi(0),
\]
contrary to (4.8) for $c \neq 0$.

The Corollary implies that, in the limit $n \to \infty$, the decision-maker with a strictly positive reference point strictly prefers the risky sequential choice problem to receiving zero gain/loss for sure. The intuition is that zero for sure is a certain loss relative to a positive reference point, which makes it unattractive. A positive reference point $c$ also reduces the limiting value $\lim_{n \to \infty} V_n$, because it reduces all gains and increases all losses ($\varphi(x) \leq c$ for all $x$), but to a lesser degree because of the flexibility afforded by switching actions. Similarly, a negative reference point implies the preference for the certain zero outcome. In this sense, a higher benchmark or aspiration level leads to more participation in risky endeavors.

### 4.2 Strategies and the absence of learning

One would like to determine strategies that support the limiting values discussed above. That is, we seek a strategy $s^*$ such that
\[
\lim_{n \to \infty} E_{P^{s^*}}[\varphi(\Sigma^n_i X_i/\sqrt{n})] = \lim_{n \to \infty} V_n. \tag{4.9}
\]
We do this for the special case where
\[
P_i^s(\cdot | \omega^{(i-1)}) = P_i^s \quad \text{for all } i \geq 1 \text{ and } s \in S \text{ such that } s_i(\omega^{(i-1)}) = s_1. \tag{4.10}
\]
Recall that $P_i^s$ gives prior beliefs for each action $s_1$. Thus (4.10) stipulates that for each given action ($s_1$ above), beliefs do not change with history. The implication is that for each fixed action $a$, the joint probability distribution over outcomes given repeated choice of $a$ is i.i.d. For each $s$, the collection $\{P_i^s : i \geq 1\}$ induces a unique product measure $P^s$ on $(\Pi^\infty \Omega_i, \mathcal{G})$.

Define
\[
\sigma_a^2 = E_{P_i^a} [X_i^2], \quad a \in A.
\]
Then
\[
\overline{\sigma} = \max_{a \in A} \sigma_a \quad \text{and} \quad \underline{\sigma} = \min_{a \in A} \sigma_a.
\]

For simplicity, we focus on $c = 0$ and then indicate at the end of this subsection how to accommodate $c \neq 0$.

---

8In this case, the rectangular set $\overline{P}$ in Lemma 4.4 can be described explicitly: it consists of all probability measures whose 1-step-ahead conditionals at every history lie in $\{P_i^a : a \in A\}$. 

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Theorem 4.7. Define strategy $s^*$ by $s^*_1 = \pi$ and, for $n > 1$,
\begin{equation}
    s^*_n = \begin{cases} 
    \bar{a} & \text{if } \Sigma_1^{n-1} X_i \leq 0 \\
    a & \text{if } \Sigma_1^{n-1} X_i > 0 
    \end{cases}
\end{equation}
(4.11)
where $\sigma_\pi = \bar{\sigma}$ and $\sigma_a = \sigma$. Let $c = 0$. Then: (i) $s^*$ satisfies (4.9).

(ii) For every $N > 0$,
\begin{align*}
    P^{s^*} (\cap_{n=N}^{\infty} \{\Sigma_1^n X_i \leq 0\}) < 1 \quad \text{and} \quad P^{s^*} (\cap_{n=N}^{\infty} \{\Sigma_1^n X_i > 0\}) < 1. \\
\end{align*}

(i) confirms that $s^*$ is (for large horizons approximately) optimal, while (ii) states that $s^*$ implies switching between actions indefinitely with positive probability according to the measure corresponding to $s^*$. A proof of (i) is given in Appendix B, while (ii) follows from Corollary 3.4, which implies that, for any $N > 0$,
\begin{align*}
    P^{s^*} (\cap_{n=N}^{\infty} \{\Sigma_1^n X_i \leq 0\}) \leq \lim_{n \to \infty} \sup_{s \in S} P^s (\Sigma_1^n X_i / \sqrt{n} \leq 0) = \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} < 1, \\
    P^{s^*} (\cap_{n=N}^{\infty} \{\Sigma_1^n X_i > 0\}) \leq \lim_{n \to \infty} \sup_{s \in S} P^s (\Sigma_1^n X_i / \sqrt{n} > 0) = \frac{\bar{\sigma}}{\sigma + \bar{\sigma}} < 1. \\
\end{align*}

The fact that $s^*$ involves indefinite switching between actions indicates a difference between our model with loss aversion and many bandit models. Commonly in the bandit literature, learning (or exploration) provides the reason for switching, and eventually it is decided that one arm is superior and experimentation ceases. Here, in contrast, switching is optimal even in the absence of learning and (with positive probability) persists indefinitely. This is because loss aversion implies that the identity of the more attractive action or arm depends on whether one is in a region of cumulative gains ($\Sigma_1^n X_i > 0$) or cumulative losses ($\Sigma_1^n X_i < 0$).

It is striking that $s^*$ involves only extreme actions, those corresponding to the largest and smallest variances, and that the same strategy works regardless of further properties of $\varphi_1$ and $\{P^a : a \in A\}$. (Indeed, it follows from Remark 3.3, that the adequacy of extreme actions is valid without the restriction (3.3), assuming only suitable continuity for $\varphi$.) Further, assuming that these actions are unique, we can derive the following limiting relative frequencies given $s^*$:
\begin{equation}
    \lim_{n \to \infty} \frac{P^{s^n} (s_n^* = \bar{a})}{P^{s^n} (s_n^* = a)} = \frac{\sigma}{\bar{\sigma}}. 
\end{equation}
(4.12)

\footnote{A global risk averter would choose the low variance action $\bar{a}$ at every stage.}
In particular, \( \overline{\sigma} \) is chosen less frequently in the limit.\(^{10}\)

It is interesting to compare the present special model with the general model. Consider a decision-maker with conditional beliefs \( \{P_i^s : i \geq 1, s \in S\} \), which in general includes learning from history. Compare her with an individual, with the same \( c, \varphi_1, A \) and initial beliefs \( \{P_i^s\} \), but whose subsequent beliefs are modeled as in (4.10) and thus do not vary with history. Then these two specifications imply the same variance bounds \( \overline{\sigma} \) and \( \overline{\sigma} \). Therefore, the limiting values (\( \lim_{n \to \infty} V_n \)) for both decision-makers are identical.\(^{11}\) (It follows from (3.5) that they both equal \( E_{P^*}[\varphi(W_1^\infty)] \), which notably depends only on the variance bounds, \( c, \varphi_1 \) and \( A \)).

For interpretation, take all probabilities as objective and as defining two sequential decision problems, and give a single decision-maker the choice between these two problems. Then, for sufficiently large horizons, she would be approximately indifferent between them. In particular, the correlation (or lack thereof) between outcomes in distinct trials of action (or experiment) \( a \) is a matter of indifference in the limit. This is true even if the two decision problems have different prior probability distributions (say \( \{P_i^n : a \in A\} \) and \( \{Q_i^n : a \in A\} \)), as long as the zero mean and variance bounds remain common. Such robustness is not true for fixed finite horizons. It can be understood as reflecting the robustness, or universality, of our CLT that, in turn, extends the following property of the classical martingale CLT: If (1.1) and (1.2) obtain with \( P = \{P\} \), then they are also satisfied for \( Q \) equal to the i.i.d. product of the marginal \( P_1 \in \Delta(\Omega_1) \), and \( P \) and \( Q \) imply the same limiting probability distribution for \( \Sigma_i^n X_i/\sqrt{n} \).

Finally, we describe how the preceding can be adjusted to accommodate \( c \neq 0 \).

For that purpose, relax the requirement surrounding (4.9) and seek a sequence \( s^n = (s_i^n) \) of strategies, where, for each \( n \), \( s^n \in S \) is thought of as a strategy used in the \( n \) stage problem (4.5). (Accordingly, components \( s_i^n \) with \( i > n \) are irrelevant.) The counterpart of (4.9) is

\[
\lim_{n \to \infty} V_n = \lim_{n \to \infty} E_{P^n} [\varphi (\Sigma_i^n X_i/\sqrt{n})].
\]

(4.13)

Then, arguing as in the proof of the above theorem, one can show that (4.13) is satisfied by \( s^n \) defined as follows: For each \( n \geq 1 \), and \( 1 \leq i \leq n \),

\[
s_i^n = \begin{cases} 
\overline{\sigma} & \text{if } \Sigma_i^{i-1} X_j/\sqrt{n} \leq c \\
\overline{\sigma} & \text{if } \Sigma_i^{i-1} X_j/\sqrt{n} > c.
\end{cases}
\]

\(^{10}\)Since the two probabilities sum to 1, the corresponding limits of absolute frequencies are obvious. To derive (4.12), argue first, as in Corollary 3.4, that the indicator for \([0, \infty)\) can be approximated by a function \( \varphi \) satisfying conditions of the CLT and the bandit application. Therefore, (4.9) applies also to the indicator. Finally, apply the closed-form expressions in the noted corollary.

\(^{11}\)Though limiting values may be equal for the learner and non-learner, that does not imply that there is an "optimal" strategy common to both. Theorem 4.8 in the next section describes an optimal strategy under learning for a special case.
(s^n_i can be defined arbitrarily if either n = 1 or i > n.)

4.3 A two-armed bandit example

We build on the trinomial example in section 2 to illustrate some of the preceding by providing a more concrete special case.

Consider two arms, a and b. For each stage, the individual chooses which arm to pull taking into account what she has learned about the arms from past experience. Thus \( A = \{a, b\} \). The set of possible outcomes for each arm and stage is \( \Omega = \{1, -1, 0\} \), and outcomes are governed, both ex ante and for any history, by the following probabilities:

\[
\text{arm } a: \quad \Pr (1) = \Pr (-1) = p_a/2 \\
\text{arm } b: \quad \Pr (1) = \Pr (-1) = p_b/2.
\]

For each arm, outcomes follow a random walk with zero mean and with variance equal to the appropriate value of \( p \). The probabilities \( p_a \) and \( p_b \) are unknown, but it is known that

\[
\{p_a, p_b\} = \{p, \bar{p}\}, \tag{4.14}
\]

where \( 0 < p < \bar{p} < 1 \) are known. Thus there is uncertainty only about which of \( p \) and \( \bar{p} \) describes arm a and which describes arm b.\(^{12}\)

Define

\[
P_a^2 = \bar{p} \quad \text{and} \quad P_b^2 = \bar{p}.
\]

The set \( \{P_n^a : n \geq 1, s \in S\} \) is defined as follows. The decision-maker’s prior beliefs about which arm is which are completely specified by \( \mu_1 \), the probability she assigns initially to \( p_a = p \). Thus, prior probabilities of the outcomes from choosing arm \( \alpha, \alpha = a, b \), are given by

\[
P_n^a (1) = \mu_1 \bar{p}/2 + (1 - \mu_1)\bar{p}/2 = P_n^a (-1) \\
P_n^b (1) = (1 - \mu_1)\bar{p}/2 + \mu_1 \bar{p}/2 = P_n^b (-1).
\]

Similarly, for each \( n > 1 \) and history \( \omega^{(n-1)} = (\omega_1, ..., \omega_{n-1}) \),

\[
P_n^a (1) = \mu_n \bar{p}/2 + (1 - \mu_n)\bar{p}/2 = P_n^a (-1) \tag{4.15} \\
P_n^b (1) = (1 - \mu_n)\bar{p}/2 + \mu_n \bar{p}/2 = P_n^b (-1),
\]

\(^{12}\)Uncertainty about "which arm is which" in a 2-arm setting is a classical version of the bandit problem (Bradt, Johnson and Karlin 1956). It typically assumes a finite horizon and maximization of the expected value of the sum of payoffs, (in particular, means rather than variances are the focus).
where \( \mu_n \) is the posterior probability that \( p_a = p \), is defined inductively by

\[
\log \left( \frac{\mu_{n+1}/(1 - \mu_{n+1})}{\mu_n/(1 - \mu_n)} \right) = \left[ I_a(s_n) - I_b(s_n) \right] \left( (1 - I_0(\omega_n)) \log \left( \frac{p}{P} \right) + I_0(\omega_n) \log \left( \frac{1 - p}{1 - P} \right) \right).
\]

The assumptions implicit in this specification are clarified by considering the probability measure \( P^* \) on \( \Omega \), induced for any strategy \( s \). For any \( \omega^{(n)} = (\omega_1, ..., \omega_n) \), the outcomes of the first \( n \) trials, and the given \( s \), define the induced frequency vector \( f^*(\omega^{(n)}) \),

\[
f^*(\omega^{(n)}) = (f_a^*(\omega^{(n)}), f_b^*(\omega^{(n)}), f_{a,0}^*(\omega^{(n)}), f_{b,0}^*(\omega^{(n)})),
\]

where: for \( \alpha \in \{a, b\} \), \( f_{\alpha}^*(\omega^{(n)}) \) and \( f_{\alpha,0}^*(\omega^{(n)}) \) give, respectively, the number of trials of arm \( \alpha \) and the number of those that yield the outcome 0. Then the expected probability of the above outcomes are given by

\[
P^*(\omega_1, ..., \omega_n) = \mu_1 \left[ (p/2) f_a^* - f_{a,0}^* (p/2) f_b^* - f_{b,0}^* (1 - p) f_{a,0}^* (1 - p) f_{b,0}^* \right] + (1 - \mu_1) \left[ (\bar{p}/2) f_a^* - f_{a,0}^* (\bar{p}/2) f_b^* - f_{b,0}^* (1 - \bar{p}) f_{a,0}^* (1 - \bar{p}) f_{b,0}^* \right].
\]

The two terms on the right correspond to the two possible scenarios, \( p_a = p \) or \( \bar{p} \), weighted by their prior probabilities. Conditional on each scenario the expression reflects two assumptions: (i) independence between distinct trials, whether conducted with the same arm or with different arms; and (ii) all trials with a given arm are viewed as similar (or interchangeable) so that the probability of any (finite) sequence of outcomes for that arm is invariant to any reordering (accordingly, for each arm, the probability of a set of outcomes depends only on the number of occurrences of 0 and \{1, -1\}). This latter assumption of "symmetry" within each arm is known as partial exchangeability, a property introduced by de Finetti (1938), who also showed that it implies conditional independence as in (i), and, in fact, that it characterizes a representation such as in (4.18).

This example satisfies all the assumptions of the more general case above and thus all the preceding results apply. Moreover, the added structure assumed herein

---

13 For \( \alpha \in \{a, b\} \), \( I_\alpha(s_n) = 1 \) if \( s_n = \alpha \) and = 0 otherwise. \( I_0(\omega_n) = 1 \) if \( \omega_n = 0 \) and = 0 otherwise.

14 Recall that the conditionals \( P_\alpha^* \) are just the 1-step-ahead conditionals of \( P^* \).

15 The stronger property of exchangeability, which is better known, assumes interchangeability also across distinct arms and thus views the two arms as being identical, which is excluded in our case because of (4.14) and \( \bar{p} \neq p \). See Link (1980) and Diaconis and Freedman (1982) for more on partial exchangeability and Kallenberg (2005) for a comprehensive treatment of probabilistic symmetries.
permits sharper results, specifically regarding supporting strategies and what is learned asymptotically. Below we assume \( c = 0 \).

Define the strategy \( s^* \) by \( s^*_1 = a \) and, for \( n > 1 \),

\[
s^*_n = \begin{cases} 
  a & \text{if } \frac{\sum_{i=1}^{n-1} X_j}{\sum_{i=1}^{n-1} X_j} \leq 0, \mu_n < \frac{1}{2} \text{ OR } \\
  b & \text{if otherwise} 
\end{cases}
\]

According to \( s^* \), arm \( a \) is used at stage \( n > 1 \) if (and only if) there are cumulative losses and it is more likely that \( a \) has higher variance (\( \mu_n < \frac{1}{2} \)), or there are cumulative gains and it is more likely that \( a \) has lower variance (\( \mu_n > \frac{1}{2} \)). This intuition argues convincingly for this choice of arm at stage \( n \) if there are no later trials remaining, but may seem myopic more generally. Nevertheless, we show that \( s^* \) is approximately optimal for large horizons.

**Theorem 4.8.** Let \( c = 0 \) and \( \mu_1 \in [0, 1] \). Then \( s^* \) is approximately optimal for large horizons, that is, it satisfies (4.9).

The result extends Theorem 4.7(i), which corresponds to the case \( \mu_1 \in \{0, 1\} \), but only for the special setting of this section.

Conclude with observations about the process of posteriors \( \{\mu_n\} \) that confirm for our setting properties familiar from Bayesian learning theory.

**Remark 4.9.** Let \( s \in S \) be any strategy. Then:

(i) Posteriors converge to certainty, that is, for any prior \( \mu_1 \),

\[
\lim_{n \to \infty} \mu_n \in \{0, 1\} \quad P^s\text{-a.s.} \tag{4.19}
\]

(ii) Suppose that, unknown to the decision-maker, the truth is that \( p_a = p \). Consequently, given any strategy \( s \), outcomes are governed by the probability law \( Q^s \in \Delta(\Pi_{i=1}^\infty \Omega_i, \mathcal{G}) \), whose 1-step-ahead conditionals are \( Q^s_i, i \geq 1 \), given by

\[
Q^s_i(1) = Q^s_i(-1) = \begin{cases} 
  p/2 & \text{if } s_i(\omega^{(i-1)}) = a \\
  b/2 & \text{if } s_i(\omega^{(i-1)}) = b 
\end{cases}
\]

Then, for every \( \mu_1 > 0 \),

\[
\lim_{n \to \infty} \mu_n = 1 \quad Q^s\text{-a.s.} \tag{4.20}
\]

Think of \( \{\mu_n\} \) as representing subjective beliefs. Then (4.19) expresses the decision-maker’s ex ante complete confidence that asymptotically she will know "which arm is which." In (ii), \( Q^s \) is the true probability law over outcome sequences when strategy \( s \) is adopted, and hence (4.20) is an expression of "Bayesian consistency". Both results are valid for any strategy, and thus reflect Bayesian updating alone and not (approximately) optimizing behavior.

\[\text{In fact, convergence to certainty is valid for every } P^s, s \in S.\]
5 Related literature

Chen and Epstein (2020) establish CLTs assuming, contrary to (1.1)-(1.2), that conditional means lie in an interval $[\mu, \bar{\mu}]$ while all conditional variances equal a constant $\sigma^2$. In common with this paper, rectangularity is a key assumption. However, their theorems are substantially different, for example, limits have a different form and proofs are much different. There exist other generalizations of the classical CLT that are motivated by robustness to ambiguity. In Epstein, Kaido and Seo (2016), experiments are not ordered and the analysis is intended for a cross-sectional, rather than sequential, context. Another difference is that their limiting distribution is normal. Peng (2009, 2019) and Fang et al (2019) also assume that experiments are ordered. Comparison with Theorem 3.2 of the latter is representative. It is more general than our results, for example, in permitting ambiguity about both mean and variance, but this comes arguably at the cost of reduced tractability, even if one limits attention to the special case where there is ambiguity about variance only. In particular, limits are more complicated and there is no counterpart of Theorem 3.1. None of the above papers recognize the potential application to sequential decision problems such as MAB.

The connection between sets of priors and decision-making with action-dependent probabilities (or moral hazard) has been recognized in the decision theory literature (Dreze 1987, Kelsey and Milne 1999, and Karni 2011, for example). These papers are concerned primarily with axiomatic foundations, extending those for subjective expected utility, while our motivation in studying the MAB is more applied. We differ also in our focus on sequential choice and in the connection to a CLT.

A Appendix: Main Proofs

The notation and assumptions in Theorem 3.1 are adopted throughout this appendix. Let $(B_t)$ be the standard Brownian motion under a probability space $(\Omega^*, \mathcal{F}^*, P^*)$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by $(B_t)_{t \geq 0}$.

A.1 Lemmas

For a small fixed $h > 0$, and any fixed $(t, x, \alpha, \beta, c) \in [0, 1 + h] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, $(Y_{t,x,\alpha,\beta,c}^s)_{s \in [t, 1 + h]}$ denotes the solution of the SDE

\[
\begin{cases}
    dY_{t,x,\alpha,\beta,c}^s = \sigma(Y_{s,x,\alpha,\beta,c}^s) \, dB_s, & s \in [t, 1 + h] \\
    Y_{t,x,\alpha,\beta,c}^t = x,
\end{cases}
\]  

(A.1)

where $\sigma(y) = \alpha I_{[-\infty,c]}(y) + \beta I_{(-\infty,-c]}(y)$, $\forall y \in \mathbb{R}$. 

23
By Keilson and Wellner (1978, Theorem 1), (see also Chen and Zili (2015)), the transition probability density of \((Y_t^{s,x,\alpha,\beta,c})_{s \in [t,1+h]}\) is given by, for any \(t < s \leq 1+h\) and \(y \in \mathbb{R}\),

\[
q^{\alpha,\beta,c}(t, x; s, y) = \frac{1}{\sqrt{2\pi(s-t)}} \frac{1}{\sigma(y)} \exp \left( -\frac{(x-c)^2}{2(s-t)} \right) + \frac{\beta - \alpha}{\beta + \alpha} \frac{1}{\sqrt{2\pi(s-t)}} \frac{\text{sgn}(y-c)}{\sigma(y)} \exp \left( -\frac{\left(\frac{x-c}{\sigma(x)} - \frac{y-c}{\sigma(y)}\right)^2}{2(s-t)} \right).
\]

(A.2)

Given \(\varphi_1 \in C^3_b(\mathbb{R})\), define

\[
\varphi(x) = \begin{cases} 
\varphi_1(x-c) & x \geq c \\
\frac{\beta}{\alpha} \varphi_1(-\frac{\beta}{\alpha}(x-c)) + (1 + \frac{\beta}{\alpha}) \varphi_1(0) & x < c.
\end{cases}
\]

(A.3)

It can be checked that

\[
\varphi \in C^4_b(\mathbb{R}) \text{ and } \varphi''(z+c) = -\frac{\alpha}{\beta} \varphi''(-\frac{\alpha}{\beta} z + c), \quad \forall z < 0.
\]

Define the set of functions \(\{H_t\}_{t \in [0,1+h]}\) by

\[
H_t(x) = E_{P^*} \left[ \varphi \left( Y_{t+h}^{1,x,\alpha,\beta,c} \right) \right], \quad \forall x \in \mathbb{R}.
\]

(A.4)

Then

\[
H_{1+h}(x) = \varphi(x), \quad H_0(0) = E_{P^*} [\varphi(Y_{1+h}^{0,0,\alpha,\beta,c})] = E_{P^*} [\varphi(W_{1+h}^{\alpha,\beta,c})].
\]

The following lemma describes some properties of the functions \(\{H_t\}_{t \in [0,1+h]}\).

**Lemma A.1.** The functions \(\{H_t\}\) defined by (A.4) satisfy:

1. For any \(t \in [0,1]\), \(H_t \in C^2_b(\mathbb{R})\), and the first and second derivatives of \(H_t\) are bounded uniformly in \(t \in [0,1]\).

2. There exists a constant \(L\) such that, for any \(x_1, x_2 \in \mathbb{R}\) and \(t \in [0,1]\),

\[
|H_t''(x_1) - H_t''(x_2)| \leq L|x_1 - x_2|.
\]
(3) (a) If $\phi''(x) \geq 0$ for $x > c$, then
\[
\begin{align*}
H''_t(x) &\geq 0 \quad \text{for } x \geq c \\
H''_t(x) &\leq 0 \quad \text{for } x \leq c.
\end{align*}
\]
(b) If $\phi''(x) \leq 0$ for $x > c$, then
\[
\begin{align*}
H''_t(x) &\leq 0 \quad \text{for } x \geq c \\
H''_t(x) &\geq 0 \quad \text{for } x \leq c.
\end{align*}
\]

(4) For any $r \in [0, 1 + h - t]$, 
\[
H_t(x) = E_{p^*} \left[ H_{t+r} \left( Y_{t+r,x}^{\alpha,\beta,c} \right) \right], \forall x \in \mathbb{R}.
\]

(5) (a) If $\phi''(x) \geq 0$ for $x > c$, then
\[
\lim_{n \to \infty} \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left| H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\alpha^2}{2n} \left[ H''_{\frac{m}{n}}(x) \right]^+ + \frac{\beta^2}{2n} \left[ H''_{\frac{m}{n}}(x) \right]^-ight| = 0.
\]
(b) If $\phi''(x) \leq 0$ for $x > c$, then
\[
\lim_{n \to \infty} \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left| H_{\frac{m-1}{n}}(x) - H_{\frac{m}{n}}(x) - \frac{\beta^2}{2n} \left[ H''_{\frac{m}{n}}(x) \right]^+ + \frac{\alpha^2}{2n} \left[ H''_{\frac{m}{n}}(x) \right]^-ight| = 0.
\]

(6) There exists a constant $C_0$ such that
\[
\sup_{x \in \mathbb{R}} |H_1(x) - \phi(x)| \leq C_0 \sqrt{\alpha^2 + \beta^2 h}.
\]

**Proof:** (1) Given the transition probability density in (A.2), we have, for $t \in [0, 1]$,
\[
H_t(x) = \int_{-\infty}^{\infty} \phi(y) q^{\alpha,\beta,c}(t, 1 + h, y) dy, \quad \forall x \in \mathbb{R}.
\]
For $T = 1 + h$, we have
\[
H'_t(x) = \begin{cases} 
\frac{1}{\alpha \sqrt{2\pi(T-t)}} \left[ \int_{-\infty}^{0} \left( \frac{\alpha - \beta}{\alpha + \beta} \phi_1(-y) + \frac{\beta \alpha}{\alpha + \beta} \phi_2\left( \frac{\beta}{\alpha} y + c \right) \right) e^{-\frac{(x-y)^2}{2n^2(T-t)}} dy \\
+ \int_{0}^{\infty} \phi'_1(y) e^{-\frac{(x-y)^2}{2n^2(T-t)}} dy \right] & \text{if } x \geq c \\
\frac{1}{\beta \sqrt{2\pi(T-t)}} \left[ \int_{-\infty}^{0} \left( \frac{\beta - \alpha}{\alpha + \beta} \phi'_2(-y + c) + \frac{\alpha \beta}{\alpha + \beta} \phi'_1\left( \frac{\beta}{\alpha} y \right) \right) e^{-\frac{(x-y)^2}{2n^2(T-t)}} dy \\
+ \int_{0}^{\infty} \phi'_2(y + c) e^{-\frac{(x-y)^2}{2n^2(T-t)}} dy \right] & \text{if } x \leq c
\end{cases}
\]
Since

The assertion follows from \( \varphi_1 \in C^3_b(\mathbb{R}) \) and the definition of \( \varphi \) in (A.3).

For any \( x > c \), \( H''_t(x) = \)

and, for \( x < c \), \( H''_t(x) = \)

Since \( \varphi_1 \in C^3_b(\mathbb{R}) \), there exists a constant \( L \) such that

The assertion follows by the Mean Value Theorem.

(3) It follows from the explicit form of \( H''_t(x) \) given above.

(4) Since \( (Y_s^{t,x,\alpha,\beta,c}) \) is a time-homogeneous Markov process, for any \( r \in [0,1+h-t] \),

(5) We prove (a); the proof of (b) is similar. It follows from part (4) that, for any \( 1 \leq m \leq n \),

Apply Itô’s formula to \( H_m \left( Y_{m-n}^{x,\alpha,\beta,c} \right) \) to derive

Using parts (3) and (4), we have

\[ H_{m-1} \left( Y_{m-n}^{x,\alpha,\beta,c} \right) = E_{P^*} \left[ H_m \left( Y_{m-n}^{x,\alpha,\beta,c} \right) \right] = \]
Thus

\[
\sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \left| H_{n}^{m-1}(x) - H_{n}^{m}(x) - \frac{\alpha^{2} + \beta^{2}}{2n} H_{n}^{m}(x) + \frac{\beta^{2}}{2n} H_{n}^{m}(x) \right|
\leq \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} E_{P^{\times}} \left[ \frac{\alpha^{2} + \beta^{2}}{2n} \int_{m-1}^{m} H_{n}^{m}(Y_{s}^{m-1,x,\alpha,\beta}) - H_{n}^{m}(x) \right] \leq \sum_{m=1}^{n} \sup_{x \in \mathbb{R}} \frac{C}{n} \left( \int_{m-1}^{m} \left( \sigma(Y_{s}^{m-1,x,\alpha,\beta}) \right)^{2} \right)^{1/2} \leq \frac{C}{\sqrt{n}} \sqrt{\alpha^{2} + \beta^{2}},
\]

where \( C \) is a constant that depends only on \( \alpha, \beta, L \).

(6) Since \( \varphi \in C_{0}^{1}(\mathbb{R}) \), \( C_0 \equiv \| \varphi' \| = \sup_{x \in \mathbb{R}} |\varphi'(x)| < \infty \), and

\[
\sup_{x \in \mathbb{R}} |H_{1}(x) - \varphi(x)| = \sup_{x \in \mathbb{R}} \left| E_{P^{\times}} \left[ \varphi(Y_{1,1,\alpha,\beta}) - \varphi(x) \right] \right| \leq \sup_{x \in \mathbb{R}} \left| E_{P^{\times}} \left[ \varphi(Y_{1,1,\alpha,\beta}) - \varphi(x) \right] \right| \leq \sup_{x \in \mathbb{R}} C_{0} E_{P^{\times}} \left[ \int_{1}^{1+h} \sigma(Y_{s}^{1,1,\alpha,\beta}) dB_{s} \right] \leq \sup_{x \in \mathbb{R}} C_{0} \left( E_{P^{\times}} \left[ \int_{1}^{1+h} \sigma(Y_{s}^{1,1,\alpha,\beta})^{2} ds \right] \right)^{1/2} \leq C_{0} \sqrt{\alpha^{2} + \beta^{2}} \sqrt{h}.
\]

**Lemma A.2.** Let \( \{H_{t}\}_{t \in [0,1]} \) be the functions defined in (A.4), and define the family of functions \( \{L_{t}\}_{t \in [0,1]} \) by

\[
L_{t}(x) = H_{t}(x) + \frac{\sigma^{2}}{2n} [H_{t}(x)]^{+} - \frac{\sigma^{2}}{2n} [H_{t}(x)]^{-}.
\]

Then

\[
\lim_{n \to \infty} \sum_{m=1}^{n} \left| \mathbb{E} \left[ H_{n}^{m} \left( \frac{\sum_{i=1}^{n} X_{i}}{\sqrt{n}} \right) \right] - \mathbb{E} \left[ L_{n}^{m} \left( \frac{\sum_{i=1}^{n-1} X_{i}}{\sqrt{n}} \right) \right] \right| = 0.
\]
Proof: It suffices to prove

\[
\lim_{n \to \infty} \sum_{m=1}^{n} \left| \mathbb{E} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \right] - f(m, n) \right| = 0 \text{ and } \tag{A.7}
\]

\[
\lim_{n \to \infty} \sum_{m=1}^{n} f(m, n) - \mathbb{E} \left[ L_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \right] = 0, \tag{A.8}
\]

where

\[
f(m, n) = \mathbb{E} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) + H'_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H''_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \frac{X^2_m}{2n} \right].
\]

By Lemma A.1, there exists \( L > 0 \) such that

\[
\sup_{t \in [0, 1]} \sup_{x \in \mathbb{R}} |H''_{t}(x)| \leq L, \quad \sup_{t \in [0, 1]} \sup_{x, y \in \mathbb{R}, x \neq y \atop |x - y|} \left| \frac{H''_{t}(x) - H''_{t}(y)}{|x - y|} \right| \leq L.
\]

By the Taylor expansion of \( H_t \in C^2_b(R) \), \( \forall \epsilon > 0 \exists \delta > 0 \) (\( \delta \) depends only on \( L \) and \( \epsilon \)), such that, for any \( x, y \in R \) and \( t \in [0, 1] \),

\[
\left| H_t(x + y) - H_t(x) - H'_t(x)y - \frac{1}{2}H''_{t}(x)y^2 \right| \leq \epsilon |y|^2 I_{|y| < \delta} + L |y|^2 I_{|y| \geq \delta}. \tag{A.9}
\]

Let \( x = \sum_{1}^{m-1} X_i / \sqrt{n} \) and \( y = X_m / \sqrt{n} \) in (A.9) to derive

\[
\sum_{m=1}^{n} \left| \mathbb{E} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \right] - f(m, n) \right| \leq \sigma^2 \epsilon + \frac{L}{n} \sum_{m=1}^{n} \mathbb{E} \left[ |X_m|^2 I_{|X_m| \geq \sqrt{n} \delta} \right].
\]

By the arbitrariness of \( \epsilon \) and the Lindeberg condition (2.1), we obtain (A.7).

By Lemma 2.1, we have

\[
f(m, n) = \mathbb{E} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) + H'_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H''_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \frac{X^2_m}{2n} \right] = \mathbb{E} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) + \frac{1}{2n} \mathbb{E} \left[ H''_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) X^2_m |\mathcal{G}_{m-1} \right] \right] = \mathbb{E} \left[ L_{\frac{m}{n}} \left( \frac{\sum_{1}^{m-1} X_i}{\sqrt{n}} \right) \right].
\]

This implies (A.8) and completes the proof. \( \blacksquare \)
A.2 Proof of the CLT (Theorem 3.1)

We prove (3.5). Proofs of other parts are similar.

Let \((\alpha, \beta) = (\sigma, \varpi)\). Then the definitions of \(\varphi\) in (A.3) and (3.3) coincide. For \(h > 0\) sufficiently small, let \(\{H_t\}_{t \in [0,1+h]}\) be the corresponding functions defined by (A.4).

First prove

\[
\lim_{n \to \infty} \left| \mathbb{E} \left[ H_1 \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ \varphi \left( W_{i+h}^c \right) \right] \right| = 0.
\]

We have

\[
\mathbb{E} \left[ H_1 \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ \varphi \left( W_{i+h}^c \right) \right] = \sum_{m=1}^n \left\{ \mathbb{E} \left[ H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[ L_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] \right\}
\]

where \(L_m(x) = H_m(x) + \frac{\sigma^2}{2n} \left[ H_m''(x) \right]^+ - \frac{\sigma^2}{2n} \left[ H_m''(x) \right]^-, \ 1 \leq m \leq n\).

By Lemma A.2,

\[
|I_{1n}| \leq \sum_{m=1}^n \mathbb{E} \left| \mathbb{E} \left[ H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[ L_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] \right| \to 0 \quad \text{as} \quad n \to \infty.
\]

Furthermore, by Lemma A.1(5), as \(n \to \infty\),

\[
|I_{2n}| \leq \sum_{m=1}^n \mathbb{E} \left| \mathbb{E} \left[ L_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right| \leq \sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| L_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) - H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right|
\]

\[
= \sum_{m=1}^n \sup_{x \in \mathbb{R}} \left| H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) - H_m \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right| \to 0.
\]
By Lemma A.1(6),
\[
\lim_{n \to \infty} \left| \mathbb{E} \left[ \varphi \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi (W_1^c)] \right| \leq \\
\lim_{n \to \infty} \left| \mathbb{E} \left[ \varphi \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] - \mathbb{E} \left[ H_1 \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] + \\
\lim_{n \to \infty} \left| \mathbb{E} \left[ H_1 \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] - E_{P^*} [\varphi (W_{1+h}^c)] \right| + \left| E_{P^*} [\varphi (W_{1+h}^c)] - E_{P^*} [\varphi (W_1^c)] \right| \\
\leq \sup_{x \in \mathbb{R}} |H_1 (x) - \varphi (x)| + C_0 \sqrt{\sigma^2 + \sigma_h^2} \leq 2C_0 \sqrt{\sigma^2 + \sigma_h^2}.
\]
Since \( h \) is arbitrary, the proof is complete.

### A.3 Proof of Corollary 3.4

We prove the result for upper probability in (3.11). The other proofs are similar.

For any \( c \in \mathbb{R} \) and \( \varepsilon > 0 \), suppose that \( f_1, g_1 \in C_0^3 (\mathbb{R}) \) satisfy
\[
\begin{align*}
\begin{cases}
\quad f_1(x) = 1 & \text{for } x \geq \frac{\varepsilon}{\sigma} \\
f''_1(x) \leq 0 & \text{for } x \geq 0 \\
f_1(0) = \frac{\sigma}{\sigma + \varepsilon}
\end{cases}
\quad & \quad \\
\begin{cases}
\quad g_1(x) = 1 & \text{for } x \geq \varepsilon \\
g''_1(x) \leq 0 & \text{for } x \geq 0 \\
g_1(0) = \frac{\sigma}{\sigma + \varepsilon}
\end{cases}
\end{align*}
\]  
Define \( f_\varepsilon \) and \( g_\varepsilon \) by
\[
\begin{align*}
f_\varepsilon(x) &= \begin{cases}
\quad f_1(x - c - \varepsilon) & \text{for } x \geq c + \varepsilon \\
-\frac{\sigma}{\sigma + \varepsilon} f_1 \left( -\frac{\sigma}{\sigma + \varepsilon} (x - c - \varepsilon) \right) + \frac{\sigma}{\sigma + \varepsilon} & \text{for } x \geq c + \varepsilon \end{cases} \\
g_\varepsilon(x) &= \begin{cases}
\quad g_1(x - c + \varepsilon) & \text{for } x \geq c - \varepsilon \\
-\frac{\sigma}{\sigma + \varepsilon} g_1 \left( -\frac{\sigma}{\sigma + \varepsilon} (x - c + \varepsilon) \right) + \frac{\sigma}{\sigma + \varepsilon} & \text{for } x \geq c - \varepsilon \end{cases}
\end{align*}
\]  
It can be checked that
\[
g_\varepsilon(x) \geq I_{[c, \infty)} (x) \geq f_\varepsilon(x) \quad \text{and} \\
|g_\varepsilon(x) - f_\varepsilon(x)| \leq I_{[-1 + \frac{\varepsilon}{\sigma}, c + 1 + \frac{\varepsilon}{\sigma}]^c} (x), \ \forall x \in \mathbb{R}.
\]

Consider the solution \( \left( \tilde{W}_t^\varepsilon \right)_{t \geq 0} \) of the SDE
\[
\begin{align*}
\begin{cases}
d\tilde{W}_t^\varepsilon = \left( \sigma I_{[0, \infty)} \left( \tilde{W}_t^\varepsilon \right) + \sigma I_{(-\infty, 0)} \left( \tilde{W}_t^\varepsilon \right) \right) dB_t, & t \geq 0 \\
\tilde{W}_0^\varepsilon = x.
\end{cases}
\end{align*}
\]  
Then \( W_1^c \) and \( c + \tilde{W}_1^{c-} \) are described by the same law, and
\[
\left| \sup_{Q \in \mathcal{P}} Q \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \geq c \right) - P^* (W_1^c \geq c) \right|
\]
\[ \begin{align*}
& \leq \mathbb{E} \left[ f_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ g_x (W_1^c) \right] + \mathbb{E} \left[ g_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ f_x (W_1^c) \right] \\
& \leq \mathbb{E} \left[ f_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ f_x (W_1^{c+\epsilon}) \right] + \mathbb{E} \left[ g_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ g_x (W_1^{-c}) \right] \\
& \quad + |E^* [f_x (W_1^{c+\epsilon}) - f_x (W_1^c)]| + |E^* [g_x (W_1^{-c}) - g_x (W_1^c)]| \\
& \quad + 2|E^* [f_x (W_1^c) - g_x (W_1^c)]| \\
& \leq \mathbb{E} \left[ f_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ f_x (W_1^{c+\epsilon}) \right] + \mathbb{E} \left[ g_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ g_x (W_1^{-c}) \right] \\
& \quad + E^* \left[ |f_x (c + \epsilon + \bar{W}_1^{c+\epsilon}) - f_x (c + \bar{W}_1^{-c})| + |g_x (c - \epsilon + \bar{W}_1^{c+\epsilon}) - g_x (c + \bar{W}_1^{-c})| \right] \\
& \quad + 2E^* \left[ f_x (W_1^c) - g_x (W_1^c) \right] \\
& \leq \mathbb{E} \left[ f_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ f_x (W_1^{c+\epsilon}) \right] + \mathbb{E} \left[ g_x \left( \frac{\sum_i X_i}{\sqrt{n}} \right) \right] - E^* \left[ g_x (W_1^{-c}) \right] \\
& \quad + C_0 E^* \left[ 2\epsilon + |\bar{W}_1^{c+\epsilon} - \bar{W}_1^{-c}| + |\bar{W}_1^{c+\epsilon} - \bar{W}_1^{-c}\epsilon| \right] + 2P^* \left( c - \left( 1 + \frac{\sigma}{\vartheta} \right) \epsilon \leq W_1^c \leq c + \left( 1 + \frac{\sigma}{\vartheta} \right) \epsilon \right),
\end{align*} \]

where \( C_0 \) is a constant that depends on \( \|f'_x\|, \|g'_x\| \). With Le Gall (1984, Theorem 1.5) and Theorem 3.1, the upper probability equation in (3.11) is proven.

### A.4 Closed-form expressions

Let \( W_{t, \beta, \epsilon} \) be the \( t = 1 \) value of the oscillating Brownian motion defined by (3.1)-(3.2). Keilson and Wellner (1978, Theorem 1) give the following expression for its pdf: For \( c \geq 0 \),

\[
q(y) = \begin{cases}
\frac{1}{\alpha \sqrt{2\pi}} e^{-\frac{(y - \epsilon \bar{q})^2}{2}} + \frac{\beta - \alpha}{\alpha + \beta} e^{\frac{(\epsilon + \bar{q})^2}{2}} & y \geq c \\
\frac{1}{\beta \sqrt{2\pi}} e^{-\frac{(y - \epsilon \bar{q})^2}{2}} - \frac{\beta - \alpha}{\alpha + \beta} e^{\frac{(\epsilon + \bar{q})^2}{2}} & y < c
\end{cases} \tag{A.13}
\]

and for \( c < 0 \),

\[
q(y) = \begin{cases}
\frac{1}{\alpha \sqrt{2\pi}} e^{-\frac{(y - \epsilon \bar{q})^2}{2}} + \frac{\beta - \alpha}{\alpha + \beta} e^{\frac{(\epsilon + \bar{q})^2}{2}} & y \geq c \\
\frac{1}{\beta \sqrt{2\pi}} e^{-\frac{(y - \epsilon \bar{q})^2}{2}} - \frac{\beta - \alpha}{\alpha + \beta} e^{\frac{(\epsilon + \bar{q})^2}{2}} & y < c
\end{cases} \tag{A.14}
\]

Using these expressions, it is straightforward to derive both (1.5) and (3.10).
B Appendix: Proofs for bandits

B.1 Proof of Corollary 4.6

We include the proof for \( c \geq 0 \). The proof for \( c < 0 \) is similar.

In light of (4.6) and (3.5), it suffices to compute \( E_{P^*}[\varphi(W_1^c)] \). Use the pdf of \( W_1^c \) in (A.13), with \((\alpha, \beta) = (\sigma, \overline{\sigma})\), to deduce that, for \( c \geq 0 \),

\[
E_{P^*}[\varphi(W_1^c)] = \int_{-\infty}^{\infty} q(y) \varphi_1(y - c) \, dy + \int_{c}^{\infty} q(y) \left[ -\frac{\overline{\sigma}}{\sigma} \varphi_1 \left( -\frac{\overline{\sigma}}{\sigma}(y - c) \right) \right] \, dy
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{c} \varphi_1(y - c) \left[ \frac{2\overline{\sigma}}{\sigma + \overline{\sigma}} \right] e^{-\frac{(y-c)^2}{2}} \, dy
\]

\[
- \frac{1}{\sqrt{2\pi} \sigma} \int_{c}^{\infty} \varphi_1 \left( -\frac{\overline{\sigma}}{\sigma}(y - c) \right) \left( e^{-\frac{(y-c)^2}{2}} - \frac{\overline{\sigma}}{\sigma + \overline{\sigma}} e^{-\frac{(c-y)^2}{2}} \right) \, dy
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma} \int_{0}^{\infty} \varphi_1(y) \left[ \frac{2\overline{\sigma}}{\sigma + \overline{\sigma}} \right] e^{-\frac{(y-c)^2}{2}} \, dy
\]

\[
- \frac{1}{\sqrt{2\pi} \overline{\sigma}} \int_{0}^{\infty} \varphi_1(z) \left( e^{-\frac{(y+m)^2}{2\overline{\sigma}^2}} - \frac{\overline{\sigma}}{\sigma + \overline{\sigma}} e^{-\frac{(z+y)^2}{2}} \right) \, dz
\]

\[
= \frac{1}{\sqrt{2\pi} \sigma} \int_{0}^{\infty} \varphi_1(y) \frac{\overline{\sigma}}{\sigma} \left[ e^{-\frac{(y+m)^2}{2\overline{\sigma}^2}} - e^{-\frac{(y-m)^2}{2\overline{\sigma}^2}} \right] \, dy
\]

where \( m = \frac{\sigma}{\overline{\sigma}} c \). Thus we want to prove that

\[
\frac{1}{\sqrt{2\pi} \sigma} \int_{0}^{\infty} \varphi_1(y) \frac{\overline{\sigma}}{\sigma} \left[ e^{-\frac{(y+m)^2}{2\overline{\sigma}^2}} - e^{-\frac{(y-m)^2}{2\overline{\sigma}^2}} \right] \, dy \geq \frac{\overline{\sigma}}{\sigma} \varphi_1 \left( \frac{\sigma}{\overline{\sigma}} c \right),
\]

with equality if and only if \( c = 0 \).

It is evident that \( E_{P^*}[\varphi(W_1^c)] = 0 = \varphi(0) \) if \( c = 0 \). Henceforth, take \( c > 0 \) and prove that

\[
\int_{0}^{\infty} \varphi_1(y) \left[ e^{-\frac{(y-m)^2}{2\sigma^2}} - e^{-\frac{(y+m)^2}{2\sigma^2}} \right] / \sqrt{2\pi} \sigma \, dy < \varphi_1(m).
\]
Denote by \( f (y) \) the expression in the square bracket, (thus \( f (y) > 0 \) for all \( y > 0 \)), and let \( F = \int_0^\infty f (y) \, dy \), \( 0 < F < 1 \). Then \( f / F \) is a density. If its mean is \( \mu \), then, by strict concavity of \( \varphi_1 \),
\[
\int_0^\infty \varphi_1 (y) \, f (y) \, dy < F \varphi_1 (\mu).
\] (B.1)

Next we prove that \( F \mu = m \):
\[
F \mu = \int_0^\infty y \left[ \left( e^{-\frac{(y-m)^2}{2\sigma^2}} - e^{-\frac{(y+m)^2}{2\sigma^2}} \right) / \sqrt{2\pi \sigma} \right] \, dy
\]
\[
= \int_{-m}^\infty (z + m) \left[ e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi \sigma} \right] \, dz - \int_{-m}^\infty (z - m) \left[ e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi \sigma} \right] \, dz
\]
\[
= \int_{-m}^m z \left[ e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi \sigma} \right] \, dz + m \int_{-m}^\infty \left[ e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi \sigma} \right] \, dz + m \int_{m}^\infty \left[ e^{-\frac{z^2}{2\sigma^2}} / \sqrt{2\pi \sigma} \right] \, dz
\]
\[
= 0 + m \left[ \Pr (Z > -m) + \Pr (Z > m) \right] = m,
\]
where probabilities are computed according to \( N (0, \sigma^2) \).

Finally, \( F \mu = m \implies F \varphi_1 (\mu) = F \varphi_1 (m / F) \leq \varphi_1 (m) \), by \( F < 1 \), \( \varphi_1 (0) = 0 \), and the concavity of \( \varphi_1 \). Combine with (B.1) to complete the proof.

**B.2 Proof of Theorem 4.7**

(ii) is proven in the text. It remains to prove (i).

Let \( c = 0 \), \( \varphi_1 \in C^2_0 (R) \), \( \varphi_1'' (x) \leq 0 \) for \( x \geq 0 \), and let \( \varphi \) be defined by (3.3), or by (A.3) with \( (\alpha, \beta) = (\varphi, \sigma) \). For small enough \( h > 0 \), let \( \{ H_t \}_{t \in [0, 1+h]} \) be the corresponding functions defined by (A.4).

First prove
\[
\lim_{n \to \infty} \left| E_{P^*} \left[ H_1 \left( \frac{\sum^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ \varphi \left( \frac{W_{1+h}^0}{\sqrt{n}} \right) \right] \right| = 0 \quad \text{(B.2)}
\]

We have
\[
E_{P^*} \left[ H_1 \left( \frac{\sum^n X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ \varphi \left( \frac{W_{1+h}^0}{\sqrt{n}} \right) \right]
\]
\[
= E_{P^*} \left[ H_1 \left( \frac{\sum^n X_i}{\sqrt{n}} \right) \right] - H_0 (0)
\]
\[
= \sum_{m=1}^n \left\{ E_{P^*} \left[ H_m \left( \frac{\sum^m X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ H_{m-1} \left( \frac{\sum^{m-1} X_i}{\sqrt{n}} \right) \right] \right\}
\]

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\[ \sum_{m=1}^{n} \left\{ E_{p^*} \left[ H_{m} \left( \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \right) \right] - E_{p^*} \left[ L_{m} \left( \frac{\sum_{i=1}^{n-1} X_i}{\sqrt{n}} \right) \right] \right\} \\
+ \sum_{m=1}^{n} \left\{ E_{p^*} \left[ L_{m} \left( \frac{\sum_{i=1}^{n-1} X_i}{\sqrt{n}} \right) \right] - E_{p^*} \left[ H_{m-1} \left( \frac{\sum_{i=1}^{n-1} X_i}{\sqrt{n}} \right) \right] \right\} = : J_{1n} + J_{2n}, \]

where \( L_{m}(x) = H_{m}(x) + \frac{\sigma^2}{2n} \left[ H_{m}'(x) \right]^+ - \frac{\sigma^2}{2n} \left[ H_{m}'(x) \right]^-, \ 1 \leq m \leq n. \)

By a similar argument to that in the proof of Lemma A.2, (using Lemma A.1(3) and the fact that \( E_{p^*} \left[ X_m | G_{m-1} \right] = I_{\{\sum_{i=1}^{m-1} X_i \leq 0\}} \sigma^2 + I_{\{\sum_{i=1}^{m-1} X_i > 0\}} \sigma^2 \)), deduce that

\[ \lim_{n \to \infty} |J_{1n}| = 0. \]

On the other hand, by Lemma A.1(5), (argue as in the proof that \(|I_{2n}| \to 0 \) in Appendix A.2), we have \( \lim_{n \to \infty} |J_{2n}| = 0. \) Thus we obtain (B.2).

By the definition of functions \( \{H_t\} \) and Lemma A.1(6), and arguing as at the end of Appendix A.2, the proof of (i) is complete.

### B.3 Proof of Theorem 4.8 and Remark 4.9

**Theorem 4.8**: Bayesian updating implies that \( \{\mu_n\} \) is a \( P^* \)-martingale adapted to \( \{G_n\} \). Since \( \{\mu_n\} \) is uniformly bounded, there exists a random variable \( \mu \) such that

\[ \lim_{n \to \infty} \mu_n = \mu \quad P^* \text{-a.s.} \]

Step 1: \( \mu = 0 \) or 1 \( P^* \)-a.s., which implies (4.19): Purely for simplicity, we give the argument when \( \mu + \bar{\mu} = 1 \); the proof for the general case will be evident.

We have \( P^*(\hat{\Omega}) = 1 \), where \( \hat{\Omega} = \{\omega \in \Omega \mid \lim_{n \to \infty} \mu_n(\omega) = \mu(\omega)\} \). For any \( \omega \in \hat{\Omega} \),

\[ \mu_n(\omega) = \frac{p \mu_{n-1}(\omega)}{p \mu_{n-1}(\omega) + \bar{p}(1 - \mu_{n-1}(\omega))} \quad \text{or} \quad \frac{\bar{p} \mu_{n-1}(\omega)}{\bar{p} \mu_{n-1}(\omega) + \mu(1 - \mu_{n-1}(\omega))}. \]

Thus, without loss of generality, there exists a subsequence \( \{\mu_{k_n}\} \) satisfying

\[ \mu_{k_n}(\omega) = \frac{p \mu_{k_n-1}(\omega)}{p \mu_{k_n-1}(\omega) + \bar{p}(1 - \mu_{k_n-1}(\omega))}, \]

which implies that

\[ \mu(\omega) = \frac{p \mu(\omega)}{p \mu(\omega) + \bar{p}(1 - \mu(\omega))}. \]
Thus \( \mu(\omega) = 0 \) or 1.

Step 2: For \( n \geq 1 \), define

\[
M_n = \min\{\mu_n, 1 - \mu_n\}, \quad \overline{M}_n = \max\{\mu_n, 1 - \mu_n\}
\]

Then, by the dominated convergence theorem,

\[
\lim_{n \to \infty} E_{P^*} \left[ M_n \right] = E_{P^*} \left[ \lim_{n \to \infty} M_n \right] = E_{P^*} \left[ \min\{\mu, 1 - \mu\} \right] = 0.
\]

For small enough \( h > 0 \), let \( \{H_t\}_{t \in [0,1+h]} \) be the functions defined in (A.4), and let \( \{L_t\}_{t \in [0,1]} \) be the functions defined in (A.5). We prove below that

\[
\lim_{n \to \infty} \sum_{m=1}^{n} \left| E_{P^*} \left[ H^m_n \left( \frac{\Sigma_1^m X_i}{\sqrt{n}} \right) \right] - E_{P^*} \left[ L^m_n \left( \frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| = 0. \tag{B.3}
\]

This is the counterpart for the present setting of the limit result (A.6) in the proof of our CLT (Lemma A.2), where instead of the expectation with respect to the single measure \( P^* \), one has the upper expectation \( E \) corresponding to the set of measures \( \mathcal{P} \). The proof of (B.3) roughly parallels the earlier arguments but the difference between \( E_{P^*} \) and \( E \) necessitates some adjustments (notably in Step 4).

Define

\[
d(m, n) = E_{P^*} \left[ H^m_n \left( \frac{\Sigma_1^m X_i}{\sqrt{n}} \right) + H^m_n \left( \frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m}{\sqrt{n}} + H^m_n \left( \frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \frac{X_m^2}{2n} \right].
\]

It suffices for (B.3) to prove that

\[
\sum_{m=1}^{n} \left| E_{P^*} \left[ H^m_n \left( \frac{\Sigma_1^m X_i}{\sqrt{n}} \right) \right] - d(m, n) \right| \to 0 \quad \text{and} \quad \sum_{m=1}^{n} \left| d(m, n) - E_{P^*} \left[ L^m_n \left( \frac{\Sigma_1^{m-1} X_i}{\sqrt{n}} \right) \right] \right| \to 0. \tag{B.4}
\]

Step 3: Prove (B.4). The argument is similar to that for (A.7).

Step 4: Prove (B.5). By (4.15), for any \( m \geq 1 \), \( E_{P^*}[X_m|\mathcal{G}_{m-1}] = 0 \), and

\[
E_{P^*}[X_m^2|\mathcal{G}_{m-1}] = \begin{cases} 
\sigma^2 M_m + \sigma^2 M_m & \text{if } \Sigma_1^{m-1} X_i \leq 0 \\
\sigma^2 M_m + \sigma^2 M_m & \text{if } \Sigma_1^{m-1} X_i > 0
\end{cases} \tag{B.6}
\]
Therefore, for \( C_1 \) equal to the uniform bounded of \(|H_1''(x)|\),

\[
\sum_{m=1}^{n} \left| d(m, n) - E_{P^s^*} \left[ L_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{m}} \right) \right] \right|
\leq \sum_{m=1}^{n} E_{P^s^*} \left[ \frac{1}{2n} \left( H''_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{m}} \right) \right) \right] + (\sigma^2 - \sigma^2 M_m - \sigma^2 M_m) \n + \sum_{m} E_{P^s^*} \left[ \frac{1}{2n} \left( H''_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{m}} \right) \right) - (\sigma^2 M_m + \sigma^2 M_m - \sigma^2) \right] \n \leq \frac{C_1 (\sigma^2 - \sigma^2)}{n} \sum_{m=1}^{n} E_{P^s^*} [M_m] \rightarrow 0 \quad \text{(by Step 2)}.
\]

**Remark B.1.** Step 4 involves a departure from the arguments of the CLT. In the latter, we had by assumption (1.2) that upper and lower conditional variances were constant and equal to \( \sigma^2 \) and \( \sigma^2 \) respectively, while here the relevant conditional variances are under \( P^s^* \) and are stochastic as shown in (B.6). Also noteworthy is that, while all other steps in the argument are valid for all strategies \( s \), Step 4 relies explicitly on \( s = s^* \).

**Step 5:** Complete the proof. It can be checked that,

\[
E_{P^s^*} \left[ H_1 \left( \frac{\sum_{i=1}^{m} X_i}{\sqrt{n}} \right) \right] - H_0 (0)
\]

\[
= \sum_{m=1}^{n} \left\{ E_{P^s^*} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m} X_i}{\sqrt{n}} \right) \right] - E_{P^s^*} \left[ H_{\frac{m-1}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{n}} \right) \right] \right\}
\]

\[
= \sum_{m=1}^{n} \left\{ E_{P^s^*} \left[ H_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m} X_i}{\sqrt{n}} \right) \right] - E_{P^s^*} \left[ L_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{n}} \right) \right] \right\}
\]

\[
+ \sum_{m=1}^{n} \left\{ E_{P^s^*} \left[ L_{\frac{m}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{n}} \right) \right] - E_{P^s^*} \left[ H_{\frac{m-1}{n}} \left( \frac{\sum_{i=1}^{m-1} X_i}{\sqrt{n}} \right) \right] \right\}
\]

\[
= \hat{J}_{1n} + \hat{J}_{2n}.
\]

By (B.3), we have \( \lim_{n \to \infty} |\hat{J}_{1n}| = 0 \). By Lemma A.1(5), (argue as in the proof that \( |I_{2n}| \to 0 \) in Appendix A.2), we have \( \lim_{n \to \infty} |\hat{J}_{2n}| = 0 \). Therefore,

\[
\lim_{n \to \infty} E_{P^s^*} \left[ H_1 \left( \frac{\sum_{i=1}^{m} X_i}{\sqrt{n}} \right) \right] - H_0 (0) = 0.
\]

By the definition of functions \( \{H_t\} \), with arguments similar to those at the end of Appendix A.2, we have

\[
\left| E_{P^s^*} \left[ \varphi \left( \frac{\sum_{i=1}^{m} X_i}{\sqrt{n}} \right) \right] - E_{P^s^*} [\varphi(W^0_t)] \right| \rightarrow 0 \quad \text{as} \quad n \to \infty. \quad \blacksquare
\]
Remark 4.9: (i) is proven in Step 1 above. It is assumed there that \( s = s^{*} \), but the identical arguments apply to any \( s \).

Consider (ii). Let \( \nu_n = \mu_n/(1 - \mu_n) \) and apply (4.16) to derive, for any \( s \),

\[
\log \nu_{n+1} - \log \nu_1 = \left[ (f^{s}_a(\omega^{(n)}) - f^{s}_{a,0}(\omega^{(n)})) - (f^{s}_b(\omega^{(n)}) - f^{s}_{b,0}(\omega^{(n)})) \right] \log \left( \frac{\nu}{\nu_1} \right)
\]

\[
+ \left[ f^{s}_{a,0}(\omega^{(n)}) - f^{s}_{b,0}(\omega^{(n)}) \right] \log \left( \frac{1 - \nu}{1 - \nu_1} \right).
\]

Define the sets

\[
N_a = \left\{ \omega : \lim_{n \to \infty} f^{s}_a(\omega^{(n)}) = \infty, \lim_{n \to \infty} f^{s}_b(\omega^{(n)}) < \infty \right\},
\]

\[
N_b = \left\{ \omega : \lim_{n \to \infty} f^{s}_a(\omega^{(n)}) < \infty, \lim_{n \to \infty} f^{s}_b(\omega^{(n)}) = \infty \right\},
\]

\[
N_{a,b} = \left\{ \omega : \lim_{n \to \infty} f^{s}_a(\omega^{(n)}) = \infty, \lim_{n \to \infty} f^{s}_b(\omega^{(n)}) = \infty \right\},
\]

\[
M_a = \left\{ \omega : \lim_{n \to \infty} f^{s}_{a,0}(\omega^{(n)}) = 1 - \nu \right\},
\]

\[
M_b = \left\{ \omega : \lim_{n \to \infty} f^{s}_{b,0}(\omega^{(n)}) = 1 - \nu_1 \right\}.
\]

Consider \( \omega \in N_{a,b} \cap M_a \cap M_b \): Then \( \log \nu_{n+1} - \log \nu_1 = \)

\[
- f^{s}_a \left[ \nu \log \left( \frac{\nu}{\nu_1} \right) + (1 - \nu) \log \left( \frac{1 - \nu}{1 - \nu_1} \right) \right]
\]

\[
- f^{s}_b \left[ \nu_1 \log \left( \frac{\nu}{\nu_1} \right) + (1 - \nu_1) \log \left( \frac{1 - \nu}{1 - \nu_1} \right) \right]
\]

\[
= -f^{s}_a H_1 - f^{s}_b H_2.
\]

By the concavity of \( \log \), \( H_1, H_2 < 0 \). Therefore, \( \nu_n \to \infty \), equivalently \( \mu_n \to 1 \), on \( N_{a,b} \cap M_a \cap M_b \). By the LLN, \( Q^s(N_{a,b} \cap M_a \cap M_b) = Q^s(N_{a,b}) \). Conclude that

\[
Q^s(N_{a,b} \cap \{\omega : \mu_n \to 1\}) = Q^s(N_{a,b}).
\]

Similar equations apply if \( N_{a,b} \) is replaced by either \( N_a \) or \( N_b \). Finally, since \( \{N_a, N_b, N_{a,b}\} \) is a partition of \( \Omega \), conclude that \( Q^s(\{\omega : \mu_n \to 1\}) = 1 \).
References


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