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Stationary Cardinal Utility and Optimal Growth under Uncertainty*

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In the literature on multiperiod planning under uncertainty, it is generally postulated that preferences may be represented by a von Neumann–Morgenstern utility index that is additive over time. This paper accomplishes two objectives: First, an axiomatic basis is provided for a more general class of non-additive utility indices defined over infinite consumption streams. Second, this class of utility functions is applied to extend existing results (*J. Econ. Theory* **4** (1972), 479–513; *J. Econ. Theory* **11** (1975), 329–339) on the nature of optimal growth under uncertainty. Of particular interest are the existence and stability of a stochastic steady state. *Journal of Economic Literature* Classification Numbers 022, 026.

I. INTRODUCTION

One way to understand the nature of the preferences we consider is as follows: Consider lotteries in which consumption in a subset N_1 of time periods is non-stochastic and common to all lotteries, and in which consumption in the set N_2 of remaining time periods is stochastic and varies across lotteries. Additivity of the utility index implies that for all N_1 and N_2 preferences over such lotteries are independent of the consumption levels in N_1. By adapting the terminology of [13, 17] we may express this property in the form "consumption in N_2 is risk independent of consumption in N_1, for all disjoint N_1 and N_2." We generalize preferences by requiring only that consumption in N_2 be risk independent of consumption in periods that precede all times in N_2, i.e., the future is risk independent of the past. A stationarity postulate is also imposed to derive a functional representation for the von Neumann–Morgenstern utility index.

The term stationary cardinal utility is drawn from the obvious analogy between the present analysis and the ordinal analysis in [14]. Indeed our postulates on preferences are natural extensions to a stochastic framework of those in [14]. The extended choice framework permits us to generate

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stronger results, with respect to the functional form of utility functions and the presence of impatience, than were obtained in the certainty framework. It is widely felt that the presence of uncertainty should strengthen the case for discounting. Our stronger result regarding the presence of impatience gives precise meaning to, and a rigorous basis for, this view.

In common with additive utility, stationary cardinal utility implies the following simplification of intertemporal planning: Suppose the planner is free to revise his plans at some $t > 0$ upon the arrival of new information. Then his decisions at t will depend on the past through accumulated assets but will not depend directly on past consumption activities. (In fact such behaviour is equivalent to the risk independence of future consumption from the past.) Thus standard dynamic programming techniques may be applied to solve the planning problem.

The paper proceeds as follows: Basic notation and definitions are presented next. Section 3 provides the axiomatic basis for stationary cardinal utility. Section 4 investigates optimal economic growth given such a utility index. Proofs are collected in appendices.

II. Basic Notation and Definitions

$Y \equiv \{y = (c_0, ..., c_t, ...): \ 0 \leqslant c_t \leqslant L \ \ \forall t\}$ is the set of non-stochastic consumption streams. Endow Y with the topology induced by the product norm $\|y\| \equiv \sum_0^\infty c_t/2^t$. If $y = (c_0, c_1, ...)$ and $c^1, ..., c^k \in [0, L]$, then $(c^1, ..., c^k, y)$ represents the consumption stream $(c^1, ..., c^k, c_0, c_1, ...)$. For any c, y_c denotes the constant consumption profile $(c, c, ...)$.

$M(Y)$ denotes the space of (countably additive) probability measures defined on the measurable space $(Y, R(Y))$, where $R(Y)$ is the Borel σ-field of Y. For each $y \in Y$, $p_y \in M(Y)$ denotes the element that assigns probability 1 to the set $\{y\}$. $D \equiv \{p_y: y \in Y\}$ is a subspace of $M(Y)$. Because of the obvious isomorphism we often identify D with Y and write y rather than p_y.

For any $c_0 \in [0, L]$ and $p \in M(Y)$, (c_0, p) denotes a new measure in $M(Y)$ defined as follows: For any Borel set R, the probability of R given the measure (c_0, p) is $p(R_{c_0})$, where $R_{c_0} \equiv \{(c_1, c_2, ...): (c_0, c_1, c_2, ...) \in R\}$. In words, (c_0, p) represents a consumption stream in which $t = 0$ consumption equals c_0 with certainty while consumption in later periods is stochastic with probability distribution corresponding to p.

For $p \in M(Y)$ and $y \in Y$, $(p_{0,1}, y)$ is that measure in $M(Y)$ which, to the Borel set R assigns the measure $p(R_y)$, where $R_y \equiv \{(\bar{c}_0, \bar{c}_1, \hat{c}_2, \hat{c}_3, ...):$ $(\bar{c}_0, \bar{c}_1, y) \in R, \ \hat{c}_t \in [0, L] \ \ \forall t\}$. Thus $(p_{0,1}, y)$ represents the consumption path in which consumption at time $t+2$ is certain and equals consumption at time t in y, $t \geqslant 0$, and in which consumption at $t = 0, 1$ is stochastic and corresponds to the marginal distribution defined by p.

The preference ordering $\gtrsim$ defined on $M(Y)$ is assumed to be reflexive, transitive and complete. Indifference is denoted by $\sim$ and strict preference by $\succ$. The ordering induced by $\gtrsim$ on Y (i.e., on D) is denoted $\gtrsim^Y$. A real valued function U with domain Y is called a von Neumann–Morgenstern utility index for $\gtrsim$ if $\int_Y U\,dp \geqslant \int_Y U\,dq$ is equivalent to $p \gtrsim q$ for all $p, q \in M(Y)$. Any such U is necessarily order preserving with respect to $\gtrsim^Y$, i.e., $U(y) \geqslant U(z)$ is equivalent to $p_y \gtrsim^Y p_z$ for $y, z \in Y$.

If for all $p, q \in M(Y)$ and $y, y' \in Y$, $(p_{0,1}, y) \gtrsim (q_{0,1}, y)$ is equivalent to $(p_{0,1}, y') \gtrsim (q_{0,1}, y')$, we say that consumption at $t = 0$ and 1 is *risk independent* of consumption in all other periods. A similar meaning is attached to the statement that consumption in the set of periods $N_1 \subset \{0, 1, 2, \ldots\}$ is risk independent of consumption in all other periods.

Finally, a function ψ is said to be increasing (decreasing) if $x > x'$ and $x \neq x'$ imply that $\psi(x) > (<)\, \psi(x')$. If only weak inequalities are implied we use the terms non-decreasing and non-increasing, respectively. ψ is concave if $\psi((x + x')/2) \geqslant [\psi(x) + \psi(x')]/2 \;\; \forall x, x'$ in the domain of ψ. If a strict inequality is always valid ψ is said to be strictly concave.

III. Stationary Cardinal Utility

Consider the following assumptions on $\gtrsim$, the preference ordering of probability measures:

Assumption 1. There exist $y, y' \in Y$ such that $y \succ^Y y'$.

Assumption 2. There exists $\hat{c}_0 \in [0, L]$ such that for all $p, q \in M(Y)$, $(\hat{c}_0, p) \gtrsim (\hat{c}_0, q)$ if and only if $p \gtrsim q$.

Assumption 3. For all $c_0, \bar{c}_0 \in [0, L]$ and $p, q \in M(Y)$, $(c_0, p) \gtrsim (c_0, q)$ if and only if $(\bar{c}_0, p) \gtrsim (\bar{c}_0, q)$.

The interpretation of these assumptions is clear. The first is a weak sensitivity requirement that rules out indifference between all consumption streams, and the second is a stationarity postulate. The latter states that the relative ranking of p and q is unaffected if the corresponding random consumption streams are postponed for one period and a particular certain consumption level $\hat{c}_0$ is placed into the initial period. (By the next assumption the same is true if any c_0 is substituted for $\hat{c}_0$.) Assumption 3 states that preferences over random consumption streams extending from $t = 1$ into the future are not affected by consumption at $t = 0$, i.e., consumption in periods $t > 1$ is risk independent of consumption in the initial period. (Risk independence has been investigated in [13, 17] though generally in finite horizon models and in symmetric form where consumption in any subset of periods is risk independent of consumption in the remaining periods.)

ASSUMPTION 4. There exists a von Neumann–Morgenstern utility index U for $\succsim$, which is a continuous function on its domain Y.

The existence of U may be *proven* from more basic postulates on $\succsim$. Theorem 3 of [10] may be applied directly to the present context because our choice of the product topology makes Y a separable metric space.

The following is the central result of this section:

THEOREM 1. *The preference ordering $\succsim$ satisfies Assumptions 1–4 if and only if the von Neumann–Morgenstern utility index U can be expressed in the form*[1]

$$U(y) = U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} v(c_t) \exp\left(-\sum_{\tau=0}^{t-1} u(c_\tau)\right), \tag{1}$$

where u and v are continuous real valued functions defined on $[0, L]$ such that $u > 0$ on $[0, L]$ and $v/(1 - e^{-u})$ is not constant on $[0, L]$.

Say that the pair of functions u and v represents $\succsim$ if U defined in (1) is a von Neumann–Morgenstern utility index for $\succsim$. The representing pair (u, v) is, of course, not unique.

COROLLARY. *Let $\succsim$ satisfy Assumptions 1–4. Then the two pairs (u, v) and $(\hat{u}, \hat{v})$ both represent $\succsim$ if and only if there exist constants a and b, $b > 0$, such that on $[0, L]$*

$$\hat{u} = u \qquad \textit{and} \qquad \hat{v} = a(1 - e^{-u}) + bv. \tag{2}$$

It is apparent from (1) that U and $\succsim$ exhibit some forms of discounting of the future. First, $\succsim^Y$ exhibits the following form of *impatience*: If the constant consumption stream y_c is preferred to $y_{c'}$, then the consumption in two successive periods of c, c' in that order is preferred to consumption in the reverse order. More precisely, $y_c \succ^Y y_{c'} \Rightarrow (c, c', y) \succ^Y (c', c, y)\ \forall y \in Y$. (Verification is straightforward; note that

$$U(y_c) = v(c)/[1 - e^{-u(c)}], \tag{3}$$

and substitute (3) into (1).) If more consumption is preferred to less, impatience expresses a preference for advancement of higher consumption levels.

[1] Note that stationary cardinal utility functionals may be defined in continuous time as well; they take the form $\int_0^\infty v(c) \exp(-\int_0^t u(c)\, d\tau)\, dt$. Also a minor notational point is that $\sum_0^{-1} \equiv 0$.

A measure of impatience is readily constructed as in [15, p. 97]. Assume the required differentiability and define $\rho(c)+1$ to be the marginal rate of substitution between consumption in periods 0 and 1 $((\partial U/\partial c_0)/(\partial U/\partial c_1))$ evaluated along the constant path y_c. Then ρ is the rate of time preference, $\rho > 0$, and ρ is given by

$$\rho(c) = e^{u(c)} - 1. \tag{4}$$

It is interesting to compare our findings with respect to the presence of discounting with those in [15, 14]. The latter studies establish the presence of impatience in certain zones in the program space. In contrast we have established that impatience prevails throughout Y. There are two major differences in the sets of postulates employed that might be expected to account for the difference in results. First we have topologized Y with the product topology rather than the topology generated by the sup norm. Since the former imposes the insignificance of the distant future the greater prevalence of impatience derived above would appear to be explained. In fact, however, we show in Appendix 2 that Theorem 1, suitably modified, remains valid if we substitute the sup topology for the product topology.

The other major difference is that we extend the choice environment to include uncertainty. (Then Koopman's postulate (3b) [14, p. 292] translates into our risk independence assumption.) Our stronger result regarding impatience thus seems to follow from the presence of uncertainty in the decision maker's environment. That is, we have an intriguing demonstration of the frequently expressed view that uncertainty contributes to impatience and discounting!

The positivity of ρ is a statement about $\succsim^Y$ and thus about the ordinal properties of U. But there is a cardinal property of U which corresponds to the following form of discounting in the preference ordering $\succsim$ on probability measures: Let $c \in [0, L]$, $y, y' \in Y$. Let $p \in M(Y)$ assign probability 1/2 to each of the sets $\{(c, y')\}$ and $\{y\}$, and let $p' \in M(Y)$ assign probability 1/2 to each of the sets $\{(c, y)\}$ and $\{y'\}$. Then $y \succ^Y y'$ implies that $p \succ p'$. Intuitively, p' is inferior because in it the better stream y is pushed one period into the future, while in p it is the less preferred stream y' that is receded into the future. (To establish this property, note that (1) implies

$$U(c, y) = v(c) + B(c)\, U(y), \qquad B(c) \equiv \exp(-u(c)). \tag{5}$$

Now $u > 0 \Rightarrow B < 1 \Rightarrow U(c, y) - U(c, y') = B(c)[U(y) - U(y')] < U(y) - U(y') \Rightarrow U(c, y) + U(y') < U(c, y') + U(y)$. This inequality is called (strong) time perspective in [15].)

To conclude this section we determine the restrictions on preferences implicit in the standard additive utility index beyond those corresponding to stationary cardinal utility. Additivity corresonds to the following additional risk independence assumption:

ASSUMPTION 5. For all $y, \bar{y} \in Y$ and $p, q \in M(Y)$, $(p_{0,1}, y) \gtrsim (q_{0,1}, y)$ if and only if $(p_{0,1}, \bar{y}) \gtrsim (q_{0,1}, \bar{y})$.

Recall that $(p_{0,1}, y)$ is a probability measure for which consumption is uncertain only at $t = 0, 1$. Assumption 5 states that preferences over such stochastic consumption streams are independent of consumption in periods beyond $t = 1$. Therefore, the "past" $(t = 0, 1)$ is risk independent of future consumption. Combined with our earlier assumption that the future is risk independent of the past one might expect additivity to follow. The next theorem confirms that expectation.

THEOREM 2. *Let $\gtrsim$ satisfy Assumptions* 1–4. *Then $\gtrsim$ satisfies Assumption* 5 *if and only if $\gtrsim$ has a von Neumann–Morgenstern utility index which has the form*

$$U(y) = \sum_{0}^{\infty} \alpha^t v(c_t) \tag{6}$$

for $0 < \alpha < 1$, *v continuous and not constant on* $[0, L]$.

Maintain Assumptions 1–4. Then Assumption 5 is an "efficient" characterization of additivity in the sense that weakened versions will not imply (6). For example, if it is required only that $t = 0$ consumption be risk independent of consumption in all other periods, then (1) will do if $e^{-u(c)} = a + bv(c)$ for some constants a and b. The "tightness" of our theorem differentiates it from the characterizations in [16].[2]

While Assumption 5 may apear to be a mild additional restriction given our earlier postulates, the difference between (1) and (6) is severe. First, for additive utility consumption in *any* set of time periods is risk independent of consumption in all other periods. Second, in the case of stationary cardinal utility the rate of time preference ρ, defined in (4), is not restricted to be constant. Since a variable rate of time preference constitutes an appealing generalization of (6), it is of interest to investigate whether existing results on intertemporal planning under uncertainty are robust to this generalization. Moreover, one may wonder whether any interesting new propositions may emerge. We turn now to these questions.[3]

[2] Another difference is that the ambiguity between additive and multiplicative utility functions present in [16, Theorem 2] vanishes in our infinite horizon framework. There does not exist a multiplicative utility index consistent with Assumptions 1–4.

[3] For analyses of deterministic models see [2, 3, 12, 8].

IV. Optimal Growth under Uncertainty

The model of growth under uncertainty considered in this section is similar to the models in [5, 18]; hence only essential features will be described:

A central planner solves the following problem:

$$\max EU(c_0, c_1, \ldots, c_t, \ldots) \tag{7}$$

subject to the constraints

$$c_t + x_t = f(x_{t-1}, \tilde{r}_{t-1}), \qquad t = 1, 2, \ldots,$$
$$c_0 + x_0 = s > 0, \qquad x_t, c_t \geqslant 0.$$

Here c_t and x_t are consumption and capital stock at period t, respectively. s is the initial stock. f is a production function which at time t is affected by the random variable $\tilde{r}_t$. The $\tilde{r}_t$'s are independently distributed and have the same distribution as the random variable $\tilde{r}$. The latter is defined on the probability space $(\Omega, \mathscr{F}, P)$. (Realizations of $\tilde{r}_t$ and $\tilde{r}$ are denoted by r_t and r, respectively.) The problem of the planner is to divide the available stock at the beginning of each period between consumption and investment. Decisions are determined by the preference ordering of stochastic consumption streams which is represented by the von Neumann–Morgenstern utility index U in (1).

The production function is assumed to satisfy the following properties: $f(\cdot, r)$ is an increasing, concave differentiable function for all r, with $f(0, r) = 0$ for all r. It is assumed that $f(x, \cdot)$ is increasing for all x, i.e., the production function is ordered.[4] Further properties are specified below.

Realizations of $\tilde{r}$ lie in the interval $[\alpha, \beta]$. The mapping $\tilde{r}: \Omega \to [\alpha, \beta]$ generates in the usual fashion a measure ν on the Borel subsets of $[\alpha, \beta]$.

Only minimal assumptions on $\succsim$ and U were made in the last section. In this section we add the following:

Assumption 6. U is increasing and strictly concave.

The question of existence of a solution to (7) is substantially the same as the corresponding existence question given additive utility. Henceforth assume the existence of a solution for (7). (In the case of Theorem 3 below existence follows from our assumptions as in [5, pp. 487–488]. Otherwise existence may be proven by adapting the very general analysis in [6] to stationary cardinal utility.) By the strict concavity of U the solution is unique.

[4] The role of this assumption is discussed in the proof of Theorem 3.

Denote by $J(s)$ the value of the problem (7). Since U and $f(\cdot, r)$ are concave, so is J. Because of (5) a dynamic programming argument yields

$$J(s) = \max_{0 \leqslant c \leqslant s} \{v(c) + B(c) \cdot EJ[f(s-c, \tilde{r})]\}, \qquad B \equiv \exp(-u). \tag{8}$$

Optimal behaviour in (14) is summarized by the consumption function $c = g(s)$ which also solves (8).

Two additional assumptions are adopted:

ASSUMPTION 7. $g(s)$ and $s - g(s)$ are both positive and increasing in s.

ASSUMPTION 8. u and v are continuously differentiable and $u' > 0$.

Interior solutions, i.e., $g(s)$, $s - g(s) > 0$, are easily guaranteed as shown below. The second part of Assumption 7 expresses the fact that both consumption and investment are "normal" goods in (7). In contrast with the additive utility model, that is not necessarily the case for stationary cardinal utility. Below we provide conditions on u and v sufficient for such normalcy.

The assumption that $u' > 0$ requires some comment. From (4) it requires that the rate of time preference along a constant consumption path increase with the level of consumption. A priori this hypothesis appears to be as reasonable as the opposite hypothesis that $\rho'(c) < 0$. In fact a stronger justification may be provided for our assumption—it follows from [2, 3] that if $\rho' < 0$ then, in general, in deterministic versions of the model (7), there exist many steady states and some of them are locally unstable. Thus $\rho' \geqslant 0$ ($u' \geqslant 0$) is necessary if we are to establish even local stability results in our stochastic model. When u and ρ are constant existing analyses with additive utility apply. For simplicity we rule out any points where u' and ρ' vanish.

Another perspective on Assumption 7 is possible. It is immediate from (5) and Assumption 6 that the second order partial derivative $U_{c_0 c_1}$, and all other mixed second order partials of U when they exist, have the sign of $-u'$. Thus $u' > 0$ implies $U_{c_t c_\tau} < 0$ $\forall t \neq \tau$. By [7] this corresponds to an aversion to (generalized) correlation in the random consumption in any two periods.[5] Of course, additive utility imposes indifference to correlation.

Denote by $F_t(x)$ the cumulative distribution function of x_t^*, the capital stock at time t when the planner acts optimally, i.e., according to the consumption function g. The following theorem establishes the existence and stability of a steady state distribution and generalizes [5, Theorem 4.1] and [18, Theorem 2].

[5] The $\tilde{r}_t$'s are independently distributed but along random consumption paths that are feasible in (7), consumption is not independently distributed across time periods. Thus attitudes towards correlation play a role in determining optimal consumption paths.

THEOREM 3. *In addition to the properties for f specified above, assume that $f(\cdot, r)$ is strictly concave $\forall r$ and that $f'(0, r) = \infty$, $f'(\infty, r) = 0$ $\forall r$. Then, under Assumptions* 1–4 *and* 6–8, *there exists a distribution function $F(x)$ such that $F_t(x) \to F(x)$, as $t \to \infty$, uniformly for all x. Furthermore, $F(x)$ does not depend on the initial stock s, and it does not carry an atom at $x = 0$.*

We assumed above that $f(\cdot, r)$ is strictly concave for all r, though it suffices that $\{r: f(\cdot, r)$ is strictly concave$\}$ have positive measure. Otherwise, it is clear that with additive utility and a constant rate of discount a stable steady state distribution, which does not assign all mass to zero or infinite capital stock size, does not in general exist;[6] simply think of the certainty model. But with utility specified by (1) a stable steady state "frequently" exists in the certainty version of model (7) even if f is linear in x. (See [3, 8].) Thus the following theorem should not be totally surprising:

THEOREM 4. *Consider problem* (7), *where $f(x, r) = rx$, $r \in [\alpha, \beta]$. Maintain Assumptions* 1–4 *and* 6–8 *and suppose that*

$$e^{u(0)} < \alpha, \qquad \beta < \lim_{c \to \infty} e^{u(c)}. \tag{9}$$

Then there exists a distribution function $F(x)$ satisfying the conditions in Theorem 3.

The inequality $e^{u(0)} < \alpha$ implies that for small levels of consumption c the rate of time preference $\rho(c)$ is less than $(\alpha - 1)$, the least favourable net return to investment. Similarly, the other inequality states that for large consumption levels $\rho(c)$ is less than $\beta - 1$, the most favourable net return to investment. These inequalities replace the Inada conditions in the proof of existence of a steady state for the case of a linear production function.

It remains to specify restrictions on u and v sufficient to imply Assumptions 6 and 7.

THEOREM 5. *Let the production function f be such that $f(\cdot, r)$ is increasing and concave for each r. Then U and $\succsim$ satisfy Assumptions* 6–8 *if*

$$v < 0, v' > 0, v'(0) = \infty, \qquad \log(-v) \text{ is convex}, \tag{10}$$

$$u > 0, u' > 0, \qquad u \text{ is strictly concave}, \tag{11}$$

$$v'e^u \text{ is non-increasing.}^7 \tag{12}$$

[6] Schectman and Escudero prove the existence of a steady state distribution in a model with $f(x, r) = ax + r$, where $a > 0$ is a constant. But their result is possible only because of their assumption of no borrowing against future income.

[7] If $v' \exp(u)$ is decreasing, concavity of u may be substituted for the assumption of strict concavity of u. If $f(\cdot, r)$ is strictly concave for all r, $v' > 0$ may be weakened to $v' \geqslant 0$.

The postulates for v imply that v may be written in the form

$$v = -e^{-\phi} \qquad \textit{for some concave function } \phi,\ \phi' > 0. \tag{13}$$

Thus v is increasing and concave, standard properties. In fact v is an increasing and concave transformation of a concave function ϕ and so is "more than concave." The stronger requirement for v is needed not to prove that U is concave. It is used to establish that consumption and investment are normal as in Assumption 7. The concavity of u and (12) are used for the same purpose. Given the required differentiability the latter condition states that $-v''/v' \geqslant u'$; in a sense this requires that the variability of the rate of time preference, and hence the divergence from the standard additive utility model, not be too great. But (10)–(12) leave a great deal of scope for specifying stationary cardinal utility functions. One simple example is $v(c) \equiv c^{1-A}/(1-A)$, $A > 1$ and u any positive, increasing and strictly concave function such that $u'(c) \leqslant A/c$ for all c.

The conditions in (10) and (12) are not invariant to the "admissible" transformations defined in the Corollary. Nevertheless, as shown by the Theorem, their conjunction with (11) constitute meaningful statements about the cardinal index U and hence about $\succsim$. (Individual conditions that are each invariant to such transformations are complex and difficult to determine. For example, define $w(c) \equiv U(y_c) = v(c)/[1 - B(c)]$, $B \equiv e^{-u}$. Then by (5), U is increasing if and only if $v(c_0) + B(c_0)\theta$ is increasing in c_0 for every θ in the range of w. This constitutes an invariant restriction on the (u, v) pair, which is implied by the assumptions $v < 0$ ($\Rightarrow w < 0$), $v' > 0$ and $u' > 0$. The analysis of curvature is much more complex.)

Example. It may be useful to provide an example to illustrate the above results, in particular Theorem 4. Concrete examples are difficult to construct. The following example conforms with all but one of the hypotheses of Theorem 4 but still provides an instance where a stable steady state distribution exists.

Let $v(c) = -1$ for all c and

$$\begin{aligned} u(c) &= 2\log(c^{1/2} + 1) - \log(9/4), && c \geqslant 0.36, \\ &= -\infty, && c < 0.36. \end{aligned}$$

For $c \geqslant 0.36$, u is positive, increasing and strictly concave. $c = 0.36$ is a subsistence level; if consumption is less than 0.36 in any period then lifetime utility, as defined in (1), equals $-\infty$. Thus in the planning problem (7) only feasible paths that provide $c_t \geqslant 0.36$ for all t, with probability 1, need be considered. It can be verified that Theorems 3 and 4 may be extended to include such utility functions.

The production function is $f(x, r) = rx$, where $\tilde{r}$ varies over the interval $[2, 3]$, $E\tilde{r}^{-1} = 4/9$. (The probability distribution of $\tilde{r}$ is otherwise unrestricted.) For any initial stock s, $s \geqslant 0.96$, the problem (7) possesses a solution. Optimal consumption is given by $g(s) = [-1 + \sqrt{1 + 4s}]/2$, which satisfies Assumption 7. Thus Theorem 4 (suitably modified) implies that the distribution of capital stock converges to a steady state distribution F. Capital stock evolves according to the rule $x_{t+1} = H(x_t, r)$, $H(x, r) \equiv \frac{1}{2}\{[1 + 4rx]^{1/2} - 1\}$. The support of the steady state distribution lies in the interval $[x_m, x_M]$, where $H(x_m, 2) = x_m$ and $H(x_M, 3) = x_M$. Thus $[x_m, x_M] = [1, 2]$.

To prove these assertions argue as follows: let $J(s) = -1 - 9s^{-1}/4$. Verify that $J(s) = \mathrm{Max}_{0.36 \leqslant c \leqslant s}\{-1 + e^{-u(c)}EJ(\tilde{r}(s - c))\}$ for $s \geqslant 0.96$, and that $g(s)$ yields the maximizing consumption level. Stochastic consumption streams that are feasible in (7) may be represented by the measurable functions $(c_0, \tilde{c}_1, ..., \tilde{c}_t, ...)$, where for $t \geqslant 1$, $\tilde{c}_t\colon \mathsf{X}_{i=0}^{t-1}[2, 3] \to [0.36, \infty)$; $\tilde{c}_t$ maps $(r_0, ..., r_{t-1})$ into $\tilde{c}_t(r_0, ..., r_{t-1})$ for $t > 0$, $c_0 \in [0.36, \infty)$ is a constant. The corresponding sequence of capital stocks is represented by the functions $\tilde{s}_t\colon \mathsf{X}_{i=0}^{t-1}[2, 3] \to [0.48, \infty)$, $\tilde{s}_t(r_0, ..., r_{t-1}) = r_{t-1}(\tilde{s}_{t-1} - \tilde{c}_{t-1})$, $t \geqslant 1$, and $\tilde{s}_0 \equiv s$. (Note that if capital stock is less than 0.48, then the constraint $\tilde{c}_t \geqslant 0.36$ is violated with certainty for some t.) Repeated application of the dynamic programming equation implies that

$$J(s) \geqslant E\left[-1 - \sum_{t=0}^{T-1} \exp\left(-\sum_{\tau=0}^{t} u(\tilde{c}_\tau)\right)\right]$$
$$+ E\left[J(\tilde{s}_{T+1}) \cdot \exp\left(-\sum_{t=0}^{T} u(\tilde{c}_t)\right)\right]. \tag{14}$$

But the last term is bounded above by zero and below by $J(0.48)\exp(-\sum_{t=0}^{T} u(0.36))$ which converges to zero as $T \to \infty$. Thus as $T \to \infty$ in (14) we conclude that expected lifetime utility along any feasible path cannot exceed $J(s)$. Equality holds in (14) for each T and in the limit if the random consumption sequence chosen is that corresponding to the function g, i.e., $c_0 = g(s)$, $\tilde{c}_1(r_0) \equiv g(\tilde{s}_1(r_0))$, $\tilde{c}_2(r_0, r_1) \equiv g(r_1(\tilde{s}_1(r_0) - g(\tilde{s}_1(r_0))))$, $(\tilde{s}_1(r_0) \equiv r_0(s - g(s)))$, and so on. Thus the consumption function g solves the planning problem (7) and yields lifetime utility $J(s)$. g defines the unique solution because U is strictly concave.

APPENDIX 1

Proof of Theorem 1. First prove the necessity of (1).

For all $p, q \in M(Y)$ and $c_0 \in [0, L]$, $\int_Y U(c_0, y)\, dp \geqslant \int_Y U(c_0, y)\, dq$ iff

$(c_0, p) \succsim (c_0, q)$ iff $(0, p) \succsim (0, q)$ (by risk independence), iff $\int_Y U(0, y)\, dp \geqslant \int_Y U(0, y)\, dq$, i.e., $U(c_0, \cdot)$ and $U(0, \cdot)$ are von Neumann–Morgenstern utility indices for the same preference ordering on $M(Y)$. Thus they must be related by a linear transformation, i.e., there exist functions a and b, $b > 0$ so that

$$U(c_0, y) = a(c_0) + b(c_0)\, U(0, y) \qquad \forall c_0 \in [0, L], y \in Y. \tag{15}$$

(Uniqueness of the von Neumann–Morgenstern utility index up to a positive linear transformation follows from standard arguments. See [11, pp. 221–222], for example. The infinite dimensionality of Y is of no consequence.)

Similar reasoning applied to Assumption 2 yields

$$U(0, y) = \hat{a} + \hat{b} U(y) \qquad \forall y \in Y, \tag{16}$$

where $\hat{a}$ and $\hat{b}$ are constants, $\hat{b} > 0$. Combine (15) and (16) and deduce that

$$U(c_0, y) = v(c_0) + B(c_0)\, U(y) \qquad \forall c_0 \in [0, L], y \in Y, \tag{17}$$

for some functions v and B defined on $[0, L]$, $B > 0$.

Suppose $B(c) > 1$ for some c and pick $y, y' \in Y$ such that $U(y) > U(y')$. Then repeated application of (17) implies that $U(y^n) - U(y'^n) > [B(c)]^n [U(y) - U(y')] \to \infty$ as $n \to \infty$, where

$$y^n \equiv (\overbrace{c, \ldots, c}^{n}, y) \qquad \text{and} \qquad y'^n \equiv (\overbrace{c, \ldots, c}^{n}, y').$$

But that contradicts the boundedness of U. (U is bounded because it is continuous and Y is compact in the product topology.)

Thus $B \leqslant 1$. Suppose $B(c) = 1$ for some c. From (17), $U(y_c)[1 - B(c)] = v(c) \Rightarrow v(c) = 0$, and so $U(y^n) = U(y)$ for all y, where

$$y^n \equiv (\overbrace{c, \ldots, c}^{n}, y).$$

But $y^n \to y_c \Rightarrow U(y^n) \to U(y_c) \Rightarrow U(y) = U(y_c)$ for all $y \in Y$. By Assumption 4, $p \sim q$ $\forall p, q \in M(Y)$, which contradicts Assumptions 1 and 2. Therefore $0 < B < 1$ on $[0, L]$ and

$$U(y_c) = v(c)/[1 - B(c)] \qquad \forall c \in [0, L]. \tag{18}$$

Now show that v and B are continuous. Let $c^n \to c^0$. For any y and y', $U(c^n, y) \to U(c^0, y)$ and $U(c^n, y') \to U(c^0, y')$. By (17),

$$[v(c^n) - v(c^0)] + U(y)[B(c^n) - B(c^0)] \to 0,$$

$$[v(c^n) - v(c^0)] + U(y')[B(c^n) - B(c^0)] \to 0.$$

We can pick y and y' so that $U(y) \neq U(y')$. Therefore $B(c^n) \to B(c^0)$ and $v(c^n) \to v(c^0)$.

To prove (1), define

$$u(c) \equiv -\log B(c) \qquad \text{or} \qquad B(c) = e^{-u(c)} \qquad \forall c \in [0, L]. \tag{19}$$

Apply (17) repeatedly and obtain

$$U(y) = \sum_0^T v(c_t) \exp\left(-\sum_0^{t-1} u(c_\tau)\right) + \exp\left(-\sum_0^T u(c_t)\right) \cdot U(c_{T+1}, c_{T+2}, \ldots), \tag{20}$$

for all $T > 0$. v is continuous and hence bounded on $[0, L]$ and $\min\{u(c)$: $c \in [0, L]\}$ exists and is positive. Therefore the first term on the right side of (20) converges as $T \to \infty$. The second term approaches zero since U is bounded.

Finally, suppose that $v(c) = K[1 - e^{-u(c)}]$ $\forall c$ and establish a contradiction. Equation (17) takes the form $U(c, y) - K = B(c)(U(y) - K)$. Repeated application of the latter equation implies that $\forall y \in Y$, $y = (c_0, c_1, \ldots)$, $U(y) - K = \exp(-\sum_0^T u(c_t)) \cdot [U(c_{T+1}, \ldots) - K]$. Take the limit as $T \to \infty$ and apply the boundedness of U to deduce that $U(y) = K$ $\forall y \in Y$. But this contradicts Assumption 1.

The proof of the sufficiency of (1) is straightforward. ∎

Proof of Corollary. Let $\hat{U}$ correspond to $\hat{u}$ and $\hat{v}$ as in (1). If (2), then $\hat{U} = a + bU$ so $(\hat{u}, \hat{v})$ represents $\succsim$. For the converse, suppose we are given $\hat{U} = a + bU$. We need to prove (8). Make repeated use of (17). Let $B \equiv e^{-u}$, $\hat{B} \equiv e^{-\hat{u}}$. For any $(c, y) \in Y$, $\hat{U}(c, y) = a + bU(c, y) = \hat{v}(c) + \hat{B}(c)[a + bU(y)] \Rightarrow$

$$U(c, y) = -\frac{a}{b} + \frac{\hat{v}(c)}{b} + \frac{a}{b}\hat{B}(c) + \hat{B}(c)\, U(y). \tag{21}$$

Combine (21) with the equation $U(c, y) = v(c) + B(c)\, U(y)$ to derive

$$[B(c) - \hat{B}(c)]\, U(y) = [-a + \hat{v}(c) + a\hat{B}(c) - v(c)b]/b. \tag{22}$$

Equation (22) is valid also if y is replaced by $\bar{y}$ such that $U(y) \neq U(\bar{y})$. Subtract the two versions of (22) to deduce that $[B(c) - \hat{B}(c)][U(y) - U(\bar{y})] = 0$, or $B(c) = B(\hat{c})$. Therefore $u = \hat{u}$. The rest of (2) now follows from (22). ∎

Proof of Theorem 2. The sufficiency of (6) is clear. Turn to the proof of necessity.

Suppose first that $\exists y \in Y$ such that $(c, y) \sim^Y (\bar{c}, y)$ $\forall c, \bar{c} \in [0, L]$. By stationarity and Assumption 5 it follows that $(c, y) \sim^Y (\bar{c}, y)$ $\forall c, \bar{c}$ and $\forall y \in Y$. Use (17) to deduce that $[B(\bar{c}) - B(c)][U(\bar{y}) - U(y)] = 0$ $\forall c, \bar{c}$ and $\forall y, \bar{y} \in Y$. By Assumption 1 it must be the case that $B(c)$ is constant and (6) follows from (1). Therefore we may proceed on the assumption $(*)$ that $\forall y \in Y\ \exists c, \bar{c}$ such that $(c, y) \succ^Y (\bar{c}, y)$.

When restricted to nonstochastic consumption streams, the risk independence assumptions imply separability restrictions on $\succsim^Y$. In particular, Assumption 5 implies that (c_0, c_1) is weakly separable and Assumption 3 implies that $(c_1, c_2, ...)$ is weakly separable. Apply Gorman's overlapping theorem [9, Theorem 1] to deduce that U may be expressed in the form

$$U(c_0, c_1, c_2, ...) = F[\phi^0(c_0) + \phi^1(c_1) + \phi^2(c_2, c_3, ...)], \tag{23}$$

where G is increasing and all functions are continuous. (The hypotheses in Gorman's theorem concerning essential and strictly essential sectors are satisfied because of $(*)$ above. Note that Gorman's theorem is not restricted to finite dimensional spaces; it is applicable here because Y is topologically separable and arc connected.) Since consumption at $t = 0, 1$ is risk independent of consumption in other periods, it follows that $U(c_0, c_1, y) = a(y) + b(y)\,\psi[\phi^0(c_0) + \phi^1(c_1)]$ for some functions $a, b > 0$ and ψ and for all $(c_0, c_1, y) \in Y$. Combine this equation with (23) to deduce that F satisfies the functional equation $F[\phi^0(c_0) + \phi^1(c_1) + \phi^2(y)] = a(y) + b(y)\,\psi[\phi^0(c_0) + \phi^1(c_1)] = \hat{a}(\phi^2(y)) + \hat{b}(\phi^2(y))\,\psi[\phi^0(c_0) + \phi^1(c_1)]$ for some functions $\hat{a}$ and $\hat{b}$. The functions ϕ^0, ϕ^1 and ϕ^2 are continuous by the above arguments and are not constant because of $(*)$ and Assumption 2. Thus the ranges of these functions are nondegenerate intervals and F satisfies the following functional equation for all z, z' lying in some interval on the real line: $F(z + z') = \hat{a}(z) + \hat{b}(z)\,\psi(z')$. By [1, Corollary 1, p. 150] there exist only two systems of solutions F (increasing) to this functional equation. We consider each possibility in turn.

Case 1. $F(z) = \gamma e^{\alpha z} + \hat{\gamma}$, $\alpha \neq 0$. With no loss of generality assume $\gamma = 1$, $\hat{\gamma} = 0$. By (23) we see that

$$U(c_0, c_1, y) = \exp(\alpha\phi^0(c_0) + \alpha\phi^1(c_1)) \cdot \exp(\alpha\phi^2(y)). \tag{24}$$

By stationarity (Assumption 2) it must be the case that $\exp(\alpha\phi^2(y))$ is a positive linear transformation of $U(y)$. Thus there exist $\hat{A}$ and $A > 0$ such that

$$U(c_0, c_1, y) = \exp(\alpha\phi^0(c_0) + \alpha\phi^1(c_1)) \cdot [\hat{A} + AU(y)]. \tag{25}$$

By similar reasoning (24) $\Rightarrow$

$$U(c, y) = \exp(\alpha\phi^0(c)) \cdot [\hat{G} + GU(y)], \qquad \hat{G} > 0, \forall (c, y) \in Y. \tag{26}$$

Apply (26) again to derive

$$U(c_0, c_1, y) = \hat{G}\exp(\alpha\phi^0(c_0)) + \hat{G}G\exp(\alpha\phi^0(c_1)) + G^2U(y) \cdot \exp[\alpha\phi^0(c_0) + \alpha\phi^1(c_1)]. \tag{27}$$

Equate (25) and (27) for two choices of y with different utilities according to U. Deduce that $\forall c_1$, $G^2\exp(\alpha\phi^0(c_1)) = A\exp(\alpha\phi^1(c_1))$, $\hat{G} + \hat{G}G\exp(\alpha\phi^0(c_1)) = \hat{A}\exp(\alpha\phi^1(c_1))$ and therefore that $\hat{G} = \exp(\alpha\phi^0(c_1)) \cdot [G^2\hat{A}/A - G\hat{G}]$. Since ϕ^0 is not constant, $\hat{G} = 0$ necessarily. Now (26) $\Rightarrow U(c, y) = G\exp(\alpha\phi^0(c)) \cdot U(y)$. In combination with (17) this implies $v(c) = 0$ $\forall c$, which contradicts Theorem 1. Thus Case 1 is impossible.

Case 2. F is linear, $F(z) = Az + \hat{A}$, $A > 0$. Without loss of generality suppose $A = 1$ and $\hat{A} = 0$. By (23), $U(c_0, c_1, y) = \phi^0(c_0) + \phi^1(c_1) + \phi^2(y) \Rightarrow$ (by stationarity) $U(c, y) = \phi^0(c) + [\hat{G} + GU(y)]$, $G > 0$. Combine this equation with (17) to derive $\phi^0(c) + \hat{G} + GU(y) = v(c) + B(c)\, U(y)$ $\forall (c, y) \in Y$. Apply this equation also for y' such that $U(y') \neq U(y)$ and conclude that $B(c) = G$ $\forall c \in [0, L]$. Thus B is constant and (6) is implied by (1). ∎

Proof of Theorem 3. Consider problem (7) with value $J(s)$. We argued above that J is concave and that (8) is satisfied. We proceed by adapting the arguments in [5, 18] to prove a series of preliminary results. (Alternatively, the results in [4] could be invoked. But to justify the hypotheses in the latter study much of the following argument would still be necessary, particularly in proving Theorem 4.)

First, use (5) and the argument in [18, Lemma 1] to prove that J is differentiable and that

$$J'(s) = v'(g(s)) = u'(g(s))\, e^{-u(g(s))} EJ(f(s - g(s), \tilde{r})), \qquad s > 0. \tag{28}$$

Combine (28) and the first order condition for an interior optimum in (8) to derive

$$J'(s) = e^{-u(g(s))} E[J'(f(s - g(s), \tilde{r})) \cdot f'(s - g(s), \tilde{r})], \qquad s > 0. \tag{29}$$

J concave $\Rightarrow J'$ is non-increasing. But by Assumption 7 the right side of (29) is decreasing in s as long as $u' > 0$ and f is concave. Thus J' is decreasing.

Let $h(x) \equiv x - g(x)$, $x \geqslant 0$. By the argument in [5, Lemma 1.2] g and h are continuous, $g(0) = h(0) = 0$.

Define $f_m(x) \equiv f(x, \alpha)$, $f_M(x) \equiv f(x, \beta)$, $H(x, r) \equiv h(f(x, r))$, $H_m(x) \equiv h(f_m(x))$, $H_M(x) \equiv h(f_M(x))$. Let $d(x, r) \equiv J'(f(x, r))$. Then $d(\cdot, r)$ is decreasing for all r and (29) may be rewritten in the form

$$d(x, r) = \exp[-u(g(f(x, r)))] \cdot \int_{\alpha}^{\beta} d(H(x, r), \mu) f'(H(x, r), \mu)\, \nu(d\mu). \tag{30}$$

Define $x_m \equiv \max\{x \geqslant 0: H_m(x) = x\}$ and $x_M \equiv \min\{x > 0: H_M(x) = x\}$. x_m could equal zero. (See [19].) Brock and Mirman (p. 498) prove that x_M is well defined, by using the hypothesis that the production functions are ordered. Mirman and Zilcha [18] delete the latter hypothesis but do not address the question of whether x_M is well defined. It is not clear whether their other hypotheses imply the existence of even one positive fixed point for H_M. Note that if 0 is the only fixed point of H_M and if $x_M \equiv 0$, the remaining arguments in [18] imply that the steady state distribution has support in $[x_m, x_M] = \{0\}$, i.e., capital stock converges to 0 with probability 1. Since we have made the assumption of ordered production functions we may use the argument of [5, Lemma 3.1], adapted to apply to (30) above, to prove that $x_M > 0$ is well defined.

The next critical step in the proof is to show that

$$a = H_m(a) \quad \text{and} \quad b = H_M(b) \Rightarrow a \leqslant b. \tag{31}$$

As in [5, Lemma 3.4] we can show that $a = H_m(a)$, $b = H_M(b) \Rightarrow$

$$\exp[-u(g(f(b, \beta)))] \int f'(b, r)\, \nu(dr)$$
$$\leqslant \exp[-u(g(f(a, \alpha)))] \int f'(a, r)\, \nu(dr). \tag{32}$$

Suppose that $b < a$. Then $b = H_M(b) = h(f(b, \beta)) < a = H_m(a) = h(f(a, \alpha)) \Rightarrow f(b, \beta) < f(a, \alpha) \Rightarrow -u(g(f(b, \beta))) > -u(g(f(a, \alpha)))$ since u is increasing. Since $f'(b, r) \geqslant f'(a, r)\ \forall r$, (32) cannot hold. Thus $b \geqslant a$ necessarily.

Subsequent arguments in [5, 18] rely only on the results derived above and not explicitly on the structure of the utility function. Thus they apply here unaltered to complete the proof. That there cannot be an atom at 0 was pointed out in [19, pp. 112, 127]. ∎

Proof of Theorem 4. In the proof of Theorem 3, strict concavity of f was not used at all. The Inada conditions ($f'(0, r) = \infty$ and $f'(\infty, r) = 0\ \forall r$) were used only in the proof that $x_M > 0$ is well defined. But it is straightforward to rewrite the proof when (9) is substituted for the Inada conditions and $f(x, r) = rx$. ∎

The proof of Theorem 5 requires two preliminary lemmas:

LEMMA 1. *Define the function V on consumption sequences by $U = -e^{-V}$, U defined in* (1). *If* (10) *and* (11) *are valid, then V is concave.*

Proof. From (10), $v = -e^{-\phi}$, ϕ concave. For each $T \geqslant 0$ define $U^T(c_0, \ldots, c_T) \equiv -\sum_0^T \exp\{\phi(c_t) - \sum_0^{t-1} u(c_\tau)\}$, $V^T \equiv -\log(-U^T)$.

First we prove by induction that for each T the function that maps $(c_0, \ldots, c_T)$ into $V^T(c_0, \ldots, c_T) - \phi(c_0)$ is concave. That is certainly true for $T = 0$ since $V^0(c_0) - \phi(c_0) = 0$. Assume for T and prove for $T+1$. Because of (5), $V^{T+1}(c_0, \ldots, c_{T+1}) = -\log[\exp(-\phi(c_0)) + \exp(-u(c_0) - V^T(c_1, \ldots, c_T))]$ and so

$$V^{T+1}(c_0, \ldots, c_{T+1}) - \phi(c_0) = -\log\{1 + \exp[-u(c_0) - (V^T - \phi(c_0))]\}. \quad (33)$$

By the induction hypothesis $-\log\{\exp[-u(c_0) - (V^T - \phi(c_0))]\}$ is concave. But that implies the concavity of the right side of (33). (For any function ψ, $-\log \psi$ concave $\Rightarrow$ $-\log(1 + \psi)$ concave, essentially because the logarithm function exhibits declining absolute risk aversion.) Thus the left side of (33) also defines a concave map.

Hence V^T is concave for each T. The concavity of V follows since $V(c_0, c_1, \ldots) = \lim_{T \to \infty} V^T(c_0, \ldots, c_T)$. ∎

LEMMA 2. *Assume* (10) *and* (11). *Let* $J(s)$ *denote the value of* (7). *Then* J'/J *is an increasing function.*

Proof. It is enough to prove that $G(s) \equiv \log(-J(s))$ defines a convex function G. (Note that $v < 0 \Rightarrow J < 0$.)

Let $X = \times_{i=0}^{\infty}[\alpha, \beta]$. The probability measure ν on $[\alpha, \beta]$ induces the product measure ν^∞ on X. As in the discussion of the example in the text we may represent a feasible stochastic consumption stream by the vector random variable $\tilde{y} = (c_0, \tilde{c}_1, \ldots, \tilde{c}_t, \ldots)$, where $\tilde{y}$ maps a typical element $(r_0, r_1, \ldots)$ into the infinite dimensional vector whose tth component is $\tilde{c}_t(r_0, \ldots, r_{t-1})$. Denote by $EU(\tilde{y})$ the expected value integral $\int_X U(\tilde{y}(w))\, \nu^\infty(dw)$.

Let y^1 and y^2 be two non-stochastic consumption streams. Lemma 1 implies that

$$-U((y^1 + y^2)/2) \leqslant [-U(y^1)]^{1/2}[-U(y^2)]^{1/2}. \quad (34)$$

Let $\tilde{y}^1$ and $\tilde{y}^2$ be any two stochastic consumption streams. Integration of (34) implies that $-EU(\tilde{y}^1 + \tilde{y}^2)/2) \leqslant E\{[-U(\tilde{y}^1)]^{1/2}[-U(\tilde{y}^2)]^{1/2}\}$. Now apply the Cauchy–Schwartz inequality to deduce that

$$-EU((\tilde{y}^1 + \tilde{y}^2)/2) \leqslant [-EU(\tilde{y}^1)]^{1/2}[-EU(\tilde{y}^2)]^{1/2}. \quad (35)$$

Let s^1 and s^2 be two different initial stocks and $\tilde{y}^1$ and $\tilde{y}^2$ the corresponding optimal stochastic consumption streams. It is immediate that $(\tilde{y}^1 + \tilde{y}^2)/2$ is feasible in the problem with initial stock $(s_1 + s_2)/2$. Therefore

$J((s_1+s_2)/2) \geqslant EU((\tilde{y}^1+\tilde{y}^2)/2)$. Finally, apply (35) to obtain $\log[-J((s_1+s_2)/2)] \leqslant \log[-EU((\tilde{y}^1+\tilde{y}^2)/2)] \leqslant \frac{1}{2}\log[-EU(\tilde{y}^1)] + \frac{1}{2}\log[-EU(\tilde{y}^2)] = \frac{1}{2}\log[-J(s^1)] + \frac{1}{2}\log[-J(s^2)]$. Thus $\log(-J)$ is convex. ∎

Proof of Theorem 5. By Lemma 1, $U=-e^{-V}$, where V is concave. Therefore U is strictly concave. Monotonicity is obvious. Thus Assumption 6 is satisfied. Assumption 8 is trivially true. That $g(s)$ and $s-g(s)$ are positive follows from $v'(0)=\infty$. Thus it remains only to show that $g(s)$ and $h(s) \equiv s-g(s)$ are both increasing.

Consider the dynamic programming equation (8). We wish to show that the objective function M, $M(c) \equiv v(c)+B(c)\,EJ[f(s-c,\tilde{r})]$, is strictly concave. By Lemma 2, $-\log[-J(s)]$ is concave (and increasing). Since $f(\cdot,r)$ is concave, it follows that $-\log(-J(f(s-c,r)))$ is concave in $c\ \forall r$. Use precisely the arguments employed above in passing from (34) to (35) to show that $-\log(-EJ(f(s-c,\tilde{r}))) \equiv G(s-c)$, defines a concave function G. But now $M(c)=v(c)-\exp\{-[u(c)+G(s-c)]\} \Rightarrow M$ is strictly concave.

g is increasing: Let $s<\bar{s}$, $c=g(s)$. The first order condition for (8) is

$$v'(c)-B(c)\,EJ(f(s-c,\tilde{r}))[u'(c)-G'(s-c)]=0, \tag{36}$$

where G is defined above. Because of the strict concavity of M proved above, it is sufficient to show that the left side of (36) is positive if $\bar{s}$ is substituted for s. But $v'(c)-B(c)\,EJ(f(\bar{s}-c,\tilde{r}))[u'(c)-G'(\bar{s}-c)] \geqslant v'(c)-B(c)\,EJ(f(\bar{s}-c,\tilde{r}))[u'(c)-G'(s-c)]$, (since G' is non-increasing and $J<0$), $=v'(c)\{1-[EJ(f(\bar{s}-c,\tilde{r}))/EJ(f(s-c,\tilde{r}))]\}$ (by substitution of (36)), $\gneqq 0$ since $J<0$, J and f are increasing and $v'>0$.

h is increasing, where $h(s) \equiv s-g(s)$: Use the notation of the last two paragraphs. $h(s)$ is the solution to the problem $\max\{N(z): 0 \leqslant z \leqslant s\}$, where $N(z) \equiv M(s-z)$. M strictly concave $\Rightarrow$ N strictly concave. Let $s<\bar{s}$ and $z=h(s)$. The first order condition for the problem is

$$-v'(s-z)-B(s-z)\,EJ(f(z,\tilde{r}))[G'(z)-u'(s-z)]=0. \tag{37}$$

Because of the strict concavity of N it is sufficient to show that the left side of (37) is positive if $\bar{s}$ is substituted for s, since then $\bar{z}=h(\bar{s})>h(s)=z$. But $-v'(\bar{s}-z)-B(\bar{s}-z)\,EJ(f(z,\tilde{r}))[G'(z)-u'(\bar{s}-z)] \gneqq -v'(\bar{s}-z)-B(\bar{s}-z)\,EJ(f(z,\tilde{r}))[G'(z)-u'(s-z)]$, (since u is strictly concave), $=B(\bar{s}-z)\{[v'(s-z)/B(s-z)]-[v'(\bar{s}-z)/B(\bar{s}-z)]\} \geqslant 0$ by (12). ∎

APPENDIX 2

In this appendix we describe how Theorem 1 is modified if the sup topology is substituted for the product topology on Y.

Replace Assumption 4 by the following:

ASSUMPTION 4′. There exists a von Neumann–Morgenstern utility index U for $\succsim$ such that U is continuous on Y topologized with the sup topology.

Koopmans adopts a stronger assumption (Postulate 1) which imposes a form of uniform continuity of U.

Two additional assumptions are required.

ASSUMPTION 9. The utility function U in Assumption 4′ is bounded on Y.

ASSUMPTION 1′. For every $c \in [0, L]$ $\exists y \in Y$ such that (c, y) and y are not indifferent under $\succsim^Y$.

The latter assumption strengthens Assumption 1 and imposes a form of sensitivity to consumption in the initial period. Assumption 9 is *weaker* than Koopman's Postulate 5 (p. 295), which requires that there exist upper and lower bounds for U which are actually attained, i.e., there exist $\underline{y}$ and $\bar{y}$ in Y such that $\underline{y} \precsim^Y y \precsim^Y \bar{y}$ $\forall y \in Y$.

THEOREM 1′. *The preference ordering $\succsim$ satisfies Assumptions* 1′, 2, 3, 4′, *and* 9 *if and only if U can be expressed in the form* (1), *where u and v satisfy the conditions of Theorem* 1.

Refer to the discussion in the text of the greater prevalence of impatience in our analysis as compared to Koopmans'. The only sensitivity postulate adopted by Koopmans is, in our notation, that $\exists c, c'$ $\exists y$ such that $(c, y) \succ^Y (c', y')$. In contrast we have maintained Assumption 1′. But that difference appears an unlikely explanation of the differing results. The major difference in the two analyses is the consideration in this paper of choice between stochastic, in addition to certain, consumption streams.

Proof of Theorem 1′. Refer to the proof of Theorem 1. Given that U is bounded the product topology for Y is used in the argument only to rule out the existence of c such that $B(c) = 1$ and $v(c) = 0$. But in the latter case, (17) implies $U(c, y) = U(y)$ $\forall y$, contradicting Assumption 1′. ∎

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