Quadratic Social Welfare Functions

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John Harsanyi has provided an intriguing argument that social welfare can be expressed as a weighted sum of individual utilities. His theorem has been criticized on the grounds that a central axiom, that social preference satisfies the independence axiom, has the morally unacceptable implication that the process of choice and considerations of ex ante fairness are of no importance. This paper presents a variation of Harsanyi's theorem in which the axioms are compatible with a concern for ex ante fairness. The implied mathematical form for social welfare is a strictly quasi-concave and quadratic function of individual utilities.

I. Introduction

In a classic and influential article, Harsanyi (1955) describes an intriguing argument that social welfare must be a weighted sum of individual utilities. In his theorem, individual and social preferences are defined on lotteries generated over a set of social states. Harsanyi's assumptions are simply that both individual and social preferences satisfy the expected utility axioms and that society should be indifferent between a pair of lotteries when all individuals are indifferent between them, a condition known as Pareto indifference. Then social utility, defined as a von Neumann–Morgenstern index that represents the social ordering, is a weighted sum of individual von Neumann–Morgenstern indices. In a variant of the theorem, all utility weights can be chosen to be positive if a stronger Pareto condition

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(called strong Pareto below) is assumed. In this paper, the expression "Harsanyi's theorem" will generally refer to this variant of Harsanyi's original result.

Harsanyi's theorem has generated considerable controversy and has spawned a large literature. Two questions that have received attention are the validity of the theorem (Harsanyi's original proof was incomplete) and its proper interpretation. The theorem's contribution is to provide a powerful argument that justifies a particular mathematical form for a social welfare function, though there is some dispute about whether it provides an argument for the moral theory of utilitarianism (Sen 1986). Weymark (1991) clarifies the controversy regarding these questions and contains many useful references, including several rigorous proofs of the theorem and some extensions.

A third question, and one that motivates this paper, is the appropriateness of the expected utility axioms for social decision making. Diamond (1967) and Sen (1970), in particular, have argued that the independence axiom at the level of social preferences has the morally unacceptable implication that only final states matter and thus that the process of choice is of no importance. This contention is clearly illustrated by Diamond's example: Consider a society composed of two identical individuals, A and B, and a government facing the problem of how to allocate one unit of an indivisible good. The government is considering two alternative policies. The first policy is to allocate the good to A, that is, to choose the allocation (1, 0). The second policy yields the two allocation vectors (1, 0) and (0, 1) with equal probabilities, corresponding to a preliminary randomization to determine which of A and B will receive the good. When the interests of the two individuals are given equal weight and thus society is indifferent between (1, 0) and (0, 1), the two policies must be socially indifferent if the independence axiom is satisfied at the societal level. A possible justification for such indifference is that for either policy the ultimate result is a distribution in which precisely one of the parties receives the good. However, many would view the second policy as socially strictly preferable because it offers both individuals a "fair chance" or "equal opportunity."

The purpose of this paper is to describe a variation of Harsanyi's theorem that has two essential features: First, motivated by Diamond's objection, our theorem does not impose expected utility theory at the societal level. Instead, it imposes a weaker appealing form of symmetry and a strict preference for randomization in situations like those represented by Diamond's example. (In the context of that example, the two new axioms together amount to the requirement that the equiprobable randomization of the two states (1, 0) and (0, 1) be strictly socially preferred to randomizations with any other probabilities and thus also to the policy (1, 0), which involves randomization only in a degenerate sense.) In spite of the apparent weakness of our axioms, they have strong implications for the representability of social preferences. The sharp result, which is the second essential feature of our theorem, is that the social ordering can be represented by a social welfare function that is a *quadratic,* strictly increasing, and strictly quasi-concave function of individual von Neumann–Morgenstern utilities. Thus we argue that, given suitable Pareto and continuity conditions, the moral view that fairness of the social choice process matters is properly represented (only) by the adoption of a strictly quasi-concave, quadratic social welfare function.

Our axioms and theorem are presented in Sections II and III. Then in Section IV, we consider Harsanyi's (1975) cogent arguments that the criticism represented by Diamond's example is untenable since the suggestion that artificial randomization can play a useful moral role in social decision making leads to highly counterintuitive policy prescriptions. This discussion serves both to support our central axiom of a preference for randomization and to clarify the implications of our theorem. Section V presents concluding remarks.

II. Postulates

Following Harsanyi (1955), let $X = \{x_1, \ldots, x_M\}$, $M \geq 2$, denote the set of certain alternatives or social states, where each $x \in X$ provides a complete description of the situation of each agent in the economy. One possibility is that X is a subset of some finite-dimensional Euclidean space E^r, so that each $x \in X$ consists of r numerical components. However, X could be any finite set whatsoever and so could represent situations in which a social state can be described only by infinitely many real components or in which some aspects of a social state are qualitative in nature and defy numerical representation.[1] The objects of choice are lotteries with prizes drawn from X. Each lottery corresponds to a probability vector $\mathbf{p} = (p_1, \ldots, p_M)$ that offers x_i with probability p_i. The set of all lotteries, denoted L, is the $(M - 1)$–dimensional simplex

$$\left\{\mathbf{p} \in E^M: p_m \geq 0 \text{ for all } m, \sum_1^M p_m = 1\right\}.$$

[1] Indeed, X could be any compact metric space, in which case the space L of lotteries defined below should be interpreted as the space of Borel probability measures on X endowed with the weak convergence topology.

There are I individuals in society. The ith person's preference ordering on L is denoted $\succcurlyeq_i$, with indifference and strict preference denoted $\sim_i$ and $\succ_i$. We make the following assumption.

INDIVIDUAL RATIONALITY.—Each preference ordering $\succcurlyeq_i$ satisfies the axioms of expected utility theory.

Thus for each individual i, the ordering $\succcurlyeq_i$ can be represented numerically by a function of the form

$$U_i(\mathbf{p}) = \sum_{m=1}^{M} p_m u_i(x_m) \tag{1}$$

for a suitable von Neumann–Morgenstern utility index u_i.

Note that there are grounds for objecting to the hypothesis of expected utility preferences at the individual level. It may be argued that it is the descriptive accuracy, rather than the normative appeal, of the expected utility model that is relevant at the individual level, and recent research (see Machina [1987] for references) has at least cast doubt on the positive validity of the model. In addition, as a theoretical proposition, the expected utility axioms will generally be violated when the lotteries involved are temporal in the sense that some decisions must be made before the lotteries are played out (Mossin 1969; Machina 1984). For example, a draft lottery is temporal, unless it is held at birth, since several life-planning decisions will typically need to be made prior to its resolution. Nevertheless, in order to focus on the primary objective of this paper, we follow the literature on the Harsanyi theorem in hypothesizing expected utility preferences at the individual level and in adopting the "persuasive" appellation "rationality."

Next assume the existence of a social ordering $\succcurlyeq$ on L, with $\succ$ and $\sim$ denoting strict preference and indifference. The remaining postulates also concern this social ordering. The first has already been mentioned but is restated more formally here.

PARETO INDIFFERENCE.—For each pair of lotteries p and q, if all individuals are indifferent between them, then so is society, that is, $p \sim_i q$ for all $i \Rightarrow p \sim q$.

Note that this axiom imposes a welfarist approach to ranking social alternatives; that is, information other than individual welfare rankings has no bearing on the social ordering. (See Sen [1977*b*] for a criticism of welfarism and Broome [1990] for a nonwelfarist theory of fairness.)

A stronger Pareto condition is needed for our theorem. We now impose the following condition.

STRONG PARETO.—For all lotteries p and q, if $p \succcurlyeq_i q$ for all i, then

$p \succcurlyeq q$; furthermore, if there exists an individual j such that $p \succ_j q$, then $p \succ q$.

The next axiom is purely technical. It is an implicit component of Harsanyi's Bayesian rationality postulate for social preference and is totally analogous to the standard continuity axiom used in consumer theory.

CONTINUITY.—For any lottery p, the set of preferred lotteries $\{q \in L: q \succcurlyeq p\}$ and the set of worse lotteries $\{q \in L: p \succcurlyeq q\}$ are both closed subsets of E^M.

To proceed, adopt the following standard notation. For any $\alpha \in [0, 1]$, $\alpha p + (1 - \alpha)q$ denotes the probability vector for which prize x_m occurs with probability $\alpha p_m + (1 - \alpha)q_m$. The vector $\alpha p + (1 - \alpha)q$ is referred to as a (probability) mixture or randomization of the lotteries p and q. It can be interpreted as describing the two-stage lottery whereby a random experiment such as a coin flip is conducted in the first stage, and depending on which of its two possible outcomes is realized, p or q is faced in the second stage.

MIXTURE SYMMETRY.—For each pair of lotteries p and q, if $p \sim q$, then any mixture $\alpha p + (1 - \alpha)q$ is socially indifferent to its symmetric counterpart $(1 - \alpha)p + \alpha q$.

The independence axiom imposed by Harsanyi states that for all lotteries p, q, and r and for every $\alpha \in (0, 1]$, $p \succcurlyeq q$ if and only if $\alpha p + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)r$. Clearly, mixture symmetry is *weaker* than the independence axiom since, under the latter, $\alpha p + (1 - \alpha)q$ and $(1 - \alpha)p + \alpha q$ would both be indifferent to p. In addition, the symmetry axiom has evident intuitive appeal. In the context of the Diamond example described in the Introduction, it requires that a policy in which one randomizes between the vectors (1, 0) and (0, 1) according to the probabilities $1/3$ and $2/3$ should be indifferent to the policy in which a $2/3$-$1/3$ randomization is conducted. In this case, the appropriate p and q are degenerate lotteries and are mirror images of one another: p delivers the prize (1, 0) with certainty and q delivers the prize (0, 1) with certainty. But neither feature is essential to the appeal of the axiom. *The mere fact that* p *and* q *are indifferent is a compelling reason for being indifferent between symmetric randomizations.* To elaborate, suppose that $p \sim q$, while $p \succ_i q$ for all $i \in I_1$, $q \succ_j p$ for all $j \in I_2$, $I_1 \cup I_2 \subseteq I$, and $p \sim_k q$ for all other individuals k. Implicit in the social ranking $p \sim q$ is the judgment that the competing claims of I_1 and I_2 balance. The social ranking $\alpha p + (1 - \alpha)q \succ (1 - \alpha)p + \alpha q$ for some $\alpha > 1/2$ would reflect a bias in favor of I_1 over I_2 that would "contradict" this judgment.

Mixture symmetry is a simple yet powerful axiom, as has been demonstrated in the decision-theoretic context by Chew, Epstein, and

Segal (1991*a*) and as is further evidenced by the representation theorem in the next section. Thus it merits considerable scrutiny. In particular, one may wonder whether the intuition provided for mixture symmetry suggests that one should also accept the following three-lottery version of the axiom:

> If $p \sim q \sim r$, then any probability mixture $\alpha p + \beta q + (1 - \alpha - \beta)r$ is socially indifferent to $\beta p + \alpha q + (1 - \alpha - \beta)r$.

This three-lottery axiom is not implied by mixture symmetry and, moreover (as shown below), is inconsistent with the randomization preference postulate to follow and thus also with the social welfare functions derived in our representation theorem. However, in our view this version of the axiom lacks the appeal of mixture symmetry. For example, if $r = q$, why should a switch of the weights attached to p and q be a matter of indifference given that the presence of r destroys the symmetry between p and q that prevails in the two-lottery setting? Note that, with $r = q$, the new axiom implies that if $p \sim q$, then any mixture $\alpha p + (1 - \alpha)q$ is indifferent to any other mixture $\beta p + (1 - \beta)q$ and thus $\frac{1}{2}p + \frac{1}{2}q$ is indifferent to p; that is, randomization has zero value.[2]

Our final observation regarding mixture symmetry is probably obvious to many readers but may still merit explicit mention. In trying to understand the axiom, one may contemplate its application in the context of consumer preference over commodity bundles, where, in the obvious notation, the axiom would read as follows:

> For all x and y in R^n_+, $x \sim y \Rightarrow \alpha x + (1 - \alpha)y \sim (1 - \alpha)x + \alpha y$ for all $\alpha \in [0, 1]$.

If the standard assumptions of consumer theory were also imposed (with convexity of preference strengthened to strict convexity), then, as in our representation theorem below, we could conclude that the consumer's utility function *must* be (ordinally) quadratic. This would be a troubling conclusion since it rules out many familiar and seemingly sensible utility functions, the constant elasticity of substitution function, for instance. But, of course, "mixture symmetry" in commodity space is much less appealing than in probability space for reasons entirely analogous to those raised in classical discussions of the independence axiom (Samuelson 1952, pp. 672–73; 1983, pp. 515–16). The mutual exclusivity of the two states "heads" and "tails" corresponding to the flip of a coin is critical to the intuitive appeal of

[2] Similar arguments apply to the following alternative axiom: if $p \sim q \sim r$, then $\alpha p + \beta q + (1 - \alpha - \beta)r \sim \alpha q + \beta r + (1 - \alpha - \beta)p$. (Note that the probability weights in the two mixtures of p, q, and r are cyclic permutations of one another.)

the independence axiom or the mixture symmetry axiom formulated in terms of probability mixtures. In contrast, a convex combination of two commodity bundles represents a physical combination of the bundles, quite the contrary of exclusivity. As an illustration, suppose that one is indifferent between a bottle of red wine and a steak. Then it seems sensible to be indifferent also between symmetric lotteries over which of the two prizes will be consumed, but there is no reason to expect indifference between ($\frac{2}{3}$ bottle wine, $\frac{1}{3}$ steak) and ($\frac{1}{3}$ bottle wine, $\frac{2}{3}$ steak).

The postulates described to this point are also imposed, at least implicitly, in Harsanyi's theorem. We deviate in our final postulate, which is a joint requirement on social and individual preferences.

RANDOMIZATION PREFERENCE.—For each pair of lotteries p and q, if p and q are socially indifferent and if there exists at least one individual who strictly prefers p to q and another who strictly prefers q to p, then the mixture $\frac{1}{2}p + \frac{1}{2}q$ is strictly socially preferred to p; that is, $p \sim q$, $p \succ_i q$, and $q \succ_j p$ for some i and $j \Rightarrow \frac{1}{2}p + \frac{1}{2}q \succ p$.

This axiom is proposed as a means to embody a concern for the ex ante fairness of a choice process. Its interpretation and appeal are perfectly clear in the context of Diamond's example if p and q are taken to be the degenerate lotteries described above. Note that since it is formulated in terms of strict social preference, the axiom is incompatible with the expected utility form for $\succcurlyeq$, which implies $\frac{1}{2}p + \frac{1}{2}q \sim p$. Also note that if everyone in society is indifferent between p and q, then "fairness" is not an issue in choosing between p and q, and it is natural to exclude this case as in the axiom. As a consequence, if all individual preference orderings coincide with the social ordering, then the randomization axiom is satisfied vacuously. However, generally if p and q are socially indifferent, there will be some individuals who strictly prefer p and others who strictly prefer q, and randomization will help to meet competing claims more fairly.

The randomization postulate is discussed further in Section IV, where supporting arguments other than fairness are mentioned. For now, we conclude discussion of the postulates by pointing out that, given continuity and strong Pareto, the conjunction of mixture symmetry and randomization preference is equivalent to the following requirement:[3]

> For each pair of lotteries p and q, if $p \sim q$ and if $p \not\sim_i q$ for some i, then $\frac{1}{2}p + \frac{1}{2}q$ is the unique best lottery among all randomizations of p and q.

[3] This equivalence is proved by Chew et al. (1991*a*). They propose and analyze the mixture symmetry axiom in the context of individual decision making under uncertainty, though they call the axiom "strong mixture symmetry."

Consequently, unless otherwise stated, randomization will refer to an equiprobable one.

III. Representation Theorem

We now discuss the implications of the postulates described above for the social welfare function. Call the function of I variables W a *social welfare function* (representing $\succcurlyeq$) if, for all lotteries p and q,

$$p \succcurlyeq q \Leftrightarrow W(U_1(p), \ldots, U_I(p)) \geq W(U_1(q), \ldots, U_I(q)), \tag{2}$$

where the U_i's are the expected utility functions defined in (1). The function W must be defined on the subset $D \equiv \{(U_1(p), \ldots, U_I(p)): p \in L\}$ of E^I. We call W *quadratic* if it is a polynomial of order two on its domain, that is,

$$W(u_1, \ldots, u_I) = \sum_{i=1}^{I} \sum_{j=1}^{I} a_{ij} u_i u_j + \sum_{1}^{I} b_i u_i \tag{3}$$

for some constants $\{a_{ij}\}$ and $\{b_i\}$, $a_{ij} = a_{ji}$. An additive constant term is unnecessary since only the ordinal properties of W are important. The qualification "up to ordinal equivalence" should be understood throughout, though it will not be stated explicitly.

Say that W is *strictly quasi-concave* if, for all distinct vectors $\mathbf{u}$ and $\mathbf{u}'$ in its domain D,

$$W(\mathbf{u}) = W(\mathbf{u}') \Rightarrow W(\tfrac{1}{2}\mathbf{u} + \tfrac{1}{2}\mathbf{u}') > W(\mathbf{u}),$$

which is the strict form of the usual convexity assumption for upper contour sets. Say that W is *strictly increasing* if, whenever $\mathbf{u}'$ and $\mathbf{u}$ in D satisfy $u_i' \geq u_i$ for all i with strict inequality for some i, $W(\mathbf{u}') > W(\mathbf{u})$. Note that for some domains D, these conditions are satisfied vacuously; for example, quasi concavity is vacuously satisfied if all U_i's are identical and the monotonicity property is vacuously satisfied if individual utilities are negatively related in the sense that $\Sigma\, U_i = 0$ identically.

The extension from a linear social welfare function to a quadratic is natural on mathematical grounds. The following theorem, which is the central result of this paper, shows that it is appealing also on ethical grounds. The theorem invokes the postulates described earlier to limit acceptable social welfare functions to the quadratic class (3), thus leaving only the finite number of parameters $\{a_{ij}, b_i\}$ to be specified to reflect further ethical values.

THEOREM.—Let individual orderings satisfy the individual rationality postulate with utility functions $U_1, \ldots, U_I$ as in (1). Then individual and social preference orderings satisfy the postulates strong Pareto, continuity, mixture symmetry, and randomization preference *if and only if* there exists a quadratic social welfare function W that is strictly increasing and strictly quasi-concave on the domain $D \equiv \{(U_1(p), \ldots, U_I(p)): p \in L\}$.

As explained in the Appendix, where a proof is provided, the theorem is a corollary of the representation results in Chew et al. (1991*a*). The proof that the axioms are satisfied given the indicated functional structure is simple and somewhat illuminating, particularly with regard to the properties of quadratic functions. Thus we outline such a proof here: Let $(U_1, \ldots, U_I)$ be abbreviated by U. If W is quadratic as in (3) and if U is linear in p, then one can compute that

$$\begin{aligned} &W(U(\alpha p + (1-\alpha)q)) - W(U((1-\alpha)p + \alpha q)) \\ &= W(\alpha U(p) + (1-\alpha)U(q)) - W((1-\alpha)U(p) + \alpha U(q)) \\ &= [\alpha - (1-\alpha)][W(U(p)) - W(U(q))], \end{aligned}$$

which equals zero if p and q are socially indifferent. Thus a quadratic functional form for W ensures that mixture symmetry is satisfied. In addition, the strict quasi concavity of W ensures randomization preference, since if $p \sim q$ and $p \not\sim_i q$ for some i, then $W(U(p)) = W(U(q))$ and $U(p) \neq U(q)$. Thus $W(U(\frac{1}{2}p + \frac{1}{2}q)) = W(\frac{1}{2}U(p) + \frac{1}{2}U(q)) > W(U(p))$.

The following example clarifies the range of social welfare functions admitted by the theorem and their relation to the utilitarian function supported by Harsanyi. Suppose that individual von Neumann–Morgenstern utility functions are normalized so that they assume only positive values on X; this can be arranged by suitable cardinal transformations. Define W^s on the positive orthant in E^I by the following specialization of (3):

$$W^s(u_1, \ldots, u_I) = A \sum_1^I u_i^2 + \sum_{i \neq j} u_i u_j, \tag{4}$$

where A is a parameter. Then W^s is increasing and strictly quasi-concave for each A in $[0, 1)$. Moreover, it approaches utilitarianism as $A \to 1$ in the sense that when $A = 1$

$$W^s(u_1, \ldots, u_I) = (\Sigma\, u_i)^2,$$

which is ordinally equivalent to the utilitarian function $\Sigma\, u_i$.

Further interpretation of W^s requires additional assumptions about the measurability and interpersonal comparability of utility. A difficulty with the present level of generality is that individual von Neumann–Morgenstern functions are defined only up to positive affine transformations $u_i \to \hat{u}_i \equiv a_i u_i + b_i$, and expected utility theory places no restrictions on how these transformations can vary across individuals. Moreover, if utilities are measured by $\hat{u}_i$ rather than u_i, then the social welfare function that represents the social ordering $\succcurlyeq$ changes accordingly to a different quadratic $\hat{W}$, where

$$\hat{W}(a_1 u_1 + b_1, \ldots, a_I u_I + b_I) \equiv W^s(u_1, \ldots, u_I).$$

Suppose, however, that we assume that the utilities u_i are measurable up to ratio scale and are fully comparable; that is, the only transformations of utilities that are admitted are those for which $u_i \to \hat{u}_i \equiv a u_i$, where $a > 0$ is common to all i.[4] Call a statement "meaningful" if its validity is unaffected by an admissible transformation of utilities. Thus our assumption on utilities renders meaningful statements of the form "person i in state x is twice as well off as person j in state y."

Moreover, the following appealing features of W^s are meaningful: First, it treats individuals symmetrically (which is the reason for the superscript s);[5] second, W^s is a "mean-variance" social welfare function. More precisely, summarize the population distribution of utility by the mean, $\mu(u) \equiv \Sigma\, u_i/I$, and variance, $\mathrm{var}(u) \equiv \Sigma\, (u_i - \mu)^2/I$. Then we can write W^s in the form

$$W^s(u) = I(I + A - 1)[\mu^2(u) - K\,\mathrm{var}(u)], \tag{5}$$

where $K \equiv (1 - A)/(I + A - 1) > 0$. Sen (1973, pp. 15–18) points to the inegalitarian nature of utilitarianism that derives from its exclusive concern with the mean (or sum) of utilities. He argues that the social welfare function should reflect concern also with the dispersion in these utility levels. The functional form above incorporates such concern in a simple way and to an extent measured by the single parameter A. Concern with the dispersion of utilities vanishes only in the utilitarian limit as $A \to 1$.[6]

[4] Sen (1986) describes a number of alternative assumptions regarding the measurability and interpersonal comparability of utility. Such assumptions are common in welfare economics and indeed are necessary in light of Arrow's impossibility theorem. Utilitarianism, in the form of the welfare function $W(u_1, \ldots, u_I) = \Sigma\, u_i$, requires that individual utilities be cardinally measurable and unit comparable; i.e., only transformations $u_i \to a u_i + b_i$ are admitted.

[5] In fact, W^s is the most general polynomial of order two on the positive orthant in E^I that treats agents symmetrically and that is ordinally invariant to any common rescaling of individual utilities. Thus it is a natural example to consider.

[6] One might be tempted to attach significance to the fact that K is decreasing in the population size I. However, our theorem and the functional form (4) for W relate to a given population of fixed size. If we were to contemplate using (4) to address social choice problems involving variable populations (see Blackorby and Donaldson 1984), then presumably we would allow A to depend on population size.

IV. Randomization

As pointed out in Section II, we deviate from Harsanyi's postulates by weakening the independence axiom for the social ordering and by adopting the postulate of randomization preference. It is widely perceived that randomness can promote fairness in an otherwise inherently unfair situation. For instance, as illustrated by Diamond's example, if an indivisible good (or burden) is to be allocated among a number of people having equal claims to the good, then "equality" seems desirable; it can be achieved in the sense of giving everyone an equal chance to obtain the good. The intuitive case for randomization was well expressed by Judge Baldwin, who presided over the trial of a seaman called Holmes. After the sinking of a ship in 1841, Holmes participated in the nonrandomized eviction of some passengers from an overloaded longboat. Judge Baldwin claimed that victims should have been chosen by lot. "This mode [of selection] is resorted to as the fairest mode . . . for selection of the victim. . . . In no other than this or some like way are those having equal rights put upon an equal footing, and in no other way is it possible to guard against partiality and oppression" (Broome 1984*a*, p. 38). Note the latter argument for randomization, which is distinct from fairness—the potential of randomization to limit opportunities for corruption and prejudice in social decision making. Another possible justification is that randomization enables society to avoid "playing God." This is particularly relevant when questions of life and death are involved, as in deciding who will receive an organ transplant or who will be forced to go off to war. For these and other reasons, lotteries are used in many instances to allocate resources and burdens (Elster 1989).

Nevertheless, the normative case for randomization in social decision making has been disputed (Harsanyi 1975; Broome 1984*b*). Therefore, we devote this section to consideration of some of the more important dissenting arguments, which may be more clearly understood when reexamined from the perspective of our formal framework. The discussion is intended also to clarify the interpretation and implications of our theorem.

Besides the role of randomization, there has been disagreement concerning other issues surrounding the Harsanyi theorem, largely expressed in an exchange between Harsanyi and Sen appearing in Harsanyi (1975, 1977, 1978) and Sen (1976, 1977*a*, 1986). They include (i) Harsanyi's derivation of social preferences via choice in the "original position" and the implied relation between individual attitudes toward risk and social attitudes toward inequality (see also Sen 1970, pp. 142–43), (ii) the acceptability of a social welfare function that is nonlinear in individual utilities, and (iii) ex ante versus ex post optimality (see also Myerson 1981; Hammond 1983). This paper implicitly reveals our positions on these issues. However, since our formal model has nothing special to offer these debates, we shall not mention them further here.

To structure our discussion we identify two (related) questions concerning randomization and address them in turn.

A. *How Many Times Should We Randomize?*

Suppose that one accepts Diamond's view that it is best in his example to use a coin flip to determine which of A and B receives the good. Then after the random selection is made, presumably the identical motivation would lead us to ignore the result and flip the coin once more. Moreover, this procedure would be repeated forever and a choice would never be made.

This dilemma stems from dynamically inconsistent behavior on the part of the social decision maker. The source of dynamic inconsistency is the assumption that after the first coin flip, the social decision maker would evaluate the choice problem anew, as though the coin had not been flipped at all. But surely the particular procedure leading to a decision node should matter: for example, ex ante, A and B had equal chances of receiving the good; ex post, if A has won, then A and B no longer have equal rights to the good and the argument for randomization fails. If social preferences are dynamically consistent, then after A has won the first coin flip, giving him the good is the socially best alternative.[7] We find such dynamic consistency or respect for the process compelling on normative grounds. In addition, its violation would most likely destroy the credibility of the social decision maker. In this subsection we show that our model is compatible with the dynamic consistency of social preference and thus is immune to the criticism of "repeated randomization."

To proceed formally, we are confronted with the difficulty that, strictly speaking, the issue of dynamic consistency cannot be addressed within our formal framework or Harsanyi's; for example, that framework considers only a single preference ordering for society and correspondingly addresses only problems of one-shot social choice. Dynamic consistency is potentially an issue only in dynamic or multistage choice problems in which a number of decisions are made sequentially and the preference orderings that direct choice at each decision node differ suitably from node to node.

However, it is a straightforward matter to suitably extend our for-

[7] See Broome (1984*b*, pp. 628–30) for this defense of randomization in the face of the dynamic inconsistency argument.

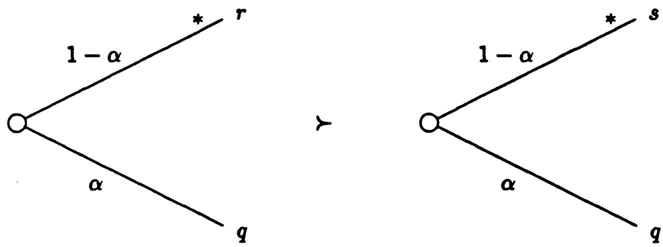

FIG. 1

mal framework.[8] To illustrate, consider society as the decision maker; individuals are discussed later. Suppose that decisions are to be made at two points in time and, correspondingly, that the initial choice is made between two-stage lotteries, two of which are depicted in figure 1. The first critical assumption about preference in this two-stage setting is that society ranks two-stage lotteries by first applying the rules of probability to reduce them to the (simple or single-stage) lotteries $\alpha q + (1 - \alpha)r$ and $\alpha q + (1 - \alpha)s$, which are elements of L, and then by applying the ordering $\succcurlyeq$. In this way, the ordering $\succcurlyeq$ determines an initial social choice such as the one indicated in figure 1.

Next let the first-stage uncertainty be resolved revealing that the event corresponding to the upper branches has occurred, and at the asterisk, before lottery r or s is played out, allow the decision maker to reconsider and possibly choose s. Whether or not that option will be exercised depends on the preference ordering $\succcurlyeq^*$ that dictates choice at this intermediate stage. The specification of $\succcurlyeq^*$ is the second critical assumption regarding dynamic preference and choice behavior.

One possibility is to specify that

$$r \succcurlyeq^* s \Leftrightarrow r \succcurlyeq s, \tag{6}$$

which in Hammond's (1983) terminology corresponds to the assumption of consequentialism. As noted by Hammond (pp. 201–3), consequentialism is incompatible with ex ante notions such as fairness because $\succcurlyeq^*$ is independent of forgone alternatives such as q. In the context of the coin flip example, the specification (6) assumes that society's preference at the intermediate stage * is unaffected by the

[8] We limit ourselves to an outline of such an extension and leave it to the interested reader to supply the complete formal analysis. For relevant literature on extending preference orderings from single-stage lotteries to multistage lotteries, see Machina (1989) and Epstein (1992).

fact that the coin has already been flipped once, thereby giving A and B an equal chance, ex ante, to receive the prize. It is evident that the specification (6) will lead to a desire to revise the initial choice as long as $\succcurlyeq$ violates the independence axiom (see Machina 1989). We argue below that Harsanyi implicitly assumes consequentialism in his criticisms of the potential role of randomization in social choice.

For the reasons given earlier, we assume in lieu of (6) that $\succcurlyeq^*$ is consistent with $\succcurlyeq$ in the sense that

$$r \succcurlyeq^* s \Leftrightarrow \alpha q + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)s. \tag{7}$$

This specification guarantees that the initial choice will not be overturned since $\alpha q + (1 - \alpha)r \succ \alpha q + (1 - \alpha)s$ implies (indeed is equivalent to) $r \succ^* s$. In particular, in Diamond's example, let $\alpha = 1/2$, let r and q be the lotteries that produce the respective social states (1, 0) and (0, 1), and let $s = \frac{1}{2}r + \frac{1}{2}q$. Then the preference in figure 1 corresponds to the ex ante superiority of the equiprobable randomization to the mixture $\frac{1}{4}r + \frac{3}{4}q$. Under the specification (7), the decision maker will not wish to flip again in the event that the outcome (1, 0) is indicated by the first coin flip. Obviously, the same can be said also in the event of the other first-stage outcome.

For each individual i, we follow a similar procedure: first $\succcurlyeq_i$ is extended to multistage lotteries via the standard rules for reducing compound lotteries and, second, we use the appropriate form of (7) to define $\succcurlyeq_i^*$, the preference ordering at the intermediate position. However, the procedure is simpler for individuals than for social preference, since each $\succcurlyeq_i$ is assumed to satisfy the independence axiom. Consequently, the corresponding forms of (6) and (7) are equivalent and $\succcurlyeq_i^*$ is independent of the forgone lottery q.

Since social preferences violate the independence axiom, social rankings at * generally depend on alternatives that were possible ex ante but were never realized. In particular, the social weight given to an individual in interim decision making depends on his opportunities ex ante. To see this more concretely, consider the social utility functions V and V^* corresponding to $\succcurlyeq$ and $\succcurlyeq^*$. The relation (7) may be translated in the form

$$V^*(p) = V(\alpha q + (1 - \alpha)p) \quad \text{for each lottery } p \text{ in } L. \tag{7'}$$

Suppose, as a result of our theorem applied to ex ante social and individual preferences, that $V(\cdot) \equiv W(U_1(\cdot), \ldots, U_I(\cdot))$, where W is a quadratic social welfare function with coefficients $\{a_{ij}, b_i\}$ as in (3). Then we compute that $V^*(p) = W^*(U_1^*(p), \ldots, U_I^*(p))$, where the interim social welfare function W^* is quadratic with coefficients $\{a_{ij}^*,$

$b_i^*\}$, and $U_i^* = U_i$ as remarked above. Moreover (if $\alpha < 1$),

$$a_{ij}^* = a_{ij}, \quad b_i^* = (1-\alpha)^{-1}\left[b_i + 2\alpha \sum_j a_{ij} U_j(q)\right]. \tag{8}$$

This updating of the parameters of the quadratic functional form describes the updating of individual welfare weights as intermediate lotteries are resolved.

In particular, it is easy to see how previous randomizations can lead to ex post unequal weights in spite of equal weights ex ante. Let the ex ante social welfare function be the symmetric function W^s defined in (4). Then (8) becomes

$$\begin{aligned} a_{ii}^* &= A, \quad a_{ij}^* = 1 \quad \text{if } i \neq j, \\ b_i^* &= 2\alpha(1-\alpha)^{-1}\left[(A-1)U_i(q) + \sum_j U_j(q)\right]. \end{aligned} \tag{9}$$

Only in the special case in which $U_i(q)$ is the same for all i (or $A = 1$, which is the utilitarian case) will individuals be treated symmetrically by W^*. More generally, individuals for whom the forgone lottery q was more favorable ex ante receive less weight ex post, that is, $U_i(q) > U_j(q) \Rightarrow b_i < b_j$. This deviation of weights from equality is needed in order that another flip of the coin be rejected at * or, more generally, in order that the ex ante social choice be carried out there.[9]

It merits emphasis that the fact that W^* must be quadratic if W is reflects the "dynamic consistency" of our model; that is, if our postulates are satisfied at the ex ante position, then they are also satisfied at * if (7) is used to update utilities. Put another way, our theorem is perfectly compatible with dynamically consistent preferences for society and individuals, in spite of the fact that social preferences violate the independence axiom. Thus the theorem is immune to attacks, such as Harsanyi's described in the next subsection, that are based on the argument that society will always wish to "flip the coin again."

[9] It may help to consider a simple example. There are two people, 1 and 2, and two "goods," health and money. There is one indivisible unit of health, and two units of the perfectly divisible money are available. Each person's von Neumann–Morgenstern utility function is $u_i(x, y) = 10x + f(y)$, where x and y denote i's consumption of health and money and f is strictly concave, with $f(0) = 0$, $f(1) = 2$, and $f(2) = 3$. Finally, social welfare has the Cobb-Douglas form $W(u_1, u_2) = u_1 u_2$. The ex ante optimal allocation is to flip a fair coin over who will receive the unit of health and to give each person one unit of money. Let * be a point in time after the coin flip when health has been allocated to the winner—person 1, say—but before the money has been distributed. If W is used again at *, then in an attempt to equalize ex post utility, society will give all the money to 2 rather than carry out the original plan of dividing the money equally ($10 \cdot 3 > 12 \cdot 2$). Society will be dynamically consistent if it updates its social welfare function according to (9), in which case $W^*(u_1, u_2) = u_1 u_2 + 12u_1 + 2u_2$.

This immunity, of course, depends on the adoption of (7), rather than the consequentialist rule (6), for updating social preference. We find the appeal of (7), based on concern with ex ante fairness and respect for the process, to be compelling in the setting of social preference.[10]

B. *Doesn't Nature Randomize for Us?*

Harsanyi (1975) rejects the value of lotteries for promoting fairness. We have nothing further to say about his view except that we do not accept it, and, as we try to demonstrate in this paper, it is not necessary for a coherent and sensible model of social choice. Here we focus on a second criticism of Diamond's position that is presented by Harsanyi; namely, that even if randomization were of value, artificial randomization would nevertheless be superfluous given the randomness of individual circumstances produced by "accidents of birth and personal life history." He asks (p. 317): "Why should a bureaucratic lottery be regarded as being a 'fairer' allocative mechanism than the great biological lottery produced by nature?" Harsanyi (pp. 316–18) offers a number of examples to fortify his criticism (see also Broome 1984*b*, pp. 630–31). Here we argue that these examples do not diminish the intuitive appeal of our randomization-preference postulate when the latter is properly interpreted. In fact our model is in agreement with Harsanyi about the *relative* merits of natural and artificial randomization. We differ from him in our view of the desirability of *some* form of randomization.

Our principal point is that randomization between the lotteries p and q is called for by the randomization preference postulate only if p and q are socially indifferent. We offer two remarks relevant to this observation. First, our model is generally agnostic about whether $p \sim q$. To elaborate, recall the updating rule (8). Given "initial conditions," in the form of a "starting time" t_0 and the welfare weights prevailing then, the weights $\{a_{ij}, b_i\}$ that apply at any $t > t_0$ are uniquely determined by the history of intermediate randomizations via repeated application of (8). However, our model is agnostic about the precise specification of t_0 and the associated welfare weights. Consequently, it is agnostic about whether $p \sim q$ at any given t. Second, even given a specification of the initial conditions, the dynamic consistency of social preference implies that the indifference required by the randomization preference postulate may not hold in situations in which

[10] Machina (1989) argues in favor of (7) over (6) in the context of individual preference violating the independence axiom. We find the case for (7) to be much stronger in our setting of social preference.

nature has randomized. Thus, as in the discussion in the previous subsection, if nature has flipped the coin once, another flip by society may not be welfare improving and in our model may actually reduce social welfare. The counterintuitive implications of Diamond's point of view described by Harsanyi can be understood to be a product primarily of Harsanyi's implicit modeling of social preferences as dynamically *inconsistent,* and thus these implications are neither surprising nor disturbing. On the other hand, the examples below seem to point to the counterintuitive nature of the policy implications of utilitarianism!

One hypothetical example offered by Harsanyi follows:[11] Each of us is born with a set of genes that determines, or at least influences, our intellectual and physical capabilities and talents. Policy I is to employ techniques of genetic engineering to randomly reallocate gene pools among individuals; policy II is to accept nature's allocation. Most of us would probably agree with Harsanyi that policy I is not morally superior to policy II. One possible underlying reason is that we implicitly view the situation from an ex ante, prebirth perspective, where individuals have names but have not yet been assigned genes. From that perspective we presumably are indifferent as a society about which name is ultimately assigned which genes, but individuals are not indifferent. If our preference postulates are adopted, then a randomized allocation of genes is optimal. However, nature conducts the randomization for us, and thus there is no gain from a second "flip of the coin" after birth. In other words, once nature has randomized, we do not view everyone as having equal "claims" to the best gene pool, and so a reallocation of genes is *not* called for by the postbirth social ordering. (The situation is formally identical to that portrayed in fig. 1, with * being the postbirth position.) On the other hand, as pointed out by Sen (1977*a*, p. 298), utilitarianism would view such a reallocation of genes to be a matter of indifference!

Next consider a slightly different example involving an accident of birth, in which the issue is not whether to revise nature's allocation but rather how the allocation should be related to other societal decisions. This is Diamond's example made more concrete. Society consists of two people who are identical except that one (G) has green eyes and the other (B) blue eyes. One must be drafted into the army and sent to war. Our preference postulates imply that *if* society is

[11] The examples to follow reflect a particular metaphysical view corresponding to Harsanyi's "original position." We emphasize that our model does not depend on acceptance of this view. Our objective here is to show that our model is immune to arguments based on the "great lottery of life."

indifferent about drafting G or B, then an equiprobable lottery is strictly preferable. One possible objection to conscription by lottery is to argue that a lottery is no better than drafting G, since with such a rule fixed and from a prebirth perspective, the individual who is conscripted is randomly chosen by nature when it assigns eye color. But if such a rule exists in society and if social preferences are dynamically consistent, then society would not be indifferent subsequently about drafting B or G, in contrast to our hypothesis. In other words, our axioms imply the superiority of randomization but not necessarily that it be artificial. The choice between artificial randomization and the draft G rule depends on whether or not society is indifferent about drafting G or B, a condition that is exogenous to our model.

Harsanyi (1975) and Broome (1984*b*, pp. 630–31) also describe examples involving accidents of life history, and they can be understood in a similar way. We offer the following final example, a slight variation of our first example. Imagine that medical science has reached the stage at which the process of organ transplantation is a riskless and costless activity. If an individual suffers an organ failure, one possibility for society is to let that individual bear the burden of the failure, either through enduring the resulting disability or death or through waiting for a donor organ (which we assume to be scarce). But in the state of technological advance described above, it would also be possible to randomly select a person in society who will be forced to exchange the specific organ with the initially afflicted individual. One could adopt an ex ante perspective and view the first policy as the choice to accept life's randomization, whereas the second policy insists on additional artificial randomization. With a utilitarian social welfare function, randomization of any sort between indifferent alternatives is a matter of indifference. In particular, society would be indifferent between these two policies. However, given our model of social choice and the ex ante perspective, once life's randomization has occurred, individuals no longer have equal "rights" to healthy organs. Thus, in contrast to the utilitarian prescription but in conformity with intuition, the additional artificial randomization reduces social welfare and the policy of accepting life's randomization is strictly preferable.

V. Concluding Remarks

For the reasons represented by Diamond's example, we are unconvinced by Harsanyi's argument in support of a linear social welfare function. At an "operational" level, a concern for ex ante fairness can be incorporated into social decision making by adopting *any* social welfare function that is strictly quasi-concave in individual utilities.

The contribution of this paper is to propose and justify axiomatically a *specific* alternative to Harsanyi's additive form: the strictly quasi-concave, quadratic functional form for social welfare. Given our axiom of randomization preference, which leads to the strict quasi concavity of the social welfare function, our justification for the further restriction to quadratic functions is based primarily on the intuitively appealing axiom of mixture symmetry.

Randomization has played a prominent role in our analysis and discussions but, as we now clarify in concluding, not to the extent that a superficial reading of the paper might suggest. Our motivating examples involved indivisible goods in which generally some ex post inequality is unavoidable but ex ante equality may be achievable by means of randomization. In such circumstances, randomization (natural or artificial) may be part of an optimal social choice, and we defended such policies against Harsanyi's criticisms. However, the axiomatic justification for a quadratic social welfare function is valid and intuitively appealing even in environments in which social optimality is achievable without any randomization. For example, interpret X as the feasible set of social states and suppose that the utility possibility set $D^0 \equiv \{(u_1(x), \ldots, u_I(x)) \colon x \in X\} \subset E^I$ is a convex set, for example, as in a standard private-goods exchange economy setting in which individual utility functions are concave (risk averse). Then the set of feasible utility allocations is not enlarged by admitting lotteries over social states (i.e., the set D defined following [2] coincides with D^0), and hence nontrivial randomization over social states is unnecessary for social optimality. In that sense, preference orderings only on X, rather than L, need be of concern. Nevertheless, it is still sensible to hypothesize that social and individual preference orderings are defined for all lotteries, that is, on L, and that they satisfy reasonable conditions there. Our axioms lose none of their appeal when viewed from this slightly different perspective, and the conclusion that social welfare must be quadratic remains intact.

Appendix

We provide a proof of our theorem.

Necessity of axioms.—Part of the proof was provided in the text. The remaining details are obvious.

Sufficiency of the axioms.—Let V be a continuous utility function that represents $\succcurlyeq$ in the sense that

$$p \succcurlyeq q \Leftrightarrow V(p) \geq V(q).$$

Such a function exists by Debreu (1964). Let $D \subset E^I$ be the subset $\{U(p) = (U_1(p), \ldots, U_I(p)) \colon p \in L\}$. Then D is compact and convex; in fact it is the convex polyhedral region having the finite set of extreme points $\{(u_1(x_m), \ldots,$

$u_I(x_m))$: $m = 1, \ldots, M\}$. We define the social welfare function W on D by

$$W(u_1, \ldots, u_I) \equiv V(p) \quad \text{for } any\ p \text{ in } L \text{ such that } U_i(p) = u_i, \text{ for all } i. \tag{A1}$$

The function W is well defined because of Pareto indifference, which is implied by strong Pareto.

The following properties can be verified on D:

$$W \text{ is continuous and strictly increasing;} \tag{A2}$$

$$\begin{aligned} &\text{if } W(u) = W(v), \text{ then } W(\alpha u + (1 - \alpha)v) \\ &= W((1 - \alpha)u + \alpha v) \quad \text{for all } \alpha \in [0, 1]; \end{aligned} \tag{A3}$$

$$W \text{ is strictly quasi-concave.} \tag{A4}$$

Strong Pareto implies that W is strictly increasing. The continuity of W follows from that of V and the compactness of L. To see this, let $\{u^n\}$ be a sequence in D that converges to $u \in D$. Then there exist $\{p^n\}$ and p in L such that $u^n = U(p^n)$ and $u = U(p)$. Since L is compact, we can assume (by taking a suitable subsequence) that $p^n \to p' \in L$. Since U is continuous, $U(p^n) \to U(p')$. Hence, $U(p')$ must equal $U(p)$ and, by Pareto indifference, $V(p') = V(p)$. Finally, $W(u^n) = V(p^n) \to V(p') = V(p) = W(u)$, where we have used the continuity of V.

Condition (A3) is mixture symmetry for W; it follows immediately from the mixture symmetry axiom for $\succcurlyeq$ since

$$\begin{aligned} W(\alpha u + (1 - \alpha)v) &= W(\alpha U(p) + (1 - \alpha)U(q)) \quad \text{for some } p \text{ and } q \\ &= W(U(\alpha p + (1 - \alpha)q)) = V(\alpha p + (1 - \alpha)q) \\ &= V((1 - \alpha)p + \alpha q) = \ldots = W((1 - \alpha)u + \alpha v). \end{aligned}$$

Condition (A4) follows from the conditions randomization preference and strong Pareto: Let $W(U(p)) = W(U(q))$ and $U(p) \neq U(q)$. Then by strong Pareto there exist i and j, $U_i(p) > U_i(q)$ and $U_j(q) > U_j(p) \Rightarrow p \succ_i q$, and $q \succ_j p \Rightarrow \frac{1}{2}p + \frac{1}{2}q \succ p$ by randomization preference $\Rightarrow W(\frac{1}{2}U(p) + \frac{1}{2}U(q)) > W(U(p))$.

It remains to show that the properties (A2)–(A4) imply that W is quadratic (up to ordinal equivalence). Denote by dim D the dimension of the linear subspace of E^I spanned by D. If dim $D > 1$, the desired property follows from Chew et al. (1991*a*, theorem 5; 1991*b*, app. 2, proposition). (To elaborate, there exists a linear transformation T: $E^I \to E^{\dim D}$, which maps D homeomorphically onto $T(D)$. Moreover [and this is the reason for defining the transformation], $T(D)$ is of full dimension in $E^{\dim D}$; that is, $T(D)$ has dimension equal to the Euclidean space $E^{\dim D}$ containing it, even though dim D may be strictly less than I. Define $\hat{W}$: $T(D) \to E^1$ by $\hat{W}(Tu) \equiv W(u)$ for each u in D. Then $\hat{W}$ satisfies continuity, strict quasi concavity, and mixture symmetry on $T(D)$. If $\hat{W}$ were also increasing on $T(D)$, we could apply the cited theorem 5 to infer that $\hat{W}$ is ordinally quadratic. However, $\hat{W}$ need not be increasing on $T(D)$ even though W is increasing on D. Therefore, we apply the cited proposition, which dispenses with the monotonicity requirement, to conclude that $\hat{W}$ is ordinally quadratic. Since T is linear, it follows that W is also ordinally quadratic.)

Suppose next that dim $D = 1$. Then D is the line segment in E^I connecting some pair of points u^* and v^*. There are two possible cases: (1) for all u and v in D, $u \neq v \Rightarrow W(u) \neq W(v)$; (2) there exist $u^0 \neq v^0$ in D such that $W(u^0) = W(v_0)$.

In the first case, assume without loss of generality that $W(u^*) < W(v^*)$. Then $W(u) \geq W(v) \Leftrightarrow d(u, u^*) \geq d(v, u^*)$, where $d(\cdot, \cdot)$ is the Euclidean metric on E^I. It follows that W is ordinally equivalent on D to the quadratic function $d^2(\cdot, u^*)$.

In the second case, (A3) and (A4) imply that

> $W(\alpha u^0 + (1 - \alpha)v^0)$ is increasing in α for $\alpha \in [0, 1/2]$ and decreasing in α for $\alpha \in [1/2, 1]$, with a maximum in α at $\alpha = 1/2$.

It follows from the strict quasi concavity of W that W is increasing as one moves from either u^* or v^* toward $1/2u^0 + 1/2v^0$ along the line segment joining u^* and v^*. Consequently, for all u and v in D, $W(u) \geq W(v) \Leftrightarrow d(u, 1/2u^0 + 1/2v^0) \leq d(v, 1/2u^0 + 1/2v^0)$, and W is ordinally equivalent on D to the quadratic function $-d^2(\cdot, 1/2u^0 + 1/2v^0)$. Q.E.D.

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