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# RISK AVERSION AND ASSET PRICES*

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This paper investigates equilibrium asset prices in a Lucas-type exchange model where preferences of the representative agent are represented by a Kreps–Porteus utility functional, rather than by an expected utility functional. Such a utility specification permits the disentangling of two critical aspects of preferences – risk aversion and intertemporal substitutability. Thus a clearer understanding of the determinants of asset prices is provided.

## 1. Introduction

This paper investigates the influence of preferences on equilibrium asset prices in a modified Lucas (1978) model. The latter is an infinitely-lived, representative agent model in which preferences are represented by the expected value of an intertemporally additive von Neumann–Morgenstern index. The curvature of the within-period utility function reflects both the extent of intertemporal substitution and the degree of risk aversion. Thus, as noted by Lucas (p. 1441), there is no way to isolate the roles played by these two distinct aspects of preferences in determining asset prices. To overcome this difficulty, we modify Lucas' model by adopting a generalized recursive utility specification, based upon Kreps and Porteus (1978) as developed in Epstein and Zin (1987a), which permits the desired separation. In this way a clearer understanding of the determinants of asset prices is provided.

In the paper by Epstein and Zin (henceforth EZ) the recursive utility specification was applied to a representative agent facing exogenous stochastic returns to the available assets. The first-order conditions for the corresponding optimization problem led to a model of the structure of asset returns. In this paper a general equilibrium framework is employed, asset returns are endogenous, and asset prices are related to exogenously determined productivity changes. Ultimately we intend to explore the consequences of our utility specification for asset pricing in more general frameworks which permit capital

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accumulation. [Expected utility versions of such models appear in Brock (1982), Prescott and Mehra (1980), and Donaldson and Mehra (1984), for example.] But the simpler Lucas framework permits sharper results and thus seems better suited to highlight the advantages of our specification of utility.

It is important to note that part of the motivation for our utility function specification is derived from the empirical literature on the behaviour of asset returns and consumption over time. Representative agent optimizing models based on expected utility preferences have not performed well empirically [Hansen and Singleton (1983), Mehra and Prescott (1985)]. An often suggested explanation for this poor performance is that the maintained specification of preferences is too rigid. Indeed, in the case of the common homogeneous specification, the elasticity of substitution and the risk aversion parameter are reciprocals of one another. Some support for this explanation is provided in EZ (1987b) where it is shown that the poor performance of the standard model is significantly improved if one adopts the more general recursive utility specification employed here, thereby relaxing the aforementioned a priori parameter constraint.

Weil (1987b) describes preferences closely related to those employed below. But he does not address any of the issues explored here. In Weil (1987a), some asset pricing equations are derived for the special case of a unitary elasticity of intertemporal subsitution, but their derivation and subsequent analysis differ from those in this paper.

Besides Lucas, to our knowledge only Donaldson and Mehra (1984) have explored the comparative static/dynamic properties of asset market variables in the infinitely-lived, representative agent general equilibrium framework. Their analysis of the consequences of increased risk aversion (hitherto referred to as a comparative risk aversion analysis) considers the consequences of an increase in the curvature of the within period von Neumann–Morgenstern utility index. Consequently, it is subject to the ambiguities noted by Lucas and described above.

Another related paper is Labadie (1986) who examines the relationship between risk aversion and asset prices in an overlapping generations model. Agents live for two periods and comparative risk aversion analysis is performed by applying the general multicommodity approach of Kihlstrom and Mirman (1974). The latter approach leads, however, to implausible preference specifications in models with more than two periods [see EZ (1987a)] and so is not available to us. For the same reason, one is left wondering whether her results are valid in overlapping generations models with longer lifetimes, where a different approach to comparative risk aversion analysis would have to be adopted. Below we confirm some of her results for our infinitely-lived representative agent model.

The paper proceeds as follows: Section 2 describes the environment, utility functions and individual behaviour. Section 3 defines equilibrium prices and proves their existence and uniqueness. The roles of risk aversion and other determinants of equilibrium prices are investigated in section 4, which investigation is the major thrust of this paper.

## 2. The representative agent

### 2.1. Utility functions[1]

Consider an infinitely-lived representative agent who receives utility from the consumption of a single good in each period. For the agent making a decision in period $t$, current consumption $c_t$ is nonstochastic but future consumption levels are generally stochastic. A utility function is defined on such intertemporal consumption programs.

There are two key assumptions underlying the specification of utility. For an agent standing in period $t$, utility $U_{t+1}$ from period $t+1$ onward is random. The first assumption is that the agent computes a certainty equivalent of random future utility, $\mu_t$, where

$$\begin{aligned} \mu_t &= \left[\mathrm{E}_t U_{t+1}^{\alpha}\right]^{1/\alpha}, \qquad \text{if} \quad 0 \neq \alpha < 1, \\ &= \exp\left[\mathrm{E}_t \log U_{t+1}\right], \quad \text{if} \quad \alpha = 0. \end{aligned} \tag{1}$$

($\mathrm{E}_t$ is the expectation operator conditional upon information available at time $t$.) Second, the agent is assumed to combine $\mu_t$ with current consumption $c_t$, via an aggregator function $W$, to compute utility at $t$, i.e.,

$$U_t = W(c_t, \mu_t).$$

Moreover, $W$ is specified as the CES form

$$W(c, z) = [c^{\rho} + \beta z^{\rho}]^{1/\rho}, \quad c, z \geq 0, \quad 0 \neq \rho < 1, \quad 0 < \beta < 1. \tag{2}$$

Thus utility is defined recursively by means of[2]

$$U_t = \left[c_t^{\rho} + \beta\left(\mathrm{E}_t U_{t+1}^{\alpha}\right)^{\rho/\alpha}\right]^{1/\rho}, \qquad t \geq 0. \tag{3}$$

[1] The reader is referred to EZ (1987a) for a more rigorous and detailed presentation.

[2] The case $\rho = 0$ could be handled in the usual fashion but is ignored for simplicity. Similarly, the analysis below is largely restricted to the case $\alpha \neq 0$. The CES specification (2) is restrictive, but it is still sufficiently flexible to permit discussion of the principal issue at hand – substitution versus risk aversion. We could handle more general specifications than (1) for the certainty equivalent [see EZ (1987a)], but with no greater insight. The homogeneity of (1) and (2) is important for the derivation of equations characterizing equilibrium asset prices which involve only 'observable' variables. Farmer (1987) has independently proposed the special case of (1)–(3) corresponding to $\alpha = 1$.

Two special cases will help to clarify (3). For deterministic consumption programs we see that

$$U_0(c_0, c_1, \ldots) = \left[\sum_0^{\infty} \beta^t c_t^\rho\right]^{1/\rho}, \tag{4}$$

which is an intertemporal CES utility function with elasticity of substitution $\sigma = (1-\rho)^{-1}$. Thus we interpret $\rho$ as reflecting substitution. In the second special case take $\alpha = \rho$. Then

$$U_0 = \left(\mathrm{E}_0\left[\sum \beta^t \tilde{c}_t^\rho\right]\right)^{1/\rho}, \tag{5}$$

which is the common homothetic expected utility specification. Thus the latter is a parametric special case of (3).

Some brief comments are in order regarding (3). Firstly, note that when restricted to deterministic consumption programs, (3) implies the recursive structure introduced by Koopmans (1960) for a deterministic model. Lucas and Stokey (1984) subsequently coined the term 'aggregator' for $W$. As a result of the recursivity, (3) implies the intertemporal consistency of preferences in the sense of Johnsen and Donaldson (1985) and the stationarity of preferences in the sense of Koopmans (1960).

Finally, $\alpha$ may be interpreted as a risk aversion parameter with the degree of risk aversion increasing as $\alpha$ falls. This interpretation is suggested by the role which $\alpha$ plays in the construction of a certainty equivalent. Of course, it is clear from (4) that $\alpha$ has no effect on the ranking of deterministic programs. Moreover, the following argument was developed by EZ: For any given consumption program with initial period consumption $c_0$, let $(c_0, c, c, \ldots)$ be indifferent to it; i.e., the latter deterministic and 'nearly' constant program is a certainty equivalent for the original consumption program.[3] Then $c$ falls as $\alpha$ falls, indicating an increase in risk aversion. (Further support for this interpretation of $\alpha$ is provided at the end of this section.)

The existence of utility functions satisfying (3) on a broad domain is proven in EZ (1987a, theorem 5.1). That domain includes all bounded consumption programs and thus all programs that are feasible in equilibrium in the optimization problem described below.

### 2.2. The environment

A single perishable consumption good is produced in $n$ distinct productive units. Let $s_{it}$ be the output of unit $i$ in period $t$, $0 \leq \underline{s} \leq s_{it} \leq \bar{s}$, $i = 1, \ldots, n$,

[3] Note that here the certainty equivalent is measured in consumption units, whereas that defined in (1) is measured in utility units.

and $s_t \equiv (s_{1t}, \ldots, s_{nt})$. Suppose that $s_t$ is i.i.d.[4] The agent observes $s_t$ at the start of period $t$.

Savings are in the form of holdings of equity shares issued by the productive units. Each unit has outstanding one perfectly divisible share which entitles its owner at the beginning of a period to all of the output during that period. Shares are traded competitively at prices $p_t = (p_{1t}, \ldots, p_{nt})$. In light of the stationarity of the environment and the recursivity of preferences, equilibrium prices should be expressible as a function $p_t = p(s_t)$. Thus we suppose that the consumer's expectations of future prices are represented by such a function $p(\cdot)$, which is taken to be positive.

## 2.3. *Optimal consumption and portfolio choice*

Denote by $z_t$ the vector of share holdings at the beginning of period $t$ and by $x_t$ the beginning of period wealth. More precisely,

$$x_t \equiv (s_t + p(s_t))z_t, \qquad t \geq 0. \tag{6}$$

In light of the recursive structure of (3), intertemporal utility maximization leads to the following dynamic programming problem for $\alpha \neq 0$:[5]

$$J(x_t, s_t) = \max_{c_t,\, z_{t+1}} \left[c_t^\rho + \beta\, \mathrm{E}_t^{\rho/\alpha}[J^\alpha(x_{t+1}, s_{t+1})]\right]^{1/\rho}, \tag{7}$$

where $x_{t+1}$ is defined by the appropriate form of (6). Of course, $J(x_t, s_t)$ denotes the maximum utility achievable given initial conditions $(x_t, s_t)$.

The objective of this section is to derive some necessasy conditions for the solution of (7). These conditions [see (17)] generalize the Euler equations of the standard expected utility model. Some readers may wish to skip directly to (17) at least on first reading.

It is evident from (6) and the homogeneity of utility that $J$ may be written in the form

$$J(x, s) = A(s)x.$$

Since utility is non-negative, $A(\cdot) \geq 0$. It is always feasible to choose $z_t = z_0$

[4]More general stochastic processes for $s_t$ could be handled. But serial dependence of the $s_t$'s is not germane to the comparative statics questions upon which we focus below. Moreover, the i.i.d. assumption allows us to use elementary mathematical arguments, rather than contraction mapping techniques, in sections 3 and 4.

[5]It follows from EZ (1987a) that there exists a solution to (7) and that this dynamic programming problem corresponds, in the usual fashion, to intertemporal utility maximization subject to appropriate constraints. The latter may be found in an earlier version of this paper, available from the author upon request.

and $c_t = z_0 s_t$, $\forall t$. Since $s_{it} \geq \underline{s} \geq 0$, a positive level of utility is therefore always feasible if $z_0 \neq 0$. Thus

$$A(\cdot) > 0.$$

Define the portfolio share vector $w = (w_1, \ldots, w_n)$,

$$w_i = p_i(s)\, z_i / p(s) z,$$

and the vector of gross asset returns $r_t = (r_{1t}, \ldots, r_{nt})$,

$$r_{it} \equiv (s_{it} + p_i(s_t))/p_i(s_{t-1}), \qquad t \geq 1. \tag{8}$$

Then the Bellman equation (7) can be rewritten in the form

$$A(s_t)x_t = \max\left\{c_t^{\rho} + \beta(x_t - c_t)^{\rho}\, \mathrm{E}_t^{\rho/\alpha}\left[A(s_{t+1})\left(\sum w_i r_{i,t+1}\right)\right]^{\alpha}\right\}^{1/\rho}, \tag{9}$$

subject to $0 \leq c_t$, $w \in R_+^n$ and $\sum w_i = 1$. It is evident that portfolio choice is described by

$$\mu_t^* \equiv \max_{w \in R_+^n, \sum w_i = 1} \mathrm{E}^{1/\alpha}\left(\left[A(s_{t+1})\left(\sum w_i r_{i,t+1}\right)\right]^{\alpha}\right), \tag{10}$$

and that consumption is chosen to solve

$$A(s_t)x_t = \max_{0 \leq c_t}\left\{c_t^{\rho} + \beta(x_t - c_t)^{\rho}\mu_t^{*\rho}\right\}^{1/\rho}. \tag{11}$$

In particular, the portfolio choice problem is separable from the consumption–savings decision.

It follows from the homogeneity of (11) that optimal consumption satisfies $c_t^* = a(s_t)x_t$, $t \geq 0$. Substitute this into (11) and obtain

$$A^{\rho}(s_t) = a^{\rho}(s_t) + \beta(1 - a(s_t))^{\rho}\mu_t^{*\rho}. \tag{12}$$

This first-order condition in (11) implies

$$a^{\rho-1} = \beta(1-a)^{\rho-1}\mu_t^{*\rho}. \tag{13}$$

These latter two equations combine to yield

$$A(s_t) = (a(s_t))^{(\rho-1)/\rho} = (c_t^*/x_t)^{(\rho-1)/\rho}.$$

From the stationarity of the problem and the recursivity of utility, it follows that

$$A(s_{t+1}) = (c^*_{t+1}/x_{t+1})^{(\rho-1)/\rho}$$
$$= (c^*_{t+1}/M_{t+1})^{(\rho-1)/\rho}(x_t - c^*_t)^{(1-\rho)/\rho}, \tag{14}$$

where

$$M_{t+1} \equiv w_t^* r^*_{t+1}$$

is the gross return to the optimal portfolio $w_t^*$.

Substitute the latter equation, (10) and (14) into (13) to deduce that

$$\beta \, \mathrm{E}_t^{\rho/\alpha}\left[(c^*_{t+1}/c^*_t)^{\alpha(\rho-1)/\rho} M_{t+1}^{\alpha/\rho}\right] = 1. \tag{15}$$

Also, (14) and the first-order conditions for (10) imply (after suppressing asterisks)

$$\mathrm{E}_t\left[(c_{t+1}/c_t)^{\alpha(\rho-1)/\rho} M_{t+1}^{(\alpha-\rho)/\rho}(r_{i,t+1} - r_{j,t+1})\right] = 0, \quad \forall i, j. \tag{16}$$

Multiply this last equation by $w_{it}^*$, sum over $i$ and apply (15) to deduce that

$$\beta \, \mathrm{E}_t^{\rho/\alpha}\left[(c_{t+1}/c_t)^{\alpha(\rho-1)/\rho} M_{t+1}^{(\alpha-\rho)/\rho} r_{j,t+1}\right] = 1, \qquad j = 1,\dots,n. \tag{17}$$

These latter equations are the analogues of the familiar Euler equations of the expected utility model based on (5), to which they reduce when $\alpha = \rho$. They form the basis for our analysis of equilibrium prices which follows shortly.

Some specializations of (17) may help to clarify this key formula. First, for completeness we include the standard expected utility specialization obtained by setting $\alpha = \rho$:

$$\beta \, \mathrm{E}_t\left[(c_{t+1}/c_t)^{\rho-1} r_{jt}\right] = 1, \qquad j = 1,\dots,n. \tag{18}$$

In that case covariance of an asset's return with consumption growth determines its systematic risk. In contrast, if the equality of $\alpha$ and $\rho$ is not imposed then it is clear from (16) and (17) that consumption growth and the return on the market portfolio *jointly* determine systematic risk. Thus elements of the consumption based CAPM model [Merton (1973), Breeden (1979)] and the static CAPM model [surveyed in Jensen (1972)] are combined in (17). In this regard, it is interesting to consider another special case which in a sense is polar to (18) and which is obtained by setting $\alpha = 0$. It can be shown that the counterpart of (16) corresponds to setting $\alpha = 0$ in (16) above. Then consumption growth drops out and the market return alone determines systematic risk.

To conclude this section we provide further justification for our interpretation of $\alpha$ as a risk aversion parameter. Let $s_0$ and $p(\cdot)$ be such that $r_t$ is independently and identically distributed over time. Then maximum lifetime utility depends only on initial wealth and not separately on current output, i.e., $J = Ax$ and $A$ is constant. Thus from (10) portfolio choice in period $t$ is determined by solving

$$\max_{w \in R^n_+, \sum w_i = 1} \mathrm{E}_t^{1/\alpha}\left[\left(\sum w_i r_{i,t+1}\right)^\alpha\right],$$

which is (equivalent to) an expected utility portfolio problem with von Neumann–Morgenstern utility index of wealth $v(x) = x^\alpha/\alpha$ having constant relative risk aversion $(1-\alpha)$. In particular, $\alpha = 1$ corresponds to risk-neutral behaviour in portfolio choice in this environment. Similarly, $\alpha = 0$ (considered above) corresponds to logarithmic risk preferences.

## 3. Equilibrium

An *equilibrium* is a function $p : [\underline{s}, \bar{s}]^n \to R_{++}$ such that the corresponding utility maximum described above, given any $s_0$ and $z_0 = (1, \ldots, 1)$, has

$$c_t = \sum s_{it} \quad \text{and} \quad z_t = (1, \ldots, 1) \quad \text{for} \quad t \geq 0.$$

Thus $p(\cdot)$ is an equilibrium if decisions based on it clear all consumption and equity markets.

If $p(\cdot)$ is an equilibrium then we can substitute (8) and the market clearing conditions into (17) and conclude that

$$1 = \beta^{\alpha/\rho} \mathrm{E}_t\left[\left[\frac{\sum s_{i,t+1}}{\sum s_{it}}\right]^{\alpha(\rho-1)/\rho}\left[\frac{\sum\left(p_i(s_{t+1}) + s_{i,t+1}\right)}{\sum p_i(s_t)}\right]^{(\alpha-\rho)/\rho}\right.$$

$$\left.\times \frac{\left(p_j(s_{t+1}) + s_{j,t+1}\right)}{p_j(s_t)}\right], \qquad j = 1, \ldots, n. \tag{19}$$

This is the counterpart of eq. (6) of Lucas to which it reduces when $\alpha = \rho$ (and Lucas' model is specialized to homogeneous utility functions). To help clarify

(19) we write the single-asset ($n = 1$) version of this equation:

$$p(s_t) = \beta \mathrm{E}_t^{\rho/\alpha}\left[\left\{\left[\frac{s_{t+1}}{s_t}\right]^{\rho-1}\left(p(s_{t+1}) + s_{t+1}\right)\right\}^{\alpha/\rho}\right]. \tag{20}$$

To complete the parallel with Lucas' analysis we need to show that (i) there exists a unique solution $p(\cdot)$ to (19), and (ii) that the solution is an equilibrium. To do so, rewrite (19) in the form

$$p_j(s_0)\left(\sum p_i(s_0)\right)^{(\alpha-\rho)/\rho}\left(\sum s_{i0}\right)^{\alpha(\rho-1)/\rho}$$

$$= K_j \equiv \beta^{\alpha/\rho}\mathrm{E}\left[\left(\sum s_{i1}\right)^{\alpha(\rho-1)/\rho}\left(\sum\left(p_i(s_1) + s_{i1}\right)\right)^{(\alpha-\rho)/\rho}\right.$$

$$\left.\times\left(p_j(s_1) + s_{j1}\right)\right], \qquad j = 1,\ldots,n,$$

where $K_j$ is a positive constant, i.e., independent of $s_0$. (The notation has been simplified somewhat by setting $t = 0$ and deleting the conditionalization of the expected value operator.) Sum over $j$ in (19) to obtain

$$p_j(s) = K_j\left(\sum K_i\right)^{(\rho-\alpha)/\alpha}\left(\sum s_i\right)^{1-\rho}. \tag{21}$$

Substitute (21) into (19) and derive the following $n$ equations in the $n$ unknowns $K_j$, $j = 1,\ldots,n$:

$$K_j = \beta^{\alpha/\rho}\mathrm{E}\left[\left(\sum s_{i1}\right)^{\rho-1}\cdot\left(\left(\sum s_{i1}\right)^{\rho} + \left(\sum K_i\right)^{\rho/\alpha}\right)^{(\alpha-\rho)/\rho}\right.$$

$$\left.\times\left\{K_j\left(\sum s_{i1}\right)^{1-\rho}\left(\sum K_i\right)^{(\rho-\alpha)/\alpha} + s_{j1}\right\}\right], \qquad j = 1,\ldots,n. \tag{22}$$

It is also useful to note that the aggregate price index $\sum p_j$ satisfies

$$\sum p_j(s) = \left(\sum K_j\right)^{\rho/\alpha}\left(\sum s_i\right)^{1-\rho}, \tag{23}$$

where $\sum K_j$ solves

$$\left(\sum K_j\right) = \beta^{\alpha/\rho}\mathrm{E}\left[\left\{\left(\sum s_{i1}\right)^{\rho} + \left(\sum K_i\right)^{\rho/\alpha}\right\}^{\alpha/\rho}\right]. \tag{24}$$

*Theorem 1.* *Let the $s_t$'s be i.i.d and $\alpha \neq 0$. Then there exists a unique equilibrium $p(\cdot)$ and it satisfies (21) and (22). Moreover, the solution $(K_1, \ldots, K_n) > 0$ to (22) is unique.*

A proof may be found in the appendix.

## 4. Properties of equilibrium

Turn now to the nature of the dependence of the equilibrium on the exogenous parameters of the model, namely preferences, the current output level and the probability distribution of output. We focus on the aggregate price index $\sum p_i(\cdot)$ which is characterized by the equation which applies in a model with a single productive unit having output $\sum s_{i1}$. Thus, without loss of generality we take $n = 1$ in this section. The equilibrium price equations take the form

$$p(s) = Ks^{1-\rho}, \tag{25}$$

$$K = \beta\, \mathrm{E}^{\rho/\alpha}\left[(s'^{\rho} + K)^{\alpha/\rho}\right]. \tag{26}$$

[Note that $K$ denotes $(\sum K_j)^{\rho/\alpha}$ of (23).]

It is evident from (25) that the output (or income) elasticity of price is

$$sp'(s)/p(s) = (1 - \rho) = \sigma^{-1}, \tag{27}$$

where $\sigma$ is the elasticity of intertemporal substitution. The intuition underlying (27) is clear: In a period of high transitory income consumers attempt to increase their purchases of securities in order to transfer part of the windfall to the future. Since there is no storage, this attempted transfer must be precluded and it is by an increase in asset prices. A small increase in prices and a corresponding small decrease in the rate of return to saving suffice if $\sigma$ is large.

Eq. (27) holds also in a (homogeneous) expected utility model, but the limited flexibility of (5) precludes as clear and sharp an interpretation as above. Thus, for example, Lucas (1978, pp. 1439, 1441) must equivocate as to which aspect of utility, substitutability or risk aversion, determines the income sensitivity of prices.

Next consider the effect of a more favourable (in the sense of first-degree stochastic dominance) probability distribution for output. The right side of (26) is increasing in $K$ and is increasing (decreasing) in the distribution of $s'$ if $\rho > 0$ $(< 0)$. These observations lead immediately to the following:

*Theorem 2.* *Let $p^i(s) = K^i s^{1-\rho}$, $i = a, b$, be the equilibrium price functions corresponding to cumulative distribution functions $F^a$ and $F^b$ for output, where*

*$F^b$-strictly dominates $F^a$ by first-order stochastic dominance. Then*

$$K^b > (<) K^a \quad \text{if} \quad \sigma \equiv (1-\rho)^{-1} > (<) 1. \tag{28}$$

New optimism regarding future dividends will, for a given $p(\cdot)$ and $s_0$, increase the perceived return to savings $(s_1 + p(s_1))/p(s_0)$. If $\sigma > 1$, the substitution effect of this change dominates, the demand for securities increases and asset prices rise to restore equilibrium. If $\sigma < 1$, the income effect dominates, there is a binge in current consumption and asset prices are pushed downward.

Once again this explanation is clearer than what is possible in the standard framework [Lucas (1978, p. 1441)]. Labadie (1986, theorems 4.1, 4.2) derives a corresponding result in her model.

Next we consider the effect on equilibrium prices of an increase in the degree of risk aversion. The latter is modelled by a reduction in $\alpha$. The right side of (26) is increasing in $\alpha/\rho$. Thus the next result is implied.

*Theorem 3. Let $p^i(s) = K^i s^{1-\rho}$, $i = a, b$, be the equilibrium price functions corresponding to $\alpha^a$ and $\alpha^b$, respectively, where $\alpha^a < \alpha^b$. Then*

$$K^a < (>) K^b \quad \text{if} \quad \sigma > (<) 1. \tag{29}$$

The result (29) is intuitive. For given $p(\cdot)$, an increase in risk aversion acts to reduce the certainty equivalent return to saving. The effect on behaviour is similar to the consequence of a lower rate of return in a deterministic model. If $\sigma < (>) 1$, the dominant income (substitution) effect implies reduced (enhanced) present consumption and an increased (reduced) demand for securities. Thus asset prices are forced to rise (fall).[6]

The above intuition was validated formally by Kihlstrom and Mirman (1974, pp. 378–380) and Selden (1979) with respect to the behaviour of an individual facing exogenous prices and operating in a two-period framework. Theorem 3 represents an infinite horizon and market equilibrium counterpart of these earlier results. Of course, the approaches to comparative risk aversion analysis adopted in the above papers are distinct from (one another and from) the approach adopted here.[7]

Comparable results for an overlapping generations model are proven by Labadie (1986, theorems 2.2, 2.3), though she claims that in her model the

[6] In light of (25), (29) has clear implications for the dispersion as well as the level of asset prices. But these implications are likely not robust to generalizations of the i.i.d. assumption for $s_t$.

[7] Yet another approach may be found in Chew and Epstein (1987) which contains an analysis of individual behaviour in an infinite-horizon framework.

change in prices is ambiguous if $\sigma < 1$. A comparative risk aversion analysis was not attempted by Lucas.

To conclude, consider the issue of the circumstances under which asset prices possess the martingale property. It is well known that the latter is generally not a feature of equilibrium. For example, Lucas (1978, p. 1443) states 'that the presence of diminishing marginal rate of substitution of future for current consumption is inconsistent with this property'. Frequently [LeRoy (1973), for example] risk aversion is cited as the theoretical reason for the violation of the martingale property. In our model, even with $s_t$'s correlated across time, we see from (20) that only if $\alpha = \rho = 1$ do we obtain the simple discounting formulae

$$p(s_0) = \beta\, \mathrm{E}_0[s' + p(s')],$$

$$p(s_0) = \mathrm{E}_0\left[\sum_1^\infty \beta^t s_t\right].$$

Thus *both* perfect intertemporal substitutability ($\sigma = \infty$) *and* risk neutrality ($\alpha = 1$) are necessary for these relations to be valid.

## Appendix: Proof of Theorem 1

*Step 1:* Show that (24) possesses a unique solution $K^* = \sum K_j$. For $y \geq 0$, let $f(y) = \beta\, \mathrm{E}^{\rho/\alpha}[((\sum s_{i1})^\rho + y)^{\alpha/\rho}] - y$. Then (24) is equivalent to $f((\sum K_i)^{\rho/\alpha}) = 0$. It is evident that $f(0) > 0 > f(\infty)$. Thus $\exists y^* > 0$, $f(y^*) = 0$. Moreover, $f(y) = 0 \Rightarrow f'(y) < 0$. Thus $y^*$ is the unique zero of $f$.

*Step 2:* Show that (22) possesses a unique solution. It suffices to prove existence and uniqueness of $K_1, \ldots, K_{n-1}$ solving $f^j(K_j) = 0$, $j = 1, \ldots, (n-1)$, where

$$f^j(K_j) \equiv \beta^{\alpha/\rho}\, \mathrm{E}\Big[\Big(\sum s_{i1}\Big)^{\rho-1}\Big(\Big(\sum s_{i1}\Big)^\rho + K^{*\rho/\alpha}\Big)^{(\alpha-\rho)/\rho}$$
$$\times\Big(K_j K^{*(\rho-\alpha)/\alpha}\Big(\sum s_{i1}\Big)^{1-\rho} + s_{j1}\Big)\Big].$$

It is straightforward to verify that

$$f^j(0) > 0 > \lim_{K_j \to \infty} f^j(K_j) \quad \text{and} \quad f^j(K_j) = 0 \Rightarrow f^{j\prime}(K_j) < 0,$$

which completes this step.

*Step 3:* Any equilibrium must satisfy (21) and (22). This has already been proven in the text.

*Step 4:* Any solution of (21) and (22) defines an equilibrium. Start with such a solution and define

$$A(s)=\left[\sum s_i/\sum(s_i+p_i(s))\right]^{(\rho-1)/\rho}, \tag{A.1}$$

which is a form of 'converse' for (14). Let

$$J(z_0,s_0)=A(s_0)\left[(s_0+p(s_0))z_0\right]. \tag{A.2}$$

Then it can be verified, by checking the appropriate first- and second-order conditions, that the Bellman equation (9) is satisfied with $z_0=z^*=(1,\dots,1)$ and $c_0^*=\sum s_{i0}$. But the only solution $J$ of (9) is the value function. Thus (A.1) and (A.2) correspond to a utility maximum and $p(\cdot)$ defines an equilibrium.

It remains only to prove the asserted uniqueness of the function $A(\cdot)$ satisfying (9): Let $A(\cdot)$ and $\mu$ satisfy (10) and (11). Substitute (21) into the former and deduce that for some constant $L$,

$$\mu=L^{\rho-1}\left(\sum s_{i0}\right)^{\rho-1}.$$

From (11) and (13),

$$\left(A(s_0)^{\rho/(1-\rho)}-1\right)^{1-\rho}=\beta\mu^{\rho} \tag{A.3}$$

$$\Rightarrow A(s)^{\rho/(1-\rho)}=1+B\left(\sum s_i\right)^{-\rho} \tag{A.4}$$

for some constant $B$. Substitute the latter equation into (A.3) and (10) to derive

$$p(s_0)\left(\sum s_{i0}\right)^{\rho-1}=\beta \max_{w\in R_+^n,\sum w_i=1} F(B,W),$$

where

$$F(B,w)=\beta\,\mathrm{E}^{1/\alpha}\left[\left(B^{-1}+\left(\sum s_{i1}\right)^{-\rho}\right)^{\alpha(1-\rho)/\rho}\left\{\sum w_i(s_{i1}+p_i(s_1))\right\}^{\alpha}\right].$$

But $F(B,w)$ is monotonic in $B$. Thus $B$ is uniquely determined. By (A.4), so is $A(\cdot)$. Q.E.D.

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