THREE PARADOXES FOR THE 'SMOOTH AMBIGUITY' MODEL OF PREFERENCE*

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Abstract

Three Ellsberg-style thought experiments are described that reflect on the smooth ambiguity decision model developed by Klibanoff, Marinacci and Mukerji (2005). The first experiment poses difficulties for the model's axiomatic foundations and, as a result, also for its interpretation, particularly for the authors' claim that the model achieves a separation between ambiguity and the attitude towards ambiguity. Given the problematic nature of its foundations, the behavioral content of the model, and how it differs from multiple-priors, for example, are not clear. The other two thought experiments cast light on these questions.

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1. INTRODUCTION

Three Ellsberg-style thought experiments, or examples, are described that reflect on the smooth ambiguity decision model developed by Klibanoff, Marinacci and Mukerji [14], henceforth KMM. It is argued that the first experiment poses difficulties for KMM's axiomatic foundations for their model and, as a result, also for its interpretation, particularly for the authors' claim that the model achieves a separation between ambiguity and the attitude towards ambiguity. KMM present their model in unqualified terms as a general model, (for example, as an alternative to multiple-priors (Gilboa and Schmeidler [8])), and describe it (p. 1875) as "offering flexibility in modeling ambiguity" and as permitting "a wide variety of patterns of ambiguity." However, given the problematic nature of its foundations, the behavioral content of the model, and how it differs from multiple-priors, for example, are not clear. The other two thought experiments cast light on these questions. They demonstrate important differences from multiple-priors in when randomization between acts is valuable, and in the meaning of "stochastic independence."

We begin with an outline of the model that we refer to here as the KMM model. Let Ω be a set of states, C the set of consequences or prizes, taken here, for simplicity, to be a compact interval in the real line, and denote by $\Delta(C)$ and $\Delta(\Omega)$ the sets of probability measures on C and Ω respectively. (Technical details are standard and suppressed.) An act is a mapping $f : \Omega \to \Delta(C)$, that is, by an act we shall mean an Anscombe-Aumann act over the state space Ω.[1] The set of all acts is $\mathcal{F}$. KMM also employ *second-order acts*, which are maps $F : \Delta(\Omega) \to C$; if F is binary (has only two possible outcomes), refer to it as a *second-order bet*. The set of all second-order acts is $\mathcal{F}^2$.

KMM posit a preference order $\succeq$ on $\mathcal{F}$ and another preference $\succeq^2$ on $\mathcal{F}^2$. The corresponding utility functions, U and U^2, have the following form:

$$U(f) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(f(\omega))\, dp(\omega)\right) d\mu(p), \quad f \in \mathcal{F}, \tag{1.1}$$

and

$$U^2(F) = \int_{\Delta(\Omega)} \phi(u(F(p)))\, d\mu(p), \; F \in \mathcal{F}^2. \tag{1.2}$$

Here μ is a (countably additive) probability measure on $\Delta(\Omega)$, $u : \Delta(C) \to R$ is mixture linear, ϕ is continuous and strictly increasing on $u(C) \subset \mathbb{R}$, where C is identified with a subset of $\Delta(C)$ in the familiar way and we denote by u also its restriction to C. Finally, it is assumed that u is continuous and strictly increasing on C. Identify a KMM agent with a triple (u, ϕ, μ) satisfying the above conditions.

These functional forms suggest appealing interpretations. The utility of an Anscombe-Aumann act would simply be its expected utility if the probability law p on Ω were known. However, it is uncertain in general, with prior beliefs represented by μ, and this uncertainty

[1] KMM use Savage acts over $\Omega \times [0, 1]$ rather than Anscombe-Aumann acts. However, this difference is not important for our purposes. Below by the "KMM model" we mean the Anscombe-Aumann version outlined here, and the corresponding translation of their axioms and arguments.

about the true law matters if ϕ is nonlinear; in particular, if ϕ is concave, then

$$\begin{aligned} U(f) &\leq \phi\left(\int_{\Delta(\Omega)} \int_{\Omega} u(f(\omega))\, dp(\omega)\, d\mu(p)\right) \\ &= \phi\left(\int_{\Delta(\Omega)} u\left(\int_{\Omega} f(\omega)\, dp(\omega)\right) d\mu(p)\right) \\ &= \phi\left(u\left(\int_{\Delta(\Omega)} \int_{\Omega} f(\omega)\, dp(\omega)\, d\mu(p)\right)\right) \\ &= U(L_f(\mu)), \end{aligned}$$

where $L_f(\mu)$ is a lottery over outcomes, viewed also as a constant act,

$$L_f(\mu) = \int_{\Delta(\Omega)} \int_{\Omega} f(\omega)\, dp(\omega)\, d\mu(p) \in \Delta(C).$$

It is the lottery derived from f if one uses μ to weight probability measures over states and then reduces the resulting three-stage compound lottery in the usual way. In that sense $L_f(\mu)$ and f embody similar uncertainty, but only for f do eventual payoffs depend on states in Ω where there is uncertainty about the true law. Thus the inequality

$$U(f) \leq U(L_f(\mu)), \text{ for all } f \in \mathcal{F}, \tag{1.3}$$

is essentially KMM's behavioral definition of *ambiguity aversion*. As noted, the latter is modeled by a concave ϕ, while ambiguity (as opposed to the attitude towards it) seems naturally to be captured by μ - hence, it is claimed, *a separation is provided between ambiguity and aversion to ambiguity.* This separation is highlighted by KMM as a major advantage of their model over all others in the literature, (see also their paper [15] dealing with a dynamic model), and has been repeated often by researchers as motivation for their adoption of the KMM model. (See Hansen [11], Chen, Ju and Miao [2] and Ju and Miao [13], for example.)

The "story" of the smooth model may seem appealing, and the functional form may seem natural, perhaps more so than the multiple-priors functional form, where

$$U^{MP}(f) = \min_{p \in P} \int_{\Omega} u(f(\omega))\, dp(\omega), \; f \in \mathcal{F}, \tag{1.4}$$

for some set of priors $P \subset \Delta(\Omega)$. After all, multiple-priors utility is a limiting case - if P is the support of μ, then, up to ordinal equivalence, (1.4) is obtained in the limit as the degree of concavity of ϕ increases without bound. However, the meaningful content and relative merits of the two models depend on their predictions for behavior and not on appearances or on purely mathematical calculations. From this more meaningful perspective, a different picture emerges.

Seo [18] provides alternative foundations for the utility function (1.1) on $\mathcal{F}$, and Ergin and Gul [6] propose a closely related model. Neither Seo, nor Ergin and Gul make strong claims for their models such as made by KMM. Their papers are discussed further below. Finally, other critical perspectives on the smooth ambiguity model may be found in Baillon et al. [1] and Halevy and Ozdenoren [10].

2. THOUGHT EXPERIMENT 1

2.1. Second-Order Bets

In Ellsberg's classic 3-color experiment, you are told the following. An urn contains 3 balls, of which 1 is red (R), and the others are either blue (B) or green (G).[2] Then you are offered some bets on the color of the ball to be drawn at random from the urn. Specifically, you are asked to chose between f_1 and f_2, and also between f_3 and f_4, where these acts are defined by:

Bets on the color			
	R	B	G
f_1	100	0	0
f_2	0	100	0
f_3	100	0	100
f_4	0	100	100

The choices pointed to by Ellsberg (and by many subsequent experimental studies) are

$$f_1 \succ f_2 \text{ and } f_3 \prec f_4. \tag{2.1}$$

The well-known intuition for these choices is uncertainty about the true composition of the urn combined with aversion to that uncertainty. Refer to the above pair of choices as the "Ellsbergian choices."

Consider a simple extension of Ellsberg's experiment that tests jointly whether Ellsbergian behavior is exhibited and, if so, whether the above explanation is correct. The extension adds bets on the true composition of the urn, as described next.

First you are told more about how the above urn (referred to below as the *normal urn*) is constructed. There exists another urn, that we call a *second-order urn*, containing 3 balls. One has the label r and the others are labeled either b or g. A ball will be drawn from this urn, and the ball's label will determine the color composition of the normal urn. If the label $i \in \{r, b, g\}$ is drawn from the second-order urn, then the normal urn will have composition p_i, where

$$p_r = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \; p_b = \left(\tfrac{1}{3}, \tfrac{2}{3}, 0\right), \text{ and } p_g = \left(\tfrac{1}{3}, 0, \tfrac{2}{3}\right),$$

are three probability measures on $\{R, B, G\}$. Thus it is certain that there will be 1 red ball but there could be either 0 or 2 blue (and hence also green). You are offered some bets, and after making your choices, a ball will be drawn from the second-order urn, and from the normal urn constructed as described according to the outcome of the first draw. Finally, the two balls drawn and the bets chosen determine payoffs.

In one pair of choice problems, you can choose between f_1 and f_2, and between f_3 and f_4, the bets on the color drawn from the normal urn as in Ellsberg's experiment. In addition, you can choose between bets on the true composition of the normal urn, or

[2] Ellsberg postulated 30 red balls and 60 that are either blue or green. But the message is clearly the same.

equivalently, on the label of the ball drawn from the second-order urn. Specifically, you choose between F_1 and F_2, and between F_3 and F_4, where they are given by:

Bets on the composition			
	r	b	g
F_1	100	0	0
F_2	0	100	0
F_3	100	0	100
F_4	0	100	100

The Ellsbergian choices here are

$$F_1 \succ F_2 \text{ and } F_3 \prec F_4. \tag{2.2}$$

If the Ellsbergian choices (2.1) are due to an aversion to uncertainty about the true composition, then one might expect that aversion to be expressed directly via the bets on the urn's composition and thus to lead to Ellsbergian choices there. In other words, our intuition is that

$$\begin{aligned} &\text{Ellsbergian choices in betting on color} \\ \Longrightarrow\ &\text{Ellsbergian choices in betting on composition.} \end{aligned}$$

We refer to this as the *intuitive hypothesis*.

If we take $\Omega = \{R, B, G\}$, then bets on the color drawn from the normal urn are acts in $\mathcal{F}$, and bets on the true composition of the normal urn are second-order acts, elements in $\mathcal{F}^2$. By KMM's Assumption 2, preference on second-order acts has the subjective expected utility representation (1.2). Thus, one can see immediately that their full model contradicts either the intuitive hypothesis in our expanded experiment or Ellsbergian behavior in betting on the color. Though there is nothing mysterious in this contradiction, we elaborate shortly on why we feel that nevertheless it has significant implications for the KMM model and its interpretation.

Finally, note that our expanded Ellsberg example has nothing to say about any of the other models of ambiguity averse preferences in the literature. The smooth ambiguity model is, to our knowledge, unique in making assumptions about the ranking of second-order acts. KMM exploit these assumptions to argue for some unique strengths of their model. But these strengths are raised into question if their supporting assumptions generate counterintuitive predictions.

2.2. Why Is It Important?

KMM's declared focus (p. 1851) is the functional form (1.1) for U on the domain $\mathcal{F}$. They expand the domain to include second-order acts only in order to provide foundations for preference on $\mathcal{F}$. The focus on $\mathcal{F}$ is understandable since economically relevant objects of choice correspond to acts in $\mathcal{F}$, while choices between bets on the true probability law are not readily observed in the field. For example, the purchase of a financial asset is a bet on a favorable realization of the stochastic process generating returns, and not directly on which probability law describes that process.[3] Thus, it might seem, the domain $\mathcal{F}^2$ is of secondary importance, and counterfactual or counterintuitive predictions there are not critical. The SEU assumption on $\mathcal{F}^2$, one might think, is merely a simplifying assumption that facilitates focussing on the important behavior.

However, the model's predictions on $\mathcal{F}^2$ are not a "side issue." As illustrated by the thought experiment, the SEU assumption contradicts the model's explanation of why ambiguity averse choices are made when choosing between acts in $\mathcal{F}$. Thus it is not at all innocuous if explaining choice in $\mathcal{F}$ is the objective. Dekel and Lipman [3, Section 2] argue similarly when considering the more general question of when the refutation of a model's "simplifying assumptions" is important. In their view, this is the case when those simplifications are crucial to the model's explanation of the central observations (here, ambiguity averse behavior in $\mathcal{F}$).

Another way in which the domain $\mathcal{F}^2$ is a critical ingredient concerns uniqueness and interpretation. It is well-known that μ and ϕ appearing in (1.1) are not pinned down uniquely by preference on $\mathcal{F}$ alone. (For example, if ϕ is linear, then any two measures μ' and μ with the same mean represent the same preference on $\mathcal{F}$.) Moreover, interpretation of μ and ϕ as capturing (and separating) ambiguity and ambiguity attitude, obviously presupposes that these components are unique. KMM achieve uniqueness by expanding their domain to include second-order acts. Thus, if one is to retain the appealing interpretations that they offer, one cannot simply ignore that component of their model.

Further, note that interpretation matters. This is true for many reasons, but the specific reason we wish to emphasize is that it matters for empirical applications of the model. In a quantitative empirical exercise, one needs to judge not only whether the model matches data (moments of asset returns, for example), but also whether parameter values make sense, and this requires that they have an interpretation.

Finally, since we have been criticizing KMM's foundations for (1.1), rather than the latter itself, one may wonder about alternative foundations. Seo [18] provides an alternative axiomatic foundation for the same model of preference on normal acts. In his model, an individual can be ambiguity averse only if she fails to reduce objective (and timeless) two-stage lotteries to their one-stage equivalents (much as in Segal's [17] seminal paper). This connection has some experimental support (Halevy [9]). Nevertheless, nonreduction of (timeless) compound lotteries is arguably a mistake while ambiguity aversion is normatively at least plausible. Thus the noted connection severely limits the scope of Seo's model of ambiguity aversion.

The related axiomatic model due to Ergin and Gul [6] will be discussed in the next subsection and again in the context of the final thought experiment.

2.3. Are We Using the Wrong State Space?

The preceding critique hinges on our adoption of $\{R, B, G\}$ as the state space Ω. In defense of KMM, one might argue that the "correct" state space is $\Omega^1 = \{R, B, G\} \times \Delta(\{R, B, G\})$, (or $\{R, B, G\} \times \{p_r, p_b, p_g\}$), since the payoffs to the bets being considered

[3] One might argue that this is reason alone to be dissatisfied with KMM's axioms. That is not our criticism, however. It suffices for our purposes that the ranking of second-order acts is in principle observable in the laboratory.

are determined by the eventual realization of a pair (color, composition). Then all bets correspond to normal acts and the expected utility assumption for second-order acts does not matter. However, incorporating the true probability law into the state space in this way does not solve the problem; for example, one could design another thought experiment that uses Ω^1, instead of Ω, as the basic state space, and that also contradicts KMM. This would force one to adopt as state space $\Omega^2 = \Omega^1 \times \Delta(\Omega^1)$, and so on ad infinitum. Besides, such an argument would render KMM's Assumption 2 for second-order acts unverifiable, and also seems contrary to the spirit of their model.

Ergin and Gul [6] study a recursive subjected expected utility functional form analogous to (1.1), except that it assumes a state space $\Omega_a \times \Omega_b$; thus, for any act defined on the latter, utility has the form

$$U^{EG}(f) = \int_{\Omega_a} \phi\left(\int_{\Omega_b} u\left(f\left(\omega_a, \omega_b\right)\right) d\mu_b\left(\omega_b\right)\right) d\mu_a\left(\omega_a\right), \tag{2.3}$$

where μ_a and μ_b describe beliefs about Ω_a and Ω_b respectively. The component state spaces are interpreted as representing two "issues" underlying the uncertainty facing the individual. In these terms, KMM restrict the issues to be Ω, the set of payoff-relevant states, and $\Delta(\Omega)$, the true probability law on Ω, while Ergin and Gul leave the specification of issues to the modeler. With this added freedom in relating the formal model to any given concrete choice situation, one can accommodate the intuitive hypothesis.[4] For example, Ellsbergian choices in both urns can be rationalized if we adopt the following formalization of the thought experiment: (i) fix a numbering, 1, 2 or 3, of the three balls within each urn such that the red (R and r) ball is numbered 1; (ii) take $\Omega_a = \{bb, bg, gb, gg\}$ corresponding to the possible colors (or labels) of balls numbered 2 and 3 in the second-order urn; and (iii) take $\Omega_b = \{(i_a, j_b) : i_a, j_b = 1, 2, \text{ or } 3\}$, where the pair (i_a, j_b) gives the numbers of the balls drawn from the two urns.

Note, however, that the fact that the Ergin and Gul model is consistent with the intuitive hypothesis does not refute our criticism of the KMM model, which these authors explicitly differentiate from the former (see [14, pp. 1874-5]). In addition, the Ergin and Gul explanation comes at a cost - the model assumes, as is explicit in its foundations, that the individual ranks all acts over $\Omega_a \times \Omega_b$, including, therefore, bets on the numbers of the balls drawn. However, the numbering of balls is arguably a modeling artifact, and part of an "as if" story, rather than being germane to the bets involved in the two urns.

3. THOUGHT EXPERIMENT 2

The example presented here does not involve second-order acts. It concerns only the properties of the KMM model on $\mathcal{F}$, the declared domain of interest.

Before describing the example, we present the simple analytical observation that underlies it. As mentioned, KMM interpret concavity of ϕ as modeling ambiguity aversion. If ϕ is strictly concave, as it is in all applications of the smooth ambiguity model that

[4] Wolfgang Pesendorfer suggested this rationalization, which adapts Ergin and Gul's rationalization (pp. 900-1) of the standard Ellsberg Paradox.

we have seen, then the preference order on $\mathcal{F}$ represented by (1.1) satisfies the following condition:[5] For all AA acts f_1 and f_2,

$$[f_1 \sim f_2 \sim \tfrac{1}{2}f_1 + \tfrac{1}{2}f_2] \implies \tfrac{1}{2}f_1 + \tfrac{1}{2}h \sim \tfrac{1}{2}f_2 + \tfrac{1}{2}h \text{ for all } h \in \mathcal{F}. \tag{3.1}$$

Thus indifference to randomization between the pair of indifferent acts f_1 and f_2 implies indifference between mixtures with *any* third act h. Of course, the implication would be required by the Independence axiom, but ambiguity aversion calls for relaxing Independence. The question is whether the property (3.1) leaves the smooth ambiguity model "too close" to expected utility to capture intuitive behavior under ambiguity.[6] Note that while strict concavity of ϕ is used to derive the sharp result in (3.1), only weak concavity is assumed henceforth.

To see the force of (3.1), consider a concrete case. You are given two urns, numbered 1 and 2, each containing 50 balls that are either red or blue. Thus, $\Omega = \{R_1, B_1\} \times \{R_2, B_2\}$, and

$$R_1 + B_1 = 50 = R_2 + B_2.$$

You are told also that the two urns are generated independently, for example, they are set up by administrators from opposite sides of the planet who have never been in contact with one another. One ball will be drawn from each urn.

Consider the following bets where $c^* > c$ are outcomes in C, and $\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$ denotes the equal probability lottery over these outcomes:

	Bets for Experiment 2			
	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	c^*	c^*	c	c
f_2	c^*	c	c^*	c
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	c^*	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	c
g_1	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$
g_2	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$	c	c^*	$\left(c^*, \frac{1}{2}; c, \frac{1}{2}\right)$

Symmetry suggests indifference between f_1 and f_2. If it is believed that the compositions of the two components are unrelated, then f_1 and f_2 do not hedge one another. This leads to the rankings:

$$f_1 \sim f_2 \sim \tfrac{1}{2}f_1 + \tfrac{1}{2}f_2. \tag{3.2}$$

Ambiguity aversion suggests

$$g_1 \succ g_2. \tag{3.3}$$

(Note that

$$g_1 = \tfrac{1}{2}f_1 + \tfrac{1}{2}h \;\text{ and }\; g_2 = \tfrac{1}{2}f_2 + \tfrac{1}{2}h, \tag{3.4}$$

where $h = (c, c, c^*, c^*)$.)

[5] The (elementary) proof will be apparent after reading the proof of the next proposition.

[6] A possible response is to argue that (3.1) has little bite because indifference to randomization as in the hypothesis is to be expected only "rarely," if at all. This is the nature of the response in Klibanoff et al. [16] to the experiment in this section.

The rankings (3.2)-(3.3), for all $c^* > c$, are easily accommodated by the multiple-priors model. However, as we show next, they are inconsistent with KMM if the natural state space $\Omega = \{R_1, B_1, R_2, B_2\}$ is adopted and if ϕ is taken to be concave.

Proposition 3.1. *If preference over the set of Anscombe-Aumann acts $\mathcal{F}$ is represented by the utility function U in (1.1), where ϕ is concave, then:*

(*) $f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2$ for all $c^* > c$ implies that $\frac{1}{2}f_1 + \frac{1}{2}h \sim \frac{1}{2}f_2 + \frac{1}{2}h$ for all $c^* > c$. In particular, in light of (3.4), the rankings (3.2)-(3.3) are impossible.

Proof. It is without loss of generality (since C was taken to be a compact interval) to assume that u has range equal to $[0,1]$ and that $\phi : [0,1] \longrightarrow \mathbb{R}$. Also without loss of generality, suppose there exists $0 < \kappa < 1$ such that, for all $t < \kappa < t'$,

$$\phi\left(\tfrac{1}{2}t + \tfrac{1}{2}t'\right) > \tfrac{1}{2}\phi(t) + \tfrac{1}{2}\phi(t').$$

Otherwise, ϕ is linear and (*) is obvious. The following cases are essentially exhaustive.

Abbreviate $p(R_1 \times \{R_2, B_2\})$ by $p(R_1)$, and so on.

Case 1: $p(R_1) = p(R_2)$ with μ-probability equal to 1. Then

$$\begin{aligned}
\int_\Omega u(f_1)\, dp &= \int_\Omega u(f_2)\, dp \quad \mu\text{-a.s.} \Longrightarrow \text{ (since } u \text{ is linear)} \\
\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) dp &= \int_\Omega u\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right) dp \quad \mu\text{-a.s.} \Longrightarrow \\
\int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) dp\right) d\mu &= \int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right) dp\right) d\mu \Longrightarrow \\
U\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}h\right) &= U\left(\tfrac{1}{2}f_2 + \tfrac{1}{2}h\right).
\end{aligned}$$

Case 2: There exists $P \subset \Delta(\Omega)$, with $\mu(P) > 0$, such that

$$p(R_1) > p(R_2) \geq 0 \text{ for all } p \in P. \tag{3.5}$$

Take the special case $P = \{p^*\}$. Pick c^* and c so that $1 \geq u(c^*) > u(c) \geq 0$ and

$$p^*(R_2) < \frac{\kappa - u(c)}{u(c^*) - u(c)} < p^*(R_1).$$

Then

$$\int_\Omega u(f_2)\, dp^* < \kappa < \int_\Omega u(f_1)\, dp^*,$$

which, by definition of κ implies that

$$\begin{aligned}
&\phi\left(\int_\Omega u\left(\tfrac{1}{2}f_1 + \tfrac{1}{2}f_2\right) dp^*\right) \\
= \; &\phi\left(\tfrac{1}{2}\int_\Omega u(f_1)\, dp^* + \tfrac{1}{2}\int_\Omega u(f_2)\, dp^*\right) \\
> \; &\tfrac{1}{2}\phi\left(\int_\Omega u(f_1)\, dp^*\right) + \tfrac{1}{2}\phi\left(\int_\Omega u(f_2)\, dp^*\right).
\end{aligned}$$

Since ϕ is concave, it follows that

$$\begin{aligned} U\left(\tfrac{1}{2}f_1+\tfrac{1}{2}f_2\right) &= \int \phi\left(\int_\Omega u\left(\tfrac{1}{2}f_1+\tfrac{1}{2}f_2\right)dp\right)d\mu(p) \\ &> \int\left[\tfrac{1}{2}\phi\left(\int_\Omega u(f_1)\,dp\right)+\tfrac{1}{2}\phi\left(\int_\Omega u(f_2)\,dp\right)\right]d\mu(p) \\ &= \tfrac{1}{2}U(f_1)+\tfrac{1}{2}U(f_2)=U(f_1), \end{aligned}$$

contrary to the hypothesis in (*).

Turn to the general case of (3.5) where P need not be a singleton. Then there exists a subset $Q\subset P$, $\mu(Q)>0$, where, for some $a>0$,

$$q(R_1)>a>q(R_2)\geq 0 \quad \text{for all } q\in Q.$$

Adapt the above argument. ■

4. THOUGHT EXPERIMENT 3

The previous thought experiment begs the question "how does one model stochastic independence?" The question is clearly important more broadly in establishing the credentials of any model of choice under uncertainty. It is well known that stochastic independence is more complicated when ambiguity matters; for example, there is more than one way to form independent products of capacities or sets of priors (Hendon et al. [12], Ghirardato [7], Epstein and Seo [5]). The present experiment suggests that stochastic independence is not easily captured within the KMM model.

The first step is to specify what one means, in terms of behavior on the domain $\mathcal{F}$ of primary interest, by stochastic independence. In light of the cited literature, a unique answer seems too much to hope for. However, we propose the following behavioral criterion as an intuitively *necessary* condition for stochastic independence. Take $\Omega=S_a\times S_b$, where $S_a=S_b=S$.

Independence hypothesis: Consider bets on events $E_a\subset S_a$, $E_b\subset S_b$ and on their conjunction, that is, on $E_a\times E_b$. Let the bet on E_a be indifferent to a bet, with the same stakes, on the toss of a coin having probability of heads (the winning outcome) equal to π_a. Let π_b be defined similarly. Then the bet on $E_a\times E_b$ is indifferent to betting on a coin with probability of heads equal to $\pi_a\pi_b$. Moreover, this is the case for any pair of winning and losing stakes that is common to all bets.

The intuition for the hypothesis is clear, since betting on the coin with bias $\pi_a\pi_b$ is equivalent to betting on successive and *independent* winning tosses of the π_a and π_b coins.

The independence hypothesis is easily accommodated within the multiple-priors model. For example, let P be a subset of $\Delta(S)$, and define[7]

$$P\otimes P=\{p\otimes p' : p,p'\in P\}\subset\Delta(S_a\times S_b). \tag{4.1}$$

[7] For any p',p in $\Delta(S)$, $p\otimes p'$ denotes the product measure on S^2. Also, we ignore technical details such as "compactness" of the set of priors needed to justify the minimum in (1.4).

Use this product set as the set of priors for the utility function in (1.4). Then the hypothesis is easily verified, since

$$\min_{p,p'\in P}\left(p\otimes p'\right)\left(E_a\times E_b\right)=\min_{p\in P}p\left(E_a\right)\quad\cdot\quad\min_{p'\in P}p'\left(E_b\right).$$

Moreover, there are other product rules (in Hendon [12], for example) that satisfy the independence hypothesis. In fact, to our knowledge all notions of stochastic independence in the literature satisfy the hypothesis. The question at hand is whether it can be satisfied within the smooth ambiguity model (excluding the SEU special case). We provide a qualified negative answer.

A seemingly natural way to model stochastic independence via the KMM utility function is to restrict the prior μ to have support on a product set of the form in (4.1). It is apparent from (1.2), that this specification is the usual way to model stochastic independence within the SEU framework, and thus it would do the job if we were concerned with stochastic independence as reflected in choice between second-order bets. However, for all the reasons given in Section 2, our focus is on the choice between normal acts (the domain $\mathcal{F}$), and thus on the behavioral independence hypothesis stated above. The next Proposition shows that, under some auxiliary assumptions, the above specification of the prior μ is not compatible with the independence hypothesis unless ϕ is linear.

Though one would like to eliminate these auxiliary assumptions, the Proposition is strongly suggestive that the KMM model cannot accommodate stochastic independence as expressed in our hypothesis. An open question is whether there exists a different intuitive notion of stochastic independence that can be accommodated within the model.

It is worth noting that similar remarks apply to the Ergin and Gul model. The arguments below rely on the recursive SEU structure in (1.1), rather than the particular choice of "issue state spaces."

Proposition 4.1. *Let $\Omega=S_a\times S_b,\ S_a=S_b=S$, where (S,Σ) is a measurable space, and define utility on $\mathcal{F}$ by (1.1). Suppose that μ has support on $P\otimes P$, such that: (i) $P=\{p_1,...,p_n\}\subset\Delta\left(S,\Sigma\right)$, $n\geq 2$, where each p_i is countably additive and nonatomic; and (ii) the measures in P are linearly independent ($\Sigma_i\lambda_i p_i\left(\cdot\right)=0$ on Σ if and only if $\lambda_i=0$ for all i). Then the independence hypothesis is satisfied only in the special case of SEU (linear ϕ).*

Call a bet normalized if the winning and losing stakes are 1 and 0, denominated in utils using u. The proof shows that the independence hypothesis restricted to normalized bets is satisfied if and only if ϕ is a power function. Admitting other stakes forces ϕ to be linear.

Proof. It is without loss of generality (since C was taken to be a compact interval) to assume that u has range equal to $[0,1]$ and that $\phi:[0,1]\longrightarrow\mathbb{R}$. Given μ on $P\otimes P$, define the marginals μ_a and μ_b on P,

$$\mu_a\left(p\right)=\Sigma_q\mu\left(p\otimes q\right),\ \mu_b\left(q\right)=\Sigma_p\mu\left(p\otimes q\right).$$

Consider the lottery $(1,\pi;\ell,1-\pi)$, where the winning probability is π and the u-denominated stakes are 1 for winning and $\ell = 1-\delta$, $\delta > 0$, for losing. Then $f \sim (1,\pi;\ell,1-\pi) \iff U(f) = \phi(\delta\pi + \ell) \iff$

$$\phi^{-1}\left(\phi\left(\int_{P\otimes P} u(f)\,d(p\otimes q)\right)d\mu\right) = \delta\pi + \ell. \tag{4.2}$$

Restricted to normalized bets, where $\delta = 1$ and $\ell = 0$, the independence hypothesis requires: for all $E_a, E_b \subset S$,

$$\begin{aligned}&\phi^{-1}\left(\int_{P\otimes P} \phi\left(p\left(E_a\right)q\left(E_b\right)\right)d\mu\left(p\otimes q\right)\right)\\ =\;& \phi^{-1}\left(\int_P \phi\left(p\left(E_a\right)\right)d\mu_a\right)\cdot\phi^{-1}\left(\int_P \phi\left(q\left(E_b\right)\right)d\mu_b\right).\end{aligned}$$

We show that this implies (up to cardinal equivalence)

$$\phi(t) = t^\alpha/\alpha \text{ for some } \alpha \geq 0. \tag{4.3}$$

Define

$$D = \{(p_1(E),...,p_n(E)) : E \subset S\} \subset [0,1]^n.$$

By the Lyapunov Convexity Theorem, D is closed, convex, and contains the main diagonal $\{t\overrightarrow{\mathbf{1}} \equiv (t,...,t) : 0 \leq t \leq 1\}$. In particular, for each t, there exists E_1 such that $p_i(E_1) = t$ for all i. Thus, for all $0 < t < 1$ and $E \subset S$,

$$\phi^{-1}\left(\int_P \phi\left(tp\left(E\right)\right)d\mu_b\right) = t\cdot\phi^{-1}\left(\int_P \phi\left(p\left(E\right)\right)d\mu_b\right),$$

which can be rewritten in the form,

$$\phi^{-1}\left(\Sigma_i m_i \phi\left(tx_i\right)\right) = t\cdot\phi^{-1}\left(\Sigma_i m_i\phi\left(x_i\right)\right) \text{ for all } x = (x_1,...,x_n)\in D. \tag{4.4}$$

We show next that D satisfies:

$$t\overrightarrow{\mathbf{1}} \in int(D) \text{ for every } 0 < t < 1. \tag{4.5}$$

Suppose not. Then $t\overrightarrow{\mathbf{1}} \in bd(D)$ and there exists a hyperplane supporting D at $t\overrightarrow{\mathbf{1}}$, that is, $\exists\lambda \in \mathbb{R}^n\backslash\{0\}$, such that

$$\lambda\cdot x \geq \lambda\cdot t\overrightarrow{\mathbf{1}} = t\Sigma_i\lambda_i \text{ for all } x\in D.$$

But $x \in D$ implies that $\left(\overrightarrow{\mathbf{1}} - x\right) \in D$, and hence,

$$(1-t)\Sigma_i\lambda_i \geq \lambda\cdot x \geq t\Sigma_i\lambda_i \text{ for all } x \in D. \tag{4.6}$$

In particular, for $x = t'\overrightarrow{\mathbf{1}}$, we obtain

$$(1-t)\Sigma_i\lambda_i \geq t'\Sigma_i\lambda_i \geq t\Sigma_i\lambda_i, \text{ for all } t' \in [0,1].$$

Conclude that $\Sigma_i \lambda_i = 0$, and hence, from (4.6), that $\lambda \cdot x = 0$ for all x in D. This contradicts the assumption of linear independence, and proves (4.5).

It is well-known that (4.4), interpreted as constant relative risk aversion in the vNM model, implies (4.3) if the former is satisfied for all x in $[0,1]^n$.[8] Take any point $x^* = t\,\overrightarrow{\mathbf{1}}$, with $0 < t < 1$, on the main diagonal. Then, since x^* lies in the interior of D, there exists a rectangle containing x^* such that (4.4) is satisfied within the rectangle. This situation is isomorphic to the case where (4.4) is satisfied globally, and hence yields the power representation locally within the rectangle. By using overlapping rectangles along the main diagonal, and the continuity of ϕ on the unit cube, one can show that, for all t in $[0,1]$,

$$\phi(t) = Nt^{\alpha} + M \quad \text{for some } \alpha > 0,\ N > 0 \text{ and } M. \tag{4.7}$$

Now admit the losing stake $\ell > 0$. By (4.2), the independence hypothesis takes the form

$$\begin{aligned} &\frac{1}{\delta}\phi^{-1}\left(\int_{P\otimes P} \phi\left(\ell + \delta p_1(E_1)\, p_2(E_2)\right) d\mu\right) - \ell \\ = \;& \left[\frac{1}{\delta}\phi^{-1}\left(\int_{P\otimes P} \phi\left(\ell + \delta p_1(E_1)\right) d\mu\right) - \ell\right] \cdot \left[\frac{1}{\delta}\phi^{-1}\left(\int_{P\otimes P} \phi\left(\ell + \delta p_2(E_2)\right) d\mu\right) - \ell\right]. \end{aligned}$$

Define $\phi_\ell(t) \equiv \phi(\ell + \delta t)$, for $0 \le t \le 1$. Then this case is isomorphic to that treated above, and we deduce that

$$\phi(\ell + \delta t) = N_\ell t^{\alpha_\ell} + M_\ell \quad \text{for some } \alpha_\ell > 0.$$

But the latter is inconsistent with (4.7) unless ϕ is linear. ■

[8] Recall that ϕ is continuous on the closed interval $[0,1]$, which excludes power functions with exponent $\alpha \le 0$.

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