ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 78, No. 6 (November, 2010), 2085–2099

A PARADOX FOR THE "SMOOTH AMBIGUITY" MODEL OF PREFERENCE

LARRY G. EPSTEIN
Boston University, Boston, MA 02215, U.S.A.

Econometrica, Vol. 78, No. 6 (November, 2010), 2085–2099

A PARADOX FOR THE "SMOOTH AMBIGUITY" MODEL OF PREFERENCE

BY LARRY G. EPSTEIN[1]

Two Ellsberg-style thought experiments are described that reflect on the smooth ambiguity decision model developed by Klibanoff, Marinacci, and Mukerji (2005). The first experiment poses difficulties for the model's axiomatic foundations and, as a result, also for its interpretation, particularly for the claim that the model achieves a separation between ambiguity and the attitude toward ambiguity. Given the problematic nature of its foundations, the behavioral content of the model and how it differs from multiple priors, for example, are not clear. The second thought experiment casts some light on these questions.

KEYWORDS: Ambiguity, calibrating ambiguity aversion, multiple priors, smooth ambiguity model of preference, separation of ambiguity from ambiguity aversion.

1. INTRODUCTION

TWO ELLSBERG-STYLE THOUGHT EXPERIMENTS, or examples, are described that reflect on the smooth ambiguity decision model developed by Klibanoff, Marinacci, and Mukerji (KMM) (2005). It is argued that the first experiment poses difficulties for KMM's axiomatic foundations for their model and, as a result, also for its interpretation, particularly for the claim that the model achieves a "separation" between ambiguity and the attitude toward ambiguity. It is shown that in an important sense separation is not afforded by the model.

KMM presented their model as a general model, for example, as an alternative to multiple priors (Gilboa and Schmeidler (1989)), and describe it (p. 1875) as "offering flexibility in modeling ambiguity" and as permitting "a wide variety of patterns of ambiguity." However, because of its problematic foundations, the behavioral content of the model and how it differs from multiple priors, for example, are not clear. The second thought experiment casts light on these questions by demonstrating important differences from multiple priors as to when randomization between acts is valuable.

We begin with an outline of the model that we refer to here as the KMM model. Let Ω be a set of states, let C be the set of consequences or prizes, taken here, for simplicity, to be a compact interval in the real line, and denote by $\Delta(C)$ and $\Delta(\Omega)$ the sets of probability measures on C and Ω, respectively. (Technical details are standard and are suppressed.) An act is a mapping $f : \Omega \to \Delta(C)$, that is, by an act we mean an Anscombe–Aumann act over the

[1]This research was supported by a grant from the National Science Foundation (Award SES-0917740). I am grateful to Andrew Ellis, Yoram Halevy, Peter Klibanoff, Asen Kochov, Bart Lipman, Mark Machina, Wolfgang Pesendorfer, Martin Schneider, Uzi Segal, Kyoungwon Seo, Peter Wakker, and especially Bob Nau for helpful comments and discussions. This paper was previously titled "Three Paradoxes for the 'Smooth Ambiguity' Model of Preference."

 DOI: 10.3982/ECTA8689

state space Ω.[2] The set of all acts is $\mathcal{F}$. KMM also employed *second-order acts*, which are maps $F:\Delta(\Omega)\to C$; if F is binary (has only two possible outcomes), refer to it as a *second-order bet*. The set of all second-order acts is $\mathcal{F}^2$.

KMM posited a preference order $\succeq$ on $\mathcal{F}$ and another preference $\succeq^2$ on $\mathcal{F}^2$. The corresponding utility functions, U and U^2, have the form

$$U(f)=\int_{\Delta(\Omega)}\phi\left(\int_{\Omega}u(f(\omega))\,dp(\omega)\right)d\mu(p),\quad f\in\mathcal{F}, \tag{1.1}$$

and

$$U^2(F)=\int_{\Delta(\Omega)}\phi\big(u(F(p))\big)\,d\mu(p),\quad F\in\mathcal{F}^2. \tag{1.2}$$

Here μ is a (countably additive) probability measure on $\Delta(\Omega)$, $u:\Delta(C)\to R$ is mixture linear, and ϕ is continuous and strictly increasing on $u(C)\subset\mathbb{R}$, where C is identified with a subset of $\Delta(C)$ in the familiar way and we denote by u also its restriction to C. Finally, it is assumed that u is continuous and strictly increasing on C. Identify a KMM agent with a triple (u,ϕ,μ) satisfying the above conditions.

These functional forms suggest appealing interpretations. The utility of an Anscombe–Aumann act f in $\mathcal{F}$ would simply be its expected utility if the probability law p on Ω were known. However, it is uncertain, in general, with prior beliefs represented by μ, and this uncertainty about the true law matters if ϕ is nonlinear; in particular, if ϕ is concave, then

$$\begin{aligned}U(f)&\le\phi\left(\int_{\Delta(\Omega)}\int_{\Omega}u(f(\omega))\,dp(\omega)\,d\mu(p)\right)\\&=\phi\left(\int_{\Delta(\Omega)}u\left(\int_{\Omega}f(\omega)\,dp(\omega)\right)d\mu(p)\right)\\&=\phi\left(u\left(\int_{\Delta(\Omega)}\int_{\Omega}f(\omega)\,dp(\omega)\,d\mu(p)\right)\right)\\&=U(L_f(\mu)),\end{aligned}$$

where $L_f(\mu)$ is a lottery over outcomes, viewed also as a constant act

$$L_f(\mu)=\int_{\Delta(\Omega)}\int_{\Omega}f(\omega)\,dp(\omega)\,d\mu(p)\in\Delta(C).$$

[2]KMM used Savage acts over $\Omega\times[0,1]$ rather than Anscombe–Aumann acts. However, this difference is not important for our purposes. Below by the “KMM model” we mean the Anscombe-Aumann version outlined here, and the corresponding translation of their axioms and arguments.

It is the lottery derived from f if one uses μ to weight probability measures over states and then reduces the resulting three-stage compound lottery in the usual way. In that sense, $L_f(\mu)$ and f embody similar uncertainty, but only for f do eventual payoffs depend on states in Ω where there is uncertainty about the true law. Thus the inequality

$$U(f) \leq U(L_f(\mu)) \quad \text{for all } f \in \mathcal{F}$$

is essentially KMM's behavioral definition of *ambiguity aversion*. As noted, the latter is modeled by a concave ϕ, while ambiguity (as opposed to the attitude toward it) seems naturally to be captured by μ—hence, it is claimed, *a separation is provided between ambiguity and aversion to ambiguity*. This separation is highlighted by KMM as a major advantage of their model over all others in the literature (see also their discussion in the paper KMM (2009a, pp. 931–932) dealing with a dynamic model) and often has been cited by researchers as motivation for their adoption of the KMM model. (See Hansen (2007), Chen, Ju, and Miao (2009), Ju and Miao (2009), Collard, Mukerji, Sheppard, and Talon (2008), for example.)

Seo (2009) provided alternative foundations for the utility function (1.1) on $\mathcal{F}$ (see Section 2.5 below), and Nau (2006) and Ergin and Gul (2009) proposed related models. None made comparably strong claims for their models. Finally, other critical perspectives on the smooth ambiguity model may be found in Baillon, Driesen, and Wakker (2009) and Halevy and Ozdenoren (2008).

2. THOUGHT EXPERIMENT 1

2.1. *Second-Order Bets*

In Ellsberg's classic three-color experiment, you are told the following. An urn contains three balls, of which one is red (R) and the others are either blue (B) or green (G).[3] Then you are offered some bets on the color of the ball to be drawn at random from the urn. Specifically, you are asked to choose

[3]Ellsberg postulated 30 red balls and 60 balls that are either blue or green, but the message is clearly the same.

between f_1 and f_2, and also between f_3 and f_4, where these acts are defined by

	Bets on the Color		
	R	B	G
f_1	100	0	0
f_2	0	100	0
f_3	100	0	100
f_4	0	100	100

The choices pointed to by Ellsberg (and by many subsequent experimental studies) are

$$f_1 \succ f_2 \quad \text{and} \quad f_3 \prec f_4. \tag{2.1}$$

The well known intuition for these choices is uncertainty about the true composition of the urn combined with aversion to that uncertainty. Refer to the pair of choices (2.1) as the "Ellsbergian choices."

Consider a simple extension of Ellsberg's experiment that adds bets on the true composition of the urn. First you are told more about how the urn (subsequently referred to as the *normal urn*) is constructed. There exists another urn, that we call a *second-order urn*, containing three balls. One has the label r and the others are labeled either b or g. A ball will be drawn from this urn and the ball's label will determine the color composition of the normal urn. If the label $i \in \{r, b, g\}$ is drawn from the second-order urn, then the normal urn will have composition p_i, where

$$p_r = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad p_b = \left(\frac{1}{3}, \frac{2}{3}, 0\right), \quad \text{and} \quad p_g = \left(\frac{1}{3}, 0, \frac{2}{3}\right), \tag{2.2}$$

are three probability measures on $\{R, B, G\}$. Thus it is certain that there will be one red ball, but there could be either zero or two blue (and hence also green). You are offered some bets, and after making your choices, a ball will be drawn from the second-order urn, and from the normal urn constructed as described according to the outcome of the first draw. Finally, the two balls drawn and the bets chosen determine payoffs.

In one pair of choice problems, you choose between f_1 and f_2, and between f_3 and f_4, the bets on the color drawn from the normal urn as in Ellsberg's experiment. In addition, you choose between bets on the true composition of the normal urn or, equivalently, on the label of the ball drawn from the second-order urn. Specifically, you choose between F_1 and F_2, and between F_3

and F_4, where they are given by

	Bets on the Composition		
	r	b	g
F_1	100	0	0
F_2	0	100	0
F_3	100	0	100
F_4	0	100	100

The Ellsbergian choices here are

$$F_1 \succ F_2 \quad \text{and} \quad F_3 \prec F_4. \tag{2.3}$$

Each urn defines a setting that is qualitatively similar to that in Ellsberg's three-color experiment. (The second-order urn is identical to Ellsberg's urn, apart from the rescaling of the total number of balls, while the information given for the normal urn is different but qualitatively similar in that the proportion of one color is unambiguous and only partial information is given about the proportions of the other two.) Thus the two urns are also qualitatively similar to one another and ambiguity aversion suggests Ellsbergian choices *both* when betting on the color *and* when betting on the composition.

To apply the smooth ambiguity model, we take $\Omega = \{R, B, G\}$. Then bets on the color drawn from the normal urn are acts in $\mathcal{F}$, and bets on the true composition of the normal urn are second-order acts, elements in $\mathcal{F}^2$. By KMM's (2005) Assumption 2, preference on second-order acts has the subjective expected utility representation (1.2). Thus, the model cannot produce Ellsbergian behavior when betting on the composition. Although there is nothing mysterious in this contradiction, we elaborate shortly on why we feel that nevertheless it has significant implications for the KMM model and its interpretation.

To elaborate, note that although the two urns are qualitatively similar, the KMM model treats bets on the urns differently, imposing ambiguity neutrality on one only, because that urn is used to determine the composition of the other. But why should it matter for an individual deciding how to bet whether the urn is a second-order urn or a normal urn? Moreover, if one were to argue for a difference in behavior, then would it not make more sense to argue that ambiguity averse behavior is *more* pronounced in the case of the second-order urn? After all, for it there is *no information at all* given about the number of b versus g balls, while the details given about the construction of the normal urn give some information about the number of B versus G balls; in fact, it implies, via the usual probability calculus, that there is an objective probability of at least $\frac{1}{9}$ of drawing B and similarly for G. That information leaves much uncertainty, but surely it implies (weakly) less ambiguity than when nothing at all is known as in the second-order urn. Thus even if one grants that an asymmetry in treatment of the urns is warranted, the asymmetry in the KMM model seems to be in the wrong direction.

Finally, note that our expanded Ellsberg example has nothing to say about any of the other models of ambiguity averse preferences in the literature. The smooth ambiguity model is, to our knowledge, unique in making assumptions about the ranking of second-order acts.

2.2. *Why Is It Important?*

KMM's (2005, p. 1851) focus is the functional form (1.1) for U on the domain $\mathcal{F}$. They expand the domain to include second-order acts only to provide foundations for preference on $\mathcal{F}$. The focus on $\mathcal{F}$ is understandable since economically relevant objects of choice correspond to acts in $\mathcal{F}$, while choices between bets on the true probability law are not readily observed in the field. For example, the purchase of a financial asset is a bet on a favorable realization of the stochastic process generating returns and not directly on which probability law describes that process.[4] Thus, it might seem, the domain $\mathcal{F}^2$ is of secondary importance and counterfactual or counterintuitive predictions there are not critical. The subjective expected utility (SEU) assumption on $\mathcal{F}^2$, one might think, is merely a simplifying assumption that facilitates focusing on the important behavior.

However, the model's predictions on $\mathcal{F}^2$ are not a side issue. The way in which uncertainty about the true probability law is treated by the individual is not a side issue when trying to explain ambiguity averse behavior in the choice between acts in $\mathcal{F}$. Dekel and Lipman (2010, Section 2) argued similarly when considering the more general question of when the refutation of a model's "simplifying assumptions" is important. In their view, this is the case when those simplifications are crucial to the model's explanation of the central observations (here, ambiguity averse behavior in $\mathcal{F}$).

Another way in which the domain $\mathcal{F}^2$ is a critical ingredient concerns uniqueness and interpretation. It is well known that μ and ϕ appearing in (1.1) are not pinned down uniquely by preference on $\mathcal{F}$ alone. (For example, if ϕ is linear, then any two measures μ' and μ with the same mean represent the same preference on $\mathcal{F}$.) Moreover, interpretation of μ and ϕ as capturing (and separating) ambiguity and ambiguity attitude obviously presupposes that these components are unique. KMM achieved uniqueness by expanding their domain to include second-order acts. Thus, if one is to retain the appealing interpretations that they offer, one cannot simply ignore that component of their model. (See below for further discussion of the model's interpretation.)

[4]One might argue that this is reason alone to be dissatisfied with KMM's axioms. That is not our criticism, however. It suffices for our purposes that the ranking of second-order acts is, in principle, observable in the laboratory.

Finally, note that interpretation matters. This is true for many reasons, but the specific reason we wish to emphasize is that it matters for empirical applications of the model. In a quantitative empirical exercise, one needs to judge not only whether the model matches data (moments of asset returns, for example), but also whether parameter values make sense, and this requires that they have an interpretation.[5]

2.3. *Are We Using the Wrong State Space?*

The preceding critique hinges on our adoption of $\{R, B, G\}$ as the state space Ω. In defense of the smooth ambiguity model, one might argue that the "correct" state space is $\Omega^1 = \{R, B, G\} \times \Delta(\{R, B, G\})$, (or $\{R, B, G\} \times \{p_r, p_b, p_g\}$), since the payoffs to the bets being considered are determined by the eventual realization of a pair (color, composition). Then all bets correspond to normal acts and the expected utility assumption for second-order acts does not matter. However, this response does not seem satisfactory, since, taken to its logical conclusion, it renders KMM's Assumption 2 (the expected utility form in (1.2)) unfalsifiable.

To make the argument most clearly, suppose that, in fact, there is no physical second-order urn: it exists, but only in the mind of the decision-maker. All bets concern the normal urn, both the color of the ball drawn and the urn's composition. The second-order urn represents the decision-maker's theory of the construction of the normal urn. Naturally, it is unobservable to the modeler, but intuitively it leads to the same choices described above, including the Ellsbergian choices (2.3) when betting on the composition. How would we model this situation using the KMM model? One could adopt the large state space and argue accordingly that bets on the composition are normal acts and that observed choices are consistent with the model. However, if the SEU assumption is falsifiable, then there exist settings and behavior that would lead to its rejection, and there is no obvious reason why the behavior described here should lead to a different conclusion. We are led, therefore, to reject the expanded state space as an acceptable way to apply the model if its axioms are viewed as in principle falsifiable. Finally, there is no reason that we can see to proceed differently if the second-order urn is concrete and observable. This concludes the argument. An alternative to the last step is to dispense throughout with the concrete second-order urn and to adopt instead the subjective story sketched here: suitably translated, the discussion to follow is largely unaffected.

[5]The following quote from Lucas (2003), addressing the equity premium puzzle, expresses clearly the typical view that fitting moments is not enough: "No one has found risk aversion parameters of 50 or 100 in the diversification of individual portfolios, in the level of insurance deductibles, in the wage premiums associated with occupations with high earnings risk, or in the revenues raised by state-operated lotteries. It would be good to have the equity premium resolved, but I think we need to look beyond high levels of risk aversion to do it." This quotation was used by Barillas, Hansen, and Sargent (2009) to motivate their attempt to *reinterpret* the risk aversion parameter as capturing in part an aversion to ambiguity or model uncertainty.

2.4. *Separation*

In Section 2.2, we pointed to "nonuniqueness" as one source of difficulty for interpreting the components μ and ϕ of the model. Here we comment further on interpretation. We describe a variation of our thought experiment that illustrates a sense in which KMM's foundations do *not* support identifying μ and ϕ separately with ambiguity and attitude toward ambiguity.

You are faced in turn with two scenarios, I and II. Scenario I is similar to that in our thought experiment. In particular, it features a second-order urn and a normal urn, related as described in (2.2). The only difference here is that the second-order urn contains 90 balls, with 30 labeled r and the other 60 labeled b or g. Scenario II is similar except that you are told more about the second-order urn, namely that $b, g \geq 20$.

Consider bets on both urns in each scenario. The following rankings seem intuitive: Bets on b and g are indifferent to one another for each second-order urn, and bets on r have the same certainty equivalent across scenarios. For each normal urn, the bet on R is strictly preferable to the bet on B, and the certainty equivalent for a bet on B is strictly larger in scenario II than in I, because the latter is intuitively more ambiguous.

How could we model these choices using the smooth ambiguity model? Assume that the KMM axioms are satisfied for each scenario, so that preferences are represented by two triples (u_i, ϕ_i, μ_i), $i = \mathrm{I}, \mathrm{II}$. The basic model (1.1)–(1.2) does not impose any connection across scenarios. However, since the scenarios differ in ambiguity only and it is the same decision-maker involved in both, one is led naturally to consider the restrictions

$$u_{\mathrm{I}} = u_{\mathrm{II}} \quad \text{and} \quad \phi_{\mathrm{I}} = \phi_{\mathrm{II}}. \tag{2.4}$$

These equalities are motivated by the hypothesis that risk and ambiguity attitudes describe the individual, and, therefore, travel with him across settings. In addition, the postulated behavior implies that μ_{I} and μ_{II} are both uniform measures on $\{r, b, g\}$ and hence coincide. Thus the indicated behavior *cannot* be rationalized. On the other hand, it *can* be rationalized if we assume that the priors μ_i are fixed (and uniform) across scenarios, but allow ϕ_{I} and ϕ_{II} to differ. The preceding defies the common interpretation of the smooth ambiguity model whereby μ captures ambiguity and ϕ represents ambiguity aversion.

The meaning of "separation" is particularly important for applied work. If ϕ describes the individual's attitude alone, and thus moves with her from one setting to another, then it serves to connect the individual's behavior across different settings. Thus, in principle, one could *calibrate* ambiguity aversion in the application under study by examining choices in other situations. Such quantitative discipline is crucial for credible empirical applications; the equity premium puzzle is a classic illustration (recall the quotation from Lucas given in Section 2.2). Thus, in the context of finance applications based on the smooth ambiguity model, Collard et al. (2008), Chen, Ju, and Miao (2009) and

Ju and Miao (2009) assumed that ϕ can be calibrated. Specifically, they employed the functional form $\phi(t) = t^{1-\alpha}/(1-\alpha)$, where $\alpha \geq 0$ is viewed as an ambiguity aversion parameter, and they used the choices implied in hypothetical or experimental Ellsberg-style choice problems to determine what values of α are reasonable to adopt for their asset market applications. KMM (2009a, p. 957) explicitly support such calibration when they write, in the context of an asset pricing example, that "we may assess a plausible range for α [the ambiguity aversion parameter] by … looking at the experimental data on ambiguity premiums in Ellsberg-like experiments." We see no justification for such an exercise.

Our thought experiment in this section dealt with a fixed individual who moves across settings. Alternatively, one might wish to compare the behavior of two individuals who face identical environments, but who differ in ambiguity attitude. KMM (2005, Theorem 2) argued that such a comparative statics exercise can be conducted within their model by keeping (u, μ) fixed across individuals, while allowing ϕ to vary. We do not dispute this feature of their model. However, we emphasize that separation in this sense does *not* make possible the calibration of ambiguity aversion, which inherently concerns the comparative statics exercise with a *single* individual and two settings.[6]

2.5. *Alternative Foundations—Nonreduction*

Since we have been criticizing KMM's foundations for (1.1), rather than the latter itself, one may wonder about alternative foundations. Seo (2009) provided an alternative axiomatic foundation for the same model of preference on normal acts. In his model, an individual can be ambiguity averse only if she fails to reduce objective (and timeless) two-stage lotteries to their one-stage equivalents (much as in Segal's (1987) seminal paper). This connection has some experimental support (Halevy (2007)). Nevertheless, nonreduction of (timeless) compound lotteries is arguably a mistake, while ambiguity aversion is normatively at least plausible. Thus the noted connection severely limits the scope of Seo's model of ambiguity aversion.

Though KMM did not include two-stage lotteries in their domain and thus did not explicitly take a stand on whether these are properly reduced, there is a sense in which nonreduction is implicit also in their model, as we now describe. The argument can be made very generally, but for concreteness, we consider again two scenarios with Ellsberg-style urns as described above. Scenario I is

[6]KMM (2005, pp. 1864–1869) contribute to confusion about the meaning of "separation" in their model. Their discussion sometimes correctly focused on the second comparative statics exercise involving two individuals and one setting. But elsewhere (p. 1852) they send the conflicting message that their model affords the separation needed to conduct a comparative statics exercise in which one "hold[s] ambiguity attitudes fixed and ask[s] how the equilibrium is affected if the perceived ambiguity is varied." A similar claim is repeated on page 1877 and also in their second paper (KMM (2009a, p. 931)).

unchanged. Let $(u_{\mathrm{I}}, \phi_{\mathrm{I}}, \mu_{\mathrm{I}})$ describe the individual in that setting; symmetry calls for $\mu_{\mathrm{I}}(b) = \mu_{\mathrm{I}}(g)$. In scenario II, you are told that the composition of the second-order urn is given by μ_{I}, that is, the subjective prior is announced as being true.[7]

We would expect the announcement not to change risk preferences or preferences over acts defined *within* the second-order urn, nor to cause the individual to change his beliefs about that urn. (Think of the corresponding exercise for a subjective expected utility agent in an abstract state space setting.) Thus $(u_{\mathrm{II}}, \phi_{\mathrm{II}}, \mu_{\mathrm{II}}) = (u_{\mathrm{I}}, \phi_{\mathrm{I}}, \mu_{\mathrm{I}})$. But then preferences on all acts, over both urns, must be unchanged across scenarios. In I, there is ambiguity aversion in betting on the normal urn. In particular, the bet on R is strictly preferable to a bet on B or (normalizing so that $u_{\mathrm{I}}(100) = 1$ and $u_{\mathrm{I}}(0) = 0$)

$$\phi_{\mathrm{I}}\left(\frac{1}{3}\right) > \mu_{\mathrm{I}}(r)\phi_{\mathrm{I}}\left(\frac{1}{3}\right) + \frac{1-\mu_{\mathrm{I}}(r)}{2}\phi_{\mathrm{I}}\left(\frac{2}{3}\right) + \frac{1-\mu_{\mathrm{I}}(r)}{2}\phi_{\mathrm{I}}(0).$$

Therefore, the corresponding inequality is satisfied also in scenario II, though there is no ambiguity there. In II, the individual faces an objective two-stage lottery and the displayed inequality reflects a failure to reduce two-stage lotteries. Thus, as in Seo's model, KMM's foundations imply that ambiguity aversion is tied to mistakes in processing objective probabilities.

3. THOUGHT EXPERIMENT 2

The example presented here does not involve second-order acts. It concerns only the properties of the KMM model on $\mathcal{F}$, the declared domain of interest.

Before describing the example, we present the simple analytical observation that underlies it. As mentioned, KMM interpreted concavity of ϕ as modeling ambiguity aversion. If ϕ is strictly concave, as it is in all applications of the smooth ambiguity model that we have seen, then the preference order on $\mathcal{F}$ represented by (1.1) satisfies the following condition[8]: For all Anscombe–Aumann acts f_1 and f_2,

$$\left[f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2\right] \Longrightarrow \frac{1}{2}f_1 + \frac{1}{2}h \sim \frac{1}{2}f_2 + \frac{1}{2}h \quad \text{for all } h \in \mathcal{F}. \tag{3.1}$$

Thus indifference to randomization between the pair of indifferent acts f_1 and f_2 implies indifference between mixtures with *any* third act h. Of course,

[7]The announcer can, in principle, infer the prior from sufficiently rich data on the individual's choices between second-order acts.

[8]The (elementary) proof will be apparent after reading the proof of the next proposition.

the implication would be required by the Independence axiom, but ambiguity aversion calls for relaxing Independence. Note that while strict concavity of ϕ is used to derive the sharp result in (3.1), only weak concavity is assumed henceforth.

To see the force of (3.1), consider a concrete case. You are given two urns, numbered 1 and 2, each containing 50 balls that are either red or blue. Thus, $\Omega = \{R_1, B_1\} \times \{R_2, B_2\}$ and

$$R_1 + B_1 = 50 = R_2 + B_2.$$

You are told also that the two urns are generated independently, for example, they are set up by administrators from opposite sides of the planet who have never been in contact with one another. One ball will be drawn from each urn.

Consider the following bets, where $c^* > c$ are outcomes in C, and $(c^*, \frac{1}{2}; c, \frac{1}{2})$ denotes the equal probability lottery over these outcomes:

	Bets for Experiment 2			
	R_1R_2	R_1B_2	B_1R_2	B_1B_2
f_1	c^*	c^*	c	c
f_2	c^*	c	c^*	c
$\frac{1}{2}f_1 + \frac{1}{2}f_2$	c^*	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	c
g_1	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	$(c^*, \frac{1}{2}; c, \frac{1}{2})$
g_2	$(c^*, \frac{1}{2}; c, \frac{1}{2})$	c	c^*	$(c^*, \frac{1}{2}; c, \frac{1}{2})$

Symmetry suggests indifference between f_1 and f_2. If it is believed that the compositions of the two urns are unrelated, then f_1 and f_2 do not hedge one another. If, as in the multiple-priors model, hedging ambiguity is the only motivation for randomizing, then we are led to the rankings

$$f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2. \tag{3.2}$$

Ambiguity aversion suggests

$$g_1 \succ g_2. \tag{3.3}$$

(Note that

$$g_1 = \frac{1}{2}f_1 + \frac{1}{2}h \quad \text{and} \quad g_2 = \frac{1}{2}f_2 + \frac{1}{2}h, \tag{3.4}$$

where $h = (c, c, c^*, c^*)$.)

The rankings (3.2)–(3.3), for all $c^* > c$, are easily accommodated by the multiple-priors model. However, as we show next, they are inconsistent with KMM if the natural state space $\Omega = \{R_1, B_1, R_2, B_2\}$ is adopted and if ϕ is taken to be concave. (See the Appendix for a proof.)

PROPOSITION 3.1: *If preference over the set of Anscombe–Aumann acts $\mathcal{F}$ is represented by the utility function U in* (1.1), *where ϕ is concave*, *then*

$$(*)\qquad f_1 \sim f_2 \sim \frac{1}{2}f_1 + \frac{1}{2}f_2 \quad \text{for all } c^* > c$$
$$\Longrightarrow \quad \frac{1}{2}f_1 + \frac{1}{2}h \sim \frac{1}{2}f_2 + \frac{1}{2}h \quad \text{for all } c^* > c.$$

In particular, *in light of* (3.4), *the rankings* (3.2)–(3.3) *are impossible*.

To our knowledge, there is no relevant experimental evidence on the hypothesis in (*) that randomizing between bets on "independent" urns is of no value. In their reply, KMM (2009b) offered the contrary intuition whereby the mixture $\frac{1}{2}f_1 + \frac{1}{2}f_2$ is strictly preferable to f_1 because it reduces the variation in expected utilities across possible probability laws. The intuition does not rely on ambiguity about the true probability law; in particular, it would presumably apply also when the prior μ is based on a given objective distribution as in the example in Section 2.5. Thus this argument for the value of randomization would seem to reflect nonreduction of compound lotteries rather than ambiguity aversion. Nevertheless, the descriptive validity of (*) is an empirical question.

4. CONCLUDING REMARKS

The smooth ambiguity model is less parsimonious than multiple priors: both require specifying a set of probability laws, the support of μ in the case of the smooth model, but only the latter requires the modeler to specify also a distribution over this set and a function ϕ. Typically, less parsimonious models are motivated by the desire to accommodate behavior that is deemed descriptively or normatively important, and yet is inconsistent with the existing tighter model. KMM did not offer descriptive evidence as motivation. One might see their axioms as providing normative motivation for their model. However, our first thought experiment has shown that these axiomatic foundations are problematic normatively.

KMM offered two other motivating arguments. The major one is conceptual—the added degrees of freedom permit the separation of ambiguity from ambiguity aversion. The discussion surrounding the first thought experiment clarifies the limited sense in which such "separation" is achieved—calibration of ambiguity aversion is *not* justified thereby.

The other motivation offered is tractability—because utility is (under standard assumptions) differentiable, calculus techniques can be applied to characterize solutions to optimization problems, unlike the case for multiple priors. Although our thought experiments do not touch directly on this rationale, we offer two comments. First, a growing literature (surveyed in Epstein and Schneider (2010)) has fruitfully applied the multiple-priors model in finance, thus showing that differentiability is not necessary for tractability. Second, as first pointed out by Dow and Werlang (1992), differentiability, or the lack thereof, has economic significance. The cited survey describes several ways in which the "first-order uncertainty aversion" generated by nondifferentiability helps to account for asset market behavior that is qualitatively puzzling in light of smooth models such as subjective expected utility and the KMM model. There is also experimental evidence (see Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010) and Ahn, Choi, Gale, and Kariv (2009)) that first-order effects are important in portfolio choice.

It remains unclear what the smooth ambiguity model adds to the arsenal of ambiguity averse preference models in terms of explanatory power. Our second thought experiment demonstrates some of the behavioral differences between the smooth and multiple-priors models, but obviously the picture is still incomplete.

APPENDIX

PROOF OF PROPOSITION 3.1: It is without loss of generality (since C was taken to be a compact interval) to assume that u has range equal to $[0, 1]$ and that $\phi : [0, 1] \longrightarrow \mathbb{R}$. Also without loss of generality, suppose there exists $0 < \kappa < 1$ such that, for all $t < \kappa < t'$,

$$\phi\left(\frac{1}{2}t + \frac{1}{2}t'\right) > \frac{1}{2}\phi(t) + \frac{1}{2}\phi(t').$$

Otherwise, ϕ is linear and (*) is obvious. The following cases are essentially exhaustive.

Abbreviate $p(R_1 \times \{R_2, B_2\})$ by $p(R_1)$ and so on.

Case 1. $p(R_1) = p(R_2)$ with μ-probability equal to 1. Then

$$\begin{aligned}
&\int_\Omega u(f_1)\,dp = \int_\Omega u(f_2)\,dp \quad \mu\text{-a.s.} \\
&\implies \quad \text{(since } u \text{ is linear)} \quad \int_\Omega u\left(\frac{1}{2}f_1 + \frac{1}{2}h\right)dp \\
&\qquad = \int_\Omega u\left(\frac{1}{2}f_2 + \frac{1}{2}h\right)dp \quad \mu\text{-a.s.} \\
&\implies \quad \int \phi\left(\int_\Omega u\left(\frac{1}{2}f_1 + \frac{1}{2}h\right)dp\right)d\mu \\
&\qquad = \int \phi\left(\int_\Omega u\left(\frac{1}{2}f_2 + \frac{1}{2}h\right)dp\right)d\mu \\
&\implies \quad U\left(\frac{1}{2}f_1 + \frac{1}{2}h\right) = U\left(\frac{1}{2}f_2 + \frac{1}{2}h\right).
\end{aligned}$$

Case 2. There exists $P \subset \Delta(\Omega)$, with $\mu(P) > 0$, such that

$$p(R_1) > p(R_2) \geq 0 \quad \text{for all } p \in P. \tag{A.1}$$

Take the special case $P = \{p^*\}$. Pick c^* and c so that $1 \geq u(c^*) > u(c) \geq 0$ and

$$p^*(R_2) < \frac{\kappa - u(c)}{u(c^*) - u(c)} < p^*(R_1).$$

Then

$$\int_\Omega u(f_2)\, dp^* < \kappa < \int_\Omega u(f_1)\, dp^*,$$

which, by definition of κ implies that

$$\begin{aligned} &\phi\left(\int_\Omega u\left(\frac{1}{2}f_1 + \frac{1}{2}f_2\right) dp^*\right) \\ &\quad = \phi\left(\frac{1}{2}\int_\Omega u(f_1)\, dp^* + \frac{1}{2}\int_\Omega u(f_2)\, dp^*\right) \\ &\quad > \frac{1}{2}\phi\left(\int_\Omega u(f_1)\, dp^*\right) + \frac{1}{2}\phi\left(\int_\Omega u(f_2)\, dp^*\right). \end{aligned}$$

Since ϕ is concave, it follows that

$$\begin{aligned} &U\left(\frac{1}{2}f_1 + \frac{1}{2}f_2\right) \\ &\quad = \int \phi\left(\int_\Omega u\left(\frac{1}{2}f_1 + \frac{1}{2}f_2\right) dp\right) d\mu(p) \\ &\quad > \int \left[\frac{1}{2}\phi\left(\int_\Omega u(f_1)\, dp\right) + \frac{1}{2}\phi\left(\int_\Omega u(f_2)\, dp\right)\right] d\mu(p) \\ &\quad = \frac{1}{2}U(f_1) + \frac{1}{2}U(f_2) = U(f_1), \end{aligned}$$

contrary to the hypothesis in (*).

Turn to the general case of (A.1), where P need not be a singleton. Then there exists a subset $Q \subset P$, $\mu(Q) > 0$, where, for some $a > 0$,

$$q(R_1) > a > q(R_2) \geq 0 \quad \text{for all } q \in Q.$$

Adapt the above argument. *Q.E.D.*

REFERENCES

AHN, D., S. CHOI, D. GALE, AND S. KARIV (2009): "Estimating Ambiguity Aversion in a Portfolio Choice Experiment." [2097]

BAILLON, A., B. DRIESEN, AND P. P. WAKKER (2009): "Relative Concave Utility for Risk and Ambiguity." [2087]

BARILLAS, F., L. P. HANSEN, AND T. J. SARGENT (2009): "Doubts, or Variability?" *Journal of Economic Theory*, 144, 2388–2418. [2091]

BOSSAERTS, P., P. GHIRARDATO, S. GUARNASCHELLI, AND W. R. ZAME (2010): "Ambiguity in Asset Markets: Theory and Experiment," *Review of Financial Studies*, 23, 1325–1359. [2097]

CHEN, H., N. JU, AND J. MIAO (2009): "Dynamic Asset Allocation With Ambiguous Return Predictability." [2087,2092]

COLLARD, F., S. MUKERJI, K. SHEPPARD, AND J. M. TALON (2008): "Ambiguity and the Historical Equity Premium." [2087,2092]

DEKEL, E., AND B. LIPMAN (2010): "How (Not) to Do Decision Theory," *Annual Review of Economics* (forthcoming). [2090]

DOW, J., AND S. R. WERLANG (1992): "Uncertainty Aversion, Risk Aversion and the Optimal Choice of Portfolio," *Econometrica*, 60, 197–204. [2097]

EPSTEIN, L. G., AND M. SCHNEIDER (2010): "Ambiguity and Asset Markets," *Annual Review of Financial Economics* (forthcoming). [2096]

ERGIN, H., AND F. GUL (2009): "A Theory of Subjective Compound Lotteries," *Journal of Economic Theory*, 144, 899–929. [2087]

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility With Non-Unique Priors," *Journal of Mathematical Economics*, 18, 141–153. [2085]

HALEVY, Y. (2007): "Ellsberg Revisited: An Experimental Study," *Econometrica*, 75, 503–536. [2093]

HALEVY, Y., AND E. OZDENOREN (2008): "Ambiguity and Compound Lotteries: Calibration." [2087]

HANSEN, L. P. (2007): "Beliefs, Doubts and Learning: Valuing Macroeconomic Risk," *American Economic Review*, 97, 1–30. [2087]

JU, N., AND J. MIAO (2009): "Ambiguity, Learning and Asset Returns." [2087,2093]

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A, Smooth Model of Decision Making Under Ambiguity," *Econometrica*, 73, 1849–1892. [2085,2089,2090,2093]

——— (2009a): "Recursive Smooth Ambiguity Preferences," *Journal of Economic Theory*, 144, 930–976. [2087,2093]

——— (2009b): "On the Smooth Ambiguity Model: A Reply." [2096]

LUCAS, R. E., JR. (2003): "Macroeconomic Priorities," *American Economic Review*, 93, 1–14. [2091]

NAU, R. F. (2006): "Uncertainty Aversion and Second-Order Utilities and Probabilities," *Management Science*, 52, 136–145. [2087]

SEGAL, U. (1987): "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach," *International Economic Review*, 28, 175–202. [2093]

SEO, K. (2009): "Ambiguity and Second-Order Belief," *Econometrica*, 77, 1575–1605. [2087,2093]

Dept. of Economics, Boston University, 270 Bay State Road, Boston, MA 02215, U.S.A.; lepstein@bu.edu.

Manuscript received July, 2009; final revision received July, 2010.