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## SUBSTITUTION, RISK AVERSION, AND THE TEMPORAL BEHAVIOR OF CONSUMPTION AND ASSET RETURNS: A THEORETICAL FRAMEWORK<sup>1</sup>

BY LARRY G. EPSTEIN AND STANLEY E. ZIN

This paper develops a class of recursive, but not necessarily expected utility, preferences over intertemporal consumption lotteries. An important feature of these general preferences is that they permit risk attitudes to be disentangled from the degree of intertemporal substitutability. Moreover, in an infinite horizon, representative agent context these preference specifications lead to a model of asset returns in which appropriate versions of both the atemporal CAPM and the intertemporal consumption-CAPM are nested as special cases. In our general model, systematic risk of an asset is determined by covariance with both the return to the market portfolio and consumption growth, while in each of the existing models only one of these factors plays a role. This result is achieved despite the homotheticity of preferences and the separability of consumption and portfolio decisions. Two other auxiliary analytical contributions which are of independent interest are the proofs of (i) the existence of recursive intertemporal utility functions, and (ii) the existence of optima to corresponding optimization problems. In proving (i), it is necessary to define a suitable domain for utility functions. This is achieved by extending the formulation of the space of temporal lotteries in Kreps and Porteus (1978) to an infinite horizon framework.

A final contribution is the integration into a temporal setting of a broad class of atemporal non-expected utility theories. For homogeneous members of the class due to Chew (1985) and Dekel (1986), the corresponding intertemporal asset pricing model is derived.

**KEYWORDS:** Intertemporal substitution, risk aversion, asset returns, recursive utility, non-expected utility theory, temporal lotteries.

### 1. INTRODUCTION

THIS PAPER DEVELOPS a class of recursive, but not necessarily expected utility, preferences over intertemporal consumption lotteries. An important feature of these general preferences is that they permit risk attitudes to be disentangled from the degree of intertemporal substitutability. Moreover, in an infinite horizon, representative agent context these preference specifications lead to a model of asset returns in which appropriate versions of both the temporal CAPM and the intertemporal consumption-CAPM are nested as special cases. In our general model, systematic risk of an asset is determined by covariance with both the return to the market portfolio and consumption growth, while in each of the existing models only one of these factors plays a role. This result is achieved despite the homotheticity of preferences and the separability of consumption and portfolio decisions. Two other auxiliary analytical contributions which are of independent interest are the proofs of (i) the existence of recursive intertemporal utility functions, and (ii) the existence of optima to corresponding optimization problems. In proving (i), it is necessary to define a suitable domain for utility

<sup>1</sup> Three referees provided detailed and helpful comments. We have also benefited from discussions with Chew Soo Hong, Roger Farmer, and Angelo Melino. The first author gratefully acknowledges the financial support of the Social Sciences and Humanities Research Council of Canada.

functions. This is achieved by extending the formulation of the space of temporal lotteries in Kreps and Porteus (1978) to an infinite horizon framework.

Our general class of preferences contains three noteworthy subclasses. In increasing order of generality, they are: (a)—expected utility preferences with an intertemporally additive and homogeneous von Neumann-Morgenstern utility index; (b)—an infinite horizon extension of homogeneous versions of the Kreps and Porteus (1978, 1979a, 1979b) structure; and (c)—a further generalization which embeds the atemporal non-expected utility theory of Chew (1989) and Dekel (1986) into a multiperiod framework. An unattractive feature of the standard specification (a) is that the two distinct aspects of preference, intertemporal substitutability and relative risk aversion, are intertwined; indeed the elasticity of substitution and the risk aversion parameter are reciprocals of one another. In contrast, our specifications, including (b) and (c), are sufficiently flexible to permit those two aspects of preference to be separated.

The advantage of such a separation has been highlighted by the empirical literature on the behavior of asset returns and consumption over time. Expected utility, representative agent, optimizing models have not performed well empirically (Hansen and Singleton (1983), Mehra and Prescott (1985)). One possible explanation for this poor performance is the above noted inflexibility of the expected utility specification. Grossman and Shiller (1981), Hall (1985), and Zin (1987) have made this suggestion and the latter two papers have attempted to remedy the specification problem by adopting Selden's (1978) OCE preferences. But the latter are intertemporally inconsistent and the equations estimated by these authors are applicable only to a naive consumer who continually ignores the fact that plans formulated at any given time will generally not be carried out in the future. In contrast, all utility functions considered in this paper are based on a recursive structure and so are intertemporally consistent. The empirical performance of our recursive utility specifications is explored in Epstein and Zin (1989).

Apart from the empirical literature noted above, a second primary motivation for this paper is provided by the recent literature on non-expected utility theories of preference. (See Machina (1982) for a survey and Dekel (1986) and Yaari (1987) for recent contributions.) These theories and the surrounding applications have been formulated largely in atemporal frameworks. Moreover, empirical support for them has been based exclusively on behavioral-experimental evidence, e.g., they can explain the Allais paradox. (See Machina (1982) for a discussion of this evidence and for an argument that violations of the independence axiom of expected utility theory are both widespread and systematic.) In contrast, this paper integrates a broad class of these non-expected utility theories, including the homogeneous members of the Chew-Dekel class, into a recursive temporal framework and derives their implications for the structure of asset returns and the temporal behavior of consumption. Moreover, our companion paper (1988) shows that the resulting model, in the Chew-Dekel case, can be estimated by available econometric techniques. In particular, we can test statistically whether the Chew-Dekel generalization of expected utility provides a

significant improvement in the explanation of asset returns. We suspect that many economists would view this as a more relevant test of the importance of this generalization of expected utility. It is thus hoped that this pair of papers will convince skeptical readers that some of the recently formulated non-expected utility theories have empirical content and that their usefulness in explaining market data could and should be explored.

An unusual feature of our preference orderings that merits some attention in these opening remarks is that they generally imply nonindifference to the way in which uncertainty about consumption resolves over time, where temporal resolution is intended in the sense of Kreps and Porteus (1978). (These authors are henceforth KP.) It is recognized (KP (1979a), Machina (1984)) that early resolution is generally preferable when considering a preference ordering for random income streams induced from preference for consumption streams, since in such contexts earlier resolution can improve planning. But the case for nonindifference to timing is not clear at the primitive level of consumption. Indeed, indifference is generally assumed, since it is implied by the usual expected utility specifications. On the other hand, we offer three comments in defense of our approach. First, it is perfectly "rational" to care about the way in which consumption uncertainty resolves over time. (See Chew and Epstein (1989) for some elaboration and examples; see also KP (1979a, p. 82).) One could attempt to employ introspection to determine whether such nonindifference is present and whether it is likely to be empirically significant. But clearly a preferable route to resolving the issue is to let the data speak. Thus, our second point is that we can test (see Epstein and Zin (1989)) whether nonindifference is revealed by the data to be statistically significant. Finally, note that if indifference to timing and the intertemporal consistency of preferences are both assumed, then (Chew and Epstein (1989)) an expected utility ordering is implied. One of these axioms must be weakened in light of the empirical evidence cited earlier and in light of the difficulty of separating risk aversion from substitution within the expected utility framework. For elaboration on the latter point, see Section 4 below and also Chew and Epstein (1987). The option of weakening consistency is pursued in the latter paper.

The paper proceeds as follows: The consumption space and recursive utility functions are formulated in Sections 2 and 3 respectively. The central properties of utility functions are considered next. Section 5 treats the consumption-portfolio choice problem for an agent with recursive utility. In Section 6, the corresponding Euler equations are employed to infer equilibrium relations for asset returns in a representative agent model. Some concluding remarks on potential applications and extensions are offered in Section 7. Proofs are relegated to appendices.

## 2. CONSUMPTION PROGRAMS

This section defines the consumption space. It will be important to model carefully the information structure facing our agent, or in other words, the way in

which consumption uncertainty is resolved over time. A formal structure which includes such detail is the space of temporal lotteries defined by KP (1978). Their definition is restricted to a finite horizon framework. Here (and in Appendix 1) we describe an extension to an infinite horizon.<sup>2</sup>

For the benefit of those readers who are anxious to move on to the structure of utility functions and the behavioral analysis, we present first an informal description of the space of temporal lotteries, which description extends to (2.5) below. The remainder of this section and Appendix 1 complete the definition of the consumption space but are not essential to the understanding of the crux of the paper.

The following notation is adopted: For any metric space  $X$  denote by  $\mathcal{B}(X)$  the Borel  $\sigma$ -algebra and by  $M(X)$  the space of Borel probability measures on  $X$  endowed with the weak convergence topology. The probability measure which assigns unit mass to  $\{x\}$  is denoted  $\delta_x$ .

Each temporal lottery  $d$  can be pictured as an infinite probability tree in which each branch corresponds to a deterministic consumption stream  $y \in R_+^\infty$ . Denote by  $D$  the space of such lotteries endowed with some metric. The lottery  $d$  can be identified with a pair  $(c_0, m)$  where  $c_0 \geq 0$  denotes the nonstochastic period 0 level of consumption and  $m$  is a probability measure over the set of  $t = 1$  nodes in the tree. But each such node may be identified with the probability tree emanating from it. Thus  $m$  can be thought of as an element of  $M(D)$ . We conclude that the space of temporal lotteries should satisfy

$$(2.1) \quad D \text{ is homeomorphic to } R_+ \times M(D).$$

The construction of  $D$ , described in detail below, is based on the following intuition: Picture an infinite probability tree  $d$  and for each  $t$  imagine "collapsing" everything beyond  $t$  in the sense that all uncertainty which in  $d$  resolves at  $t$  or later is now completely resolved at  $t$ . This transformation generates a new tree  $d_t$ . As  $t$  increases,  $d_t$  provides a better approximation to the initial tree  $d$  and the approximation error vanishes asymptotically. Thus the infinite sequence of such approximations  $(d_1, \dots, d_t, \dots)$  accurately represents the infinite horizon lottery  $d$  and we identify  $d$  with the infinite sequence. The  $d_t$ 's prescribe a common period 0 consumption level  $c_0$  and they induce the same probability distribution for consumption in period 1 and beyond. They differ only in the way in which the uncertainty about future consumption is resolved over time. In particular, given  $d = (d_1, \dots, d_t, \dots)$ , then  $d_1 = (c_0, m_1)$  where  $m_1 \in M(R_+^\infty)$  and all uncertainty in  $d_1$  is resolved in period 1. Refer to  $m_1$  as representing the *atemporal distribution* of uncertain future consumption.

The space  $D$  is too broad to serve as a domain for the class of utility functions of interest here; e.g., the uncertainty regarding future consumption need not be limited to bounded support for any normal definition of boundedness. Thus we

<sup>2</sup> After completion of this paper we learned of a very similar mathematical construction by Mertens and Zamir (1985) for the space of infinite hierarchy of beliefs in the context of Bayesian games. See also Myerson (1985).

will proceed to define a subspace of  $D$  which will serve as a domain for utility functions.

First, for any  $b \geq 1$  and  $l > 0$  define the set of deterministic consumption sequences

$$(2.2) \quad \begin{aligned} Y(b; l) &\equiv \{y = (c_0, c_1, \dots) \in R_+^\infty : \sup c_t/b^t \leq l\}, \quad b < \infty, \\ Y(\infty; l) &\equiv R_+^\infty. \end{aligned}$$

(Endow these sets, as well as  $R_+^\infty$ , with the metrizable product topology.) Of course,  $b$  bounds the rate of growth of consumption while  $l$  bounds consumption levels. We take the consumption space to be  $D(b) \subset D$  consisting of those temporal lotteries in which the atemporal distribution of future consumption is represented by some  $m_1$  in  $M^b(R^\infty) \equiv \cup_{l>0} M(Y(b; l))$ , i.e.,

$$(2.3) \quad D(b) \equiv \{d \in D : d_1 = (c_0, m_1), m_1 \in M^b(R_+^\infty)\}.$$

Elements of  $D(b)$  may be called *consumption programs* as well as temporal lotteries.

This definition of the consumption space satisfies two desiderata. First, by construction, each  $m_1$  has compact support. Second, the homeomorphism (2.1) survives in the modified form whereby

$$(2.4) \quad D(b) \text{ is homeomorphic to } R_+ \times \hat{M}(D(b)),$$

where  $\hat{M}(D(b))$  is a subspace of  $M(D(b))$  defined precisely below. Thus each  $d \in D(b)$  can be identified with a pair  $(c_0, m)$  where  $c_0 \geq 0$  is period 0 consumption and  $m \in \hat{M}(D(b))$  represents the uncertain future. The significance of (2.4) is that as one moves along a probability tree corresponding to a lottery in  $D(b)$ , the subtree emanating from any intermediate node necessarily lies in the domain  $D(b)$ . Such "stationarity" of the choice space is essential for the investigation of recursive (and hence also stationary) utility functions. The parallel requirement for a domain  $Y \subset R_+^\infty$  in a deterministic analysis would be that

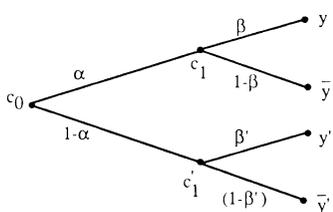
$$(c_0, c_1, c_2, \dots) \in Y \Rightarrow (c_1, c_2, \dots) \in Y.$$

Finally, we will also be interested in subspaces of the form

$$(2.5) \quad D(b; l) \equiv \{d \in D : d_1 = (c_0, m_1), m_1 \in M(Y(b; l))\},$$

i.e., the subspace of temporal lotteries in which the atemporal distribution of consumption has support in  $Y(b; l)$ . Choice sets in the optimization problems below will be contained in some  $D(b; l)$  and the latter lies in  $D(b)$ , the domain of utility functions.

Some readers may wish at this point to skip to Section 3. We continue here with a more thorough and formal analysis. We adopt the following conventions:  $X$  is identified as a subspace of  $M(X)$  in the usual fashion. We write  $X \subseteq X'$  if  $X$  is homeomorphic to a (Borel) subspace of  $X'$ . In that case, we can identify  $M(X)$  with a subspace of  $M(X')$  via the map which takes  $m \in M(X)$  into  $m' \in M(X')$ ,  $m'(B) \equiv m(B \cap X)$  for all Borel subsets  $B$  of  $X'$ .



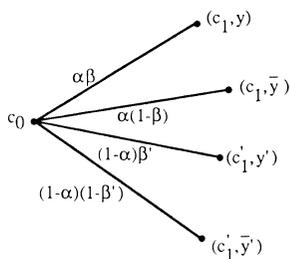
$$d \in D_2 \subset D, d = (c_0, m)$$

$$m = \alpha \delta_{(c_1, d_1)} + (1-\alpha) \delta_{(c_1, d'_1)}$$

$$d_1 = (c_1, \beta \delta_y + (1-\beta) \delta_{\bar{y}}) \in D_1$$

$$d'_1 = (c'_1, \beta' \delta_{y'} + (1-\beta') \delta_{\bar{y}'}) \in D_1$$

$$y, \bar{y}, y' \text{ and } \bar{y}' \in R_+^\infty$$



$$(c_0, m_1) \in D_1, m_1 = f(m),$$

$$f \text{ defined in (2.7)}$$

FIGURE 1

For  $t \geq 1$  define the spaces  $D_t$  inductively as follows:

$$(2.6) \quad D_1 \equiv R_+ \times M(R_+^\infty), \quad D_t \equiv R_+ \times M(D_{t-1}), \quad t \geq 2.$$

For each  $t$ ,  $D_t$  can be interpreted as the set of (temporal) consumption lotteries in which all uncertainty is resolved at or before time  $t$ . KP show how elements of  $D_t$  can be represented by probability trees (see Figure 1). Since  $R_+^\infty \subset D_1$ , one can show by induction that  $D_t \subset D_{t+1}$  for all  $t \geq 1$ . Each  $D_t$  is a separable metric space (Parthasarathy (1967, Theorem 6.2, p. 43)). Denote by  $\mathcal{B}_t$  the Borel  $\sigma$ -algebra for  $D_t$  ( $t \geq 1$ ).

We wish to define formally what it means for  $d_t \in D_t$  and  $d_{t+1} \in D_{t+1}$  to induce the identical probability measure on  $R_+^\infty$  and thus to differ only in the temporal resolution of the common uncertainty. For that purpose define  $f: M(R_+ \times M(R_+^\infty)) \rightarrow M(R_+^\infty)$  by

$$(2.7) \quad f(m)(B) \equiv E_m T_B(\cdot, \cdot), \quad B \in \mathcal{B}(R_+^\infty),$$

where

$$T_B: R_+ \times M(R_+^\infty) \rightarrow R_+,$$

$$T_B(c, \nu) \equiv \nu \{ y \in R_+^\infty : (c, y) \in B \}.$$

For each “two-stage lottery”  $m$ ,  $f(m)$  is the probability measure induced on  $R_+^\infty$  by having all uncertainty resolve at the “first stage.”

Define the functions

$$f_t: M(D_t) \rightarrow M(D_{t-1}), \quad D_0 \equiv R_+^\infty,$$

$$g_t: D_{t+1} \rightarrow D_t, \quad t \geq 1,$$

inductively as follows:

$$(2.8) \quad f_1 = f,$$

$$g_t(c_0, m) = (c_0, f_t(m)), \quad t \geq 1,$$

$$f_t(m)(B) = m(g_{t-1}^{-1}(B)), \quad \forall B \in \mathcal{B}_{t-1}, \quad t \geq 2.$$

Then for any  $d_{t+1} \in D_{t+1}$ ,  $g_t(d_{t+1}) \in D_t$  induces the same uncertainty regarding  $(c_1, c_2, \dots)$  as does  $d_{t+1}$  but the uncertainty is resolved earlier. (See Figure 1 again.) Therefore, if in  $d_{t+1}$  all uncertainty is resolved by period  $t$ , then the operation of  $g_t$  has no effect. In fact,

$$(2.9) \quad g_t(d_{t+1}) = d_{t+1} \Leftrightarrow d_{t+1} \in D_t.$$

We are now ready to define the space of temporal lotteries  $D$ . The intuition, provided above, that each infinite probability tree can be identified with the infinite sequence of the “collapsed” finite horizon trees, leads to the following formal definition:

$$(2.10) \quad D = \{(d_1, \dots, d_t, \dots) : d_t \in D_t \text{ and } d_t = g_t(d_{t+1}) \forall t \geq 1\}.$$

(This construction is an inverse limit in the sense of Parthasarathy (p. 135).) The topology of  $D$  is that induced by the product topology on the Cartesian product  $D_1 \times \dots \times D_t \times \dots$ .

Each  $D_t$  is embedded in  $D$  by the map which takes  $d_t \rightarrow (d_1, \dots, d_{t-1}, d_t, d_t, \dots)$ , where  $d_i = g_i(d_{i+1})$  for  $i = 1, \dots, t-1$ . Given the product topology on  $D$  it is clear that  $\bigcup_0^\infty D_t$  is dense in  $D$ . This denseness corresponds to the fact that for lotteries in  $D$  all uncertainty is resolved asymptotically.

We can now prove the following Theorem (see Appendix 1):

**THEOREM 2.1:** *The space  $D$  defined by (2.10) is a separable metric space and  $D$  is homeomorphic to  $R_+ \times M(D)$ . Moreover,  $\bigcup_0^\infty D_t$  is dense in  $D$ .*

For the reasons given above, we define the consumption space to be the subspace  $D(b)$  from (2.3). To state the counterpart of Theorem 2.1 for  $D(b)$ , as well as for later uses, it is convenient to introduce some further notation. For each  $t$ , let  $\pi_t: D \rightarrow D_t$  be the projection map. Denote by  $\mathcal{B}_t$  the Borel  $\sigma$ -algebra for  $D_t$  and by  $\pi_t^{-1}(\mathcal{B}_t)$  the family  $\{\pi_t^{-1}(B_t) : B_t \in \mathcal{B}_t\}$  of subsets of  $D$ . The

projection maps induce, for each  $t \geq 1$ , the map

$$(2.11) \quad P_{t+1}: M(D) \rightarrow M(D_t), \quad P_{t+1}m(B_t) \equiv m(\pi_t^{-1}B_t), \quad \forall B_t \in \mathcal{B}_t.$$

Finally, let  $D_t(b) \equiv D_t \cap D(b)$  be the set of consumption programs in which all uncertainty is resolved by period  $t$ . Then  $D_t(b) \subset D_{t+1}(b)$  and  $\bigcup_0^\infty D_t(b) \subset D(b)$  is the set of consumption programs in which all uncertainty is resolved in finite time.

**THEOREM 2.2:** *The space  $D(b)$  defined by (2.3) is a separable metric space which is homeomorphic to  $R_+ \times \hat{M}(D(b))$ , where*

$$\hat{M}(D(b)) = \left\{ m \in M(D(b)) : f(m_2) \in \bigcup_{l>0} M(Y(b;l)), \quad m_2 = P_2 m \right\}.$$

Moreover,  $\bigcup_0^\infty D(b)$  is dense in  $D(b)$ .

The interpretation of the Theorem is evident in light of the preceding discussion. (The difference between  $\hat{M}(D(b))$  and  $M(D(b))$  is clarified in Appendix 1 following the proof of the theorem.)

### 3. RECURSIVE UTILITY

All utility functions are defined on  $D(b)$  and are recursive there. It may be helpful, therefore, to begin by considering briefly the structure of recursive utility over deterministic consumption streams. If  $V$  is such a utility function, then (Koopmans (1960)) in the obvious notation

$$(3.1) \quad V(c_0, c_1, \dots) = W(c_0, V(c_1, c_2, \dots)),$$

for some function  $W$ . This structure has been explored also by Lucas and Stokey (1984) and Boyd (1987) where  $W$  is termed an *aggregator*, as it combines current consumption and future utility to determine current utility.

In contemplating an extension of (3.1) to the stochastic case we note that future utility is random. It seems natural, in that case, to compute a certainty equivalent for random future utility and then to combine the certainty equivalent utility level with  $c_0$  via an aggregator. Thus we are led to consider certainty equivalent (or generalized mean value) functionals  $\mu$ . Each such mean value is a map,

$$\mu: \text{dom } \mu \subseteq M(R_+) \rightarrow R_+,$$

which is consistent with first and second degree stochastic dominance and satisfies

$$(3.2) \quad \mu(\delta_x) = x \quad \forall x \in R_+,$$

i.e., if a gamble yields the outcome  $x$  with certainty, then  $x$  is the certainty equivalent of the gamble.

Given a utility function  $V: D(b) \rightarrow R_+$  and  $(c_0, m) \in D(b)$ , denote by  $V[m]$  the probability measure for future utility implied by  $V$  and  $m \in \hat{M}(D(b)) \subset$

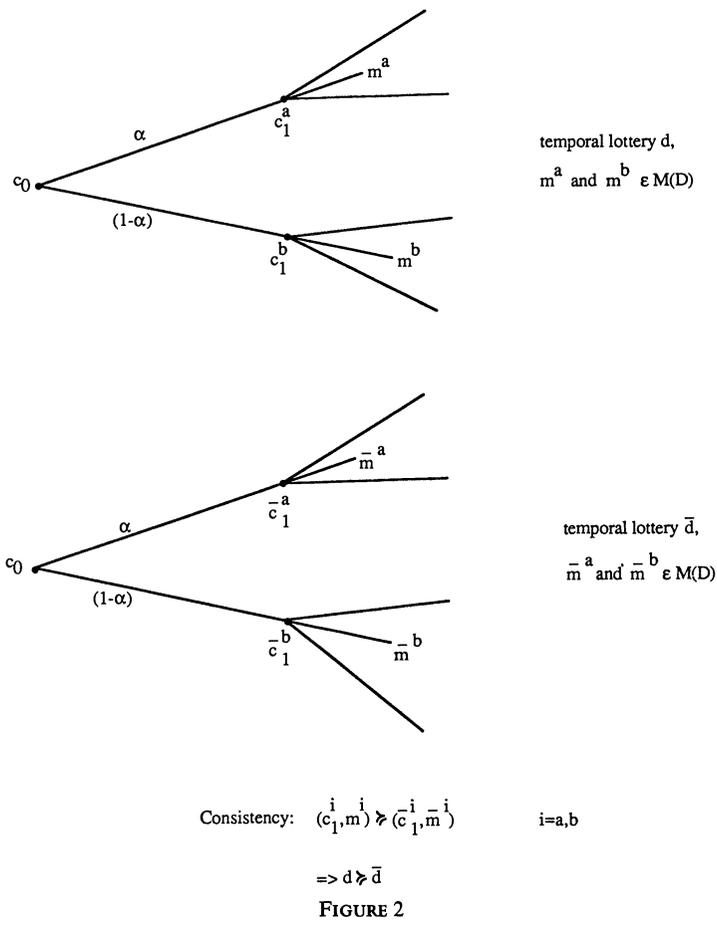


FIGURE 2

$M(D(b))$ , i.e.,

$$(3.3) \quad V[m](Q) = m \{ d \in D : V(d) \in Q \}, \quad Q \in \mathcal{B}(R_+).$$

The utility function  $V$  is called *recursive* if it satisfies the following equation on its domain:

$$(3.4) \quad V(c_0, m) = W(c_0, \mu(V[m]))$$

for some increasing aggregator function  $W: R_+^2 \rightarrow R_+$  and some certainty equivalent  $\mu$ .

This relation is the cornerstone of our analysis. Of course, it generalizes the more familiar structure (3.1). Note also that the recursive structure immediately implies the intertemporal consistency of preference (in the sense of Johnsen and Donaldson (1985) or Figure 2) and the stationarity of preference (in the sense of Koopmans (1960), for example).

The question which immediately arises is whether and under what circumstances there exist utility functions  $V$  satisfying (3.4).<sup>3</sup> To answer this question we restrict the admissible aggregators and certainty equivalents. First, we require that  $W$  have the CES form

$$(3.5) \quad W(c, z) = [c^\rho + \beta z^\rho]^{1/\rho}, \quad 0 \neq \rho < 1, \quad 0 < \beta < 1.$$

(The  $\rho = 0$  case is ignored for simplicity.) In conjunction with (3.1), this implies that when restricted to deterministic consumption programs,  $V$  is an intertemporal CES utility function with elasticity of substitution  $\sigma = (1 - \rho)^{-1}$ . Though restrictive, the CES specification for  $W$  is still sufficiently flexible to permit the issue of separation of substitutability from risk aversion to be addressed.

In contrast, a broad class of mean value functionals will be allowed. In this section they will be required to satisfy the following:

MV.1: If  $\{p_n\}$  and  $p$  are in  $M([0, a]) \subset M(R_+)$ , then

- (a)  $\lim \int f dp_n = \int f dp \quad \forall f: R_+ \rightarrow R_+ \text{ increasing}$   
 $\Rightarrow \lim \mu(p_n) = \mu(p), \text{ and}$
- (b)  $\limsup \int f dp_n \leq \int f dp \quad \text{for all } f: R_+ \rightarrow R_+ \text{ increasing}$   
 $\Rightarrow \limsup \mu(p_n) \leq \mu(p).$

MV.1 is a form of continuity. The convergence criteria implicit in the hypotheses of (a) and (b) are stringent in the sense that the functions  $f$  are not required to be bounded or continuous, but MV.1(a) (resp. (b)) is not comparable with the assumption that  $\mu$  is continuous (resp. u.s.c.) on  $M(R_+)$ .<sup>4</sup>

We can now prove the existence of recursive utility functions.

**THEOREM 3.1:** *Let  $W$  be defined by (3.5) and let  $\mu$  be a mean value functional satisfying MV.1. Then there exists a solution  $V$  to (3.4) if (a)  $\rho > 0$  and  $\beta b^\rho < 1$ , in which case  $V$  is defined on  $D(b)$  and u.s.c. on  $D(b; l)$  for all  $l > 0$ ; (b)  $\rho < 0$ , in which case  $V$  is defined on  $D(\infty)$  and is u.s.c. on any  $D(b; l)$ .*

When  $\rho < 0$ ,  $W(c_0, 0) = 0$  for all  $c_0$  so that the zero function solves (3.4). The solution  $V$  provided by the theorem is nontrivial. In particular, on  $\cup_l Y(b; l)$ ,

<sup>3</sup> Thus our approach, similar to that followed by Lucas and Stokey (1984), is to begin with  $W$  and  $\mu$  and to show that a utility function is implied. It would be interesting also to work in the opposite direction of specifying axioms for intertemporal utility and deriving  $W$  and  $\mu$ . A related axiomatic analysis in a finite horizon framework may be found in Chew and Epstein (1989).

<sup>4</sup> Sequences in  $M(R_+)$  satisfying the hypothesis in MV.1(b) arise here in the following way: Let  $m_n \rightarrow m$  in  $M(D)$ ,  $g: D \rightarrow R_+$  u.s.c. and bounded above. Then Lemma A2.1 implies that  $\limsup \int f(g(\cdot)) dm_n(\cdot) \leq \int f(g(\cdot)) dm(\cdot)$  for all increasing  $f$  since  $f(g(\cdot))$  is u.s.c. and bounded above on  $D$ . See Case 2 in the proof of Theorem 3.1 for the role played by MV.1(a). It essentially ensures that  $\mu$  satisfies an extension of the monotone convergence theorem of integration.

which is a subspace of  $D(b)$ ,  $V$  coincides with the common specification  $[\sum \beta^t c_t^\rho]^{1/\rho}$ . This coincidence applies also if  $\rho > 0$ .

A proof of the theorem is provided in Appendix 3 but some comments are in order here. In deterministic frameworks the existence of recursive utility functions has been proven by application of the Contraction Mapping Theorem (Lucas and Stokey (1984)). The commonly used form of this theorem requires bounded utility and aggregator functions, which is violated by (3.5). Thus we apply a Weighted Contraction Theorem developed by Boyd (1987) to deal with unbounded aggregators. His theorem does not apply in all cases below since the stochastic structure introduces some complications. Nevertheless, for those cases where the contraction mapping technique fails, we are able to prove existence of recursive utility by means of a "partial sum" or "monotone convergence" technique which is also adapted from Boyd.

Turn now to some subclasses of recursive utility functions based on particular specifications for the certainty equivalent functional. We argue at the end of Section 4 that each of these subclasses is of theoretical interest and not merely a parametric example of a recursive utility function.

In all of the subclasses to follow  $\mu$  satisfies MV.1 and the homogeneity property MV.2.

MV.2:  $\mu(p_{\lambda\tilde{x}}) = \lambda\mu(p_{\tilde{x}})$  for all  $\lambda > 0$ , where  $p_{\lambda\tilde{x}}$  and  $p_{\tilde{x}}$  are probability measures in  $\text{dom } \mu$  corresponding to the random variables  $\tilde{x}$  and  $\lambda\tilde{x}$  respectively.

This homogeneity plays a large role in the behavioral analysis below; particularly in the derivation of equilibrium asset return relations which involve only market, and thus presumably observable, variables.

CLASS 1 (Expected Utility): Let

$$\mu(p) \equiv \left( \int x^\rho dp(x) \right)^{1/\rho} \equiv (E_p \tilde{x}^\rho)^{1/\rho}, \quad p \in M(R_+),$$

where  $\rho$  is the parameter appearing in the aggregator (3.5). This specification for  $\mu$  leads to the common intertemporal utility function

$$(3.6) \quad V(c_0, m) = \left[ c_0^\rho + E_{m_1} \sum_1^\infty \beta^t \tilde{c}_t^\rho \right]^{1/\rho},$$

where  $m_1$  denotes the temporal probability measure on consumption streams induced by  $m$  as described in Section 2. (More precisely,  $(c_0, m_1) = \pi_1(d) \equiv \pi_1(c_0, m)$  where  $\pi_1$  is defined prior to Theorem 2.2.)

CLASS 2 (Kreps/Porteus): Let

$$(3.7) \quad \mu(p) \equiv (E_p \tilde{x}^\alpha)^{1/\alpha}, \quad p \in M(R_+),$$

where  $0 \neq \alpha < 1$ , in which case  $V$  satisfies the recursive relation

$$(3.8) \quad V(c_0, m) = \left[ c_0^\rho + \beta (E_m V^\alpha(\cdot))^{\rho/\alpha} \right]^{1/\rho}.$$

(The  $\alpha = 0$  case is ignored for simplicity.) The previous class is obtained by the parametric restriction  $\alpha = \rho$ .<sup>5</sup>

We can rewrite (3.8) in the form

$$(3.9) \quad U(c_0, m) = H(c_0, E_m U(\cdot)),$$

where  $U \equiv V^\alpha/\alpha$  is ordinally equivalent to  $V$  and where

$$H(c, z) \equiv [c^\rho + \beta(\alpha z)^\rho]^\alpha / \alpha.$$

The recursive relation (3.9) is a special case (due to the particular specification of  $H$ ) of the structures studied by KP (1978). Those authors point out that the utility functions defined by (3.9) conform with expected utility theory when ranking timeless gambles, i.e., those in which uncertainty is resolved before further consumption takes place. In fact, for ranking wealth gambles which are timeless and whose outcomes reveal nothing regarding future asset rates of return, an agent with recursive utility would use the objective function  $\mu(p_{\tilde{x}})$ , where  $\tilde{x}$  is random wealth. (See the end of Section 5.) Thus the certainty equivalent functional  $\mu$  which was introduced to evaluate *utility* distributions is also relevant for evaluating timeless *wealth* lotteries. In particular, if  $\mu$  is specified by (3.7), then timeless wealth gambles are ranked by an expected utility ordering. Since the bulk of the experimental evidence against expected utility theory is based on choices amongst timeless gambles, the KP specification (3.8) is inconsistent with that evidence. Thus we are led to seek non-expected utility based mean value functions and consequently generalizations of the KP intertemporal utility specifications.

CLASS 3 (Chew/Dekel): The mean value functional  $\mu$  is defined on  $M(R_+)$  *implicitly* by an equation of the form

$$(3.10) \quad \int F(x, \mu(p)) dp(x) = 0, \quad p \in M(R_+),$$

where  $F: R_+^2 \rightarrow R$  is continuous, increasing (decreasing) in its first (second) argument,  $F(\cdot, z)$  is concave, and  $F(x, x) \equiv 0$ . Chew (1989) and Dekel (1986) show that such a functional  $\mu$  is consistent with first and second degree stochastic dominance and can explain the Allais paradox.<sup>6</sup> Condition (3.2) is immediate. In Appendix 2 we show that such a functional  $\mu$  satisfies MV.1. In order that  $\mu$  satisfy the homogeneity condition MV.2 we require that  $F$  be linearly homogeneous. In that case, by defining  $\phi(x) \equiv F(x, 1)$ , we can rewrite (3.10) in the form

$$(3.11) \quad \int \phi(x/\mu(p)) dp(x) = 0, \quad p \in M(R_{++}).$$

<sup>5</sup> Farmer (1987) employs the  $\alpha = 1$  specialization of (3.8) adapted to a finite horizon framework. Also, Weil (1987a, 1987b) has independently proposed functional forms similar to (3.8). But their presentation is not based upon the general recursive structure (3.4). Moreover, his papers do not contain any of the analysis provided here.

<sup>6</sup> See also Chew (1983) which deals with the case  $F(x, z) = w(x)[v(x) - v(z)]$ , where the explicit representation  $\mu(p) = v^{-1}\{E_p[w(\cdot)v(\cdot)]/E_p w(\cdot)\}$  exists. In general, however, (3.10) does not admit an explicit solution for  $\mu(p)$ .

Then  $\phi$  is continuous, increasing, concave, and  $\phi(1) = 0$ . Note that if

$$(3.12) \quad \phi(x) = (x^\alpha - 1)/\alpha, \quad 0 \neq \alpha < 1,$$

then  $\mu(p)$  reduces to the expected utility based certainty equivalent (3.7). But, of course, other specifications for  $\phi$  are possible. For example, one that is investigated empirically in Epstein and Zin (1989) is

$$(3.13) \quad \phi(x) = (x^\alpha - 1)/\alpha + a(x - 1), \quad 0 \neq \alpha < 1, \quad a \geq 0.$$

If the functional  $\mu$  defined in (3.11) is substituted into (3.4) then a recursive relation is obtained for intertemporal utility  $V$ . Theorem 3.1 may be applied to establish the existence of  $V$ . Since (3.11) generalizes (3.7), the class of Chew-Dekel based intertemporal utility functions generalizes the KP class. An appeal of this generalization is that it can potentially provide a *unified explanation* of both *market* consumption and asset return data (via the representative agent framework described below) and *experimental* data. With regard to the latter, Machina (1982) formulates a property of functionals  $\mu$ , called Hypothesis II, which he argues is both sufficient and in a sense also necessary for an explanation of Allais and other paradoxes. In terms of (3.11), Hypothesis II is equivalent to the straightforward restriction on  $\phi$  that  $-x\phi''(x)/\phi'(x)$  is (strictly) decreasing. This restriction can be readily incorporated into specifications for  $\phi$ , e.g., it is satisfied by (3.13) if  $a > 0$ .

The above examples do not exhaust the class of recursive intertemporal utility functions covered by Theorem 3.1. Other specifications for  $\mu$ , taken from the atemporal non-expected utility literature for example, could be adopted if the seemingly mild continuity condition MV.1 is satisfied. Thus Theorem 3.1 should permit the integration into a temporal setting of a substantial portion of the non-expected utility literature.

4. SUBSTITUTION, RISK AVERSION, AND TIMING

The key properties of recursive utility functionals will be discussed here.

It has already been noted that the specification (3.5) for the aggregator implies that deterministic consumption sequences are ranked by an intertemporal CES utility function with elasticity of substitution  $\sigma = (1 - \rho)^{-1}$ . Thus we interpret  $\rho$  as a parameter reflecting substitutability.

Next turn to risk aversion and in particular to comparative risk aversion. Let  $V$  and  $V^*$  be two recursive utility functions with possibly distinct aggregators  $W$  and  $W^*$  conforming to (3.5).<sup>7</sup> We wish to define what it means for  $V^*$  to be more risk averse than  $V$ . To do that, define  $c = \lambda(m)$  for any  $(c_0, m)$  by

$$V(c_0, m) = V(c_0, c, c, \dots).$$

<sup>7</sup> The ensuing discussion could be carried out without restricting aggregators to conform with (3.5). But such generality would complicate the exposition somewhat for the following reason: If  $V$  is recursive with aggregator  $W$  and certainty equivalent functional  $\mu$ , then any monotonic transform of  $V$ , say  $h(V)$ , is also recursive with aggregator  $\bar{W}$ ,  $\bar{W}(c, z) \equiv hW(c, h^{-1}(z))$  and certainty equivalent functional  $\bar{\mu}$ ,  $\bar{\mu}(p_x) \equiv h\{u(p_{h^{-1}(x)})\}$ . Thus a given preference ordering of temporal lotteries can be represented by many  $(W, \mu)$  pairs. By fixing the representation (3.5) for the aggregator, we avoid the need to refer to the entire class of "equivalent"  $(W, \mu)$  pairs.

Interpret the “nearly” constant and deterministic path  $(c_0, c, c, \dots)$  which is indifferent to  $(c_0, m)$  as the certainty equivalent for the latter. (Note that  $c$  depends only on  $m$  and not on  $c_0$ .) The path  $(c_0, \lambda^*(m), \lambda^*(m), \dots)$  is defined analogously given  $V^*$ . It is natural to say that  $V^*$  is more risk averse than  $V$  if and only if  $\lambda^*(m) \leq \lambda(m)$  for all  $(c_0, m)$  in some common domain for  $V$  and  $V^*$ .

Evidently, if  $V$  and  $V^*$  are comparable in the above sense, then they must rank nonstochastic consumption programs identically, i.e.,  $W = W^*$  or equivalently,  $\rho = \rho^*$  and  $\beta = \beta^*$ . Moreover,  $V^*$  is more risk averse than  $V$  if and only if  $W = W^*$  and

$$(4.1) \quad \mu^*(\cdot) \leq \mu(\cdot)$$

on the appropriate domain. (Necessity of these conditions is obvious. For sufficiency, note that they imply, given the construction in the proof of Theorem 3.1, that  $V^*(\cdot) \leq V(\cdot)$ .) Thus the certainty equivalent functional determines the degree of risk aversion of the corresponding intertemporal utility function, at least for comparative purposes. Further support for this interpretation of  $\mu$  is provided at the end of Section 5.

Since, by assumption, mean value functionals  $\mu^*$  exhibit risk aversion in the sense of second degree stochastic dominance, it follows that

$$\mu^*(\cdot) \leq E(\cdot),$$

where  $E(\cdot)$  denotes the expected value operator. Thus the least risk averse intertemporal utility function is the one for which  $\mu(\cdot) = E(\cdot)$ . Moreover, there is a sense in which the latter specification implies risk neutrality, e.g., in the context of timeless wealth gambles or in the portfolio choice context described at the end of Section 5. It is apparent, therefore, that “low” or “moderate” risk aversion can coexist with a small elasticity of substitution, which is impossible in the expected utility specification.

In the case of KP functionals (3.8), the condition (4.1) is equivalent to  $\alpha^* \leq \alpha$ . Thus we interpret  $\alpha$  as a measure of risk aversion for comparative purposes with smaller  $\alpha$ 's indicating greater risk aversion. A separation between the risk aversion parameter ( $\alpha$ ) and the substitution parameter ( $\rho$ ) is achieved.

For the Chew-Dekel class based on (3.11), (4.1) is equivalent to

$$\phi^{*''}(\cdot)/\phi^{*'}(\cdot) \leq \phi''(\cdot)/\phi'(\cdot).$$

This is satisfied in the parametric class defined by (3.13) if  $\alpha^* \leq \alpha$  and  $a^* \leq a$ .

A comparable separation between risk aversion and substitution does not appear possible within the expected utility model. To see this, consider consumption programs  $(c_0, m)$ , with  $m \in M(Y(b; l)) \subset M(R_+^\infty)$  and let

$$V(c_0, m) = E_m \left\{ c_0^\rho / \rho + \sum_1^\infty \beta^t \tilde{c}_t^\rho / \rho \right\},$$

which is ordinally equivalent to (3.6). The general multicommodity analysis of Kihlstrom and Mirman (1974) suggests that we take a monotonic transform  $h$  of

the von Neumann-Morgenstern utility index and define

$$(4.2) \quad V^*(c_0, m) = E_m h \left[ c_0^\rho / \rho + \sum_1^\infty \beta^t \bar{c}_t^\rho / \rho \right].$$

Then  $V^*$  is more risk averse than  $V$  in the sense defined above, or equivalently in the sense of Kihlstrom and Mirman (1974), if and only if  $h$  is concave.

At first glance, therefore, it would seem that comparative risk aversion analysis is feasible within an expected utility framework. But in a temporal setting, this familiar approach encounters serious difficulties. To see this, consider an individual with the utility function (4.2) who arrives at period  $T$  and contemplates the remaining future. If past consumption levels were  $\bar{c}_0, \dots, \bar{c}_{T-1}$ , then the utility function for the remaining future is presumably<sup>8</sup>

$$(4.3) \quad V^{*T}(c_T, m; \bar{c}_0, \dots, \bar{c}_{T-1}) \\ \equiv E_m h \left[ \sum_0^{T-1} \beta^t \bar{c}_t^\rho / \rho + \beta^T \left( c_T^\rho + \sum_{T+1}^\infty \beta^{t-T} \bar{c}_t^\rho \right) / \rho \right].$$

Thus the preference ordering at  $T$  depends upon past consumption values unless  $h$  has constant absolute risk aversion. Dependence upon the past is in principle sensible but the form which this dependence takes above is implausible, since (4.3) and  $0 < \beta < 1$  imply that the dependence on past consumption is *greater* as the past becomes more *distant*. (For example, denote by  $H(\bar{c}_0, \dots, \bar{c}_{T-1}; c_T, c_{T+1}, \dots)$  the von Neumann-Morgenstern utility index on the right side of (4.3). If derivatives are evaluated at a point where  $\bar{c}_0 = \dots = \bar{c}_{T-1}$ , then

$$\frac{\partial}{\partial \bar{c}_{T-1}} \left( -H_{c_{T+1}c_{T+1}} / H_{c_{T+1}} \right) = \beta^{T-1} \frac{\partial}{\partial \bar{c}_0} \left( -H_{c_{T+1}c_{T+1}} / H_{c_{T+1}} \right).$$

Thus the risk premium for a small gamble in period  $(T+1)$  consumption is affected more by a small change in  $\bar{c}_0$  than by a small change in  $\bar{c}_{T-1}$ . On the other hand, if  $h(z) = -\exp(-Az)$ ,  $A > 0$ , then the nonstationarity of preferences is implied as period  $T$  preferences are represented by  $V^T(c_T, m) = E_m - \exp(-A\beta^T u(c_T, \bar{c}_{T+1}, \dots))$ , where  $u$  is the additive functional  $\sum_0^\infty \beta^t c_t^\rho / \rho$ . Declining risk aversion with  $T$  is imposed a priori. Though such a "changing tastes" specification may be appropriate in some modelling exercises, it would appear to be a hindrance rather than a help to exploring the questions outlined in the introduction and in Section 7. More importantly, if attitudes towards future gambles are changing with the passage of time as above, then plans will generally

<sup>8</sup> Otherwise tastes are changing through time. This is unappealing as an a priori specification and, moreover, implies that preferences are intertemporally inconsistent. Hall's (1981) specification suffers from these problems.

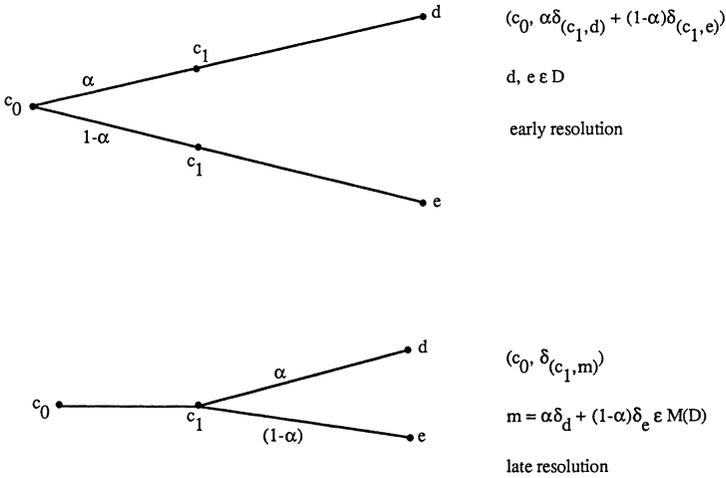


FIGURE 3

not be intertemporally consistent. The above unappealing features of the familiar Kihlstrom and Mirman approach to comparative risk aversion are not restricted to the case where the von Neumann-Morgenstern utility index is additively separable. For example, they may be confirmed also for the nonadditive indices axiomatized in Epstein (1983) which feature variable discount rates.

Finally, with regard to attitudes towards the timing of the resolution of uncertainty, see Figure 3. The two temporal lotteries portrayed there differ precisely in the timing of the resolution of uncertainty as defined by KP (1978). A recursive utility function is indifferent to the timing of resolution (in all such pairs of lotteries) if and only if it is an expected utility functional such as (3.6). (This follows by a straightforward extension of the finite horizon arguments in Chew and Epstein (1989).) In particular, for the KP class, the curvature of  $H(c_0, \cdot)$  defined in (3.9) is the determinant of attitudes towards timing with indifference towards timing prevailing only if  $H(c_0, \cdot)$  is linear (KP (1978)). We can conclude, in fact, that given (3.8) early (late) resolution is preferred if  $\alpha < (>) \rho$ .

For more general recursive utility functions, we have not found a characterization in terms of  $W$  and  $\mu$  of the conditions under which early or late resolution is preferred. But the characterization for the KP class raises an issue which we suspect is relevant more generally and which calls for some attention. We have interpreted  $\alpha$  as a risk aversion parameter. But with  $\rho$  fixed, a reduction in  $\alpha$  not only increases risk aversion but also may transform a preference for late resolution into a preference for early resolution. One is left wondering how to interpret the comparative statics effects of a change in  $\alpha$ . Similarly, a change in  $\rho$  for given  $\alpha$  affects both substitutability and attitudes towards timing. Thus the latter aspect of preference seems intertwined with both substitutability and risk aversion.

We offer three comments in response. First, from the perspective of potential empirical applications, the specifications (3.8) and a fortiori (3.4) are still more flexible than the common expected utility functional form. Second, the behavioral analysis in the next section will provide further support for our interpretation of  $\alpha$ , or more generally  $\mu$ , as a risk aversion parameter since  $\alpha$  or  $\mu$  will determine the degree of risk taking in certain portfolio choice problems. Finally, we suspect that the lack of separation noted above reflects the inherent inseparability of these three aspects of preference rather than a deficiency of our theoretical framework. Further study of this issue is required.

To conclude this section, we observe that attitudes towards timing can be used to distinguish, within the family of recursive utility functions, each of the three subclasses defined in Section 3. It has already been pointed out that timing indifference implies an expected utility ordering. Next consider the Chew-Dekel subclass. Suppose that  $V$  is such that the lotteries in Figure 3 are indifferent to one another whenever  $V(d) = V(e)$ ; that is, the timing of resolution is a matter of indifference if the two future prospects regarding which information is being provided, are themselves indifferent. Refer to this property as *quasi-timing indifference* (QTI). A straightforward extension of the finite horizon arguments in Chew and Epstein (1989) shows that the only recursive utility functions satisfying QTI are those based on (3.10). If an appropriate homotheticity assumption is imposed on  $V$ , then  $\mu$  must satisfy MV.2 and (3.11) is obtained. (See Chew and Epstein (1989) for the basis for a comparable argument for the KP class (3.8) and for discussion of QTI. An alternative basis for an axiomatization of KP preferences may be found in KP (1978).) Thus a *theoretical* case can be made for interest in the KP and Chew-Dekel subclasses of recursive utility functions. Accordingly, we do not apologize for the fact that some of the discussion of the asset pricing implications of our framework is limited to these subclasses.

##### 5. THE REPRESENTATIVE AGENT

The remainder of this paper derives relations between aggregate consumption and real rates of return which must hold in a competitive equilibrium. The procedure adopted is that of the rational expectations literature on aggregate consumption (Hall (1978)). In this section we determine the optimal consumption and portfolio behavior of an individual who faces exogenous rates of return to saving. Then, in the next section, we take the individual to be a representative agent in the economy so that the Euler equations corresponding to his intertemporal plan define relations between aggregate consumption and rates of return that must hold in equilibrium. We deviate from earlier literature in the specification of a recursive (but not necessarily expected utility) specification for preferences. Consequently, the derivation of the Euler equations is nonstandard.

Our representative agent operates in a standard environment. There are  $K$  assets. The gross return to holding the  $k$ th asset between  $t$  and  $(t+1)$  is described by the random variable  $\tilde{r}_{kt}$ ,  $-\infty < t < \infty$ , where each  $\tilde{r}_{kt}$  has support

in  $[r, \bar{r}]$ ,  $r > 0$ . Let  $\tilde{r}_t = (\tilde{r}_{1t}, \dots, \tilde{r}_{Kt})$ . We assume that  $(\tilde{r}_t, \tilde{z}_t)_{-\infty}^{\infty}$  is a stationary stochastic process. The role of  $z_t \in R^Z$  is to provide information regarding the future. It is assumed that  $(\tilde{r}_t, \tilde{z}_t)$  is observed at the start of period  $(t + 1)$ , just prior to the time at which period  $(t + 1)$  consumption and portfolio decisions are made. Without loss of generality we may take the underlying probability space to be  $(\Omega, \mathcal{B}(\Omega), \Psi)$  where  $\Omega$  is the set of doubly infinite sequences with  $t$ th component  $(r_t, z_t)$  and  $\mathcal{B}(\Omega)$  is the (product) Borel  $\sigma$ -algebra.

The state of the world at time  $t$  is defined by the existing wealth level  $x_t$  and by the history of realized past values of  $(\tilde{r}_i, \tilde{z}_i)$ 's. Thus let

$$\Omega_t = [0, \infty) \times \prod_{i=-\infty}^{t-1} ([r, \bar{r}]^K \times R^Z) \equiv [0, \infty) \times I_t,$$

where the (product) Borel  $\sigma$ -algebra is adopted for  $I_t$  and  $\Omega_t$ . A consumption-portfolio plan (from  $t = 0$  onwards) is a sequence  $(h_0, \dots, h_t, \dots)$  of measurable functions  $h_t: \Omega_t \rightarrow [0, \infty) \times S^K$ , where  $S^K$  is the unit simplex in  $R^K$ . The interpretation of  $h_t(x_t, I_t) = (c_t, w_t)$  is that given period  $t$  wealth  $x_t$  and history  $I_t \in I_t$ , the agent consumes  $c_t$  and invests the proportion  $w_{kt}$  in the  $k$ th asset,  $w_t = (w_{1t}, \dots, w_{Kt})$ .

A plan is *homogeneous* if  $\forall t > 0$  and  $\forall (x, I_t) \in \Omega_t$ ,  $h_t(1, I_t) = (c_t, w_t) \Rightarrow h_t(x, I_t) = (c_t x, w_t)$ . Because of the homotheticity of preferences and the linearity of "technology," it is natural to restrict oneself to homogeneous plans which are henceforth simply plans.

A plan is *stationary* if  $\exists h$  such that  $h_t = h \forall t$ . Finally, a plan is *feasible* if  $\forall t \geq 0$  and  $\forall (x_t, I_t) \in \Omega_t$ ,  $c_t \leq x_t$  where  $c_t$  is the first component of  $h_t(x_t, I_t)$  and where wealth evolves according to

$$(5.1) \quad x_t = (x_{t-1} - c_{t-1})w'_{t-1}\tilde{r}_{t-1}, \quad t \geq 1,$$

$x_0 > 0$  given.<sup>9</sup>

Each plan implies an infinite probability tree in consumption levels. Moreover, if the plan is feasible, then the corresponding probability tree itself corresponds to a temporal lottery in  $D(\bar{r}; x_0) \subset D(\bar{r})$ . Formally, denote by FP the set of feasible plans for a given  $(x_0, I_0)$ . Appendix 4 shows that FP can be embedded in a "natural fashion" in  $D(\bar{r})$  by the map  $e$ . Moreover,  $D(\bar{r})$  is a subspace of  $D(b)$  for any  $b > \bar{r}$ . Thus, if  $V$  is defined on  $D(b)$ , the problem

$$(5.2) \quad J(I_0, x_0) \equiv \sup\{V(d) : d \in e(\text{FP})\}$$

is well-defined if  $\bar{r} < b$ . The conditions under which the supremum is attained are specified in the next theorem.

**THEOREM 5.1:** *Let  $V$  be the recursive utility function constructed in Theorem 3.1, defined on  $D(b)$  and having aggregator (3.5) and a mean value functional satisfying*

<sup>9</sup> This budget constraint excludes exogenous sources of income and labor income. This exclusion is important for the homogeneity property (5.4) below and subsequently for our derivation of Euler equations. Some discussion of the modifications necessary to accommodate these sources of income may be found in Epstein and Zin (1989).

*MV.1 and MV.2. Then (5.2) possesses a maximum, achieved by a stationary and homogeneous plan if (a)  $\rho > 0$ , and  $\bar{r} < b$ , or (b)  $\rho < 0$  and  $\beta r^\rho < 1$ . In either case,  $J > 0$ .*

Note that the homogeneity of  $\mu$  (MV.2) has been added as an assumption. Also note that the strict positivity of the value function of  $J$  is used below.

We now turn to the implications of optimality and more particularly to the appropriate set of Euler equations for interior optima. The recursive structure of utility functions immediately implies the "Bellman equation"

$$(5.3) \quad J(I_0, x_0) = \max_{c_0 \geq 0, w_0 \in S^K} \left[ c_0^\rho + \beta \mu^\rho \left[ p_{J(\tilde{I}_1, (x_0 - c_0)w_0' \tilde{r}_0) / I_0} \right] \right]^{1/\rho},$$

where the argument of  $\mu$  is the probability measure for  $J(\tilde{I}_1, (x_0 - c_0)w_0' \tilde{r}_0)$  conditional on  $I_0$ . Moreover, the maximizing values of  $c_0$  and  $w_0$  correspond to the utility maximizing plan in the customary fashion.

It is evident from the homotheticity of utility, that  $J$  can be expressed in the form

$$(5.4) \quad J(I, x) = A(I)x.$$

Thus (5.3) can be written

$$(5.5) \quad A(I_0)x_0 = \max_{c_0 \geq 0, w_0 \in S^K} \left[ c_0^\rho + \beta (x_0 - c_0)^\rho \mu^\rho \left[ p_{A(\tilde{I}_1)w_0' \tilde{r}_0 / I_0} \right] \right]^{1/\rho}.$$

An immediate implication is the portfolio separation property and more particularly that the portfolio decision is determined by the solution to

$$(5.6) \quad \max_{w_0 \in S^K} \mu \left[ p_{A(\tilde{I}_1)w_0' \tilde{r}_0 / I_0} \right].$$

Write  $c_0^* = a_0 x_0$ , where an asterisk denotes the maximizing value. Substitution into (5.5) yields

$$A^\rho(I_0) = a_0^\rho + \beta(1 - a_0)^\rho \mu^{*\rho},$$

and the first order condition for consumption in (5.5) yields

$$(5.7) \quad a_0^{\rho-1} = (1 - a_0)^{\rho-1} \beta \mu^{*\rho}.$$

These last two equations can be combined to yield  $A(I_0) = a_0^{(\rho-1)/\rho} = (c_0^*/x_0)^{(\rho-1)/\rho}$ . From the stationarity of the problem and the recursivity of utility, it follows that

$$(5.8) \quad A(\tilde{I}_1) = (\tilde{c}_1^*/\tilde{x}_1)^{(\rho-1)/\rho}.$$

Substitute (5.8) and (5.1) into (5.7), write  $\tilde{M}_0$  for the market portfolio return  $w_0' \tilde{r}_0$  and suppress asterisks to deduce that

$$(5.9) \quad \beta^{1/\rho} \mu \left[ p_{(\tilde{c}_1/c_0)^{(\rho-1)/\rho} \tilde{M}_0^{1/\rho} / I_0} \right] = 1,$$

which is the first order condition for consumption expressed in terms of presum-

ably observable market variables. Finally, use (5.8) to rewrite (5.6) in the form

$$(5.10) \quad \max_{w_0 \in S^K} \mu \left[ P_{(\tilde{c}_1/\bar{M}_0)^{(\rho-1)/\rho}(w_0^0 \bar{r}_0)} / I_0 \right].$$

The relations (5.9) and (5.10) are the principal implications of intertemporally optimizing behavior and are the basis for the model's predictions regarding the temporal behavior of aggregate consumption and asset returns. (See Section 6.) To conclude this section we provide justification for two assertions made earlier in the paper.

First, we show that the above behavioral analysis provides further support for our interpretation of the mean value functional  $\mu$  as the embodiment of risk aversion. Consider an environment where the  $\tilde{r}_t$ 's are independently and identically distributed and the  $\tilde{z}_t$ 's are absent. Then  $J(I, x)$  is independent of  $I$  and from (5.6) we see that the portfolio decision is determined by

$$\max_{w_0 \in S^K} \mu (P_{w_0^0 \bar{r}_0}).$$

The latter is an atemporal portfolio choice problem with the utility of wealth distributions being given by  $\mu$ . We see that the specification in which  $\mu$  equals the expected value operator implies risk neutral behavior in this i.i.d. environment. In fact, even in the more general environment the specification  $\mu(\cdot) = E(\cdot)$  implies a form of risk neutrality since then the objective function in the appropriate form of (5.6) is linear in  $w$  and thus assets are perfect substitutes.

It was asserted in Section 3 in the discussion of KP utility functionals, that the functional  $\mu$  is relevant to the ranking of timeless wealth lotteries. We can now explain why, and in what sense, this assertion is valid. The consumption programs underlying timeless wealth gambles have  $c_0$  random while we have defined utility functions only for lotteries where  $c_0$  is deterministic. But the domain of recursive utility functions is extended in a natural fashion by using  $\mu$  to compute the certainty equivalent utility value associated with any  $m \in M(R_+ \times M(D))$ . An agent with this extended utility function would use the objective function  $\mu(P_{J(I_0, \tilde{x}_0)}) = \mu(P_{A(I_0, \tilde{x})})$  to rank timeless wealth gambles, where  $\tilde{x}$  is random wealth. If the outcome of the wealth lottery provides no information regarding future asset rates of return, then  $A(I_0)$  is a constant in  $\mu(\cdot)$  and, by the homogeneity property MV.2, it can be taken outside. This leaves  $\mu(p_{\tilde{x}})$  as the basis for evaluating the timeless wealth lottery corresponding to the random variable  $\tilde{x}$ .

### 6. CONSUMPTION AND ASSET RETURNS

Assume the existence of a representative agent for our economy. (Since preferences are homothetic, if they are common to all consumers and if consumers are similar in all other respects with the possible exception of differing wealths, then the representative agent assumption can be justified in the usual way.) Then the Euler equations generated by the agent's intertemporal optimization problem imply equilibrium relations between aggregate consumption and asset returns. These relations will be now considered.

Apply the analysis of last section to the representative agent. Then, if (5.10) were replaced by its first order conditions, we would obtain a complete set of “Euler equations” for the intertemporal optimization problem. Moreover, the first order conditions emanating from (5.10) would contain a model of asset returns. One feature of that model which is evident from (5.10) even without specifying the model in detail, is that *both* consumption  $\tilde{c}_1$  and the return to the market portfolio will play a role in explaining differences in the expected returns to any two assets. This is in contrast to both the static CAPM, where covariance with the market portfolio alone determines the systematic risk of any asset, and also to the consumption-based CAPM where systematic risk is measured by covariance with consumption.

To derive first order conditions for (5.10) some smoothness properties must be assumed for  $\mu$ , e.g. Fréchet differentiability as in Machina (1982). We leave it to the interested reader to analyze (5.10) further using Machina’s local utility functions. Here we proceed under the assumption of Chew-Dekel preferences. They are not necessarily Fréchet differentiable but they satisfy a weaker smoothness property (Chew, Epstein, and Zilcha (1988)) which suffices for our purposes.

We adopt the following argument: Denote by  $\mu^*$  the maximum value in (5.6). When  $\mu$  is given by (3.11), the solution of (5.6) also solves<sup>10</sup>

$$\max_{w_0 \in S^K} E \left[ \phi \left( A(\tilde{I}_1) w_0' \tilde{r}_0 / \mu^* \right) / I_0 \right].$$

The first order conditions for this problem are

$$(6.1) \quad E \left[ \phi' \left( A(\tilde{I}_1) \tilde{M}_0 / \mu^* \right) \cdot A(\tilde{I}_1) \cdot (\tilde{r}_{k0} - \tilde{r}_{10}) / I_0 \right] = 0,$$

$k = 2, \dots, K$ . From (5.7),

$$\begin{aligned} \mu^* &= \left[ \frac{1 - a_0}{a_0} \right]^{(1-\rho)/\rho} \beta^{-1/\rho} \\ \Rightarrow A(\tilde{I}_1) / \mu^* &= a(\tilde{I}_1)^{(\rho-1)/\rho} \left[ \frac{1 - a_0}{a_0} \right]^{(\rho-1)/\rho} \beta^{1/\rho} \\ &= \left[ \frac{\tilde{c}_1}{c_0} \right]^{(\rho-1)/\rho} \beta^{1/\rho} \tilde{M}_0^{(1-\rho)/\rho}. \end{aligned}$$

Substitute into (6.1) to obtain

$$(6.2) \quad E \left[ \phi' \left( \left[ \frac{\tilde{c}_1}{c_0} \right]^{(\rho-1)/\rho} \tilde{M}_0^{1/\rho} \beta^{1/\rho} \right) \cdot \left[ \frac{\tilde{c}_1}{\tilde{M}_0} \right]^{(\rho-1)/\rho} \cdot (\tilde{r}_{k0} - \tilde{r}_{10}) / I_0 \right] = 0,$$

$k = 2, \dots, K$ .

<sup>10</sup> Let  $S$  be a set of probability measures and  $\mu^* \equiv \max \{ \mu(p) : p \in S \}$  where  $\mu$  is given by (3.10). Then  $\mu^* = \mu(p^*) \Rightarrow p^*$  solves  $\max (E_p F(\cdot, \mu^*) : p \in S)$ . See Epstein (1986, Section 3).

Also, substitute (5.9) into (3.11) to deduce

$$(6.3) \quad E \left[ \phi \left( \beta^{1/\rho} \left[ \frac{\tilde{c}_1}{c_0} \right]^{(\rho-1)/\rho} \tilde{M}_0^{1/\rho} \right) / I_0 \right] = 0.$$

Equations (6.2) and (6.3) constitute the Euler equations for the model based on Chew-Dekel preferences. The former reveal more clearly than (5.10) the joint role played by consumption and the market return in determining systematic risk.

A further specialization to KP preferences is obtained by imposing (3.12). The resulting Euler equations take the form

$$(6.4) \quad E \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho} \cdot (\tilde{r}_{k0} - \tilde{r}_{10}) / I_0 \right] = 0 \quad (k = 2, \dots, K),$$

$$(6.5) \quad \beta E^{\rho/\alpha} \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{\alpha/\rho} / I_0 \right] = 1.$$

Alternatively, these equations are equivalent to

$$(6.6) \quad \beta E^{\rho/\alpha} \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho} \tilde{r}_{k0} / I_0 \right] = 1 \quad (k = 1, \dots, K).$$

If the further specialization  $\alpha = \rho$  is adopted, then we obtain

$$(6.7) \quad \beta E \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\rho-1} \tilde{r}_{k0} / I_0 \right] = 1 \quad (k = 1, \dots, K),$$

which are the familiar Euler equations of the expected utility model (see Hansen and Singleton (1983), for example).

Earlier we pointed out that one consequence of the generalization from expected utility to recursive utility is the emergence of the market return as a factor in explaining excess mean returns. The significance of the market return is apparent from (6.4), which can be rewritten

$$E [\tilde{r}_{k0} - \tilde{r}_{10} / I_0] = \frac{\text{cov} \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho}, (\tilde{r}_{k0} - \tilde{r}_{10}) / I_0 \right]}{E \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho} / I_0 \right]}.$$

Thus both consumption and the market return enter into the covariance that defines systematic risk. The consumption-CAPM is obtained if  $\alpha = \rho$ . But if the substitution  $\alpha = 0$  is adopted instead, then covariance with the market return

alone determines systematic risk which is the prediction of the static CAPM.<sup>11</sup> (Though  $\alpha = 0$  lies outside the scope of our formal analysis, one can show that the  $\alpha = 0$  version of (6.4) is the model of mean excess returns that corresponds to the  $\alpha = 0$  version of KP preferences.)

Another specialization of (6.4) which yields the market based CAPM model is  $\rho = 1$ , or infinite elasticity of substitution. This case was excluded from the preceding analysis because it would generally rule out interior optima for consumption (see (5.5)). But the appropriate form of (6.4) is still valid. Intuitively, the emergence of the static CAPM here is presumably due to the perfect substitutability of consumption across time.

We have described some relations between aggregate consumption and asset returns which must hold in a competitive equilibrium, but we have not demonstrated the consistency of our analysis with a general equilibrium framework such as Lucas' (1978) stochastic pure endowment economy. Such an extension would need to confront the questions of existence and uniqueness of equilibrium asset prices. Moreover, Lucas' contraction mapping techniques would not suffice for the same reasons that those techniques were inadequate in establishing Theorem 3.1. Thus we leave such an extension to a separate paper. However, we conclude this section by describing some asset pricing implications of our analysis which are valid for any general equilibrium extension.

For simplicity, consider KP preferences, though comparable formulae may be derived for Chew-Dekel preferences. Consider an asset which pays the dividend  $\tilde{q}_t$  in period  $t$ . The real gross return to holding the asset during period 0 is  $(\tilde{P}_1 + \tilde{q}_1)/P_0$ , where  $P_0$  and  $\tilde{P}_1$  denote the current and random period 1 prices respectively. Then substitution into (6.6) implies that the asset price satisfies the recursive relation

$$P_0 = \beta^{\alpha/\rho} E \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho} (\tilde{P}_1 + \tilde{q}_1)/I_0 \right],$$

which has a solution

$$(6.8) \quad P_0 = E \left[ \sum_{t=1}^{\infty} \beta^{(\alpha/\rho)t} \left[ \frac{\tilde{c}_t}{c_0} \right]^{\alpha(\rho-1)/\rho} (\tilde{M}_0 \dots \tilde{M}_{t-1})^{(\alpha-\rho)/\rho} \tilde{q}_t / I_0 \right],$$

if the right-hand side is finite. Price equals the discounted expected value of

<sup>11</sup> The  $\rho = 0$  version of the expected utility model, i.e., logarithmic within period utility function, leads to  $E[\tilde{M}_0^{-1}(\tilde{r}_{k0} - \tilde{r}_{10})/I_0] = 0 \forall k$ , a CAPM model. But then it is also true that

$$E \left[ \left[ \frac{\tilde{c}_1}{c_0} \right]^{-1} (\tilde{r}_{k0} - \tilde{r}_{10}) / I_0 \right] = 0 \quad \forall k$$

so that the consumption-CAPM also applies. In contrast, our specialization  $\alpha = 0$  implies that *only* the market return determines systematic risk. A related observation is that we generate a CAPM model without restricting the intertemporal elasticity of substitution in consumption. See Grossman and Laroque (1987) for an alternative intertemporal model which leads to the static CAPM.

future dividends where the discount factors involve both consumption and market returns.

In the case of a one-period pure discount bond,  $\tilde{q}_1 \equiv 1$  and  $\tilde{q}_t \equiv 0$  for  $t > 1$ ,

$$P_0 = \beta^{\alpha/\rho} E \left[ \left[ \begin{array}{c} \tilde{c}_1 \\ c_0 \end{array} \right]^{\alpha(\rho-1)/\rho} \tilde{M}_0^{(\alpha-\rho)/\rho} / I_0 \right].$$

Given fixed marginal distributions for  $\tilde{c}_1$  and  $\tilde{M}_0$ , the bond price increases with the covariance between  $\tilde{c}_1^{\alpha(\rho-1)/\rho}$  and  $\tilde{M}_0^{(\alpha-\rho)/\rho}$ . The two exponents have the same signs if and only if  $0 < \alpha < \rho$  or  $\rho < \alpha < 0$ . In those cases, the bond price increases as consumption  $\tilde{c}_1$  and the market return  $\tilde{M}_0$  become more correlated.<sup>12</sup> Otherwise, the bond price falls in response to an increase in correlation.

### 7. CONCLUDING REMARKS

The intertemporal utility functions we have formulated have three very appealing features: (1) intertemporal substitution and risk aversion are disentangled; (2) they integrate atemporal non-expected utility theories into a temporal framework; and (3) they generate implications for the temporal behavior of consumption and asset returns. Moreover, these implications may be investigated empirically by existing econometric techniques as demonstrated in Epstein and Zin (1989).

Some empirical work is done in the latter paper but further empirical investigation, exploring alternative data sets and functional forms, would be worthwhile. A promising application on the theoretical front is to recursive dynamic GE modelling (Sargent (1987)). For example, we have already mentioned the need to integrate our model into a general equilibrium framework such as Lucas' (1978). Because of the inseparability of substitution and risk aversion in his expected utility model, Lucas is unable to provide a clear interpretation for some of his comparative statics results. Our utility functions should clarify those results and thus provide a clearer understanding of the determinants of asset prices. (See Epstein (1988) for such a comparative statics analysis in a stochastic pure endowment economy where endowments are i.i.d.). In addition, the separation which they provide should make them useful in exploring the role played by differences in risk aversion in influencing the distribution of wealth across agents. Such an investigation would complement existing theories of distribution that are based on differences in time preference (Epstein (1987)).

In these and other theoretical developments, the specific functional form (3.5) for the aggregator could be useful in providing some initial insights. Indeed many of the early multiperiod expected utility models of consumption/portfolio behavior are based on the homogeneous parameterization (3.6). But it would clearly be desirable to generalize to a larger class of recursive utility functions. It is hoped that several elements of this paper, such as the proofs of Theorems 3.1 and 5.1

<sup>12</sup> A notion of "greater correlation" that suffices here is described in Epstein and Tanny (1980).

and especially our formalization of the space of infinite horizon temporal lotteries, will be useful in developing such generalizations.

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APPENDIX 1

In this appendix we prove Theorems 2.1 and 2.2. The notation of Section 2 is adopted.

LEMMA A1.1: *The function  $f$  defined in (2.7) is continuous.*

PROOF: By Parthasarathy (1967, Theorem 6.1, p. 40) it suffices to show that

$$m_n \rightarrow m \Rightarrow f(m_n)(B) \rightarrow f(m)(B)$$

$\forall B \in \mathcal{B}(R_+^{\infty})$  such that  $f(m)(\partial B) = 0$ . But the latter condition implies  $\nu\{y \in R_+^{\infty}: (c_0, y) \in \partial B\} = 0$  a.e.  $[m]$  and hence (Billingsley (1968, Theorem 5.5)) that  $T_B$  is continuous a.e.  $[m]$ . By Billingsley (1968, Theorem 5.1),  $E_{m_n}T_B \rightarrow E_mT_B$  as desired. Q.E.D.

LEMMA A1.2: *For all  $t \geq 1$  the functions  $f_t$  and  $g_t$  defined in (2.8) are continuous and satisfy (2.9).*

PROOF: Continuity follows from Lemma A1.1 and Billingsley (1968, p. 29). Condition (2.9) is readily verified. Q.E.D.

Recall the projection maps  $\pi_t$  defined in Section 2.

LEMMA A1.3: (a) *For each  $t \geq 1$  and  $B_t \in \mathcal{B}_t$ ,  $\pi_t^{-1}(B_t) = \pi_{t+1}^{-1}(g_t^{-1}(B_t))$ .* (b)  $\pi_t^{-1}(B_t) \subset \pi_{t+1}^{-1}(B_{t+1})$ ,  $t \geq 1$ . (c)  $\bigcup_1^{\infty} \pi_t^{-1}(B_t)$  *is an algebra.* (d)  $\mathcal{B} = \sigma(\bigcup_1^{\infty} \pi_t^{-1}(B_t)) \equiv$  *the smallest  $\sigma$ -algebra containing  $\bigcup_1^{\infty} \pi_t^{-1}(B_t)$ .*

PROOF: (a)–(c) are evident. (d) follows from Parthasarathy (Theorem 1.10, p. 6). Q.E.D.

PROOF OF THEOREM 2.1: If  $d = (d_1, \dots, d_t, \dots) \in D$ ,  $d_t = (c_0, m_t)$ ,  $\exists$  unique  $m \in M(D)$  such that

$$(A1.1) \quad m(\pi_t^{-1}B_t) = m_{t+1}(B_t) \quad \forall B_t \in \mathcal{B}_t, \quad t \geq 1.$$

(To see this, argue as follows: Use (A1.1) to define  $m$  on  $\bigcup_1^{\infty} \pi_t^{-1}(B_t)$ .  $m$  is well-defined and countably additive in the latter algebra by Lemma A1.3. By part (d) of that lemma and by the extension theorem (Billingsley (1979, Theorem 3.1))  $m$  can be extended uniquely to an element of  $M(D)$ .)

Define  $\Theta: D \rightarrow R_+ \times M(D)$  by setting  $\Theta(d) = (c_0, m)$ .

$\Theta$  is one-to-one: If  $\Theta(d) = \Theta(d')$ , then  $c_0 = c'_0$  and by (A1.1)  $m_t = m'_t \forall t \geq 2$ . But  $m_1 = f(m_2) = f(m'_2) = m'_1$  and so  $m_t = m'_t \forall t \geq 1$ . Thus  $d = d'$ .

$\Theta$  maps  $D$  onto  $R_+ \times M(D)$ : obvious.

$\Theta$  is open: follows from the nature of weak convergence (Parthasarathy (Theorem 6.1, p. 40)).

$\Theta$  is continuous: If  $d^n \rightarrow d$ , then  $m_t^n \rightarrow m_t \forall t \geq 1$  and so by (A1.1)  $m^n(A) \rightarrow m(A)$  for any  $A \in \bigcup_1^{\infty} \pi_t^{-1}(B_t)$  such that  $m(\partial A) = 0$ .

But the algebra  $\bigcup_1^\infty \pi^{-1}(B_i)$  is convergence determining in the sense of Billingsley (1968, exercise 7, p. 22) and thus  $m^n \rightarrow m$  in  $M(D)$ .

This proves that  $\Theta$  is a homeomorphism.

$D$  is a metric space by construction. It is also a closed subspace of the separable space  $X_1^\infty D$ , by the continuity of the  $g_t$ 's. Thus  $D$  is separable. Finally, the desired denseness is immediate. *Q.E.D.*

PROOF OF THEOREM 2.2: One can show using Billingsley (1968, Theorems 6.2 and 6.4) that  $D(b; l)$  from (2.5) is a compact metric space and hence separable. Moreover,  $D(b) = \bigcup_l D(b; l)$  and so  $D(b)$  is separable.

It remains to prove the asserted homeomorphism. Let  $\Theta$  be the map constructed in the proof of Theorem 2.1. We need to show that

$$\Theta(D(b)) \subseteq R_+ \times M(D(b)),$$

or equivalently, that  $m \in M(D)$  defined in (A1.1) satisfies  $m(D(b)) = 1$ .

Let  $d = (d_1, \dots, d_t, \dots) \in D(b)$ ,  $d_t = (c_0, m_t) \forall t$ . It suffices to show that

$$(A1.2) \quad m_{t+1}(D_t(b)) = 1 \quad \forall t \geq 1.$$

Since  $d \in D(b)$ , we know that

$$(A1.3) \quad m_1 \in \bigcup_l M(Y(b; l)) \quad \text{and} \quad f_t(m_{t+1}) = m_t \quad \forall t \geq 1.$$

Prove the  $t = 1$  version of (A1.2), i.e.,

$$(A1.4) \quad m_2(R_+ \times M^b(R_+^\infty)) = 1.$$

Of course,  $m_2 \in M(R_+ \times M(R_+^\infty))$  is given. By (A1.3) and the definition (2.7) of  $f$ , there exists  $\bar{l} \geq 0$  such that

$$\begin{aligned} f(m_2)(Y(b; \bar{l})) &= 1 \Rightarrow \\ \nu \{ y \in R_+^\infty : (c, y) \in Y(b; \bar{l}) \} &= 1 \quad \text{a.e. } [m_2] \Rightarrow \\ m_2 \{ (c, \nu) \in R_+ \times M(R_+^\infty) : \nu \{ y \in R_+^\infty : (c, y) \in Y(b; \bar{l}) \} = 1 \} &= 1 \Rightarrow \\ m_2 \{ (c, \nu) \in R_+ \times M(R_+^\infty) : \nu(Y(b; \bar{l})) = 1 \} &= 1 \Rightarrow \\ m_2(R_+ \times M(Y(b; \bar{l}))) &= 1 \Rightarrow (A1.4). \end{aligned}$$

In a similar fashion (A1.2) can be proven by induction.

Finally, note that the subspace  $\hat{M}(D(b))$  in the statement of the theorem satisfies

$$\Theta(D(b)) = R_+ \times \hat{M}(D(b)). \qquad \qquad \qquad \text{Q.E.D.}$$

To clarify the difference between  $\hat{M}(D(b))$  and  $M(D(b))$ , consider the following: Let  $D_0(b) \equiv \{(c_0, y) : c_0 > 0, y \in \bigcup_{l>0} Y(b; l)\}$ , which can be identified with a subspace of  $D(b)$ . Then a given  $m \in M(D_0(b)) \subset M(D(b))$  will also lie in  $\hat{M}(D(b))$  if and only if  $\exists l > 0$  for which the support of  $m$  lies in  $Y(b; l)$ , i.e.,  $m \in M(Y(b; l))$ . That will not generally be the case for the same reason that  $\bigcup_{l>0} M(Y(b; l))$  is a proper subset of  $M(\bigcup_{l>0} Y(b; l))$ .

APPENDIX 2

Two lemmas are provided here. The first is used in the proof of Theorem 3.1 and the second establishes that the Chew-Dekel mean value (3.10) satisfies MV.1.

LEMMA A2.1: *Let  $X$  be a metric space and  $m_n \rightarrow m$  in  $M(X)$ . Then  $\limsup \int f dm_n \leq \int f dm$  for all  $f : X \rightarrow R$  u.s.c. and bounded above.*

PROOF: When  $f$  is bounded below, see Billingsley (1968, exercise 7, p. 17). In general, for each integer  $K < 0$ , define

$$f^K(x) \equiv \max\{f(x), K\}.$$

Then  $f^K$  is bounded and u.s.c. Thus (by above)  $\phi^K: M(X) \rightarrow R$  is u.s.c. where  $\phi^K(p) \equiv \int f^K dp$ . Since  $f^K(x) \downarrow_K f(x)$ ,  $\phi(p) \equiv \int f dp = \inf \phi^K(p)$  and so  $\phi$  is u.s.c. Q.E.D.

LEMMA A2.2: *The implicit weighted functional  $\mu$  in (3.10) satisfies MV.1.*

PROOF: (a) Let  $\mu_n = \mu(p_n)$ ,  $\mu = \mu(p)$ . The  $\mu_n$ 's and  $\mu$  all lie in  $[0, a]$ . Suppose (for some subsequence)  $\mu_n \rightarrow \delta$ . Then since  $F$  is uniformly continuous on  $[0, 1]^2$ ,

$$\int |F(\cdot, \mu_n) - F(\cdot, \delta)| dp_n < \epsilon \quad \forall n > N(\epsilon).$$

Since  $F(x, z)$  is increasing in  $x$ ,

$$\int F(\cdot, \delta) dp_n \rightarrow \int F(\cdot, \delta) dp.$$

It follows that  $0 = \int F(\cdot, \mu_n) dp_n \rightarrow \int F(\cdot, \delta) dp$ , which implies  $0 = \mu_n \rightarrow \int F(\cdot, \delta) dp$ . But then  $\delta = \mu$  since  $F(x, z)$  is decreasing in  $z$ .

Similarly for (b). Q.E.D.

### APPENDIX 3

We prove Theorem 3.1 regarding the existence of recursive utility functions.

Denote by  $S^+(D(b))$  the set of functions from  $D(b)$  into  $R_+$ . Let  $h \in S^+(D(b))$  be strictly positive and define

$$S_h^+(D(b)) \equiv \{v \in S^+(D(b)) : \|v\|_h \equiv \sup v(d)/h(d) < \infty\}.$$

With the norm  $\|\cdot\|_h$ ,  $S_h^+(D(b))$  is a complete metric space. A transformation  $T: S_h^+(D(b)) \rightarrow S_h^+(D(b))$  is a strict contraction if  $\|Tv - Tu\|_h \leq \Theta \|v - u\|_h$  with  $\Theta < 1$ . Every strict contraction on a complete metric space has a unique fixed point. The following is an immediate corollary which is adapted from Boyd.

WEIGHTED CONTRACTION MAPPING THEOREM (WCMT): *Let  $T: S_h^+(D(b)) \rightarrow S^+(D(b))$  be such that (WCM.1)  $u \leq v \Rightarrow T(u) \leq T(v)$ ; (WCM.2)  $T(0) \in S_h^+(D)$ ; (WCM.3)  $T(u + Ah) \leq T(u) + \Theta Ah$  for some constant  $\Theta < 1$  and  $\forall A > 0$ . Then  $T$  has a unique fixed point  $v^*$ . Moreover,  $T^N 0 \rightarrow v^*$  in  $S_h^+(D)$ .*

We are able to base the following proof on WCMT in the KP case when  $\alpha$  and  $\rho$  have identical signs. An advantage of such a proof is that it leads to the uniqueness of the solution to (3.4) and also to stronger continuity properties for  $V$  than described in the theorem. Moreover, it facilitates the proof of existence of optimal plans. But we could not find a way to apply WCMT to the remaining cases for the KP functional or to more general specifications for  $\mu$ . Thus we use it below only where absolutely necessary, namely in Case 1, and otherwise we present a shorter, simpler argument.

PROOF OF THEOREM 3.1: CASE 1: KP functional,  $0 < \rho \leq \alpha = 1$ ,  $\beta b^\rho < 1$ . Define  $h: D(b) \rightarrow R_{++}$  by

$$h(d) \equiv \left[ 1 + c_0 + E_{m_1} \sum_1^\infty \lambda^i (\tilde{c}_i/b^i) \right]^p,$$

where  $\lambda \in (0, 1)$  is to be determined below and  $d = (c_0, m)$ ,  $m_1 = f(P_2 m)$ . (See (2.3) or (2.7) and (2.11);  $m_1$  is the "atemporal" probability measure on future consumption induced by  $m$ .) The above expected value exists since  $\sum_0^\infty \lambda^i (c_i/b^i)$  is bounded on each  $Y(b; l)$  and since  $m_1 \in \bigcup_{l>0} M(Y(b; l))$ .

Rewrite (3.8) in terms of  $U \equiv V^\rho/\rho$  as follows:

$$(A3.1) \quad \begin{aligned} U(c_0, m) &= H\left(c_0, [E_m(\rho U(\cdot))^\gamma]^{1/\gamma}/\rho\right), \\ H(c_0, z) &\equiv \frac{c_0^\rho}{\rho} + \beta z, \quad \gamma \equiv 1/\rho \geq 1. \end{aligned}$$

Use (A3.1) to define the transformation  $T: S_h^+(D(b)) \rightarrow S^+(D(b))$ , where

$$Tu(c_0, m) \equiv H\left(c_0, [E_m u^\gamma(\cdot)]^{1/\gamma}\right).$$

(Note that

$$[E_m u^\gamma(\cdot)]^{1/\gamma} \leq \|u\|_h \cdot [E_m h^\gamma(\cdot)]^{1/\gamma}$$

if  $u \in S_h^+(D(b))$ ). Moreover,

$$E_m h^\gamma(\cdot) = 1 + c_0 + E_{m_1} \sum_1^\infty \lambda \tilde{c}_i / b^i$$

which is finite for the same reasons provided above for the finiteness of  $h(d)$ .)

We wish to prove that  $T$  has a fixed point and so we verify the conditions of WCMT. The first two conditions are immediate. For the third, note that

$$\begin{aligned} T(u + Ah)(c_0, m) &= H\left(c_0, [E_m(u(\cdot) + Ah(\cdot))^\gamma]^{1/\gamma}\right) \\ &\leq H\left(c_0, [E_m u^\gamma(\cdot)]^{1/\gamma}\right) + \beta A [E_m h^\gamma(\cdot)]^{1/\gamma} \\ &\hspace{15em} \text{(by Minkowski's inequality and } \gamma \geq 1) \\ &= Tu(c_0, m) + \beta A [E_m g(\cdot)]^{1/\gamma} \quad (g \equiv h^\gamma) \\ &\leq Tu(c_0, m) + \beta \frac{b^\rho}{\lambda^\rho} A [E_m g(c_0, \cdot)]^{1/\gamma} \quad (\lambda^\rho g(d) \leq b^\rho g(c_0, d)) \\ &= Tu(c_0, m) + \beta \frac{b^\rho}{\lambda^\rho} A [g(c_0, m)]^{1/\gamma} \\ &\hspace{10em} (g(c, \cdot) \text{ satisfies the independence axiom on } M(D(b))) \\ &= Tu(c, m) + \beta b^\rho Ah(c_0, m) / \lambda^\rho. \end{aligned}$$

Thus WCM.3 is satisfied with  $\Theta = \beta b^\rho / \lambda^\rho$  if  $\lambda$  is any number such that  $\beta b^\rho < \lambda^\rho$ .

By WCMT,  $T^N 0 \rightarrow U = V^\rho/\rho$  in the  $\|\cdot\|_h$  topology on  $S_h^+(D(b))$ . Moreover,  $h$  is bounded on any  $D(b; l)$ . Thus  $\sup\{|T^N 0(d) - U(d)|: d \in D(b; l)\} \rightarrow 0$  as  $N \rightarrow \infty$ . Each  $T^N 0(\cdot)$  is continuous on  $D(b; l)$ . Thus  $U$  and  $V$  are continuous there.

It can be shown that  $D(b; l)$  is compact. Thus

$$(A3.2) \quad \max\{V(d): d \in D(b; l)\} < \infty,$$

a fact which is used below.

CASE 2: General  $\mu, \rho > 0, \beta b^\rho < 1$ . Refer to the previous case and let  $v^*$  be the corresponding utility functional which satisfies

$$(A3.3) \quad v^*(c_0, m) = W(c_0, E_m v^*(\cdot)).$$

Define  $T: S_h^+(D(b)) \rightarrow S^+(D(b))$  by

$$Tv(c_0, m) = W(c_0, \mu(v[m])).$$

(See the notation introduced in (3.3).) Then  $T^N v^* \geq 0$  so the sequence  $\{T^N v^*(c_0, m)\}$  is bounded

below. It is also declining in  $N$ :

$$\begin{aligned} T^N v^*(c_0, m) &\leq W(c_0, E_m v^*(\cdot)) \quad (\text{since } \mu \text{ is risk averse}) \\ &= v^*(c_0, m) \quad (\text{by (A3.3)}). \end{aligned}$$

By induction,  $T^N v^*(c_0, m) \leq T^{N-1} v^*(c_0, m) \forall N \geq 1$ . Define  $V(c_0, m) \equiv \lim T^N v^*(c_0, m)$ .

It remains to show that  $V$  solves (3.4). Since

$$T^N v^*(c_0, m) = W(c_0, \mu(T^{N-1} v^*[m]))$$

and the left side converges to  $V(c_0, m)$ , it suffices to show that

$$\mu(T^{N-1} v^*[m]) \rightarrow \mu(V[m]).$$

But this is true by (A2.2), MV.1, and the monotone convergence theorem (Billingsley (1979, p. 179)), the latter of which implies

$$\int f(T^{N-1} v^*(\cdot)) dm(\cdot) \rightarrow \int f(V(\cdot)) dm(\cdot) \quad \text{for all increasing } f.$$

Since  $v^*$  was shown to be continuous, on  $D(b; l)$  each  $T^N v^*$  is continuous there by induction. Thus, as the infimum of continuous functions,  $V$  is u.s.c.

CASE 3: General  $\mu, \rho < 0$ . Define the sequence  $\{u^n\}$  of nonpositive extended real valued functions on  $D(\infty)$  inductively:

$$\begin{aligned} u^1(c_0, m) &= c_0^0 / \rho, \\ u^n(c_0, m) &= c_0^0 / \rho + (B/\rho) \mu^\rho \left( (\rho u^{n-1})^{1/\rho} [m] \right), \quad n \geq 2. \end{aligned}$$

(Recall that  $(\rho u^{n-1})^{1/\rho} [m]$  denotes the distribution of  $(\rho u^{n-1}(\cdot))^{1/\rho}$  induced by  $m$ . Define  $\mu = \infty$  if there is positive probability that  $(\rho u^{n-1})^{1/\rho} = \infty$ .) Then  $u^n \leq u^{n-1}$ . So the extended real valued function  $U, U(d) \equiv \lim u^n(d)$ , is well-defined. Let  $V$  solve  $V^\rho = \rho U$ . That  $V$  solves (3.4) is demonstrated as in Case 2.

Each  $u^n$  is u.s.c. on  $D(b; l)$  by recursive applications of Lemma A2.1 and MV.1(b). Thus the same is true of the infimum. Q.E.D.

#### APPENDIX 4

This appendix defines the map  $e$  employed in (5.2) and then proves Theorem 5.1.

Denote by  $F_{t+}$  the  $\sigma$ -algebra generated by  $(\tilde{r}_t, \tilde{z}_t)_{t'}^\infty$ . We assume that associated with the stochastic process  $(\tilde{r}_t, \tilde{z}_t)_{t'}^\infty$  are conditional probability measures  $\Psi(\cdot/I)$  satisfying, for each  $t \geq 0, B \in F_{t+}$  and  $I_t \in I_t$ ,

(A4.1)  $\Psi(\cdot/I_t)$  is a probability measure on  $(\Omega, F_{t+})$  and

(A4.2)  $\Psi(B/\cdot)$  is  $F_{t-}$ -measurable.

The fact that  $\Psi$  does not vary with  $t$  reflects the assumed stationarity of the process.

Each plan, in conjunction with the wealth accumulation equation, defines a random variable (r.v.)  $\tilde{y}: \Omega \rightarrow Y(b) \equiv \bigcup_{l>0} Y(b; l)$  such that  $\tilde{y} = (\tilde{c}_0, \dots, \tilde{c}_t, \dots)$  and  $\tilde{c}_t$  is measurable with respect to  $F_{t-}$ , the  $\sigma$ -algebra generated by  $(\tilde{r}_t, \tilde{z}_t)_{t'}^{t-1}$ . For given  $I_0$  and  $x_0, \tilde{c}_0$  is nonstochastic and is written simply  $c_0$ . Feasibility implies that

(A4.3)  $\sum_0^\infty \tilde{c}_t / \bar{r}^t \leq x_0 \quad \text{a.e. } [\Psi].$

For given  $I_0$  and  $x_0$ , we associate with each feasible r.v. consumption program  $\tilde{y}$  a temporal lottery  $d = (d_1, \dots, d_t, \dots), d_t = (c_0, m_t), m_t \in M(D_{t-1})$  for  $t > 2$  and  $m_1 \in M(Y(\tilde{r}; x_0))$ . Each such  $d \in D(\tilde{r}; x_0) \subset D(b; x_0) \subset D(b) \forall b > \bar{r}$ . Roughly speaking, the association from  $\tilde{y}$  to  $d$  is as follows. The measure  $m_1$  is that implied by the function  $\tilde{y}$  and the probability distribution for  $(\tilde{r}_t, \tilde{z}_t)_{t'}^\infty$  by treating all of the latter as though they were realized at  $t = 1$ . In this way one obtains a consumption lottery in which all uncertainty is resolved by  $t = 1$ , i.e.,  $m_1 \in M(Y(\tilde{r}; x_0)) \subset M(R_+^\infty)$ . Similarly, if we

recognize that  $(\tilde{r}_0, \tilde{z}_0)$  is realized at  $t = 1$  and regard  $(\tilde{r}_t, \tilde{z}_t)$ ,  $t \geq 1$ , as being realized at  $t = 2$ , we obtain  $m_2 \in M(R_+ \times M(R_+^\infty)) = M(D_1)$ ; and so on for the other  $m_t$ 's. These measures differ only in the way the (common) uncertainty regarding future consumption is resolved through time. Thus an element  $d \in D$  is obtained.

Formally, define  $m_1$  by

$$m_1(Q/I_0) \equiv \Psi(\{\omega \in \Omega: \tilde{y}(\omega) \in Q\}/I_0), \quad Q \in \mathcal{B}(R_+^\infty).$$

By (A4.3),  $m_1 \in M(Y(\tilde{r}; x_0))$ .

Suppose that we have constructed  $m_t(\cdot/I_t) \in M(D_{t-1})$  for some  $t \geq 1$ . (Actually, for  $t = 1$ ,  $m_1(\cdot/I_0) \in M(R_+^\infty)$ .) Then  $m_{t+1}(\cdot/I)$  is defined by

$$m_{t+1}(Q/I) \equiv \Psi(\{\omega \in \Omega: (\tilde{c}_1(\omega), m_t(\cdot/(I, \omega_0))) \in Q\}/I).$$

Here,  $\omega_0 = (r_0, z_0)$  and  $(I, \omega_0)$  represents realized history at  $t = 1$ . In this way

$$d = ((c_0, m_1), \dots, (c_0, m_t), \dots) \in D(\tilde{r}; x_0)$$

is constructed.

The desired map  $e: FP \rightarrow D(\tilde{r}; x_0)$  is the composition: plan  $\rightarrow$  r.v. consumption program  $\rightarrow$  temporal lottery.

PROOF OF THEOREM 5.1: Define Cases 1–3 as in the proof of Theorem 3.1. In those cases where a contraction mapping is applicable, existence follows in the customary fashion. For the remaining cases we employ parallel arguments to those in the proof of Theorem 3.1.

An alternative route would be to prove the compactness of  $e(FP)$  and apply the u.s.c. of utility. But we were able to show only that the closure of  $e(FP)$  is compact and not that  $e(FP)$  is closed. Thus only the finiteness of the supremum in (5.2) could be inferred in this way.

CASE 1: Since WCMT applies, the existence of a maximum follows from Denardo (1967). That it can be achieved by a stationary plan follows from Sobel (1975). Clearly,  $J > 0$  since  $c_0 = x_0$  is feasible and yields utility  $= x_0$ .

CASE 2: Let  $v^*$  and  $J^*$  be the utility and value functions respectively for Case 1. Because of the homotheticity of preferences we may write  $J^*(I, x) = A^*(I)x$ .  $A^*(\cdot)$  is bounded above since  $v^*$  is bounded on  $D(\tilde{r}; x_0)$  (see (A3.2)). Moreover, we have just seen that  $J^* \geq x_0$ . We conclude that

$$(A4.4) \quad 1 \leq A^*(\cdot) \leq a < \infty \quad \text{on dom } A^*.$$

Let

$$(A4.5) \quad (FA^*)(I) \equiv \max_{a \in [0,1], w \in S^K} W(a, (1-a)\mu [D_{A^*(\tilde{I}_1)w\tilde{r}_0}/I]),$$

and  $(F^N A^*)(\cdot) \equiv F(F^{N-1} A^*)(\cdot)$  for  $N \geq 2$ . (Apply (A4.4) and the Lebesgue dominated convergence theorem (Billingsley (1979, p. 180)) to deduce that the objective function in (A4.5) is u.s.c. in  $w$  and ultimately that the maximum exists.) Since  $\mu$  is risk averse,

$$(FA^*)(I) \leq \max_{a \in [0,1], w \in S^K} W(a, (1-a)E[A^*(\tilde{I}_1)w\tilde{r}_0/I]),$$

which in turn equals  $A^*(I_0)$  by the definition of  $A^*$ . Thus

$$(FA^*)(I) \leq A^*(I).$$

By induction, prove that  $F^N A^*(I) \downarrow$  in  $N$ . Since the sequence is bounded below by 0,

$$A(I) \equiv \lim_{N \rightarrow \infty} F^N A^*(I)$$

is well-defined.

We now show that  $A(I_0)x_0$  is at least as large as the supremum in (5.2): For any feasible lottery  $(c_0, m)$ ,  $v^*(c_0, m) \leq A^*(I_0)x_0$ . By induction,  $T^N v^*(c_0, m) \leq F^N A^*(I_0)x_0$ , where  $T$  is defined following (A3.3). Thus taking limits yields  $V(c_0, m) \leq A(I_0)x_0$ .

We need to show that the supremum in (5.2) is attained. For any plan  $h = (h_0, \dots, h_t, \dots)$  denote by  $e(h; I_0, x_0)$  the temporal lottery in  $D(b)$ , where  $e$  is defined above and its dependence on  $I_0$  and  $x_0$  is made explicit. Then  $V(e(h; I_0, x_0))$  is the utility of the plan  $h$  given initial conditions  $(I_0, x_0)$ . By

homogeneity,

$$(A4.6) \quad V(e(h; I_0, x_0)) = H(I_0)x_0.$$

Let  $h^{**}$  be any (not necessarily feasible) plan such that  $e(h^{**}; I_0, x_0) \in D(b; l)$  for some  $l > 0$  and

$$V(e(h^{**}; I_0, x_0)) = A^*(I_0)x_0.$$

(Note that  $h^{**}$  is not the optimal plan of Case 1 since the utility function  $V$  is not the utility function of Case 1.) For example,  $h^{**}$  could be the plan in which  $c_0 = A^*(I_0)x_0$  and consumption is set equal to zero in all subsequent periods. Denote by  $(a_0(I; H), w_0(I; H))$  the solution to (A4.5) when  $A^*(\cdot)$  in the maximand is replaced by  $H(\cdot)$ .

Define the transformation of plans  $G$  by

$$G(h_0, \dots, h_t, \dots) = (\hat{h}_0, h_0, \dots, h_t, \dots),$$

where  $\hat{h}_0(I, x) = (xa_0(I; H), w_0(I; H))$  and  $H$  is defined in (A4.6). Consider the sequence of plans  $\{G^N(h^{**}); N \geq 1\}$ . Following is a list of facts regarding this sequence. Write  $G^N(h^{**}) = (h_0^N, \dots, h_t^N, \dots)$  and  $h_t^N = (hc_t^N, hp_t^N)$ .

FACT 1:  $V(e(G^N(h^{**}); I_0, x_0)) = F^N A^*(I_0) \cdot x_0 \downarrow A(I_0)x_0$ .

FACT 2: For each  $t$  and  $N \geq t + 1$ ,  $hc_t^N(\cdot)$  is increasing in  $N$  if  $\rho > 0$  and decreasing if  $\rho < 0$ . In either case,  $hc_t(\cdot) \equiv \lim hc_t^N(\cdot)$  exists.

FACT 3: For each  $t, I$  and  $N \geq t + 1$ ,

$$hp_t^N(I) = w_0(I; F^{N-t-1}A^*(\cdot)) \xrightarrow{N \rightarrow \infty} w_0(I; A(\cdot)) \equiv hp_t(I).$$

FACT 4: The plan  $\hat{h} = ((hc_0, hp_0), \dots, (hc_t, hp_t), \dots)$  is feasible and stationary.

FACT 5:  $e(G^N(h^{**}); I_0, x_0) \rightarrow e(\hat{h}; I_0, x_0)$  in  $D(b; l)$ .

(To prove the latter use the fact that a.e. pointwise convergence of a sequence of random variables implies the weak convergence of the corresponding sequence of probability measures.)

Combine Facts 1, 5, and the u.s.c. of utility to conclude that

$$V(e(\hat{h}; I_0, x_0)) \geq A(I_0)x_0.$$

But  $A(I_0)x_0$  is no smaller than the utility supremum over feasible paths. Thus by Fact 4, equality must prevail and the supremum is achieved.

The positivity of  $J$  is clear as in Case 1.

The argument for Case 3 is similar to that for Case 2. To prove positivity when  $\rho < 0$ , note that there exists  $\lambda \geq 1$  and a sufficiently small  $\Theta > 0$  such that the plan of consuming  $\Theta \cdot (t/\lambda)^t$  in each  $t$  (and making any portfolio decisions whatsoever) is feasible. This plan yields utility

$$\Theta \cdot \left[ \sum_{t=0}^{\infty} (\beta(t/\lambda)^\rho)^t \right]^{1/\rho} > 0$$

if  $\beta r^\rho < \lambda^\rho \leq 1$ .

*Q.E.D.*

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