JOURNAL OF ECONOMIC THEORY 59, 183–188 (1993)

A Unifying Approach to Axiomatic Non-expected Utility Theories: Correction and Comment

CHEW SOO HONG

Department of Economics, University of California, Irvine, California, 92717

LARRY G. EPSTEIN

Department of Economics, University of Toronto, Toronto, Ontario M5S 1A1, Canada

AND

PETER WAKKER

Department of Mathematical Psychology, University of Nijmegen, P.O. Box 9104, 6500 HE Nijmegen, The Netherlands

Received July 19, 1991; revised October 22, 1991

Chew and Epstein attempted to provide a unifying axiomatic framework for a number of generalizations of expected utility theory. Wakker pointed out that Theorem A, on which the central unifying proposition is based, is false. In this note, we apply Segal's result to prove that Theorem 2 is nevertheless valid with minor modifications. *Journal of Economic Literature* Classification Number: D1. © 1993 Academic Press, Inc.

1. INTRODUCTION

In Chew and Epstein [2], two of us attempted to provide a unifying axiomatic framework for a number of generalizations of expected utility theory. Our approach was to weaken the independence axiom by requiring its associated separability to hold only on suitable subsets of the domain of state-contingent outcomes. Then we invoked Theorem A of Appendix 3 which extended the additive utility representation results of Debreu [3] and Gorman [5] from the case of full Cartesian products to more general domains. Theorem A claims, given a continuous and strictly increasing preference ordering, that the Debreu–Gorman additive representation can be extended to connected subsets of $\mathbb{R}^N$ when indifference sets are

0022-0531/93 $5.00

connected. Unfortunately, as was first pointed out by Wakker [7], Theorem A is false.

Segal [6] proves that Theorem A holds under the additional restrictions that the domain is open, and its intersections with translations of coordinate hyperplanes are connected. Here, we use Segal's result to prove that the central unifying proposition [2, Theorem 2] is nevertheless valid with minor modifications. Wakker [8, 9] provided earlier additive representation theorems for rank-ordered subsets of a Cartesian product. These papers also provided examples to demonstrate that the question of existence of additive representations is a delicate one when the domain is not a full Cartesian product. In addition, these papers contain references to the literature where errors similar to Theorem A occur.

2. The Unifying Representation

Let the set of prizes X be an open interval and denote by $D^c(X)$ the set of c.d.f.'s with compact support in X, i.e., $D^c(X) = \bigcup \{D(K) : K \subset X$ is a compact interval$\}$. Modify Theorem 2 by requiring that the axioms for $\succcurlyeq$ are satisfied and that the desired representations apply on $D^c(X)$, rather than on $D(X)$. Difficulties with utility being driven to infinity at the boundary (see [9, Example 3.8]) are avoided in this way.

We provide the additional arguments needed to show that in (the modified) Theorem 2, the axioms imply the desired representation. Notation is as in [2] except that we often abbreviate $\sum_{i=1}^{N} (1/N)\delta_{x_i}$ and $\sum_{i=1}^{N} (1/N)\delta_{y_i}$ by $(x_1, ..., x_N)$ and $(y_1, ..., y_N)$. We provide details corresponding to Case (iv) of the proof of Theorem 2. The other cases are more elementary and analogous.

Let $\mathscr{I}_\uparrow(\bar{x}) = \mathscr{I}(\bar{x}) \cap X_\uparrow^N$. The superscript "0" applied to a subset of X^N refers to its restriction to exclude elements which have non-distinct components. For example, $\mathscr{I}_\uparrow^0(\bar{x}) = \{x \in \mathscr{I}_\uparrow(\bar{x})$: the components of x are distinct$\}$. Let V be a continuous and increasing utility function that represents $\succcurlyeq$ on $D^c(X)$.

Lemma. *For $\bar{x} \in X^N$, $\mathscr{I}_\uparrow^0(\bar{x})$ is arc connected.*

Proof. Let $x \neq y \in \mathscr{I}_\uparrow^0(\bar{x})$. We construct an arc from x to y within $\mathscr{I}_\uparrow^0(\bar{x})$ through vectors $x(=x^0), x^1, x^2, ...,$ where each subsequent vector has one or two components more in common with y than its predecessor. Let j be the smallest index for which $x_j > y_j$, i the largest index for which $x_i < y_i$. By $x_{-i,j}v_i,v_j$, we denote x with x_i, x_j replaced by v_i, v_j. Note (for $j > 1$)

that $y_j > y_{j-1} \geqslant x_{j-1}$, and (for $i < N$) that $y_i < y_{i+1} \leqslant x_{i+1}$. Suppose $x_{-i,j} y_i, y_j \succcurlyeq y$ (the case $y \succcurlyeq x_{-i,j} y_i, y_j$ is analogous). Then for each z_j between $x_j > y_j$ there is a unique z_i between $y_i > x_i$ such that $x_{-i,j} z_i, z_j \sim y$. Note that our choices of maximal j and minimal i imply inequalities such as $z_j \geqslant y_j > y_{j-1} \geqslant x_{j-1}$ and $z_i \leqslant y_i < y_{i+1} \leqslant x_{i+1}$, guaranteeing $x_{-i,j} z_i, z_j \in \mathscr{I}_{\uparrow}^{0}(\bar{x})$ (also if $j = i+1$). For $z_j = y_j$ the associated z_i is denoted as y_i'; $y_i' \leqslant y_i$. We define $x^1 := x_{-i,j} y_i', y_j$. Now $x_{-i,j} z_i, z_j$ provides an arc from x to x^1 within $\mathscr{I}_{\uparrow}^{0}(\bar{x})$, where x^1 has at least one component more in common with y than x (the ith component). We construct x^2 from x^1 as we constructed x^1 from x, again obtaining one more component in common with y. After at most N steps the constructed alternative becomes identical to y. Between each subsequent pair of vectors x^k, x^{k+1} there is an arc within $\mathscr{I}_{\uparrow}^{0}(\bar{x})$. These arcs together constitute an arc from x to y. Indeed $\mathscr{I}_{\uparrow}^{0}(\bar{x})$ is arc connected. ∎

Case (iv) IRLU. The ordering $\succcurlyeq_{(x,i)}^{N-1}$ is continuous, strictly increasing, and completely separable on

$$\mathscr{I}_{\uparrow}(\bar{x})_{-i} = \{x \in X_{\uparrow}^{N-1} : \exists z \in X \text{ such that } (x, z) \in \mathscr{I}(\bar{x}) \text{ and } z \text{ is the } i\text{th rank component of } (x, z)\}.$$

Let $\mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i} = \mathscr{I}_{\uparrow}(\bar{x})_{-i} \cap \mathscr{I}_{\uparrow}^{0}(\bar{x})$. To apply Segal's theorem, we must show:

(a) $\mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i}$ is open and connected in $\mathbb{R}^{N-1}$, and

(b) $\forall c \in X$, $\mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i} \cap \{x \in X^{N-1} : x_k = c\}$ is connected, for each $k \neq i$.

Proof of (a). The arc connectedness of $\mathscr{I}_{\uparrow}^{0}(\bar{x})$ implies that its projection $\mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i}$ is connected, using [4, p. 115]. To prove openness, let $x \in \mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i}$, i.e., $x' = (x, z) \sim \bar{x}$ and z is the ith rank component of (x, z). Then $\exists \delta > 0 \ni (x, z+\delta) \succ \bar{x} \succ (x, z-\delta) \Rightarrow \exists \varepsilon > 0 \ni \|y - x\| < \varepsilon r(y, z+\delta) \succ \bar{x} \succ (y, z-\delta) \Rightarrow \exists z' \in X \ni (y, z') \sim x'$. Thus, $\{y \in X^{N-1} : \|y - x\| < \varepsilon\} \subset \mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i}$.

Proof of (b). Path connectedness of

$$S = \mathscr{I}_{\uparrow}^{0}(\bar{x}) \cap \{x \in X^N : x_k = c\}$$

may be proven as in the lemma. Thus its projection $\mathscr{I}_{\uparrow}^{0}(\bar{x})_{-i} \cap \{x \in X^{N-1} : x_k = c\}$ is connected, completing the proof of (b).

Given the additive representation provided by Segal's theorem, we proceed as in [2, pp. 233–234] to derive $\psi : \mathrm{dom}(\psi) \subset X \times [0, 1]^* \times X \to \mathbb{R}$ such that $\psi(x, p, y)$ is continuous in x for each (p, y); $\psi(\cdot, 0, \cdot) \equiv 0$;

$$\psi(x, p, y) - \psi(x, p', y) - \psi(x', p, y) + \psi(x', p', y) > 0 \tag{1}$$

if $p > p'$ and $x > x'$;[1] such that ψ represents $\succcurlyeq$ on $D^c(X)$ in the sense that on $X_\uparrow^N$, for any N,

$$\sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i} \succcurlyeq \sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{y_i} \Leftrightarrow m\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i}\right) \geqslant m\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{y_i}\right),$$

where for $F = \sum_{i=1}^{N}(1/N)\delta_{x_i}$, $m(F)$ is defined by

$$\sum_{i=1}^{N}[\psi(x_i, i/N, m(F)) - \psi(x_i, (i-1)/N, m(F))] = 0.^2$$

Given $y \in X$, define

$$x^*(p, y) = \sup\{z' \in X\colon p\delta_z + (1-p)\delta_{z'} \sim \delta_y;\ \text{for some } z < z'\},$$

and for $x < y$,

$$p^*(x, y) = \sup\{q \in [0, 1]\colon q\delta_x + (1-q)\delta_{z'} \sim \delta_y;\ \text{for some } y < z'\}.$$

Then $\mathrm{dom}(\psi) = \bigcup_{y \in X}\mathrm{dom}(\psi(\cdot, \cdot, y))$, where

$$\mathrm{dom}(\psi(\cdot, \cdot, y)) = \{(x, p) \in X \times [0, 1]^*\colon x < x^*(p, y);\ p < p^*(x, y)\}.$$

We show that $\forall y \in X$, $\psi(\cdot, \cdot, y)$ is continuous on $\mathrm{dom}(\psi(\cdot, \cdot, y))$. Pick $F \in D^c(X)$, $F = \sum_{i=1}^{M} q_i\delta_{x_i}$ such that $y = m(F)$. For $1 < k < N$, $p := \sum_{i=1}^{k} q_i$ is the probability under F of receiving an outcome that is not greater than x_k. Let $j \neq k$. Let $x_j^n \to x_j$, $p^n \to p$, $x_k^n \to x_k$ such that $x_{j-1} < x_j^n < x_{j+1}$, $x_{k-1} < x_k^n < x_{k+1}$, and $F \sim F_n$ for all n, where F_n is derived from F by replacing (x_j, x_k, p) by (x_j^n, x_k^n, p^n). In other words, $F_n = \sum_{i=1}^{M} r_i^n\delta_{x_i^n}$, where $x_i = x_i^n$ except for $i = j, k$, and $\sum_{i=1}^{k} r_i^n = p^n$; $r_i^n = q_i$ except for $i = k, k+1$. Note that a choice of p^n, x_k^n by indifference to F uniquely determines x_j^n. That such sequences x_j^n, p^n, x_k^n, exist is straightforwardly derived from weak continuity. From indifference to F, it follows that

$$\left[\psi\left(x_i^n, \sum^{i} q_j, y\right) - \psi\left(x_i, \sum^{i} q_j, y\right)\right] + [\psi(x^n, p^n, y) - \psi(x, p, y)] = 0.$$

By the continuity of ψ in the first argument, the first bracket approaches 0. Therefore, $\psi(x_k^n, p^n, y) - \psi(x_k, p, y)$ also approaches 0. Thus, $\forall y \in X$, $\psi(\cdot, \cdot, y)$ is continuous on $\mathrm{dom}(\psi(\cdot, \cdot, y))$.

[1] For $p = k/N$ and $p' = j/N$, $\psi(x, p, y) - \psi(x, p', y) - \psi(x', p, y) + \psi(x', p', y)$ is, according the construction, given by $\sum_{i=j+1}^{k}[u_i^N(x, y) - u_i^N(x', y)] > 0$.

[2] Strictly speaking, the representation obtains only for lotteries in $D^c(X)$ which have distinct outcomes. The extension to non-distinct outcomes is accomplished via weak continuity.

For each $y \in X$, we extend $\psi(\cdot, \cdot, y)$ continuously to

$$\{(x, p, y) \in X \times [0, 1] \times X: x \leqslant x^*(p, y);\ p \leqslant p^*(x, y)\}.$$

We further extend $\psi(\cdot, \cdot, y)$ to all of $X \times [0, 1]$ by defining

$$\psi(x, p, y) = \begin{cases} \psi(x^*(p, y), p, y), & x \in (x^*(p, y), \bar{x}), \\ \psi(x, p^*(x, y), y), & p \in (p^*(x, y), 1]. \end{cases}$$

By construction, $\forall y \in X$, $\psi(\cdot, \cdot, y)$ is jointly continuous on $X \times [0, 1]$ and satisfies (1). Therefore, $\psi(\cdot, \cdot, y)$ defines a positive Lebesgue–Stieltjes measure $\lambda(\cdot; y)$ on $X \times [0, 1]$ [1, p. 149, Theorem 12.5]. Let $^*F$ denote the epigraph of F. Then the representation of the ordering on $D^c(X)$ can be expressed as: $\forall F \in D^c(X)$,

$$\lambda(^*F, m(F)) = \lambda(^*\delta_{m(F)}, m(F)).$$

Since $D^e(X)$ is dense in $D(X)$, in order to extend the representation to $D^c(X)$, it suffices to show that

$$\lambda(^*F_n \backslash {}^*F, m(F)) + \lambda(^*F \backslash {}^*F_n, m(F)) \to 0 \tag{2}$$

whenever $\{F_n\} \subset \mathscr{J}(F) \to F$ and $\exists$ compact interval $K \subset X$ such that $\operatorname{supp}(F_n) \subset K$, $\forall n$. Let 1_A denote the indicator function for a set A. Then (2) is implied by the countable additivity of λ if on $K \times [0, 1]$,

$$1_{^*F_n} \to 1_{^*F} \qquad \text{a.e. } [\lambda].$$

The above convergence can fail only for points $(x, p) \in S \cup \operatorname{gr} F$, where

$$\operatorname{gr} F = \{(x, p): p = F(x)\},$$

and

$$S = \{(x, F(x)): x \text{ is a discontinuity point of } F\}.$$

Since F has at most countably many discontinuity points and since ψ is continuous in x, $\lambda(S, m(F)) = 0$. Thus, it remains to prove that $\forall F \in D^c(X)$,

$$\lambda(\operatorname{gr} F, m(F)) = 0. \tag{3}$$

It follows from the continuity of $\psi(\cdot, \cdot, m(F))$ that $\lambda(\operatorname{gr} F, m(F)) = 0$, $\forall F \in D^e(X) \cap \mathscr{J}(F)$. Suppose $\exists F \in D^c(X) \backslash D^e(X)$ such that $\lambda(\operatorname{gr} F, m(F)) > 0$. Pick $(t, q) \in \operatorname{gr} F$ with $q \in (0, 1)$ such that the $\lambda(\cdot, m(F))$ measure of either the portion A of $\operatorname{gr} F$ above (t, q) *or* of the portion B below (t, q) is strictly positive. Without loss of generality, say $\lambda(A, m(F)) > 0$. Let $F' \in D^c(X) \cap \mathscr{J}(F)$ be such that F' coincides with F at and above (t, q) and

supp(F') to the left of t is nonempty and finite. It follows that the $\lambda(\cdot, m(F))$ measure of the portion of gr F' below (t, q) is 0.

We may assume $F' = F$. By weak continuity and monotonicity of $\succcurlyeq$, sequences (G_n) and (H_n) in $D^c(X) \cap \mathscr{J}(F)$ can be obtained that converge in distribution to F such that $d(G_n, H_n) \to 0$ (d is the Prohorov metric), and for all n, G_n and H_n each cross F once at (t, q) with $G_n(H_n)$ strictly below (above) for $x < t$ and with $G_n(H_n)$ strictly above (below) for $x > t$. Let C_n (D_n) be the difference between $^*G_n$ and $^*H_n$ above (below) the point (t, q). Since $G_n \sim H_n \sim F$,

$$\lambda(C_n, m(F)) = \lambda(D_n, m(F)). \tag{4}$$

The difference between B and $\bigcap_{n=1}^{\infty} D_n$ is contained within vertical line segments associated with the finite number of discontinuity points of F, hence has λ measure 0. Therefore, λ being countably additive, $0 = \lambda(B, m(F)) = \lambda(\bigcap_{n=1}^{\infty} D_n, m(F))$, and $\lambda(D_n, m(F)) \to 0$. But for every n, $\lambda(C_n, m(F)) \geqslant \lambda(\text{gr } F, m(F)) > 0$, which yields a contradiction to (4). Thus, (3) has been proved. ∎

To conclude, we point out that Lemma B of [2, Appendix 2], which provides sufficient conditions on ψ for the corresponding certainty equivalent functional m to be continuous, should include an additional assumption to guarantee (3). For this, continuity of $\psi_{12}(\cdot, \cdot, y)$ for all $y \in X$ suffices.

References

1. P. Billingsley, "Probability and Measures," Wiley, New York, 1979.
2. S. H. Chew and L. G. Epstein, A unifying approach to axiomatic non-expected utility theories, *J. Econ. Theory* **49** (1989), 207–240.
3. G. Debreu, Topological methods in cardinal utility theory, *in* "Mathematical Methods in the Social Sciences" (K. Arrow and P. Suppes, Eds.), Stanford Univ. Press, Stanford, 1964.
4. J. Dugundji, "Topology," Allyn and Bacon, Boston, 1966.
5. W. M. Gorman, The structure of utility functions, *Rev. Econ. Stud.* **35** (1968), 367–390.
6. U. Segal, Additive representations on nonconvex subsets, *J. Econ. Theory* **56** (1992), 89–99.
7. P. P. Wakker, "Counterexample to Additive Representation on Non-Full Cartesian Products," Duke Working Paper, 1990.
8. P. P. Wakker, Additive representation on subsets of cartesian products. I. The algebraic approach, *J. Math. Psych.* **35** (1991), 501–531.
9. P. P. Wakker, Additive representation on subsets of cartesian products. II. The topological approach, *J. Math. Econ.*, forthcoming.