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A Unifying Approach to Axiomatic Non-expected Utility Theories*

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This paper unifies the two principal thrusts in the literature on axiomatic theories of transitive preferences which generalize expected utility theory; namely, the betweenness conforming theories and the rank-dependent theories. The unification is achieved in two respects. First, new axiomatizations are provided for the existing theories based on separability restrictions in outcome space. These axiomatizations bring into clear focus both the similarities and the differences between the existing theories. Second, an axiomatization is provided for a new class of preferences which includes existing classes as special cases. *Journal of Economic Literature* Classification Numbers: 022, 026.

1. INTRODUCTION

There is a recent and growing literature on preferences beyond the received expected utility theory. The bulk of these works maintain a continuity requirement. In addition to continuity, one may impose a smoothness requirement to examine whether "expected utility analysis" may be applicable in a "local" sense (Machina [21]). Another direction is to specifically weaken certain properties of expected utility to axiomatically characterize more general preference functionals.

If transitivity is maintained, then there are two discernible approaches within the axiomatic direction.[1] One maintains the betweenness property —a probability mixture of two lotteries is intermediate in preference

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[1] For examples of axiomatic nontransitive theories, see Fishburn [14].

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between the individual lotteries—of expected utility. Recent works in this approach include Chew and MacCrimmon [7], Chew [3, 4], Fishburn [15], Nakamura [22], and Dekel [12]. We refer to these theories as *implicit linear utility* (ILU) theories.

The other may be labeled the rank-dependent or rank-linear utility (RLU) approach. It is distinguished by the rank ordering of outcomes prior to the applying of the representation. Rank-linear theories have been proposed by Quiggin [23], Yaari [26], Segal [25], Chew [5], and Green and Jullien [18]. There is no intersection between the two approaches other than expected utility theory.

Our paper is concerned with the unification of the two approaches. This is accomplished in two respects. First, new axiomatizations are provided for the ILU and RLU theories. The axiomatizations are based principally on separability restrictions in outcome space and on (a variation of) a well known result from demand theory regarding additive utilities [10] and [17].[2] Both ILU and RLU satisfy separability conditions, but on different domains. Thus both the similarities and the differences between the two theories are put into clear focus. In contrast, such a focus is not provided by the existing disparate axiomatizations.

A second contribution of the paper is the axiomatization of a class of continuous preferences called implicit rank-linear utility (IRLU). Since this class includes both ILU and RLU as special cases, it provides a unifying framework for these existing theories. (See Fig. 3, which will be explained further below.) Moreover, IRLU contains several interesting new classes of preferences which are discussed in varying degrees of detail below.

It is worth emphasizing the "practical" importance of the generality of IRLU. As things stand now, a modeler who wishes to specify a transitive non-expected utility preference ordering (possibly to explain behavioral paradoxes or for some other reason) must choose between the two alternatives of ILU and RLU. Even if he finds some appeal in the axiomatic bases for each, the modeller must make the discrete choice between the two. On the other hand, the IRLU class retains elements of both theories. Thus it provides opportunities for adoption of "intermediate" specifications.

In Section 2, we present some notation, a representation theorem for a continuous utility function on probability distributions, and several examples of non-expected utility theories which have appeared in the literature. Section 3 presents some new utility functionals and corresponding separability axioms. Section 4 presents and discusses the main representation theorem as well as further results on uniqueness and risk aversion. Some new specializations of the implicit rank-linear utility representation are provided in Section 5. Concluding remarks are offered in Section 6.

[2] For a corresponding axiomatization of expected utility theory, see [2].

2. PRELIMINARIES

We adopt the following notation. X is an interval in $\mathbb{R}$. $D(X)=\{F, G, H, ...\}$ denotes the space of c.d.f.'s on X, endowed with the weak convergence topology. A distribution function that is concentrated at a single point $s \in X$ is denoted by δ_s, where

$$\delta_s(t)=\begin{cases}1 & t \geqslant s\\ 0 & t<s.\end{cases} \tag{2.1}$$

We denote by $D^0(X)$ the set of c.d.f.'s having finite supports. Elements of $D^0(X)$ can be written as

$$F \equiv \sum_{i=1}^{n} p_i \delta_{x_i} \qquad (p_i>0), \tag{2.2}$$

where $\text{supp}(F)=\{x_1, ..., x_n\}$.

The following axioms apply to a binary relation $\preccurlyeq$ on $D(X)$.

Axiom O (Ordering). $\preccurlyeq$ is complete and transitive.

Axiom C (Continuity). $\forall F \in D(X)$, $\{G \in D(X): G \preccurlyeq F\}$ and $\{G \in D(X): F \preccurlyeq G\}$ are closed.

The existence of a numerical representation for $\preccurlyeq$ follows from Debreu [11].

THEOREM 1. *There is a continuous utility function $V: D(X) \to \mathbb{R}$ iff $\preccurlyeq$ satisfies Axioms* O *and* C.

Axiom M (Monotonicity). $\forall H \in D(X)$, $p \in (0, 1]$ and $s, t \in X, s<t$ implies

$$p\delta_s+(1-p)\,H \prec p\delta_t+(1-p)\,H.$$

It is known that, on $D^0(X)$, Axiom M is equivalent to monotonicity in the sense of first order stochastic dominance. Therefore, any continuous utility V on $D(X)$ satisfying Axiom M is monotone in the sense of first order stochastic dominance on $D(X)$. In particular, $V(\sum_{i=1}^{N} (1/N)\, \delta_{x_i})$ is continuous and increasing on X^N. (Note that "monotonicity" and "increasingness" are intended in the strict sense.)

The bulk of the literature in the economics of uncertainty restricts the utility function V further and requires that it have the form

$$V\left(\sum_{i=1}^{N} p_i \delta_{x_i}\right)=\sum_{i=1}^{N} p_i v(x_i) \tag{2.3}$$

for some v that is increasing and continuous on X. (For the remainder of this section, we restrict ourselves to simple c.d.f.'s. The formulations for general c.d.f.'s are defined in the natural fashion and will appear below.) Such *expected utility* (EU) or *linear utility* (LU) functionals imply parallel and linear indifference curves in the probability simplex corresponding to gambles with three possible outcomes (Fig. 1). It is convenient to work throughout with the certainty equivalent functional $m(\cdot)$, defined by $V(\delta_m) = V(F)$, that is ordinally equivalent to $V(\cdot)$. Thus (2.3) takes the form

$$v\left(m\left(\sum_{i=1}^{n} p_i \delta_{x_i}\right)\right) = \sum_{i=1}^{n} p_i v(x_i). \tag{2.4}$$

A more general specification, called *weighted utility* (WU), has the form (Chew [3])

$$v\left(m\left(\sum_{i=1}^{n} p_i \delta_{x_i}\right)\right) = \sum_{i=1}^{n} p_i w(x_i)\, v(x_i) \Big/ \sum_{i=1}^{n} p_i w(x_i), \tag{2.5}$$

where on X, w is positive valued and v is increasing and both are continuous. Its indifference curves in the simplex are also linear but they are all rays projected from a point O on the extension of the indifference ray through the intermediate outcome. If $w \equiv$ constant, EU is obtained and the projection point O is at infinity.

Finally, in order to provide further perspective for the analysis to follow, we define *rank-dependent expected utility* (RDEU), which takes the form

$$v\left(m\left(\sum_{i=1}^{n} p_i \delta_{x_i}\right)\right) = \sum_{i=1}^{n} v(x_i)\left[g\left(\sum_{j=1}^{i} p_j\right) - g\left(\sum_{j=1}^{i-1} p_j\right)\right], \tag{2.6}$$

where the outcomes have been arranged so that $x_1 \leqslant \cdots \leqslant x_n$, v is increasing and continuous on X, and g: $[0, 1] \to [0, 1]$ is increasing, continuous, and onto. (See Quiggin [23], Yaari [26], Segal [25], and Chew [5].) When $g(p) \equiv p$, EU is obtained. In general, however, m (or V) can be viewed as an expected utility function only with respect to the c.d.f. $g(F)$ derived from F via the transformation function g. The consequences of the latter for indifference curves in the simplex are demonstrated in Fig. 1. The curves are generally nonlinear but they retain a form of parallelism—the tangents of the indifference curves along the $\underline{x} - \bar{x}$ edge are parallel.

The unifying perspective for these and other utility functionals which we propose in this paper is via separability properties in the space of state-contingent outcomes. The EU functional (2.4) has the well-known strong or additive separability structure. The WU and RDEU functionals also involve summation in fundamental ways which suggests that their essential nature may, at least in part, correspond to appropriate separability properties. We will identify the latter in the next section where some more general classes of functionals will be introduced.

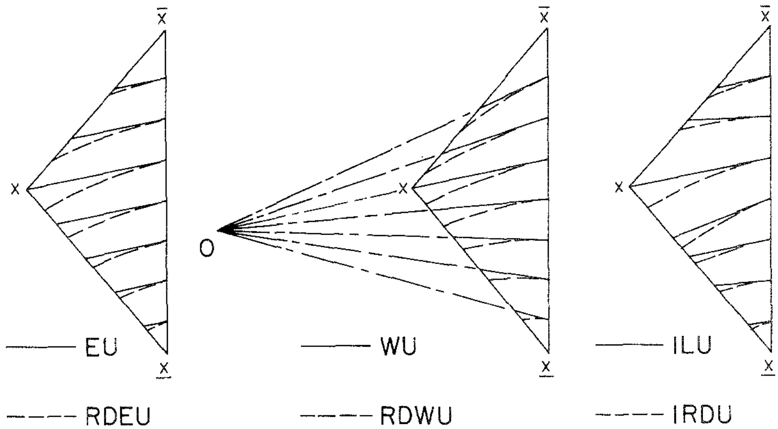

FIGURE 1.

3. UTILITY FUNCTIONALS AND SEPARABILITY AXIOMS

Three classes of utility functionals and their corresponding separability axioms are described here. In each case, the utility functional V as well as the corresponding certainty equivalent functional m is continuous and monotone. Functional forms are specified first for simple distributions. The formulations for general c.d.f.'s are given at the end of the section.

We refer to m as an *implicit linear utility* (ILU) certainty equivalent if $\exists \tau: X \times X \to \mathbb{R}$ with $\tau(\cdot, y)$ continuous and increasing $\forall y \in X$ and $\tau(x, x) \equiv 0$ such that $m(F)$ is given by the unique implicit solution y to

$$\sum_{i=1}^{n} p_i \tau(x_i, y) = 0. \tag{3.1}$$

(See Dekel [12], for example.) If $\tau(x, y) = w(x)[v(x) - v(y)]$, then (3.1) admits an explicit solution $y = m(F)$ which coincides with weighted utility (2.6).[3] ILU implies straight line indifference curves within the simplex (Fig. 1) which need not all emanate from a single point as in WU.

In order to formulate the separability axiom corresponding to ILU we need some further notation. For any partition $I_N^c \cup I_N^r$ of $I_N = \{1, ..., N\}$, let $X^N = X^{(c)} \times X^{(r)}$ be the corresponding decomposition of X^N. Let $k = \|I_N^r\|$ and $N - k = \|I_N^c\|$. Let $T = (T^1, ..., T^N)$, where T^k is a correspondence from X^N into X^k. All separability axioms in the paper have the form:

[3] Fishburn [16] axiomatized a different subclass of ILU for which τ is skew-symmetric.

Axiom TS (T-Separability). For all $x=(x_1,..., x_N)\in X^N$ and all decompositions $X^{(c)}\times X^{(r)}$ of X^N,

$$\sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i}\sim\frac{k}{N}\delta_{c*}+\sum_{i\in I^r}\left(\frac{1}{N}\right)\delta_{x_i}$$
$$\Rightarrow\sum_{i\in I^c}\left(\frac{1}{N}\right)\delta_{x_i}+\sum_{i\in I^r}\left(\frac{1}{N}\right)\delta_{y_i}\sim\frac{k}{N}\delta_{c*}+\sum_{i\in I^r}\left(\frac{1}{N}\right)\delta_{y_i},\qquad\forall y^r\in T^k(x).$$

The interpretation of the axioms is as follows: The quantity c^* is a *certainty equivalent* for x^c, *contingent* on x^r. The above condition is met if c^* is invariant with respect to the change from x^r to any y^r in $T^k(x)$. Clearly, the specification of T determines the nature of the restriction imposed by the invariance requirement. Thus, we refer to the latter as *T-Separability.*

Some examples will help to clarify this notion. First, if each T^k is defined so that $T^k(x)=X^k$, then c^* is invariant with respect to any substitution of y^r for x^r that is consistent with the domain restriction that lottery outcomes lie in X. This yields the usual additive separability to which we refer simply as separability.

Axiom S (Separability). T-Separatility where $T^k(x)\equiv X^k$.

Axiom S is satisfied by linear utility functionals. The more general utility functions considered in this paper satisfy weaker axioms. Thus, for example, for ILU the invariance of the contingent certainty equivalent c^* holds only if changes from x^r to y^r are required to preserve indifference as in the following axiom.

Axiom IS (Indifference Separability). T-separability where

$$T^k(x)=\left\{y^r\in X^k\colon \sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i}\sim\sum_{i\in I^c}\left(\frac{1}{N}\right)\delta_{x_i}+\sum_{i\in I^r}\left(\frac{1}{N}\right)\delta_{y_i}\right\}.$$

To see the motivation for the latter axiom and those that follow, consider the choices represented in Fig. 2. There are 100 equally likely states. The outcomes L, I, and H are in ascending order of preference. The contingent certainty equivalent $c_s^*(A)$ of A for states 1 to q, contingent on outcome s for states $q+1$ to 100, is always I. Axiom M implies that the contingent certainty equivalent $c_s^*(B)$ lies between L and H. Savage's sure-thing principle (STP) or Axiom S requires that $c_s^*(B)$ be invariant to changes in s. However, the standard Allais paradox corresponds to $c_s^*(B)<I$ when $s=I$ and $c_s^*(B)>I$ when $s=L$. The original parameters

		States		
		1···p	p+1 ··· q	q+1···100
Acts	A	I	I	s
	B	L	H	s

FIG. 2. The standard Allais parodox.

were $(L, I, H; p, q) = (0, 1$ million Fr, 5 million Fr; $1, 11)$.[4] Since comparisons of the state-contingent form in Fig. 2 do not involve timing or multiple stage considerations, the standard Allais paradox directly tests the invariance of $c_s^*(B)$ with respect to changes in s.

There have been several replications of the standard Allais paradox with different outcome and probability parameters (see, e.g., MacCrimmon and Larsson [20], Kahneman and Tversky [19], Chew and Waller [9]). The consistent finding from the empirical studies is that $c_s^*(B)$ is higher when $s = L$ than when $s = I$, thus violating the invariance implication of STP. This provides the motivation for our weakening of the STP by restricting the domain of invariance of c^*. The empirical evidence is however consistent with IS (and each of the weaker axioms specified below), since changing from $s = L$ to $s = I$ violates the indifference restriction in IS (and the corresponding rank restrictions in the other axioms).

The next class of utility functionals generalizes RDEU. Its representation is based on a function $\varphi: X \times [0, 1] \to \mathbb{R}$ which is continuous in each argument, $\varphi(\cdot, 0) \equiv 0$, and φ satisfies

$$\varphi(x, p) - \varphi(x, q) - \varphi(y, p) + \varphi(y, q) > 0, \qquad \forall x > y \text{ and } p > q. \quad (3.2)$$

(For example, in the differentiable case, (3.2) is equivalent to the positivity of the cross partial derivative φ_{12}.) The *rank-linear utility* (RLU) certainty equivalent m for $F \equiv \sum_{i=1}^n p_i \delta_{x_i}$ is given by the solution to

$$\varphi(m(F), 1) = \sum_{i=1}^{n} \left[\varphi\left(x_i, \sum_{j=1}^{i} p_j\right) - \varphi\left(x_i, \sum_{j=1}^{i-1} p_j\right) \right], \quad (3.3)$$

where the x_i's are arranged in ascending order. (See Segal [25] and Green and Jullien [18] for closely related functional forms.) Since $\varphi(\cdot, 1)$ is continuous and increasing, (3.3) provides an explicit representation. The LHS of (3.2) represents an increase in utility of (3.3) from $F \equiv (q - p)\,\delta_x + (1 - q + p)\,G$ to $F' \equiv (q - p)\,\delta_y + (1 - q + p)\,G$ where

[4] Our exposition of the Allais paradox is due to Savage [24, p. 103] who considered the validity of STP self-evident once alternatives are represented in terms of state-contingent outcome vectors.

$[x, y] \cap \operatorname{supp} G = \varnothing$. In this way, Condition (3.2) guarantees that the RLU certainty equivalent satisfies Axiom M on $D^0(X)$. The subclass RDEU corresponds to the case where φ has the multiplicatively separable form $\varphi(x, p) = v(x)\, g(p)$. The behavior of the indifference curves for RLU generalizes those of RDEU in that the parallelism noted earlier for the latter case is absent.

To formulate the separability requirement for RLU, we introduce the notation $X_{\uparrow}^N = \{x \in X^N : x_1 \leqslant \cdots \leqslant x_N\}$ and for any x, we denote by $x_{\uparrow} = (x_{[1]}, ..., x_{[N]})$ its increasing rearrangement. We say that x and $y \in X^N$ are *rank preserving* if $x_{[i]} \in [y_{[i-1]}, y_{[i+1]}]$ and $y_{[i]} \in [x_{[i-1]}, x_{[i+1]}]$ $\forall i$, where $x_0 = y_0 \equiv -\infty$ and $x_{N+1} = y_{N+1} \equiv \infty$. The next separability axiom, called rank separability, imposes the invariance of the contingent certainty equivalent only when the substitution of y^r for x^r does not change the rank ordering of the outcomes.[5]

Axiom RS (Rank Separability). T-separability where $\forall x \in X^N$, $\forall k$, $T^k(x) = \{y^r \in X^k : x \text{ and } (x^c, y^r) \text{ are rank-preserving}\}$.

This axiom is consistent with the evidence cited earlier regarding the Allais paradox since changing from $s = L$ to $s = I$ violates rank-preservation in RS.

The final class of utility functionals, called *implicit rank linear utility* (IRLU), generalizes all of the above. The functional form is based on a function $\psi: X \times [0, 1] \times X \rightarrow \mathbb{R}$ that is continuous in each of the first two arguments; $\forall (x, p, y) \in X \times [0, 1] \times X$, $\psi(x, 0, y) = \psi(y, p, y) = 0$; and $\psi(\cdot, \cdot, y)$ satisfies (3.2) $\forall y \in X$. For each $F \equiv \sum_{i=1}^n p_i \delta_{x_i}$, $m(F)$ is the unique solution y to

$$\sum_{i=1}^{n} \left[\psi\left(x_i, \sum_{j=1}^{i} p_j, y\right) - \psi\left(x_i, \sum_{j=1}^{i-1} p_j, y\right) \right] = 0, \tag{3.4}$$

where $x_1 \leqslant \cdots \leqslant x_n$. ILU is the special case corresponding to

$$\psi(x, p, y) = p\tau(x, y), \tag{3.5}$$

and RLU corresponds to

$$\psi(x, p, y) = \alpha(y)[\varphi(x, p) - \varphi(y, p)] \tag{3.6}$$

for some $\alpha: X \rightarrow \mathbb{R}^+$. Indifference curves in the simplex for general IRLU have no discernible special properties.

The special structure of IRLU can be characterized by the following separability axiom which, naturally, weakens both IS and RS.

[5] In a income inequality context, Ebert [13] uses the same axiom to characterize the counterpart of RLU.

Axiom IRS (Indifference Rank-Separability). T-separability where $\forall k$ $\forall x \in X^N$, $T^k(x) = \{y^r \in X^k : \sum_{i=1}^N (1/N)\, \delta_{x_i} \sim \sum_{i \in I^c} (1/N)\, \delta_{x_i} + \sum_{i \in I^r} (1/N)\, \delta_{y_i}$ and x and (x^c, y^r) are rank-preserving$\}$.

Thus, IRS requires that contingent certainty equivalents be invariant only when substitutions preserve both indifference and rank ordering.

These classes of utility functionals and their interrelationships are represented in Fig. 3. The most general class is IRLU. Its representation is based on the function ψ of the three variables x, p, and y. The specializations of this ψ indicated in the figure lead to the other classes listed above, as well as to other theories which have appeared in the literature.

We now extend the definition of IRLU utility functionals from $D^0(X)$ to

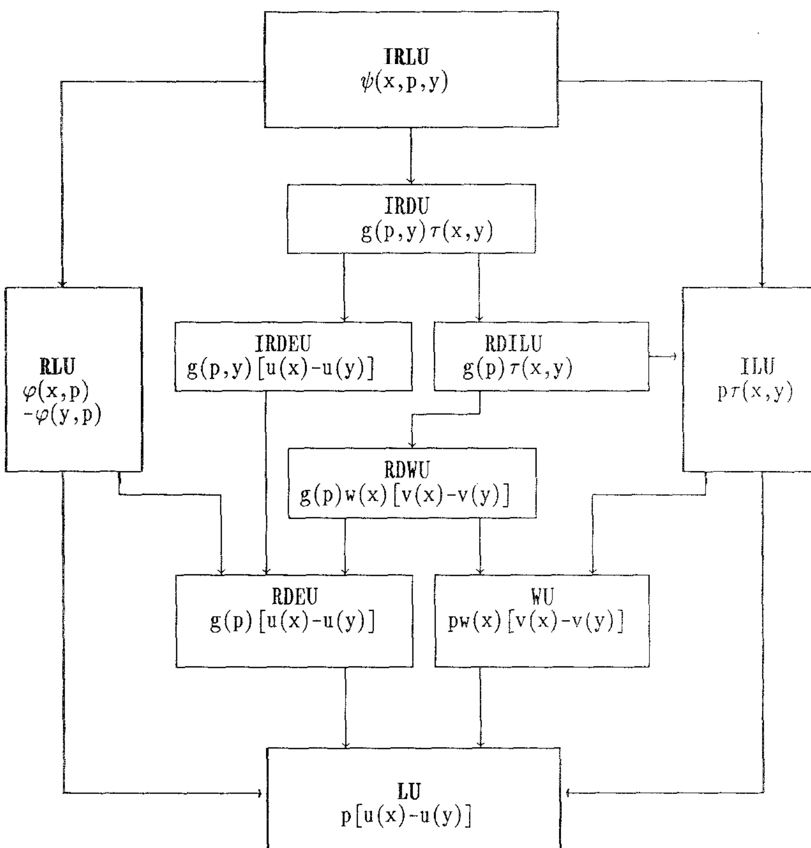

FIG. 3. Specializations of IRLU.

$D(X)$. For the ILU subclass, the extension is straightforward; simply replace (3.1) by

$$\int_X \tau(x, y)\, dF(x) = 0. \tag{3.1'}$$

That is, $m(F)$ is the unique solution y to the above equation. The LU representation is, of course,

$$v(m(F)) = \int_X v(x)\, dF(x). \tag{3.7}$$

The extension is more complicated in cases where there is sensitivity to rank. We require the following construction: For any function $\varphi\colon X \times [0, 1] \to \mathbb{R}$ which satisfies (3.2) and the conditions immediately preceding it, we define the *partial expectation* $E_2(\varphi, F)$ first on $D^0(X)$. Given a simple distribution $F \equiv \sum_{i=1}^N p_i \delta_{x_i}$, $x \in X_\uparrow^N$,

$$E_2(\varphi, F) \equiv \int_X d_2 \varphi(x, F(x)) \equiv \sum_{i=1}^N [\varphi(x_i, h_i) - \varphi(x_i, h_{i-1})],$$

where

$$h_i = \sum_{j=1}^i p_j \qquad \text{for } i \geqslant 1 \text{ and } h_0 = 0.$$

Given a sequence $\{F_n\} \subset D^0(X)$ which converges weakly to F, we prove via Lemma A in Appendix 1 that $\operatorname{Lim}_{n \to \infty} E_2(\varphi, F_n)$ exists and is independent of the choice of the sequence. Thus, for $F \in D(X) \backslash D^0(X)$, we define $E_2(\varphi, F) = \operatorname{Lim}_{n \to \infty} E_2(\varphi, F_n)$ for any sequence $\{F_n\}$ in $D^0(X)$ which converges weakly to F. Condition (3.2) ensures that $E_2(\varphi, \cdot)$ is monotone on $D(X)$ in the sense of first degree stochastic dominance.

To define RLU, for any $F \in D(X)$, replace (3.3) by

$$\varphi(m(F), 1) = \int_X d_2 \varphi(x, F(x)). \tag{3.3'}$$

If $\varphi(x, p) = v(x)\, g(p)$, then we obtain

$$v(m(F)) = \int_X v(x)\, dg(F(x)), \tag{2.6'}$$

which defines the general RDEU corresponding to (2.6).

Finally, for IRLU, if ψ is a function satisfying the conditions surrounding (3.4), then for each y, $E_2(\psi(\cdot, \cdot, y), F)$ is well defined. Thus we can define IRLU on $D(X)$ by requiring that $m(F)$ uniquely solve

$$\int_X d_2 \psi(x, F(x), y) = 0. \tag{3.4'}$$

4. Representation Theorem

The relation between the separability axioms and the classes of utility functionals defined above is summarized in Theorem 2, which is the central result in this paper. The proof, found in Appendix 4, makes use of Theorem A in Appendix 3 dealing with the existence of additive utilities. Henceforth, with the exception of the end of this section, we assume that X is the compact interval $[\underline{x}, \bar{x}]$.

Theorem 2. *Let X be a compact interval. A binary relation $\preccurlyeq$ on $D(X)$ satisfies Axioms* O, C, M, *and $\mathscr{B}$ iff it can be represented by a utility function satisfying $\mathscr{A}$, where*

$\mathscr{A}$	LU	ILU	RLU	IRLU
$\mathscr{B}$	S	IS	RS	IRS

The necessity of the axioms given the appropriate functional forms is readily verified using the definitions of the latter on the set of simple distributions. But their sufficiency is nontrivial and is the principal contribution of the theorem.

The theorem falls short of a complete description of the implications of the stated axioms in that we have not spelled out conditions for ψ that correspond to the continuity and monotonicity of $\succcurlyeq$ Similarly, the definition of IRLU asserts that $m(F)$ is uniquely defined by the solution to (3.4) or (3.4′); but we have not provided an explicit condition on ψ which is equivalent to uniqueness. (These deficiencies apply only to the ILU and IRLU classes. For LU and RLU, $m(F)$ is explicitly defined as in (3.5) and (3.3′) so that uniqueness holds trivially. The continuity and monotonicity of LU are well known. The continuity and monotonicity properties of RLU follow from the corresponding properties for $E_2(\varphi, \cdot)$ given the assumptions made for φ.)

But we do provide, in Appendix 2, two sets of simple conditions on ψ that are sufficient for the IRLU functional m to satisfy the desired continuity, monotonicity, and uniqueness of solution. They guarantee that $\int_X d_2\psi(x, F(x), y)$ is decreasing in y, which is the key to proving uniqueness and monotonicity. Both sets of conditions reduce in the ILU case to

$$\forall x \in X, \tau(x, \cdot) \text{ is continuous and decreasing.} \tag{4.1}$$

A class of IRLU functionals is described in Section 5. Another example is provided by taking

$$\psi(x, p, y) = p\tau(x, y) + g(p)[v(x) - v(y)], \tag{4.2}$$

where g: $[0, 1] \to [0, 1]$ is continuous, increasing, and onto, and where τ satisfies (4.1), $\tau(x, x) = 0$, $\tau(\cdot, y)$ continuous and increasing on X. (It is readily verified that ψ in (4.2) satisfies the conditions in Lemma B of Appendix 2.) Note that ψ is the sum of ψ_1 and ψ_2, where ψ_1 corresponds to an ILU functional and ψ_2 corresponds to an RLU functional; but the utility function corresponding to ψ does not belong to any of the subclasses of IRLU described in Fig. 4.

Given IRLU represented by ψ, it is clear, for any positive valued function α on X and $\hat{\psi}$ defined by

$$\hat{\psi}(x, p, y) = \alpha(y)\, \psi(x, p, y), \tag{4.3}$$

that $\hat{\psi}$ represents the same ordering. In fact (4.3) defines the uniqueness class of ψ.

COROLLARY 1. *The functions ψ and $\hat{\psi}$ represent the same* IRLU *ordering if and only if there exists a positive valued function α on X such that* $\forall (x, p, y) \in X \times [0, 1] \times X$, $\hat{\psi}(x, p, y) \equiv \alpha(y)\, \psi(x, p, y)$.

Proof. See Appendix 4.

Since risk aversion is a basic hypothesis in uncertainty theory, we turn now to a characterization of risk aversion for IRLU in terms of ψ. A number of definitions of risk aversion have appeared in the literature. Machina [21] provided a Fréchet based characterization of the equivalence among the three definitions of risk aversion—mean-preserving-spread, conditional risk premium, and conditional asset demand—based on the concavity of the Fréchet derivative referred to as the local utility function. This result cannot be directly applied here because IRLU is not generally Fréchet differentiable (see Chew, Karni, and Safra [6] for a demonstration in terms of RDEU).

We apply the characterization of risk aversion for continuous utility functionals in Chew and Mao [8]. Given an elementary lottery $\sum_{i=1}^{N} (1/N)\, \delta_{x_i}$ and a continuous utility functional V, V is risk averse (as defined in Machina [21]) if and only if, for each N, $V(\sum_{i=1}^{N} (1/N)\, \delta_{x_i})$ is *Schur-concave* on X^N, i.e., for $(x_1, ..., x_N) \in X_{\uparrow}^N$,

$$V\left(\frac{1}{N}\delta_{x_k - \varepsilon} + \frac{1}{N}\delta_{x_{k+1} + \varepsilon} + \sum_{i \neq k, k+1} \frac{1}{N}\delta_{x_i}\right)$$

is nonincreasing in ε for values of ε that do not alter the ranks of $x_k - \varepsilon$ and $x_{k+1} + \varepsilon$.

The following is proved in Appendix 4.

THEOREM 3. *Given an* IRLU *preference* ψ, ψ *is risk averse if and only if* $\forall x, x', y \in X$, $p, p' \in [0, 1)$, *with* $x > x'$, $p > p'$,

$$\psi(x, p, y) - \psi(x, p', y) \text{ is concave in } x, \tag{4.4}$$

and

$$\psi(x, p, y) - \psi(x', p, z) \text{ is concave in } p. \tag{4.5}$$

Note that (4.4) and (4.5) can be equivalently stated as

$$\psi(x, p, y) - \psi(x, p', y) - \psi(x', p, y) + \psi(x', p', y) \tag{4.6}$$

is concave in x and in p separately. Suppose ψ is differentiable in the first two arguments. Then the latter risk aversion condition is

$$\psi_{12}(\cdot, \cdot, y) \text{ is nonincreasing.} \tag{4.7}$$

Recall that the monotonicity condition (3.2) when applied to ψ is equivalent to $\psi_{12} > 0$ a.e. The above conditions are the counterpart for IRLU of the familiar restrictions for LU theories that marginal utility is positive (for monotonicity) and nonincreasing (for risk aversion).

In the case of ILU, (4.6) specializes to the concavity of $\tau(\cdot, y)$. If we specialize further to LU, we obtain the familiar result that concavity of the von Neumann–Morgenstern utility index characterizes risk aversion. For RDEU, (4.6) is equivalent to the concavity of v and g, which is consistent with Yaari [26] and Chew, Karni, and Safra [6].

We devote the remainder of this section to the case when X is not restricted to being a compact interval. First we observe that the separability proof of Theorem 2 does not depend on the compactness of X. Consequently, the construction of ψ can be accomplished on a noncompact X and the resulting utility functional represents the preference ordering on $D^0(X)$. Consider the following continuity requirement.

Axiom CC (Compact Continuity). For every compact interval $K \subset X$, $\preccurlyeq$ is continuous on $D(K)$.

When X is unbounded, the above is weaker than the continuity defined by Axiom C. By adopting CC, we can immediately obtain the extension below of Theorem 2 to $D^c(X)$, the set of c.d.f.'s in $D(X)$ having compact supports.

COROLLARY 2. *Theorem* 2 *holds on* $D^c(X)$ *if Axiom* C *is replaced by Axiom* CC.

In this case, we may define the certainty equivalent of $F \in D(X) \backslash D^c(X)$ by

$$m(F) = \lim_{n \to \infty} m(F_{K_n}),$$

where $\{K_n\}$ is an arbitrary increasing sequence of compact intervals which converges to X. Of course, as in the case of risk neutrality, the domain of $m(\cdot)$ may not be all of $D(X)$. This means that the corresponding preference ordering is complete on $D^c(X)$ but not necessarily complete on $D(X)$.

5. Rank-Dependent Specializations

Theorem 2 provides axiomatizations for the classes of utility functionals contained in the larger boxes in Fig. 3. Some of the remaining boxes are considered here.

In Sections 2 and 3, we referred to rank-dependent expected utility (RDEU). Here, we consider an implicit form for RDEU, called *implicit rank-dependent utility* (IRDU), which has not appeared in the literature. The relation between IRLU and IRDU parallels that between RLU and RDEU. This parallel is clear from the point of view of functional representation since both specializations correspond to ψ's being multiplicatively separable in x and p. In other words, for IRLU,

$$\psi(x, p, y) \equiv g(p, y)\, \tau(x, y) \tag{5.1}$$

and for RLU,

$$\varphi(x, p) - \varphi(y, p) \equiv g(p)[u(x) - u(y)], \tag{5.2}$$

where $\forall y \in X$, $g(\cdot, y)$: $[0, 1] \to [0, 1]$ is continuous, increasing and onto. In the former case, the mean value functional $m(\cdot)$ satisfies

$$\int_X \tau(x, m(F))\, dg(F(x), m(F)) = 0, \tag{5.3}$$

while the latter case, of course, leads to (3.6).

For IRDU, the condition (4.6) for risk aversion specializes to

$$g(\cdot, y) \text{ and } \tau(\cdot, y) \text{ are both concave,} \qquad \forall z \in X. \tag{5.4}$$

This condition reduces easily to the risk aversion conditions for all other boxes in Fig. 3 with the exception of RLU.

In the case of IRDU, we can also obtain an appealing condition for the unique existence of the implicit solution to (3.4′). IRDU is given by the solution to

$$\int_X \tau(x, y)\, dg(F(x), y) = 0. \tag{5.5}$$

Suppose that τ satisfies (4.1) and

$$g(p, y) \leqslant g(p, y') \qquad \text{whenever } y < y'. \tag{5.6}$$

Then unique existence as well as continuity and monotonicity of the solution to (5.5) follows from Lemma C in Appendix 2.

When $\tau(x, y) = v(x) - v(y)$ (we label the corresponding utility functional as *implicit rank-dependent expected utility* (IRDEU)), the certainty equivalent functional m solves

$$v(m(F)) = \int_X v(x)\, dg(F(x), m(F)). \tag{5.7}$$

This functional generalizes RDEU by permitting the transformation function g to depend on the level of utility. Viewed in these terms, Condition (5.6) may be interpreted as the decision maker's displaying a greater degree of 'pessimism' in transforming the given probability distribution F by $g(\cdot, y)$ when comparing gambles to which he assigns higher levels of utility. This interpretation is compatible with the Allais type behavior discussed in Section 3 and also below.

The following is a simple class of g functions for IRDEU. Given continuous, increasing, and onto functions a: $[0, 1] \to [0, A]$ and b: $[0, 1] \to [0, B]$, define for $y \in [0, \infty)$

$$g(p, y) = [a(p) + yb(p)]/[A + yB].$$

It is clear that (5.6) is satisfied as long as $a(p)/A \leqslant b(p)/B$. As y increases from 0 to ∞, the probability transformation function changes from $a(\cdot)/A$ asymptotically towards $b(\cdot)/B$.

A different subclass of IRDU, which is in a sense polar to (5.7), is called *rank-dependent implicit linear utility* (RDILU). In it, the function τ, but not the transformation function g, may depend on the utility level. Thus $m(F)$ satisfies

$$\int_X \tau(x, m(F))\, dg(F(x)) = 0. \tag{5.8}$$

If g is the identity function, ILU is obtained. Since WU is the only explicit theory within ILU, it is of interest to consider the weighted utility specialization of RDILU class given by $\tau(x, y) = w(x)[v(x) - v(y)]$ which yields an explicit theory that integrates elements of RDEU and ILU. We term this *rank-dependent weighted utility* (RDWU); $m(F)$ satisfies

$$v(m(F)) = \int_X v(x)\, w(x)\, dg(F(x)) \Big/ \int_X w(x)\, dg(F(x)). \tag{5.9}$$

The behavior of the specializations of IRLU on a three-outcome probability simplex is presented in Fig. 1. The three outcomes $\underline{x}$, x, and $\bar{x}$ are arranged in ascending order of preference. We have already observed that the ILU specializations have straight indifference curves. The indifference curves for WU are projected from a point O on the indifference curve through x extended, and those of EU are parallel in alignment.

The rank-dependent specializations have nonstraight indifference curves except when they coincide with ILU. (We sketch the indifference curves corresponding to a risk averse agent.) We can identify a property which is shared by the ILU and IRLU theories. The tangents of the indifference curves along the $\underline{x}-\bar{x}$ edge behave exactly as their ILU counterparts. For IRDU, the only restriction is that these initial tangents do not intersect even though they are not themselves indifference curves. In general, IRLU does not satisfy such a restriction.

It is known that Allais type choice behavior corresponds to indifference curves in the simplex becoming steeper as one moves along a direction of increasing preference. (This is often called the "fanning" property.) In Fig. 1, the natural classes of preferences displaying this property consist of WU and RDWU depending on whether having straight indifference curves is desirable. While RDEU has been shown to be compatible with the Allais paradox, the parallelism of the initial tangents makes fanning a possibility only if g is concave, which is compatible with risk aversion.

The preceding discussion suggests that there is enough information based on the behavior on simplexes to characterize further specializations of IRLU. We do not pursue the possibility here. But we do provide an axiomatization, in a different vein, of the IRDU class. The axiom also serves to axiomatize RDEU in a parallel fashion, thus supporting the parallelism drawn at the beginning of this section.

Axiom MS (Multiplicative Separability). $\forall p, q, r, s \in (0, 1)$ and $m \in X$, $\exists \alpha \in (0, 1)$ such that $\forall F \in D(X)$, $x, x', y, y' \in J$, an open interval of X with $F(J)=0$, $F(\inf J)=r$, $x<x'$, $y<y'$,

$$\delta_m \sim \frac{p}{2}\delta_x + \frac{p}{2}\delta_{x'} + (1-p)F \sim \frac{p}{2}\delta_y + \frac{p}{2}\delta_{y'} + (1-p)F,$$

and $\forall G \in D(X)$ with $G(J)=0$, $G(\inf J)=s$, if

$$\delta_m \sim \alpha q\delta_x + (1-\alpha)q\delta_{x'} + (1-q)G,$$

then

$$\delta_m \sim \alpha q\delta_y + (1-\alpha)q\delta_{y'} + (1-q)G$$

THEOREM 4. *Under Theorem* 2, IRLU (RLU) *is of the form* (6.2) ((6.3)) *if and only if* $\prec$ *satisfies additionally Axiom* MS.

Proof. See Appendix 5.

6. CONCLUDING REMARKS

The major contribution of this paper is the unification it provides, as illustrated in Fig. 3, of a number of generalizations of expected utility. The paper suggests several promising directions for future research, First, we have introduced a number of new classes of utility functions whose properties and usefulness need to be explored further. For a number of specifications of IRLU, axiomatizations need to be provided. In a different vein, we may investigate the possibility of other generalizations of LU based on different variations of the separability axioms than those considered in this paper.

There is a well-known link between mean values and indices of income inequality via the notion of the representative income. For example, the special case of RDEU with $\psi(x, p, y) \equiv [1-(1-p)^2][x-y]$ corresponds to the Gini index, while Chew [3] applied WU to income inequality measurement. Our separability axioms have clear interpretations in this context. We intend to explore further the usefulness of IRLU and its various subclasses for inequality measurement.

APPENDIX 1: PARTIAL EXPECTATION ON $D(X)$

LEMMA 1. *Let* φ *be as in Section* 3 *and* $X=[\underline{x}, \bar{x}]$. *Suppose* $\{F_n\}$ *in* $D^0(X)$ *converges weakly to* $F \in D(X)$. *Then* $E_2(\varphi, F)=\mathrm{Lim}_{n\to\infty} E_2(\varphi, F_n)$ *exists and is independent of the choice of* $\{F_n\}$. *Moreover,* $E_2(\varphi, \cdot)$ *is monotone in the sense of first-degree stochastic dominance if* φ *satisfies* (3.2).

Proof. Define a finite signed measure λ on the Borel sets of $X \times [0, 1]$ by $\lambda((y, x] \times (q, p]) = \varphi(x, p) - \varphi(x, q) + \varphi(y, p) - \varphi(y, q)$. Define ${}^*F = \{(a, b) \in X \times [0, 1]: b \geqslant F(a)\}$. Observe that $E_2(\varphi, F)=\lambda({}^*F)+\varphi(\underline{x}, 1)$ on $D^0(X)$. Consider a sequence $\{F_n\}$ in $D^0(X)$ which converges weakly (in distribution) to $F \in D(X) \backslash D^0(X)$. Let 1_A denote the indicator function for $A \subset X \times [0, 1]$. Then $\lambda({}^*F \backslash {}^*F_n)=\int_{X\times[0,1]}[1_{{}^*F}-1_{{}^*F_n}]\,d\lambda \to 0$ since $1_{{}^*F_n}$ converges to $1_{{}^*F}$ a.e. with respect to λ and $1_{X\times[0,1]}$ is integrable. We define $E_2(\varphi, F)=\mathrm{Lim}_{n\to\infty} E_2(\varphi, F_n)=\lambda({}^*F)+\varphi(\underline{x}, 1)$. This limit is clearly not dependent on the choice of the convergent sequence. Thus, we have extended $E_2(\varphi, \cdot)$ continuously from $D^0(X)$ to $D(X)$.[6]

[6] The observation that RLU can be viewed as a measure on epigraphs of distribution functions is due to Segal [25].

If φ satisfies (3.2), then λ is a positive measure. Thus, monotonicity of $E_2(\varphi, \cdot)$ follows from the observation that $^*F \subset {}^*G$ if G dominates F in the first degree. Q.E.D.

It is clear that Lemma A can be extended to an unbounded X by requiring $\varphi(\cdot, p)$ to be bounded for each $p \in [0, 1]$.

APPENDIX 2: *Sufficient Conditions for* IRLU

We provide two lemmas which describe sufficient conditions for ψ to define an IRLU functional. The first lemma is applicable to Example (4.2). Numerous other such examples can be constructed, since if ψ_1 and ψ_2 each satisfies the conditions of Lemma B, then so does any positive linear combination. Lemma C is applicable to the IRDU class discussed in Section 5.

LEMMA B. *ψ defines an* IRLU *functional if*

(S1) *$\psi: X \times [0, 1] \times X \to \mathbb{R}$ is continuous in each of the first two arguments, and $\forall$ $(x, p, y) \in X \times [0, 1] \times X$, $\psi(x, 0, y) = \psi(y, p, y) = 0$, and $\psi(\cdot, \cdot, y)$ satisfies* (3.2);

(S2) *$\forall x \in X$ and $p, q \in [0, 1]$, $p > q$, $\psi(x, p, \cdot) - \psi(x, q, \cdot)$ is continuous and decreasing.*

Proof. Note that for $F \equiv \sum_{i=1}^{N} (1/N)\, \delta_{x_i}$ in $D^0(X)$, $\int_X d_2\psi(x, F(x), y)$ is given by

$$\sum_{i=1}^{N} \{\psi(x_i, h_i, y) - \psi(x_i, h_{i-1}, y)\}, \tag{A.2.1}$$

where $h_i = \sum_{j=1}^{i} p_j$. Existence of a solution on $D^0(X)$ follows from the continuity of $\int_X d_2\psi(x, F(x), y)$ in y and its assuming opposite signs at the extreme points of the support of F. Uniqueness of the solution to (3.4) on $D^0(X)$ follows from the decreasingness in y of (A.2.1).

To prove monotonicity on $D^0(X)$, note that a small increase in x_i increases the ith term within the summation (A.2.1). In order to restore equality with 0, y has to be increased since each term in the summation is decreasing in y.

Now, consider distributions in $D(X)$. For $y \in X$, define as in Lemma A a finite measure $\lambda(\cdot, y)$ on Borel subsets of $X \times [0, 1]$ by using Expression (3.2) for $\psi(\cdot, \cdot, y)$. Denote $\mathscr{L}(y) = \{(a, b) \in X \times [0, 1]: a \leqslant y\}$. Observe from Lemma A that

$$\int_X d_2\psi(x, F, y) = \lambda(1_{^*F}, y) - \lambda(1_{\mathscr{L}(y)}, y). \tag{A.2.2}$$

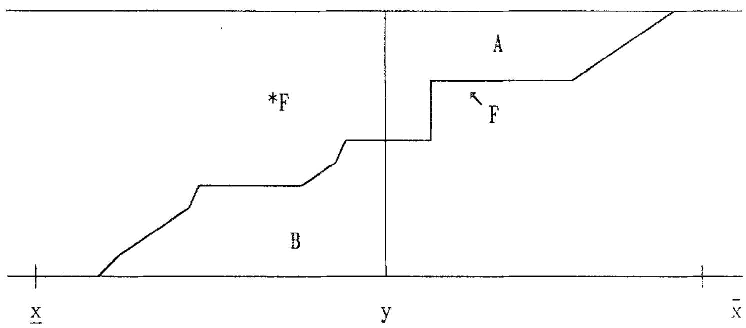

FIGURE 4.

In terms of Fig. 4, the RHS of (A.2.2) equals the difference between the measure of A, the epigraph $^*F$ to the right of y, and the measure of the region B which lies to the left of y and below the given distribution F. IRLU is defined by the value of y which equates $\lambda(A, y)$ and $\lambda(B, y)$.

Let $F \in D(X)$ and let $\{F_n\} \subset D^0(X)$ be a decreasing sequence which converges weakly to F such that $m_n = m(F_n)$ converges to m'. Let $h_n^+ = 1_{*F_n}$, $h_n^- = 1_{\mathscr{L}(m_n)}$, $h^+ = 1_{*F}$, $h^- = 1_{\mathscr{L}(m')}$, $P_n(\cdot) \equiv \lambda(\cdot, m_n)$, and $P(\cdot) \equiv \lambda(\cdot, m')$. Clearly, P_n converges weakly to P given the continuity of ψ in (S2). By Theorem 5.5 in Billingsley [1, p. 34],

$$P_n(1_{*F_n}) \to P(1_{*F}) \qquad \text{and} \qquad P_n(1_{\mathscr{L}(m_n)}) \to P(1_{\mathscr{L}(m')}).$$

Note that $P_n(1_{*F_n}) = P_n(1_{\mathscr{L}(m_n)})$ for each n. It follows that

$$0 = P^+(1_{*F}) - P^-(1_{\mathscr{L}(m')}) = \int_X d_2\psi(x, F(x), m'),$$

which implies that m' solves (3.4′) for F. Uniqueness of the solution follows since $\int_X d_2\psi(x, F(x), y)$ decreases in y.

Continuity of $m(\cdot)$ on $D(X)$ follows from applying the preceding argument without restricting the sequence $\{F_n\}$ to lie in $D^0(X)$. Monotonicity on $D(X)$ is evident. Q.E.D.

LEMMA C. *ψ defines an* IRLU *functional if it satisfies* (S1) *and the following*: $\exists\tilde{\psi}\colon X \times [0, 1] \times X \to \mathbb{R}$ *such that* $\psi(x, p, y) = \psi(x, 1, y)\,\tilde{\psi}(x, p, y)$; $\forall (x, p, y) \in X \times [0, 1] \times X$,

(S3) $\psi(x, 1, \cdot)$ *and* $-\tilde{\psi}(x, p, \cdot)$ *are nonincreasing and at least one is decreasing*,

(S4) $\tilde{\psi}(x, 1, x) = 1$ *and* $\tilde{\psi}(x, 0, x) = 0$,

(S5) $\tilde{\psi}(x, \cdot, y)$ *is increasing.*

(S6) $\psi(x, p, \cdot)$ *is continuous.*

Proof. We will prove that $\int_X d_2\psi(x, F(x), \cdot)$ decreases. The remaining arguments are similar to those in the preceding proof.

First, let $F \in D^0(X)$ and let $y < y'$. Then

$$\begin{aligned}
&\int_X d_2\psi(x, F(x), y) - \int_X d_2\psi(x, F(x), y') \\
&\quad = \sum_{i=1}^{N} [\psi(x_i, h_i, y)) - \psi(x_i, h_{i-1}, y) - \psi(x_i, h_i, y') + \psi(x_i, h_{i-1}, y')] \\
&\quad = \sum_{i=1}^{N} \{\psi(x_i, 1, y)[\tilde{\psi}(x_i, h_i, y) - \tilde{\psi}(x_i, h_{i-1}, y)] \\
&\qquad - \psi(x_i, 1, y')[\tilde{\psi}(x_i, h_i, y') - \tilde{\psi}(x_i, h_{i-1}, y')]\} \\
&\quad = I_1 + I_2,
\end{aligned} \tag{A.2.3}$$

where

$$I_1 = \sum_{i=1}^{N} [\psi(x_i, 1, y) - \psi(x_i, 1, y')][\tilde{\psi}(x_i, h_i, y) - \tilde{\psi}(x_i, h_{i-1}, y)]$$

and

$$\begin{aligned}
I_2 = \sum_{i=1}^{N} \psi(x_i, 1, y') \cdot [&\tilde{\psi}(x_i, h_i, y) - \tilde{\psi}(x_i, h_{i-1}, y) \\
&- \tilde{\psi}(x_i, h_i, y') + \tilde{\psi}(x_i, h_{i-1}, y')].
\end{aligned}$$

We know that $I_1 \geqslant 0$ by (S3) and (S5). By applying $\tilde{\psi}(x, 1, z) \equiv 1$, $\tilde{\psi}(x, 0, z) \equiv 0$, and summation by parts, we can rewrite

$$\begin{aligned}
I_2 = \sum_{i=2}^{N} &[\psi(x_{i-1}, 1, y') - \psi(x_i, 1, y')] \\
&\times [\tilde{\psi}(x_{i-1}, h_{i-1}, y) - \tilde{\psi}(x_{i-1}, h_{i-1}, y')].
\end{aligned}$$

Set $p = 1$ and $q = 0$ in (2.6) to deduce that $\psi(\cdot, 1, y')$ is increasing. Then $I_2 \geqslant 0$ by (S3). Moreover, $I_1 + I_2 > 0$.

For $F \in D(X)$, we can pass to the limit in the above argument and conclude that the difference in (A.2.3) is given by

$$\int_X d_2\theta(x, F(x)) + \int_X [\tilde{\psi}(x, F(x), y') - \tilde{\psi}(x, F(x), y)]\, d\psi(x, 1, y'),$$

where $\theta(x, p) \equiv \tilde{\psi}(x, p, y)[\psi(x, 1, y) - \psi(x, 1, y')]$ and the second term is a Stieltjes integral with respect to $\psi(\cdot, 1, y')$. Since $\theta(x, \cdot)$ is nondecreasing, the first term is nonnegative. The second term is nonnegative by (S3) and the increasingness of $\psi(\cdot, 1, y')$. Moreover, the sum of the two terms is strictly positive by (S3). Q.E.D.

APPENDIX 3: ADDITIVE UTILITY ON A SUBSET OF X^N

The additive utility theorem of Debreu [10] and Gorman [17] is usually stated in the setting of a product of arc-connected topological spaces. Here, we establish a variant for Euclidian domains that are not Cartesian products. The terminology—complete strict separability—is adopted from Gorman. For each $i = 1, ..., N$, π_i denotes the ith coordinate projection map.

THEOREM A. *Let $\Omega \subset X^N$ $(N > 2)$ have a nonempty and connected interior and let V be a continuous and increasing utility function on Ω. Suppose that for each $\bar{x} \in \Omega$, the corresponding indifference surface $\{x \in \Omega\colon V(x) = V(\bar{x})\}$ is connected. Then V is completely strictly separable if and only if there exist continuous, increasing functions*

$$u^i\colon \pi_i(\Omega) \to \mathbb{R}$$

and an increasing function

$$\zeta\colon \sum_{i=1}^{N} \operatorname{Rng}(u^i) \to \mathbb{R}$$

such that

$$V(x) = \zeta\left(\sum_{i=1}^{N} u^i(x_i)\right).$$

Proof. Necessity is obvious. We observe that the validity of the theorem for the case when $\Omega = \prod_{i=1}^{N} J_i$ is an interval in X) is well known. Therefore, for an open rectangle $O = \prod_{i=1}^{N} (a_i, b_i) \subset \Omega$, there are

$$u^i(\cdot, O)\colon (a_i, b_i) \to \mathbb{R}$$

such that $V(\cdot, 0) = \sum_{i=1}^{N} u^i(\cdot, O)$ is a utility function on O.

Suppose $O \cap O' \neq \phi$, where O' is another open rectangle in Ω. We can extend $V(\cdot, O)$ to $O \cup O'$ by selecting those $u^i(\cdot, O')$'s such that $u^i(\cdot\, O) \equiv u^i(\cdot, O')$ on $O \cap O'$. This can be done because the u^i's are unique up to similar affine transformations (i.e., u^i's and $\hat{u}^i$'s are equivalent if there are $c > 0$ and d_i's such that $\hat{u}^i \equiv cu^i + d_i$). We have

$$u^i(x, O \cup O') = \begin{cases} u^i(x, O) & x \in O \\ u^i(x, O') & x \in O' - O. \end{cases}$$

Hence, $V(x, O \cup O') = \sum u^i(x_i, O \cup O')$ is a utility function on $O \cup O'$.

By repeated applications of the above 'piecing together' of locally defined additive utility functions, we can construct, for each $\bar{x} \in \Omega$, an additive representation on a strip of minimum thickness $\eta > 0$ centered on the indifference surface containing $\bar{x}$. We can further piece together these additive functions to obtain an additive function U, $U(x) = \sum_{i=1}^{N} u^i(x_i)$, defined on all of Ω.

By the nature of the above construction, V and U are ordinally equivalent locally, i.e., on a sufficiently small neighborhood of any point in Ω. Since indifference surfaces of V are connected, it follows that U is constant on each such indifference surface. Thus, U and V are ordinally equivalent on Ω. Q.E.D.

APPENDIX 4: REPRESENTATION, UNIQUENESS, AND RISK AVERSION

Denote by $D^e(X)$ the set of c.d.f.'s in $D^0(X)$ for which the probability of each outcome is rational. Each such c.d.f. can be expressed in the form $F = \sum_{i=1}^{N} (1/N)\, \delta_{x_i}$ for some N.

Proof of Theorem 2. First, we make the following observation. For $x \in X^N$, if $y \in X^N$ is obtained from x by a permutation of the components of x, then

$$\sum_{i=1}^{N} \left(\frac{1}{N}\right) \delta_{y_i} \equiv \sum_{i=1}^{N} \left(\frac{1}{N}\right) \delta_{x_i}$$

so that $V_N(x) \equiv V(\sum_{i=1}^{N} (1/N)\, \delta_{x_i})$ is symmetric on X^N.

(Sufficiency) First, we establish the result for IRLU. Observe that $m(\sum_{i=1}^{N} \delta_{x_i})$, $x \in X_\uparrow^N$, is given by the unique solution to

$$\sum_{i=1}^{N} \{\psi(x_i, i/N, y) - \psi(x_i, [i-1]/N, y)\} = 0.$$

It follows that m is completely separable on the intersection of $X_\uparrow^N$ and the indifference surface in X^N corresponding to $m(\sum_{i=1}^{N} (1/N)\, \delta_{x_i})$. Thus, IRS is satisfied.

The verification of sufficiency for LU, RLU, and ILU is similar.

(Necessity) *Case* i $(\mathscr{A}, \mathscr{B}) = (\text{LU}, \text{S})$. For $x \in X^N$ $(N > 2)$, Axiom S and monotonicity imply that V_N is completely strictly separable in the sense of [17]. By Gorman [17, pp. 388–389], there exist continuous and increasing functions

$$u_N^i : X \to \mathbb{R}, \qquad i = 1, \ldots, N$$

and an increasing function

$$\zeta_N \colon \sum_{i=1}^{N} \operatorname{Rng}(u_N^i) \to \mathbb{R}$$

such that

$$V_N(x) = \zeta_N\left(\sum_{i=1}^{N} u_N^i(x_i)\right).$$

Symmetry of V_N implies that $u_N^i \equiv u_N^j \equiv u_N$.

Define the certainty equivalence functional m by

$$V(\delta_{m(F)}) = V(F).$$

Then, for $x \in X^N$,

$$V(\delta_{m(\sum(1/N)\,\delta_{x_i})}) = V_N(m\mathbf{1}) = V_N(x)$$

yields

$$Nu_N(m) = \sum_{i=1}^{N} u_N(x_i).$$

Observe that each u_N is unique up to affine transformations and any two different u_N and u_K are equivalent to each other since they are each equivalent to u_{NK}. Therefore, we can pick $u \equiv u_3$ and define

$$u\left(m\left(\sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i}\right)\right) = \sum_{i=1}^{N}\left(\frac{1}{N}\right)u(x_i)$$

for any $x \in X^N$, and for $N = 3, 4, 5, \ldots$.

The extension of m from $D^e(X)$ to $D(X)$ is standard under continuity.

Case ii $(\mathscr{A}, \mathscr{B}) = (\text{ILU, IS})$: For $\bar{x} \in X^N$ $(N > 3)$, define

$$\mathscr{I}(\bar{x}) = \left\{x \in X^N \colon \sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{x_i} \sim \sum_{i=1}^{N}\left(\frac{1}{N}\right)\delta_{\bar{x}_i}\right\}.$$

Let

$$\mathscr{I}(\bar{x})_{-1} = \{x \in X^{N-1} \colon (x, z) \in \mathscr{I}(\bar{x}) \text{ for some } z \in X\}$$

be the projection of $\mathscr{I}(\bar{x})$ on X^{N-1}. Since V is symmetric, the $\mathbb{R}^{N-1}$ projection of $\mathscr{I}(\bar{x})$ is not dependent on the location of the component deleted. This is why we denote the projection by $\mathscr{I}(\bar{x})_{-1}$ rather than $\mathscr{I}(\bar{x})_{-i}$.

Define an ordering $\preccurlyeq_{\bar{x}}^{N-1}$ on $\mathscr{I}(\bar{x})_{-1}$ by

$$x \preccurlyeq_{\bar{x}}^{N-1} y \Leftrightarrow c_{\bar{x}}(x) \leqslant c_{\bar{x}}(y),$$

where $c_{\bar{x}}(x)$ is defined by

$$\sum_{i=1}^{N-1} \left(\frac{1}{N}\right) \delta_{x_i} + \frac{1}{N} \delta_z \sim \frac{N-1}{N} \delta_{c_{\bar{x}}(x)} + \frac{1}{N} \delta_z \sim \sum_{i=1}^{N} \left(\frac{1}{N}\right) \delta_{\bar{x}_i}.$$

Axiom IS implies that $\preccurlyeq_{\bar{x}}^{N-1}$ is completely strictly separable on $\mathscr{I}(\bar{x})_{-1}$ and satisfies the other hypotheses in Theorem A in Appendix 3. Therefore, there are continuous, increasing functions

$$u_{N-1}^{i}(\cdot\,; \bar{x}) \colon \pi_i(\mathscr{I}(\bar{x})_{-1}) \to \mathbb{R}$$

(where π_i is the ith-coordinate projection operator) and an increasing function

$$\zeta_N \colon \sum_{i=1}^{N-1} \operatorname{Rng}(u_{N-1}^{i}) \to \mathbb{R}$$

such that $\preccurlyeq_{\bar{x}}^{N-1}$ is represented by $V_{N-1}(\cdot\,; \bar{x})$, where

$$V_{N-1}(x; \bar{x}) \equiv \zeta_N \left(\sum_{i=1}^{N} u_{N-1}^{i}(x_i; \bar{x}) \right).$$

Since $V_{N-1}(\cdot\,; \bar{x})$ must be symmetric, $u_{N-1}^{i}(\cdot\,; \bar{x}) \equiv u_{N-1}^{j}(\cdot\,; \bar{x}) \equiv u_{N-1}(\cdot\,; \bar{x})$.

It remains to show that the ordering represented by $\sum_{i=1}^{N} u_{N-1}(x_i; \bar{x})$ has $\mathscr{I}(\bar{x})$ as an indifference set. Suppose there exists $x \in X^N$ with $x \sim \bar{x}$ such that

$$\sum_{i=1}^{N} u_{N-1}(x_i, \bar{x}) > \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x}).$$

(The reverse inequality may be handled similarly.) This is only possible if x and $\bar{x}$ have no common component (since if they have a common component, say the Nth, then $x \sim \bar{x} \Rightarrow x_{-N} \sim_{\bar{x}}^{N-1} \bar{x}_{-N} \Rightarrow$

$$\sum_{i=1}^{N} u_{N-1}(x_i, \bar{x}) = \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})).$$

Pick (x_k, x_j) such that $x_k > \bar{x}_k$ and $x_j < \bar{x}_j$. Replace x_k by $x_k^1 > \bar{x}_k$ such that $\sum_{i \neq k} u_{N-1}(x_i, \bar{x}) + u_{N-1}(x_K^1, \bar{x}) = \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})$. We can rule out $x_k^1 \leqslant \bar{x}_k$. Otherwise, even if $x_k^1 = \bar{x}_k$,

$$\sum_{i \neq k} u_{N-1}(x_i, \bar{x}) + u_{N-1}(x_K^1, \bar{x}) < \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x}) \tag{A.4.1}$$

since $(x_1, ..., x_k^1, ..., x_N) \prec \bar{x}$.

Next we replace x_j by $x_j^2 < \bar{x}_j$ such that

$$\sum_{i \neq k, j} \frac{1}{N} \delta_{x_i} + \frac{1}{N} \delta_{x_k^1} + \frac{1}{N} \delta_{x_j^2} \sim \sum_{i=1}^{N} \left(\frac{1}{N}\right) \delta_{\bar{x}_i}. \tag{A.4.2}$$

Again, we rule out the possibility that $x_j^2 \geqslant \bar{x}_j$. Otherwise, at $x_j^2 = \bar{x}_j$, LHS of (A.4.2) $\succ$ RHS of (A.4.2) since $\sum_{i \neq k, j} u_{N-1}(x_i, \bar{x}) + u_{N-1}(x_k^1, \bar{x}) + u_{N-1}(x_j^2, \bar{x}) > \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})$.

We continue this process in order to construct a sequence $\{y^k\} \subset X^N$ with the property that, when m is odd, $y^m \prec \bar{x}$ and $\sum_{i=1}^{N} u_{N-1}(y_i^m, \bar{x}) = \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})$; when m is even, $y^m \sim \bar{x}$ and $\sum_{i=1}^{N} u_{N-1}(y_i^m, \bar{x}) > \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})$. Both x_k^m (m odd) and x_j^m (m even) converge since they are monotone bounded sequences. Moreover, $x_k^m \downarrow \bar{x}_k$ and $x_j^m \uparrow \bar{x}_j$. Therefore, $\lim_{m \to \infty} y^m = y \sim \bar{x} \sim x$. Since y shares two common components with $\bar{x}$ and $N-2$ common components with x, we have $\sum_{i=1}^{N} u_{N-1}(x_i, \bar{x}) = \sum_{i=1}^{N} u_{N-1}(y_i, \bar{x}) = \sum_{i=1}^{N} u_{N-1}(\bar{x}_i, \bar{x})$, which is a contradiction.

Moreover, we can conclude that (A.4.1) holds if and only if $x \succ \bar{x}$ since we can decrease the components of x until it is indifferent to $\bar{x}$, at which point we have equality.

As in the proof of Case (i), all the u_{N-1} $(N > 3)$ are equivalent. We can choose $u \equiv u_3$ to generate the implicit equality

$$u\left(m\left(\sum \frac{1}{N} \delta_{x_i}\right), m'\right) = \frac{1}{N} \sum u(x_i, m'),$$

where $\sum (1/N)\, \delta_{x_i} \sim \sum (1/N)\, \delta_{\bar{x}_i} \sim \delta_{m'}$. A standard extension argument again yields $\forall F \in D(X)$, $m(F)$ as an implicit solution of

$$u(m, m) = \int_X u(x, m)\, dF(x). \tag{A.4.3}$$

The above becomes

$$\int_X \tau(x, m)\, dF(x) = 0 \tag{A.4.4}$$

if we define

$$\tau(x, y) \equiv u(x, y) - u(y, y). \tag{A.4.5}$$

The uniqueness of the solution $m(F)$ follows as in Case (iv) below.

Case iii $(\mathscr{A}, \mathscr{B}) = (\text{RLU, RS})$. Axiom RS implies that V_N is completely strictly separable on $X_\uparrow^N$. Theorem A in Appendix 3 is applicable and implies the existence of continuous increasing functions

$$u_N^i : X \to \mathbb{R}, \qquad i = 1, ..., N$$

and an increasing function

$$\zeta_N\colon \sum_{i=1}^{N} \operatorname{Rng}(u_N^i) \to \mathbb{R}$$

such that $\leqslant$ is represented by V_N where

$$V_N(x) \equiv \zeta_N\left(\sum_{i=1}^{N} u_N^i(x_i)\right).$$

Moreover,

$$\sum_{i=1}^{N} u_N^i\left(m\left(\sum \frac{1}{N}\delta_{x_i}\right)\right) = \sum_{i=1}^{N} u_N^i(x_i).$$

Observe that $\sum_{i=1}^{N} u_N^i(m) = \sum_{j=1}^{K} u_N^j(m) = \sum_{l=1}^{KN} u_N^l(m)$ for any $K, N > 2$. Define $\varphi(x, p)$: $X \times [0, 1]^* \to \mathbb{R}$ (where $[0, 1]^* =$ (rational numbers in $[0, 1]\}$) by

$$\varphi\left(x, \frac{i}{N}\right) \equiv \sum_{j=1}^{i} u_N^j(x) \qquad \text{and} \qquad \varphi(x, 0) \equiv 0.$$

Then, given N, for any $x \in X_{\uparrow}^N$,

$$\begin{aligned}\varphi\left(m\left(\sum \frac{1}{N}\delta_{x_i}\right), 1\right) &= \sum_{i=1}^{N}\left[\varphi\left(x_i, \frac{i}{N}\right) - \varphi\left(x_i, \frac{i-1}{N}\right)\right] \\ &= \int_X d_2\varphi\left(z, \sum \frac{1}{N}\delta_{x_i}(z)\right).\end{aligned} \tag{A.4.6}$$

We extend the domain of φ continuously to $X \times [0, 1]$. Consequently, we can extend the domain of m from $D^e(X)$ to $D(X)$ as in Lemma A in Appendix 1 so that $\forall F \in D(X)$,

$$\varphi(m(F), 1) = \int_X d_2\varphi(z, F(z)). \tag{A.4.7}$$

Finally, let $\psi(x, p, y) = \varphi(x, p) - \varphi(y, p)$.

Case iv $(\mathscr{A}, \mathscr{B}) =$ (IRLU, IRS). For $\bar{x} \in X_{\uparrow}^N$ $(N > 3)$, define

$$\mathscr{I}_{\uparrow}(\bar{x}) = \mathscr{I}(\bar{x}) \cap X_{\uparrow}^N$$

and

$$\mathscr{I}_{\uparrow}(\bar{x})_{-i} = \{x \in {}_{\uparrow}^{N-1}\colon (x, z) \in \mathscr{I}(\bar{x}) \text{ and } z \text{ is the } i\text{th-rank component of } (x, z)\}.$$

Define $c(\cdot, \bar{x})_i \colon \mathscr{I}_\uparrow(\bar{x})_{-i} \to \mathbb{R}$ by

$$\sum_{i=1}^{N-1} \left(\frac{1}{N}\right) \delta_{x_i} + \frac{1}{N} \delta_z \sim \frac{N-1}{N} \delta_{c(x, \bar{x})_i} + \frac{1}{N} \delta_z \sim \sum_{i=1}^{N} \left(\frac{1}{N}\right) \delta_{\bar{x}_i}.$$

Axiom IRS implies that the $\preccurlyeq^{N-1}_{(x,i)}$ ordering defined by $c(\cdot, \bar{x})_i$ is completely strictly separable on $\mathscr{I}_\uparrow(\bar{x})_{-i}$ for each i. By applying Theorem A in Appendix 3, we have that, for each i, there are continuous, increasing functions $u^j_{N-1,i} \colon \pi_j(\mathscr{I}_\uparrow(\bar{x})_{-i}) \to \mathbb{R}$ such that

$$\sum_{j \neq i} u^j_{N-1,i}(c(x, \bar{x})_i) = \sum_{j \neq i} u^j_{N-1,i}(x_i, \bar{x}). \tag{A.4.8}$$

Extend the $\preccurlyeq^{N-1}_{(x,i)}$ ordering trivially to comparisons of x and y in $X^N_\uparrow$, where $x_{-i}, y_{-i} \in \mathscr{I}_\uparrow(\bar{x})_{-i}$ and $x_i = y_i$, by requiring $x \preccurlyeq y$ if and only if $c(x_{-i}, \bar{x})_i \leqslant c(y_{-i}, \bar{x})_i$. In the above extended sense, we observe that $\preccurlyeq^{N-1}_{(x,i)}$ and $\preccurlyeq^{N-1}_{(x,j)}$ coincide over comparisons of x, $y \in X^N_\uparrow$, where $x_{-i}, y_{-i} \in \mathscr{I}_\uparrow(\bar{x})_{-i}$, $x_{-j}, y_{-j} \in \mathscr{I}_\uparrow(\bar{x})_{-j}$, and $(x_i, x_j) = (y_i, y_j)$. We have

$$u^k_{N-1,i}(\cdot, \bar{x}) = u^{k'}_{N-1,j}(\cdot, \bar{x}),$$

where k and k' refer to the same rank relative to $\bar{x}$ (i.e., $k = k'$ if $i \leqslant (\geqslant)$ $k, j \leqslant (\geqslant)\, k$; $k = k' + 1$ if $i \leqslant k$, $j \geqslant k'$; $k = k' - 1$ *if* $i \geqslant k, j \leqslant k'$). In other words, the $u^k_{N-1,i}$ $(\cdot, \bar{x}) \equiv u^k_{N-1}(\cdot, \bar{x})$ functions do not depend on i. Construct

$$u^j_N(\cdot, \bar{x}) = \begin{cases} u^j_{N-1,N}(\cdot, \bar{x}), & j = 1, ..., N-1 \\ u^{N-1}_{N-1,1}(\cdot, \bar{x}), & j = N. \end{cases} \tag{A.4.9}$$

Then $\sum u^j_N(\cdot, \bar{x})$ represents the conditional ordering $\preccurlyeq^{N-1}_{(x,i)}$ for each i.

We need to show that the above representation coincides with $\mathscr{I}_\uparrow(\bar{x})$ whenever $\sum_{i=1}^N (1/N)\, \delta_{x_i} \sim \sum_N^N)\, \delta_{\bar{x}_i}$. Suppose there exists $x \in X^N_\uparrow$ with $x \sim \bar{x}$ such that

$$\sum_{i=1}^{N} u^i_N(x_i, \bar{x}) > \sum_{i=1}^{N} u^i_N(\bar{x}_i, \bar{x}). \tag{A.4.10}$$

(The reverse inequality may be treated similarly.) This is only possible if $x_i \neq \bar{x}_i$ for each i. Pick $x_k > \bar{x}_k$ and $x_j < \bar{x}_j$ such that $k = 1$ or $x_{k-1} < \bar{x}_k$ and $j = N$ or $x_{j+1} > \bar{x}_j$. Replace x_k by $x^1_k > \bar{x}_k$ such that $\sum_{i \neq k} u^i_N(x_i, \bar{x}) + u^i_N(x^1_K, \bar{x}) = \sum_{i=1}^N (\bar{x}_i, \bar{x})$. Next, replace x_j by $x^2_j < \bar{x}_j$ such that $\sum_{i \neq k,j}(1/N)\, \delta_{x_i} + (1/N)\, \delta_{x^1_k} + (1/N)\, \delta_{x^2_j} \sim \sum_{i=1}^N (1/N))\, \delta_{\bar{x}_i}$. Continuing the process, we obtain a sequence $\{y^k\} \subset X_\uparrow^N$ with the property that, when m is odd, $y^m \prec \bar{x}$ and $\sum_{i=1}^N u_N^i(y_i^m, \bar{x}) = \sum_{i=1}^N u_N^i(\bar{x}_i, \bar{x})$; when m is even, $y^m \sim \bar{x}$ and $\sum_{i=1}^N u_N^i(y_i^m, \bar{x}) > \sum_{i=1}^N u_N^i(\bar{x}_i, \bar{x})$. Observe that $x_k^{2m-1} \downarrow \bar{x}_k$ and $x_j^{2m} \uparrow \bar{x}_j$ so that $\mathrm{Lim}_{m\to\infty} y^m = y \sim \bar{x} \sim x$. This yields a contradiction. Note as in the proof of Case (ii) that (A.4.10) holds if and only if $x \succ \bar{x}$.

For each $m \in X$, we define $\mathscr{I}(m) = \{F \in D(X): F \sim \delta_m\}$. For each N, we can go through the preceding construction to obtain the $u_N^i(\cdot, m)$'s which represent the ordering on $\mathscr{I}(m) \cap X_\uparrow^N$. Construct $\psi: X \times [0, 1]^* \times X \to \mathbb{R}$ by

$$\psi\left(x, \frac{i}{N}, m\right) = \sum_{j=1}^{i} u_N^j(x, m) - \sum_{j=1}^{N} u_N^j(m, m) \tag{A.4.11}$$

and extend its domain for the second argument continuously to $[0, 1)$. It follows that, $\forall F \in D^e(X)$, the certainty equivalent $m(F)$ is given by an implicit solution of

$$\int_X d_2\, \psi(x, F(x), y) = 0. \tag{A.4.12}$$

To see that $m(F)$ is the unique implicit solution, suppose there exists $y' \in X$ satisfying $\int_X d_2 \psi(x, F, y') = 0$ but $y' \neq m(F)$. Let $F' \in D^e(X)$ be such that $y' = m(F')$ and $\int_X d_2 \psi(x, F', y') = 0$. We conclude that $\int_X d_2 \psi(x, F', y')) = \int_X d_2 \psi(x, F, y'))$. This implies $F \sim F'$ which is a contradiction.

We have shown that ψ represents $m(\cdot)$ on $D^e(X)$ in the sense that, $\forall F \in D^e(X)$, $m(F)$ is the unique solution to the equation $\int_X d_2 \psi(x, F, y) = 0$. We now show that in the same sense ψ represents $m(\cdot)$ on all of $D(X)$. Let $\{F^n\}$ be an increasing sequence in $D^e(X)$ which converges to $F \in D(X)$ so that $y^n = m(F^n)$ converges to y. Suppose

$$\int_X d_2 \psi(x, F(x), y) < 0.$$

Then the above would hold under a small lateral shift η in F denoted by $F_{+\eta}$; i.e., $\int_X d_2 \psi(x, F_{+\eta}(x), y) < 0$. This implies that for n sufficiently large, $\int_X d_2\, \psi(x, F_{+\eta}^n(x), y) < 0$, i.e., $F_{+\eta}^n \prec \delta_y$ since $\{F_{+\eta}^n\} \subset D^e(X)$. This in turn implies that $F_{+\eta} \preccurlyeq F$ which is a contradiction.

Finally, we observe that the following properties of ψ, which appear in the definition of IRLU, hold. The continuity of ψ in x and in p holds by construction. Condition (2.7) for $\psi(\cdot, \cdot, y)$ is equivalent to the increasingness of the branch utility functions $u_N^i(x, y)$. Q.E.D.

Proof of Corollary 1. Sufficiency of (4.3) is obvious. To prove necessity, suppose ψ and $\hat{\psi}$ both represent the same IRLU ordering. For each N, we construct $u_N^j, \hat{u}_N^j: X \times X \to \mathbb{R}$ via

$$u_N^j(x, y) = \psi\left(x, \frac{j}{N}, y\right) - \psi\left(x, \frac{j-1}{N}, y\right),$$

$$\hat{u}_N^j(x, y) = \hat{\psi}\left(x, \frac{j}{N}, y\right) - \hat{\psi}\left(x, \frac{j-1}{N}, y\right).$$

Since u_N^j's and $\hat{u}_N^j$'s provide additive utility representations of the same ordering over X^N, they are related by the similar positive affine transformation

$$\hat{u}_N^j(\cdot, y) = \alpha_N(y)\, u_N^j(\cdot, y) + \beta_N^j(y), \tag{A.4.13}$$

so that

$$\hat{\psi}\left(x, \frac{i}{N}, y\right) = \sum_{j=1}^{i} [\hat{u}_N^j(x, y) - \hat{u}_N^j(y, y)]$$
$$= \alpha_N(y) \sum_{j=1}^{N} [u_N^j(x, y) - u_N^j(y, y)] = \alpha_N(y)\, \psi\left(x, \frac{i}{N}, y\right).$$

Clearly, α_N cannot depend on N in the above expression. Hence, the conclusion follows by continuous extension of ψ in the second argument from the rationals to $[0, 1]$. Q.E.D.

Proof of Theorem 3. It is easier to proceed using the following statement of Condition (4.6): Given $\varepsilon, \gamma > 0$, $y \in X$.

$$\Delta(x, p, \varepsilon, \gamma, y) = \psi(x + \varepsilon, p + \gamma, y) - \psi(x + \varepsilon, p, y)$$
$$- \psi(x, p + \gamma, y) + \psi(x, p, y) \tag{4.6'}$$

is nonincreasing in (x, p), for $[x, x + \varepsilon] \times [p, p + \gamma] \subset X \times [0, 1]$. In view of the equivalence between MRA and Schur-concavity on $D^e(X)$ for a continuous, monotone utility functional on $D(X)$ [8, Corollary 2], it suffices to prove

(i): (4.6′) implies that IRLU is Schur-concave on $D^e(X)$, and

(ii): MRA implies that (4.6′) holds.

Case (i). To prove the necessity of (4.6′) for MRA, suppose $\exists p, \varepsilon, \gamma \in (0, 1)$ and $x, x', y \in X$ $(x' > x)$ such that $\Delta(x', p, \varepsilon, \gamma, y) > \Delta(x, p, \varepsilon, \gamma, y)$. Let $\kappa = mi$,

$$f\left(\frac{i}{N}\right) = \Delta\left(x, p, \kappa, \frac{i}{N}\gamma, y\right),$$

and

$$f'\left(\frac{i}{N}\right) = \Delta\left(x', p, \kappa, \frac{i}{K}\gamma, y\right).$$

Note that $f(0) = f'(0) = 0$ and $f(1) < f'(1)$.

For each N, we will show that

$$f\left(\frac{i}{N}\right) - f\left(\frac{i-1}{N}\right) \geqslant f'\left(\frac{i+1}{N}\right) - f'\left(\frac{i}{N}\right) \qquad (i = 1, ..., N-1). \quad \text{(A.4.14)}$$

Pick any N and $1 < i < N$. Consider

$$\frac{\gamma}{N}\delta_{x+\kappa} + \frac{\gamma}{N}\delta_{x'} + \left(1 - \frac{2}{N}\gamma\right)F \sim \delta_y, \qquad \text{(A.4.15)}$$

where $F(x + \kappa) = p + ((i - 1/N)\gamma$, $F(x') = p + (i/N)\gamma$, and $[x, x' + \kappa] \cap \operatorname{supp} F = \phi$. Then the IRLU of (A.4.15) is given by an expression having the form

$$A + \theta\left(x + \kappa, p + \frac{i-1}{N}\gamma, \frac{\gamma}{N}, y\right) + \theta\left(x', p + \frac{i}{N}\gamma, \frac{\gamma}{N}, y\right) + B = 0, \quad \text{(A.4.16)}$$

where

$$\theta(x, p, \gamma, y) \equiv \psi(x, p+\gamma, y) - \psi(x, p, y). \qquad \text{(A.4.17)}$$

Since $\preccurlyeq$ satisfies MRA,

$$\frac{\gamma}{N}\delta_x + \frac{\gamma}{N}\delta_{x'+\kappa} + \left(1 - \frac{2}{N}\gamma\right)F \preccurlyeq \frac{\gamma}{N}\delta_{x+\kappa} + \frac{\gamma}{N}\delta_{x'} + \left(1 - \frac{2}{N}\gamma\right)F \sim \delta_y.$$

In terms of IRLU, the above yields

$$A + \theta\left(x, p + \frac{i-1}{N}\gamma, \frac{\gamma}{N}, y\right) + \theta\left(x', p + \frac{i}{N}\gamma, \frac{\gamma}{N}, y\right) + B \leqslant 0. \quad \text{(A.4.18)}$$

Subtracting (A.4.16) from (A.4.18) yields

$$f\left(\frac{i}{N}\right) - f\left(\frac{i-1}{N}\right) \geqslant f'\left(\frac{i+1}{N}\right) - f'\left(\frac{i}{N}\right).$$

It follows that $\forall N$,

$$f\left(\frac{N-1}{N}\right) \leqslant f'(1) - f'\left(\frac{1}{N}\right).$$

Taking the limit as N tends to ∞ and by continuity of Δ in γ, $f(1) \leqslant f'(1)$ which is a contradiction. (The other case where Δ is strictly increasing in p is essentially the same.)

Case (ii). Now we prove the sufficiency of (4.6′) for the Schur-concavity of V on $D^{e}(X)$. For $(x_1, ..., x_N) \in X_{\uparrow}^{N}$, consider $m(\sum_{i=1}^{N} (1/N)\, \delta_{x_i}) = y$. Let $\varepsilon > 0$ be such that $x_{k-1} \leqslant x_k - \varepsilon$, $x_{k+1} + \varepsilon \leqslant x_{k+2}$. Then (4.6′) implies that

$$\Delta\left(x_k, \frac{k}{N}, \varepsilon, \frac{1}{N}, y\right) - \Delta\left(x_k - \varepsilon, \frac{k}{N}, \varepsilon, \frac{1}{N}, y\right) + \Delta\left(x_k - \varepsilon, \frac{k}{N}, \varepsilon, \frac{1}{N}, y\right)$$
$$- \Delta\left(x_k - \varepsilon, \frac{k-1}{N}, \varepsilon, \frac{1}{N}, y\right) \geqslant 0.$$

Rewriting the above yields

$$\begin{aligned}
&\psi\left(x_{k+1} + \varepsilon, \frac{k+1}{N}, y\right) - \psi\left(x_{k+1} + \varepsilon, \frac{k}{N}, y\right) \\
&\quad + \psi\left(x_k - \varepsilon, \frac{k}{N}, y\right) - \psi\left(x_k - \varepsilon, \frac{k-1}{N}, y\right) \\
&\leqslant \psi\left(x_{k+1}, \frac{k+1}{N}, y\right) - \psi\left(x_{k+1}, \frac{k}{N}, y\right) \\
&\quad + \psi\left(x_k, \frac{k}{N}, y\right) - \psi\left(x_k, \frac{k-1}{N}, y\right),
\end{aligned} \tag{A.4.19}$$

which is equivalent to

$$m\left(\frac{1}{N}\delta_{x_k - \varepsilon} + \frac{1}{N}\delta_{x_{k+1}+\varepsilon} + \sum_{i \neq k,\, k+1} \frac{1}{N}\delta_{x_i}\right) \leqslant y. \qquad \text{Q.E.D.}$$

Appendix 5: Multiplicative Separability

Proof of Theorem 4. We provide the proof for the IRLU case. The result for the RLU case follows. Necessity of the RLU case is implied by the existing proof. For sufficiency, observe that iff ψ is RLU and satisfies (5.1), then $\tau(x, y) = \varphi(x, 1) - \varphi(y, 1)$. Thus (5.2) is implied.

To prove the necessity of the IRLU case, suppose $\psi(x, p, y) \equiv g(p, y)\, \tau(x, y)$. Define

$$v(x, p, r, y) = \psi(x, r + p, y) - \psi(x, r, y). \tag{A.5.1}$$

Then $v(x, r, y(\equiv [g(r+p, y) - g(r, y)]\, \tau(x, y)$. Note that $v(\cdot, p, r, y)$ is proportional to $v(\cdot, q, s, y)$ for $(q, s) \neq (p, r)$.

Given (p, q, r, s, y), the α in the hypothesis of Axiom MS will be given by the unique solution of $b_2/a_2 = b_1/a_1$, where

$$b_2 = g(s+q, m) - g(s+\alpha q, m), \qquad a_2 = g(s+\alpha q, m) - g(s, m),$$
$$b_1 = g(r+p, m) - g(r+p/2, m), \qquad a_1 = g(r+p/2, m) - g(r, m).$$

Consider $v(x, p/2, r+p/2, m)$, $v(x, p/2, r, m)$, $v(x, (1-\alpha) q, s+\alpha q, m)$ and $v(x, \alpha q, s, m)$. The result follows from the observation that the certainty equivalent m given by

$$\frac{p}{2}\delta_x + \frac{p}{2}\delta_{x'} + (1-p) F \sim \delta_m \sim \alpha q \delta_x + (1-\alpha) q \delta_{x'} + (1+q) G, \qquad \text{(A.5.2)}$$

where $F(x) = r$, $G(x) = s$, and $x, x' \notin \operatorname{supp} F \cup \operatorname{supp} G$, is respectively of the form

$$A_1 + a_1 \tau(x, m) + b_1 \tau(x', m) + B_1 = 0$$

and

$$A_2 + a_2 \tau(x, m) + b_2 \tau(x', m) + B_2 = 0.$$

To prove sufficiency, define

$$v(x, p, r, m) = \psi(x, r+p, m) - \psi(x, r, m).$$

Note that multiplicative separability of ψ is equivalent to $v(\cdot, p, r, m)$'s being proportional to $v(\cdot, q, s, m)$ for $(p, r) \neq (q, s)$. Given $x \in J$, $p \in (0, 1]$, $r \in [0, 1)$, define the function c by

$$v(x-\varepsilon, p, r+p/2, m) + v(x + c(\varepsilon, x, p, r), p/2, r, m) = v(x, p, r, m).$$

Axiom MS implies that for any $q \in (0, 1]$, $s \in [0, 1)$, $\exists \alpha \in (0, 1)$ such that

$$v(x-\varepsilon, q, s+\alpha q, m) + v(x + c(\varepsilon, x, p, r), \alpha q, s, m) = v(x, q, s, m),$$

i.e., c is not dependent on (q, s). Thus

$$v(\cdot, q, s+\alpha q, m) + v(\cdot, \alpha q, s, m) \qquad \text{(A.5.3)}$$

and

$$v(\cdot, p, r+p/2, m) + v(\cdot, p/2, r, m) \qquad \text{(A.5.4)}$$

define the 'same' indifference curve through $(x, x) \in J^2$. By varying the choice of $x \in J$ and $F, G \in D(X)$ while maintaining the constancy of m, we conclude that (A.5.3) and (A.5.4) are equivalent additive utility representations for the orderings induced on $(x, x') \in J^2$ via (A.5.2). Since additive utility is unique up to a positive affine transformation and $v(m, p, m) = 0 \ \forall m$, it follows that

$$v(\cdot, q, s + \alpha q, m)/v(\cdot, p, r + p/2, m) \equiv v(\cdot, \alpha q, s, m)/v(\cdot, p/2, r, m)$$
$$\equiv \text{constant}. \qquad \text{Q.E.D.}$$

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