Cyclic Rhythm and Rhythmic Qualities: Introduction to the DFT and an Application to Steve Reich's Phase Music

Jason Yust, Spring 2018

Much of Steve Reich's music is based on repetition (often *ad lib*) of 1–2 measure rhythmic patterns. This is particularly true of his "phase" music from *Piano Phase* and *Violin Phase* in 1967 to *Octet [Eight Lines]* in 1979, and also much of the music of the 1980s, such as *New York Counterpoint* and *Electric Counterpoint*. The repetition of a consistent unit, one or two measures of 4/4 or 12/8, sometimes 10/8, with a basic pulse of and eighth-note or sixteenth, creates the framework of a rhythmic cycle. Rhythmic cycles invite an analogy with pitch classes, which are also cyclic. Hence, we can think of cyclic rhythms as "beat-class sets." (This idea was first suggested by Milton Babbitt and built upon by Jeff Pressing, Richard Cohn, and John Roeder.[1]) Here, I will propose a method of analyzing cyclic rhythms using the DFT (discrete Fourier transform). This is, mathematically, the same method that is used for determining what frequencies are present in an audio signal. The differences when using this method to analyze beat-class sets are (1) Order of magnitude: the frequencies are much slower, so that they correspond to rhythmic periodicities rather than pitches, and (2) Simplifying assumptions: Rhythmic cycles include a relatively small number of eighth or sixteenth-notes (typically 6, 8, 10, 12, 16, or 24), and they are assumed to be perfectly periodic, meaning they could repeat indefinitely.

Introduction to the DFT and Music for Pieces of Wood:

Music for Pieces of Wood (1973) is a simple process piece based on three cyclic rhythms. The rhythms cycles occur within a 6/4, 4/4, and 3/4 measure where the smallest duration is an eighth-note. I will call these a 12-cycle, 8-cycle, and 6-cycle, respectively, indicating how many basic pulses (eighth-notes in this case) occur within a cycle. Let us begin by considering the simplest rhythm, working in backwards order of how they are presented in the piece.

The 3/4 rhythm starts in clave 2 in m. 47:

The rest of the piece up to m. 59 gradually builds up four transpositions of that rhythm (adding one note at a time). In m. 51 clave 3 has:

And in m. 55 clave 4 has:

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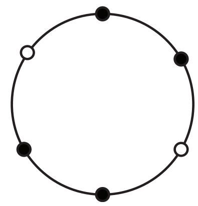
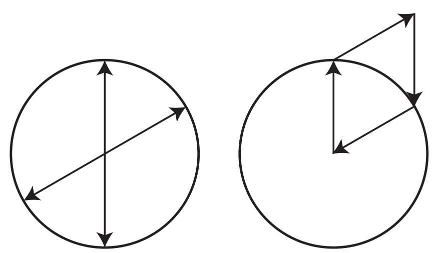
[1] Milton Babbitt "Twelve-Tone Rhythmic Structure and the Electronic Medium," *Perspectives of New Music* 1 (1): 49–79, Jeff Pressing "Cognitive Isomorphisms between Pitch and Rhythm in World Musics: West Africa, the Balkans, Thailand and Western Tonality," *Studies in Music* 17 (1983): 38–61, Richard Cohn, "Properties and Generability of Transpositionally Invariant Sets," *Journal of Music Theory* 35 (1992): 1–32, John Roeder, "Beat-Class Modulation in Steve Reich's Music." *Music Theory Spectrum* 25 (2): 275–304.

This is Reich's "phasing" process: he layers rhythms that are the same but out of phase with one another. You could also think of these as repetitive canons, where clave 3 is one ♪ behind clave 2 and clave 4 is one ♪ behind clave 3. Because the initial rhythm is cyclic, we can draw it on a circle:



Our first question about this rhythm will be, is it weighted more towards any particular part of the cycle? What we want is an average of all the positions on the circle that have onsets. The way to do this is called a "circular average." We treat each point on the circle as a vector from the center of the circle (the origin), then add all these together:

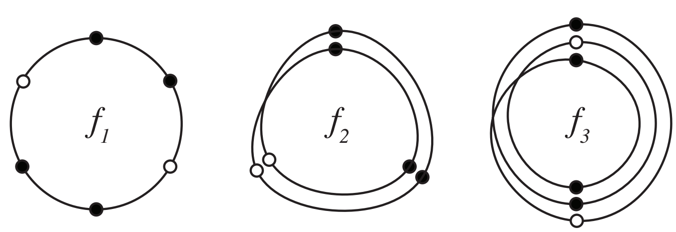


In this case, there are two pairs opposite one another that cancel one another out, and we end up back at the origin. That means that this rhythm is *perfectly balanced*. It is not weighted more towards any one part of the cycle than any other.

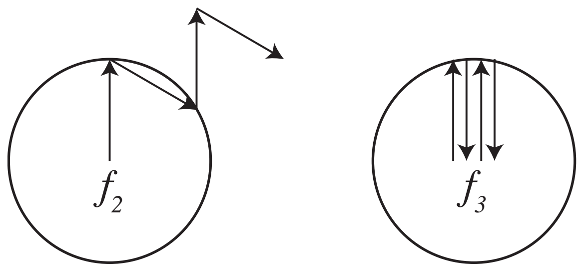
Our next question is how well the rhythm reflects a division of the whole cycle into two or three parts. In other words, if we consider a cycle the length of a dotted quarter, which divides the whole cycle into two parts, or a cycle the length of a quarter, which divides the whole cycle into three parts. To do this, we take our string of beads:

• – • – o – • – • – o –

And instead of wrapping it around the circle one time, as we did at first, we will stretch it out to wrap around the circle twice, or three times. We will call the original beat-class space the f_1 space (one time around) and call these others the f_2 (twice around) and f_3 (three times around) spaces:



Because 2 and 3 happen to divide 6 evenly, two beads can fall in the same place in the f_2 space, and there are only three places where they can end up. The f_3 space is even more limited: beads can only end up in two positions directly opposite one another. The three f_2 positions reflect the on-beat and two off-beat positions of a hypothetical 6/8 meter, where the two f_3 positions reflect the on-beat and off-beat positions of a 3/4 meter. Reich's rhythm turns out to be a perfect 6/8 rhythm: it repeats every half-cycle exactly, which is why it's perfectly balanced. The only reason it doesn't sound like a 6/8 rhythm is the presence of clave 1, which sets up a 3/4 beat before the pattern starts. If we do circular averages,

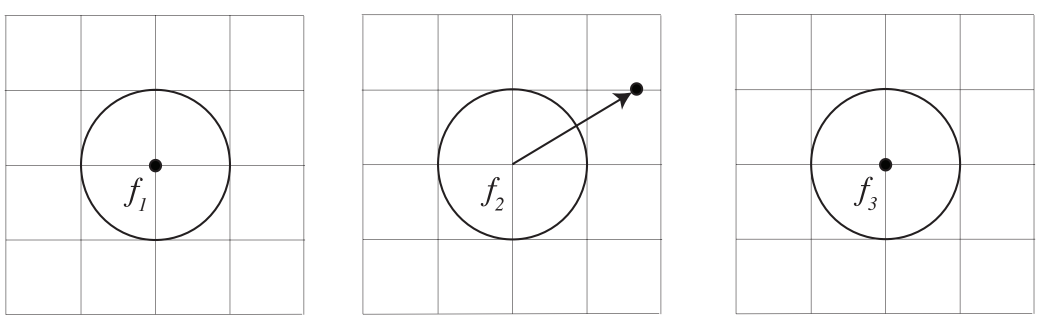


We find that the rhythm is strongly weighted in f_2 and perfectly balanced in f_3. Perfect balance in f_3 means that there are an equal number of on-beat and off-beat notes.

Reich's rhythm thus has two special properties: it is evenly distributed over the cycle (perfect balance in f_1) and it is evenly distributed over the quarter-note beat (perfect balance in f_3). Showing this by adding up vectors in three different spaces might not seem like the most straightforward way to do it, but there is an important payoff: what we have actually done is a DFT, a transformation of the rhythm from one form into another. The original form may be thought of as a string of 1s (beads) and 0s:

1 1 0 1 1 0

We transformed that into a point in each of three two-dimensional spaces:

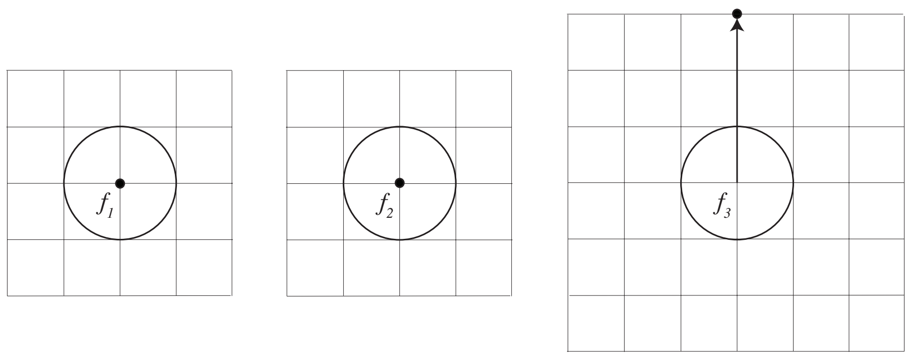


The first characterizes the rhythm on a 𝅗𝅥.-cycle (perfectly balanced), the second on a ♩.-cycle (heavily weighted), and the third on a ♩-cycle (perfectly balanced).

This transform has a lot of special properties that will turn out to be very useful. The first property is that it is a one-to-one reversible transform: the result of the transform is unique to that rhythm, and the rhythm is unique to the DFT values. That means no other rhythm is perfectly balanced on f_1 and f_3 and is weighted heavily just to the left of the downbeat in f_2 like this. The "information content" of the rhythm and its DFT are the same. A second useful property is that the "total power" of the DFT is fixed, where "power" refers to a sum of squared vector lengths. More specifically, for a given number of onsets, the following quantity is constant:

$$2|f_1|^2 + 2|f_2|^2 + |f_3|^2$$

The expression $|x|$ means "length of x," so we square the lengths of f_1, f_2, and f_3, double all except for f_3, and add these up. (f_3 is special because 3 is half of 6.) In Reich's rhythm, the f_1 and f_3 vectors have length 0, so the sum is determined entirely by f_2. In other words, the *only* way to get perfect balance in f_1 and f_3 is to have a *maximum* value for f_2. Similarly, the only way to get perfect balance in f_1 and f_2 would be to maximize f_3. In other words, it would be a DFT like this:



Which corresponds to a rhythm like this:

• – o – • – o – • – o –

Which is, in fact, the rhythm of clave 1. The spaces f_1, f_2, and f_3 can be understood as *qualities* of rhythms in the 6-cycle. The first is a general quality of balance in the cycle, the second is a kind of 6/8 quality, and the third is a kind of 3/4 quality. The properties of the DFT tell us that the qualities of a rhythm are unique to that rhythm, and a rhythm must have some mixture of qualities—either it dominated by one quality, or has some mixture of two or three qualities.

Another useful property of the DFT is that transposing (displacing) a rhythm only changes the angle of the DFT vectors, not their length. In other words, qualities (understood as the size of f_1, f_2, and f_3) are independent of transposition. Reich's phasing process therefore preserves the quality of his rhythm, while changing only its orientation in the cycle. What's more, the qualities of the combined rhythm are determined by multiplying the qualities of the original rhythm and those of the interval of transposition. In the 3/4 section of *Music for Pieces of Wood*, the rhythm is consistently translated by one eighth note. Therefore the initial canon is represented by the set:

• – • – o – o – o – o –

The elements of this set indicate where each canonic entry begins in the cycle. It has qualities (1.7, 1, 0). It has a high f_1, but the rhythm has a zero f_1, and the f_1 of the resultant canon is the multiple of these, so it is also zero. Therefore f_2 dominates the two-part canon as it does the basic ¾ rhythm. The full three part canon is represented by:

• – • – • – o – o – o –

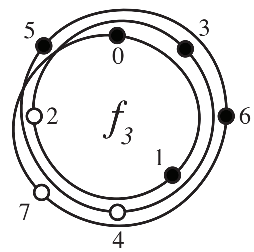
Which has qualities (2, 0, 1). Now there are zeros in all places (f_1 and f_3 in the original rhythm, and f_2 in the pattern of canonic entries). That means that the resultant rhythm is completely flat.

This illustrates two typical features of Reich's rhythmic constructions: the initial rhythm in clave 2 and the initial canon between claves 2 and 3 express a rhythmic quality that cuts against the meter (maintained by clave 1). The end goal of the process, however, is a flattening of that rhythm, when the three-part canon is completed in clave 4.

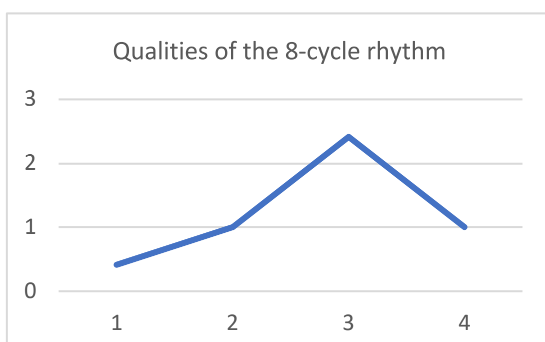
Now let's look at the 4/4 rhythm:

● – ● – o – ● – o – ● – ● – o –

This belongs to an 8-cycle, which has four qualities, f_1, f_2, f_3, and f_4. The first gives the overall balance again, while f_2 is a half-note length cycle, and f_4 is a quarter note cycle, on-beat vs. off-beat. Three, however, does not evenly divide eight, so when we wrap the necklace three times, no beads will fall in the same place:



The cycle for f_3 is the length of a triplet half-note, which is not itself available as a possible place for an onset, so no onsets can be aligned in f_3. Instead, onsets separated by a distance close to the triplet half-note, a dotted quarter, are adjacent on the f_3 cycle. As you can see above, the onsets of Reich's rhythm are all bunched together in f_3, because the rhythm can be *generated* by a dotted quarter-note interval. This means it has a *maximum* f_3, which then implies that it will be relatively balanced in the other DFT spaces. In fact, it has minimum values for all other qualities (none of which can be perfectly balanced for a rhythm with an odd number of onsets): (0.41, 1, 2.41, 1).

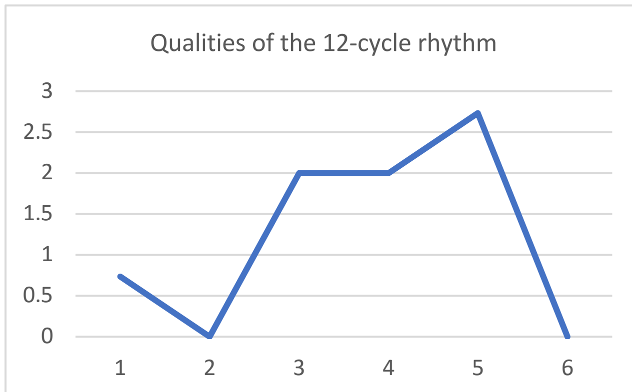


Reich's phasing process for the rhythm transposes it by a quarter note, which has qualities (1.41, 0, 1.41, 2). The combined rhythm of the two-part canon therefore has a high f_3, low f_1, and is perfectly balanced in f_2. Reich could have phased the rhythm at the dotted quarter and made f_3 even higher, but the resulting rhythms would have had a lot of overlap (with four out of five common onsets), and higher f_1 and f_2. Thus, the 8-cycle rhythm and its transpositions seem to be chosen to give the evenest possible distribution, or greatest balance, over the longer cycles, the full measure (f_1), and the half note (f_2).

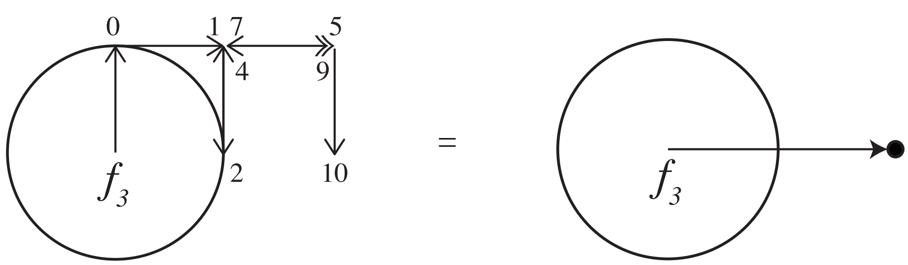
The 12-cycle of the 6/4 rhythm makes for the most direct analogy of beat-class to pitch-class. Reich's rhythm for the first section of *Music for Pieces of Wood* is analogous to an eight-note scale:

● – ● – ● – o – ● – ● – o – ● – o – ● – ● – o –

The analogous scale might be described as mixolydian with an added $\flat\hat{2}$, or the complement of a minor seventh chord. The 12-cycle has six qualities.



The rhythm has some interesting properties: it is perfectly balanced in f_2 and f_6, and its largest quality is one that does not divide the twelve cycle evenly, f_5. The other components correspond to cycles in multiples of ♪: the full 6/4 measure (f_1), the dotted half (f_2), the half (f_3), the dotted quarter (f_4), and the quarter (f_6). The rhythm has pretty strong half-note (f_3) and dotted quarter-note (f_4) qualities, in addition to the high f_5. You might think the half-note quality would produce a sense of meter in combination with the steady quarters of clave 1 (all f_6). But its phase, which is weighted towards the second eighth of each half of the measure, is not aligned with clave 1's beat:



Though this rhythm has a high f_5, Reich could have used a slightly different 8-note rhythm with a higher f_5:

This rhythm is not as balanced though, particularly on f_2, which is its next highest quality after f_5.

This 12-cycle rhythm of *Music for Pieces of Wood* is also the basis of Reich's *Clapping Music* in its entirety. In *Clapping Music*, there are only two parts and they go through every possible phase relationship, and this process constitutes the entire piece. We might say that the rhythm allows for multiple metrical feels, particularly those that divide the measure into three or four parts, and by going through all phase relationships we hear the complete kaleidoscope of possibilities.

In *Music for Pieces of Wood*, Reich's choice of phasing for the 12-cycle rhythm is a half-measure (dotted half), which is an interesting choice particularly because it can't be reiterated to produce more than one other new rhythm. The qualities of the dotted half note are (0, 4, 0, 4, 0, 4). Its zeros (in f_1, f_3, and f_5) complement the zeros of the basic rhythm (in f_2 and f_6) so that the combined rhythm has only one quality, that of f_4, or the dotted quarter. The overall effect of the combined rhythm is therefore that of a 3-against-2 cross rhythm with the foundation provided by clave 1. This is exactly the same effect as the 6-cycle rhythm, which is a pure f_2 rhythm (f_2 is also a dotted-quarter cycle). If the three rhythms of the piece thus feel very closely related or similar, that is because in fact they are. In fact, the resultant rhythms of the claves 2 and 3 in the 12-cycle (m. 10) and the 6-cycle (m. 51) are the same, counting the number of onsets for each position in the cycle:

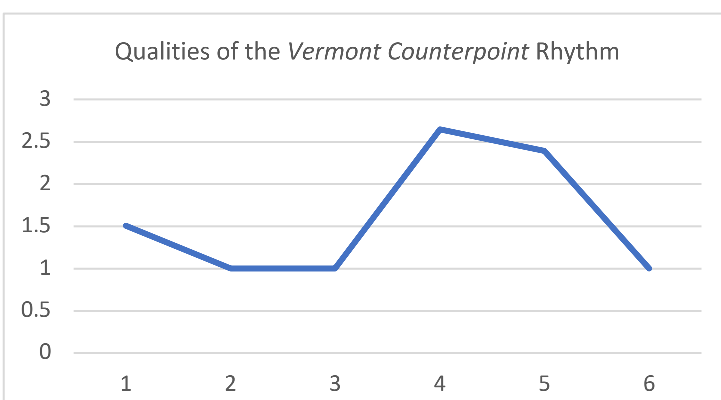
12-cycle: 1 2 1 1 2 1 1 2 1 1 2 1 6-cycle: 1 2 1 1 2 1

In the 8-cycle there is no quality corresponding precisely to a dotted quarter, but the predominant quality of Reich's rhythm is f_3, whose cycle of a triplet half-note is approximated by a dotted quarter. One might even go so far as to posit a kind of ternary form for the piece, with the irregular f_3 quality of the 8-cycle contrasting with the regular f_4 and f_2 qualities of the 12-cycle and the 6-cycle.

Reich uses a very similar rhythm as the basis of a later work, *Vermont Counterpoint*, where the phasing process forms the basis of a more complex design. The piece is written in 3/4 with the basic unit of a sixteenth, as opposed to 6/4 with a basic unit of an eighth. The rhythm is otherwise the same as one in *Music for Pieces of Wood* and *Clapping Music* except it omits the note in the sixth position (a sixteenth after beat 2 in 3/4):

• – • – • – o – • – o – o – • – o – • – • – o –

The qualities of this rhythm are similar to the 8-note rhythm, except that the change emphasizes f_4 over f_5, f_3 no longer stands out, and nothing is perfectly balanced:



Despite the lack of a prominent f_3, the contour of the line (with a high point on beat 2) initially supports the notated 3/4 meter. In the phasing process, though, Reich transposes the rhythm by a dotted eighth, which would be the equivalent of a dotted quarter in the 6/4 version (interval 3). The qualities of interval 3 are (1.41, 0, 1.41, 2, 1.41, 0). When the texture builds to three voices in canon, the relationship between the parts is:

• – o – o – • – o – o – • – o – o – o – o – o –

Which has qualities (1, 1, 1, 3, 1, 1). In other words, the phasing process preserves all the qualities of the basic rhythm except f_4, which it enhances. The effect at the beginning of *Vermont Counterpoint* is the sense of a gradual emergence of a 12/8 meter out of the 3/4.

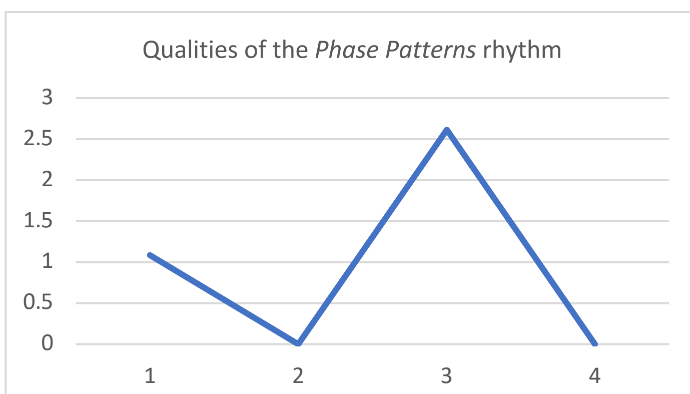
Phase Patterns

One of Reich's early phase pieces is *Phase Patterns*, 1970. The special properties of the rhythm for this piece were discussed by Richard Cohn (1992), who drew upon Babbitt's pitch-class/beat-class duality. We can reproduce Cohn's findings using the DFT.

The basic rhythm of the piece is a 4/4 (8-cycle) pattern:

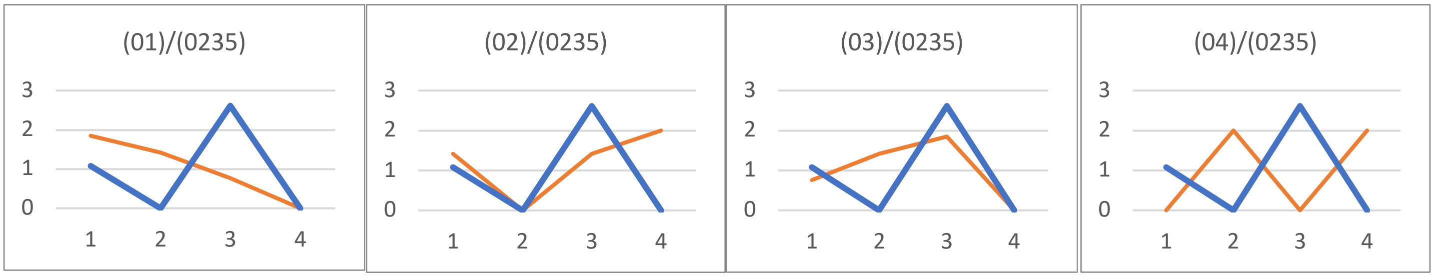
Each part plays this rhythm in the right hand, with its transposition by 4 in the left hand:

A special property of the rhythm is that its transposition by 4 is also its complement—i.e., it has an onset at every point in the eighth-note grid where the right hand has a rest. The two transpositions therefore interlock without overlap. This property is observable in the DFT:



The rhythm is perfectly balanced on even-numbered qualities f_2 and f_4, while the interval of transposition (a half-cycle) is perfectly balanced on all odd-numbered qualities (f_1 and f_3). Therefore the combination is perfectly balanced on all qualities, a perfectly flat rhythm.

The piece is one of Reich's "pure process" pieces, similar to *Clapping Music* and *Piano Phase*. The process is for organ 2 to phase shift with respect to organ 1 in a series of stages, so that all transpositional relationships are heard in order. The properties of each of these are determined by the qualities of each interval:



Qualities of the *Phase Patterns* rhythm $(0235)_8$ (blue) compared to all of the 8-cycle interval types (orange).

Transposition 1 evens out f_1 and f_3 qualities; transposition 2 reinforces both of these qualities; transposition 3 reinforces f_3 more strongly; and transposition 4 flattens the entire rhythm. Thus, the phasing process gradually brings the rhythms into greater alignment until we get to transposition 3, where the combination is close to the original pattern itself. Then at the last stage, the rhythms suddenly lock into pure complementation, so that all is heard is a flat repetition, and the original rhythm can only be seen, not heard.

Eight Lines

Reich's *Eight Lines*, a revised version of the 1969 *Octet*, is an example of a similar process using a 10-cycle, a 5/4 meter with the basic unit of an eighth-note. The initial rhythm is a single-measure cycle:

• – • – • – • – o – • – • – o – • – o |

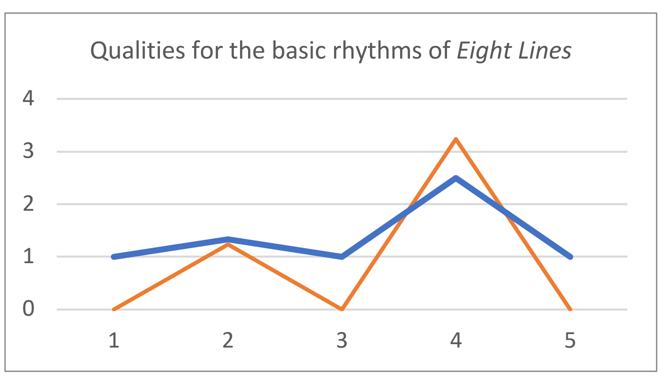
However, for most of the composition this rhythm alternates with a similar one:

• – • – o – • – o – • – • – o – • – o |

Making a 2-measure (20-cycle) pattern:

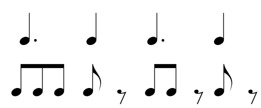
| • – • – • – • – o – • – • – o – • – o | • – • – o – • – o – • – • – o – • – o |

The two single-measure rhythms are very similar, with the main difference being that the second is perfectly symmetrical at a half-measure transposition (5♪s). If we look at the two rhythms as different 10-cycles we get the following DFTs:



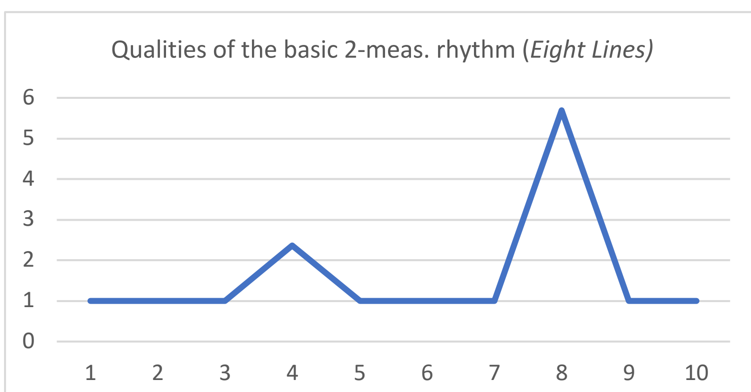
The first rhythm is in blue. It has the same prominent components as the second rhythm (in orange), f_4 first and f_2 next. The second rhythm, because it is symmetrical, has zeros for odd-numbered qualities, whereas the first rhythm has 1s for all of these.

The 10-cycle is interesting because it not only has qualities that divide the cycle perfectly (f_2 and f_5), and a quality that shares no factors with the full cycle (f_3) but also one that shares a factor, but doesn't divide the cycle perfectly (f_4). Reich's rhythmic foundation clearly focuses its energy on f_4, which is unlike any quality available in an 8-cycle or 12-cycle. An even division of the 5/4 measure into 4 parts is a ♩‿𝅘𝅥𝅯-length cycle. This lines up with the eighth-note grid at the midpoint of the measure (the "and" of three, at 𝅗𝅥‿♪). At other places, it is halfway between two points on the grid. The ♩‿𝅘𝅥𝅯 is approximated by either a ♩ or a ♩., which means that high f_4 rhythms are those that imply aksak-type meters of alternating ♩ and ♩., such as ♩. ♩ ♩. ♩ or ♩ ♩. ♩ ♩. Reich's rhythm can be understood as an elaboration of the first pattern:



However, the contour of the lines reinforces the notated meter (5/4) by emphasizing the notes on beats 1, 4, and 5 (the low note, high note, and chord) so the f_4 quality (like the prominent qualities of the rhythms in *Music for Pieces of Wood*) should be understood as giving a counter-metrical feel to rhythm, rather than determining the meter.

Another way to look at the pattern is to consider it a two measure, 20-element cycle:



This pattern reflects the same essential properties as the 10-cycle rhythms. All qualities are low though non-zero, except f_8 and f_4, which correspond to f_4 and f_2 in the 10-cycle rhythm (dividing two measures into 8 is the same as dividing one measure by 4, etc.).

Eight Lines is in five sections altogether. Throughout, the basic canons are always in the pianos, with the strings and winds reinforcing elements to create different textures. Here's a summary using Reich's rehearsal numbers:

Introduction (0–3): Two-part canon using the basic 10-cycle rhythm

Part 1 (4–8): Builds from a two- to a three-part canon on the 20-cycle rhythm
(9–13): 3-part canon continues in pianos while strings/winds pick out resulting patterns.

Transition (14–15): The lower parts of the canons fade out and a new upper part fades in, up a fifth, changing the key signature by one sharp.

Part 2 (16–27): A new part is added in [18] (with 9 onsets) and is filled out in [19] to the full (13 onset) rhythm, making a new three-part canon, which persists through the section accompanied by chords in the strings.

Transition (28–30): Lower parts of canon fade out.

Part 3 (31–39): Key goes up by fifth again and a new part is added a sixth below. Reich builds a new three-part canon gradually.
(40–43): Canon persist as strings/winds pick out resultant patterns. The parts are diatonically transposed at [43].

Transition (44–46): Lower parts fade out and a new lower part fades in.

Part 4 (47–51): A new three-part canon gradually builds.
(52–57): Canon persists accompanied by held chords in strings.

Transition (58–67): Upper parts fade out, new lower part fades in.

Part 5 (68–72): A new lower part is gradually introduced, but it is rhythmically in sync with the upper part, so that only a two-part canon results.
(73–74): Winds pick out resulting patterns.

The framework for the entire form is thus the four three-part canons of Parts 1–4, all based on the 20-cycle rhythm. This is framed by two-part canons in the introduction and Part 5, and in between the sections as one part is taken out and another introduced.

Here are the relationships between parts in all of these canons, represented by where the initial bass note falls in the 10-cycle or 20-cycle:

Introduction: ●–o–o–o–●–o–o–o–o–o (04)

Part 1: ●–o–o–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o (04)

●–o–o–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–●–o (026)

Part 2: ●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–●–o (02)

●–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–●–o (024)

Part 3: ●–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o (02)

●–o–●–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o (024)

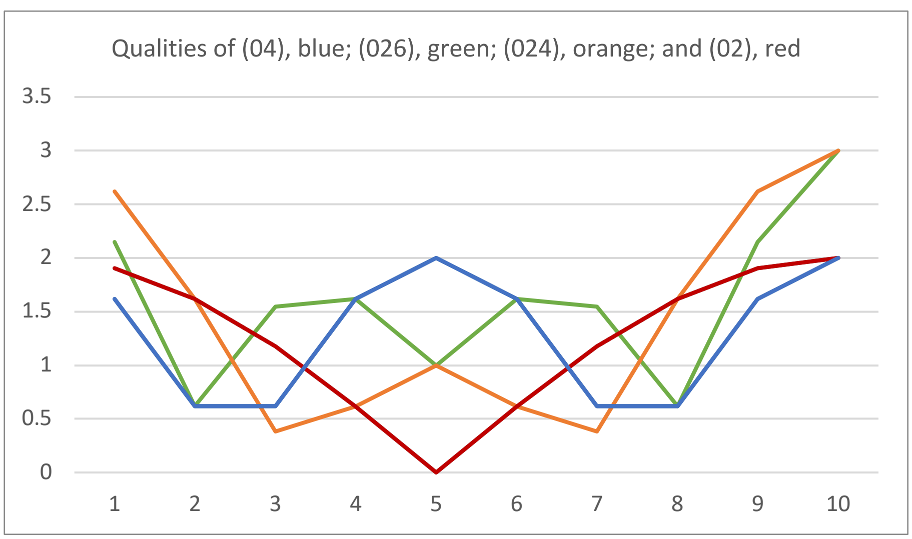
Part 4: ●–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o (02)

●–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–●–o–o–o (026)

Part 5: ●–o–o–o–●–o–o–o–o–o–o–o–o–o–o–o–o–o–o–o (04)

The rhythmic process thus represents a basic arch-form plan, gradually shifting from canons involving distances of 4 (half note) to distances of 2 (quarter note) and back. To the left are set-class labels, prime forms, for each of the canonic-entry sets.

Here are the DFT qualities of each of these sets in the 20-cycle:



One important property of DFT qualities is that their size is determined only by the intervallic properties of the set, so both of the (02) canons, both (024) canons, and both (026) canons have the same overall DFT profile. The sets all have a maximum value for f_{10}, which divides the two measures into quarter notes, because all elements fall exclusively on quarter-note beats. (The maximum is higher for the 3-element sets than the 2-element sets.) The canonic process therefore introduces a strong 5/4 rhythm not present in the basic rhythm itself (but suggested by the contour of the line).

The other points of interest are f_4 and f_8, because these are the prominent elements of the basic rhythm, and f_5, because it varies most dramatically between the different canonic arrangements.

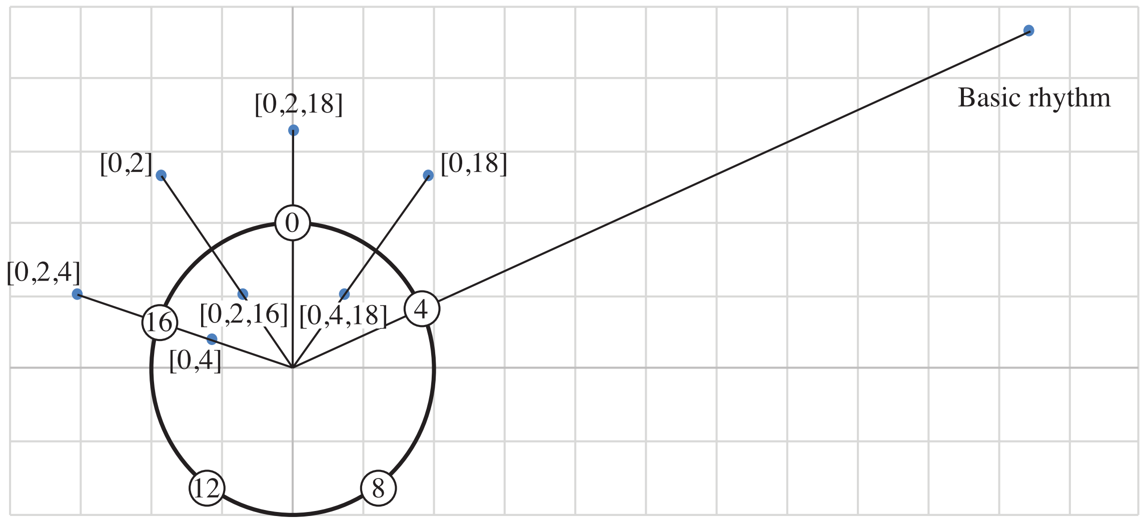
For f_4 and f_8, the two-part (04) canon behaves exactly line the three-part (026) canon, and the (02) canon behaves just like (024). The half-note-based canons ((04) and (026)) enhance f_4 and depress f_8, whereas the quarter-note-based canons ((02) and (024)) do exactly the opposite, enhancing f_8 and depressing f_4. The f_8 quality is the strongest in the basic rhythm, and the one that divides each measure into four cycles of length ♩‿𝅘𝅥𝅯. The f_4 quality is secondary and divides each measure in half (a 𝅗𝅥‿♪-length cycle). We may understand f_8 as a quality more strongly dissonant with the 5/4 meter, so the arch form of the piece is one of intensifying dissonance in the middle sections. Reich's choice of basic rhythm allows him maximum latitude to manipulate this dissonant metrical quality through the process of layering canons.

The other interesting quality is f_5, which is a division of the two measures into 5 half notes, a kind of 5/2-meter quality. The three-part canons do not effect this quality at all, but the two-part canons have divergent effects on it, with (04) bringing it out and (02) eliminating it entirely. This quality therefore only appears at the beginning and end of the piece. However, unlike f_4 and f_8, no corresponding quality exists in the 10-cycle, so the f_5 quality is totally dependent upon the difference between the two measures of the 20-cycle pattern (the extra note in the first measure). The f_5 quality also appears prominently in some of the longer lines and canons in the strings, particularly the canon in [1–3] and the rhythm that appears throughout part 2 [16–30] and part 4 [46–61].

I have focused primarily on the size of different rhythmic qualities in all of the analysis above, rather than their phases. In *Eight Lines*, certain canons have the same type of arrangement, related by transposition or retrograde. There are two transpositionally related (02) canons, two transpositionally related (024) canons, and two retrograde related (026)-type canons. While these have the same DFT profiles (allowing us to generalize above), they differ in phase, the orientation of each quality with respect to the downbeat.

The qualities of primary interest, f_4 and f_8, are always counter-metrical; the meter itself, the quarter-note beat and location of the downbeat, is never in doubt throughout the piece. Therefore, the alignment of each of these qualities with the downbeat is of primary interest. Another dimension of dissonance is added to these qualities when they are displaced from the downbeat. (Note that the basic cycle of both qualities goes in and out of alignment with the quarter-note layer once per measure, so alignment with one downbeat always implies alignment with the other.)

The graph below shows the basic rhythm of *Eight Lines* and the canon-configuration sets in f_8 space. The size of f_8 for each of these is its distance from the origin, and the phase is the place where the line from the origin crosses the unit circle.



Values of f_8 for the basic rhythm of *Eight Lines* and the canon configurations

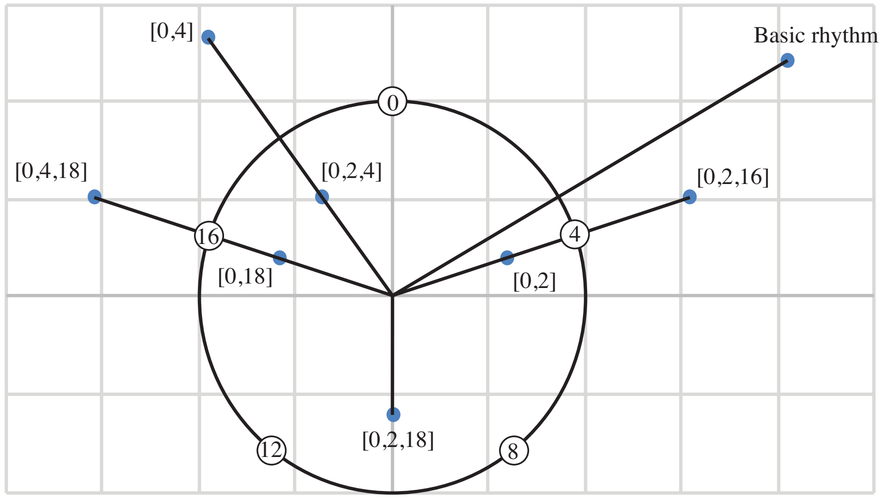
By convention, we divide the full cycle into 20 parts (corresponding to the division of the f_1 cycle). Since the f_8 cycle is of length ♩‿♪, each 4 units correspond to a sixteenth-note displacement. The original rhythm is aligned to phase number 4, because its basic pattern of ♩. ♩ is aligned with the downbeat, and the asymmetry nudges the pattern a little ahead of the beat.

We have been using something called the "convolution theorem" throughout all of these analyses, which says that combining a rhythm with its transpositions multiplies that rhythm's Fourier qualities by the qualities for the interval of transpositions or set corresponding to the transpositions. The other part of the convolution theorem says that the phase of the combination is the *sum* of the phases of the convolved sets (the basic rhythm and the set of transpositions). This fact allows us to see what the different canon-configurations will do to the basic rhythm. The [0, 2, 4] and [0, 4] canons will bring therefore the overall f_8 into perfect alignment with the downbeat (4 + 16 = 0 mod 20), while sets [0, 18] and [0, 4, 18] will throw it further ahead of the downbeat. The process over the course of the piece can then be understood from the perspective of the phase of f_8 as:

(1) Begin with [0, 4] putting the counter-metrical f_8 quality in alignment with the downbeat.
(2) Introduce position 18, throwing f_8 quality out of alignment. Removing position 4 then strengthens the displaced f_8 quality.
(3) Introduce position 2, which counteracts position 18
(4) Bring back position 4, which returns f_8 to alignment with the downbeat, first as a strong quality, in the [0, 2, 4] canon, then returning to its weak state in the [0, 4] canon.

This further enhances the arch form we found when just considering the strength of quality. In the middle sections based on the same quarter-note-type canons (the ones that strengthen f_8), the size of f_8 is consistent, but it undergoes a phase shift, from misalignment in Part 2 ([0, 18] and [0, 4, 18] canons) to realignment in Part 3 ([0, 2] and [0, 2, 4] canons).

We can draw similar conclusions about the f_4 phases:



Values of f_4 for the basic rhythm of *Eight Lines* and the canon configurations

The basic rhythm is also ahead of the beat, here by about an eighth-note (since f_4 is a 𝅗𝅥 ♪-length cycle), meaning that is aligned with beat 4 (the high note of the line). Position 4 plays the same moderating role for f_4 that it does in f_8, bringing that quality in alignment with the downbeat. The main disruptive influences in this case (throwing f_4 further ahead of the downbeat), however, are positions 2 and 16. Therefore when f_4 is prominent at the beginning of the piece (in the [0, 4] and [0, 4, 18] canons) it is aligned with the downbeat, but when it returns to prominence in Part 4, in the [0, 2, 16] canon, it is distinctly out of alignment. Therefore, this is a dissonant element to the rhythmic pattern that takes over after the dissonance of f_8 abates, and resolves only in Part 5.

In all of these pieces we see a common strategy to how Reich constructs his rhythm and phasing process. All of them establish a clear and consistent meter with a quarter-note beat. The basic rhythms, however, have a strong anti-metrical rhythmic quality. The phasing process then manipulates this anti-metrical quality, intensifying it and then, often, suddenly eliminating it entirely. In some cases the displacement of a rhythmic quality from the meter may also play a role in giving a sense of rhythmic dimensionality or dissonance.