

The Geometry of Melodic, Harmonic, and Metrical Hierarchy

Abstract. Music is hierarchically structured in numerous ways, and all of these forms of organization share essential mathematical features. A geometrical construct called the Stasheff polytope or associahedron summarizes these similarities. The Stasheff polytope has a robust mathematical literature behind it demonstrating its wealth of mathematical structure. By recognizing hierarchies that arise in music, we can see how this rich structure is realized in multiple aspects of musical organization. In this paper I define hierarchic forms of melodic, harmonic, and metrical organization in music, drawing on some concepts from Schenkerian analysis, and show how each of them exhibits the geometry of the Stasheff polytope. Because the same mathematical construct is realized in multiple musical parameters, the Stasheff polytope not only describes relationships between hierarchies on a single parameter, but also defines patterns of agreement and conflict between simultaneous hierarchies on different parameters. I give musical examples of conflict between melodic and rhythmic organization, and show how melodic and harmonic organization combine in melody and counterpoint.

1 General Characteristics of Musical Hierarchy

Hierarchies operate in a number of different modalities in tonal music, and forms of musical hierarchy generally share some essential properties. This fact opens the door to a rich application of a mathematical conception of hierarchy to music.

Hierarchical accounts in the basic modalities melody, harmony, and metrical rhythm play crucial roles in understanding tonal music. All of these types of hierarchy share principal underlying features: (1) they involve objects ordered in time, (2) they are better represented by hierarchies on intervals between successive objects rather than the objects themselves, and (3) they are accurately captured by strictly binary bracketings. Together these features imply a robust mathematical description captured by the geometry of the Stasheff polytope or associahedron.

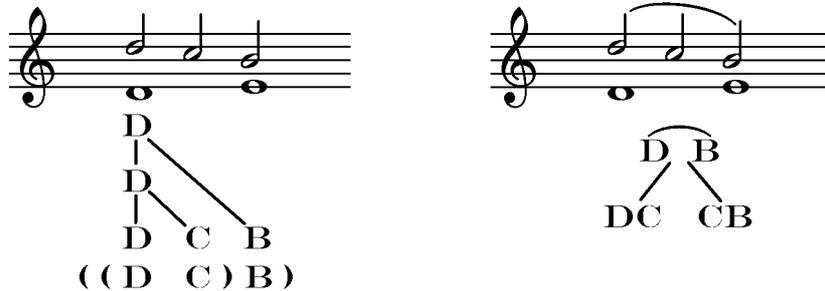
The graph-theoretical object of a *rooted tree* represents the concept of hierarchy in its most general sense. Property (1) for musical hierarchies implies that they correspond more specifically to *plane trees*, trees in which the children of each node are ordered from left to right (where for our purposes left corresponds to prior in time, et c.). Plane trees are equivalent to ways of bracketing series of objects (the objects corresponding to the leaves of the tree).

As for property (2), consider the passing-tone figure in Figure 1. Formal accounts of Schenkerian theory often represent the hierarchy of this type of musical figure with a plane tree that takes musical notes as its objects, as in Figure 2 (Cohn and Dempster 1998, Marsden 2005, Lerdahl and Jackendoff 1983, Rahn 1979, Smoliar 1980). This

type of hierarchy involves an arbitrary assignment of either of the consonant tones as the parent of the dissonant one, obfuscating the symmetry of the musical figure. Larson 1997 criticizes this aspect of the theory of Lerdahl and Jackendoff 1983, and Lerdahl's response (1997) is dismissive and inadequate. Proctor and Higgins 1988 and Yust 2006 have pointed out that such "reductionist" theories inaccurately represent Schenker's own analytical practice. In more recent computational applications that take this formal model as the standard (Marsden 2005 and 2007), the requirement for multiply-embedded arbitrary decisions results in a combinatorial explosion within the acceptable analysis of short musical phrases. Yust 2006 proposes a solution to the interpretive awkwardness of hierarchies with notes as objects: bracketing intervals between successive notes, rather than the notes themselves, as shown in Figure 3.



Fig. 1: The passing tone of second species counterpoint.



Figs. 2-3: Plane trees with musical notes or motions between notes as objects.

Meter presents a precisely analogous situation: it is not a bracketing on timepoints but on timespans (intervals between timepoints). In the rhythm of Figure 1, a whole-note duration divides into two half-note durations; the weak-beat timepoint does not group with either the preceding or following strong beat by virtue of the meter. Unlike models of tonal hierarchy proposed by music theorists, models of metrical hierarchy typically take intervals between events (durations) as their objects rather than the events themselves (timepoints). Presumably the difference treatment of parameters comes from musical notation, which reifies durations and pitches rather than timepoints and intervals. As a result, studies that discuss relationships and

conflict between meter and tonal structure (Komar 1971; Lerdahl and Jackendoff 1983; Rahn 1979; Schachter 1976, 1980, 1987; Yeston 1976) present models in which time and pitch are treated in fundamentally different ways. Construing tonal hierarchy in terms of intervals rather than notes corrects this situation, so that rhythmic and tonal patterns can be compared directly in terms of hierarchic structure.

Property (3) provides a useful heuristic for present purposes. Although non-binary relationships can exist in melodic, harmonic, and rhythmic hierarchies, binary structures are the most important and demonstrate the essential properties of hierarchy in these modalities. Therefore I will focus here on the binary plane tree, in which all nodes that are not leaves (called *internal nodes*) have exactly two children, a left child and a right child. The application of the Stasheff polytope is more general however, describing the relationships between all possible plane trees. (See the end of section 3 below).

Though the objects the musical hierarchies explored here are intervals rather than more concrete musical objects, networks visually convey their musical implications more clearly when the nodes correspond to concrete objects. The bijective equivalence of binary plane trees and *triangulations of polygons* makes it possible to create a network on musical objects that gives a hierarchy on directed intervals between them. Figure 4 illustrates this bijection—which replaces the nodes of the tree with edges—and shows how to redraw the polygon to show the left-to-right ordering on the vertices clearly. Note that internal edges in the triangulation (which correspond to internal nodes in the tree) can be described as upward- or downward-slanting.

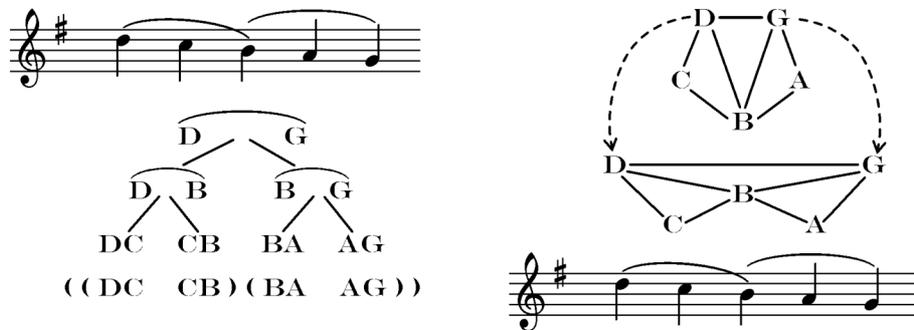


Fig. 4: The bijection between binary plane trees and triangulations of a polygon as representations of musical hierarchy

2 Harmonic, Melodic, and Metrical Forms of Hierarchy

2.1 Harmonic Hierarchy

Schenker (1987) had the fruitful insight of taking the passing-tone figure of second species counterpoint as a paradigm for hierarchical relationships in music. However, the passing-tone figure can only serve as a paradigm of tonal hierarchy if it is somehow generalized, because literal passing figures alone do not embed one another recursively. Schenker (1979, 1997) addressed this by harmonizing the dissonant passing note, as in the *Ursatz* of Figure 5, and *horizontalizing* the resulting vertical consonance so that he could then fill it with new passing motion, as shown in Figure 6. The process implies a tonal hierarchy, also shown in Figure 6, that generalizes the passing-tone figure as a *symmetrical binary division of step-class intervals*. That is, the passing tone divides the step-class interval of a third into two equal step-class intervals (seconds) and similarly, the second divides into two fifths, the harmonization of the passing tone, the fifth divides into two thirds which complete the triad that supports the passing tone, and new passing motion then fills the horizontalized thirds. (This process of division is an order-3 automorphism of the group of step-class intervals, a cyclic group of order seven; see Yust 2007)



Fig. 5: Schenker's *Ursatz* (From *Der Freie Satz*).

I will call this type of hierarchy *harmonic structure*. Because its building blocks include triads and series of fifths, it is useful for describing harmonic patterns in tonal music, particularly sequential harmonic patterns. Its main shortcoming, from the perspective of music analysis, is also one of its assets: it cuts across compositional voices and registers, collapsing tonal content into a single hierarchy that can only show one melodic progression at a time. Therefore, a description that recognizes counterpoint and the continuity of voices requires a different form of hierarchy on tones, a purely melodic type of hierarchy, to complement this one.

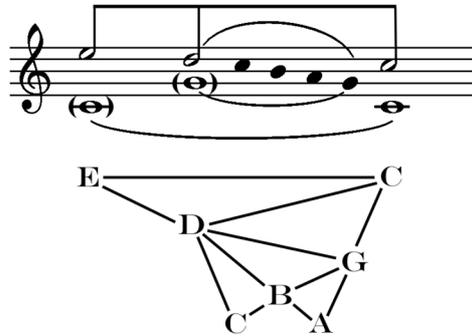


Fig. 6: Schenker's *Ursatz* with a horizontalized dominant.

2.2 Melodic Hierarchy

A melodic hierarchy is one on tones strictly ordered in time and belonging to one voice. One particularly useful melodic hierarchy emerges from the principles, first, that the line it represents should be conceived as constantly *in motion* at every level (so that the line can include no repeated tones or neighboring motions); and second, that the hierarchy reflect the process—characteristic of Schenkerian analysis—of locating stepwise connections and stepwise passing motions within a musical line at various levels.

Again, the passing-tone figure is a model, and the problem of its inability to embed itself reappears. Expanding seconds with incomplete neighbors gives a solution that prioritizes stepwise motion. Figure 7 shows two such possibilities for the descending second. We can equate these different incomplete neighbor figures under a system of equivalence classes consisting of notes separated by diatonic fourths. These equivalence classes identify the position of tones within a system of conjunct tetrachords, so we can call them *tetrachordal positions*. For melodies of restricted range, a sequence of three distinct tetrachordal positions may represent a literal stepwise passing motion, any incomplete neighbor figure involving a leap of a third, or arpeggiations of a close-position triad. Unlike harmonic hierarchies, melodic hierarchies built out of tetrachordal passing motions do not recognize octave equivalence, and so are most useful in confined registral ranges.



Fig. 7: Incomplete neighbor figures for melodic elaboration.

2.3 Metric Hierarchy

Another useful form of hierarchy is the *metric* hierarchy of an unsyncopated rhythm. For present purposes, we will consider only strictly binary metrical schemes. (See the end of the next section regarding mixed binary-triple schemes). An exemplary binary metrical scheme is 4/4 meter, possibly extending hypermetrically to regular four- and eight-bar phrases and so forth. An *unsyncopated rhythm* within such a scheme is a succession of timepoints (1) that begins and ends on the metrically strongest timepoints within its span, and (2) where no unarticulated timepoint is stronger than the articulated one immediately preceding it. Like the assumption of strict binary organization, the no-syncopation assumption sacrifices musical generality for mathematical simplicity and can be corrected after the basic mathematical framework is established.

3 Musical Realizations of the Stasheff Polytope

These three forms of hierarchy enable a musical exploration of the Stasheff polytope. The vertices of the Stasheff polytope correspond to the possible binary hierarchies, and its geometrical properties represent much of the structure implicit in the concept of hierarchy. The work of mathematician Jean-Louis Loday and others demonstrates the great depth of this structure. (Stasheff 1963; Loday 2004, 2005, 2007; Loday and Ronco 2002) This convex polytope can be constructed in any dimension. In n -dimensional space, the vertices of the polytope correspond to binary trees with $n + 2$ leaves, or triangulations of an $(n + 3)$ -gon.

For example, let us construct the two-dimensional Stasheff polytope, whose points correspond to the binary plane trees with four leaves or the triangulations of a pentagon. Figure 8 shows the possible hierarchies on series of five musical objects, both as trees and as triangulations. We define a point in three-dimensional space for each triangulation where the coordinates of the point correspond to the vertices of the triangulation from left to right (which in turn correspond to musical events in the sequence) excluding the first and last (uppermost) vertices. The value for each coordinate is given by taking the highest edges to the left and right of that point, counting the number of boundary edges below them, and multiplying the two numbers. Figure 8 shows the resulting coordinates for triangulations of the pentagon.

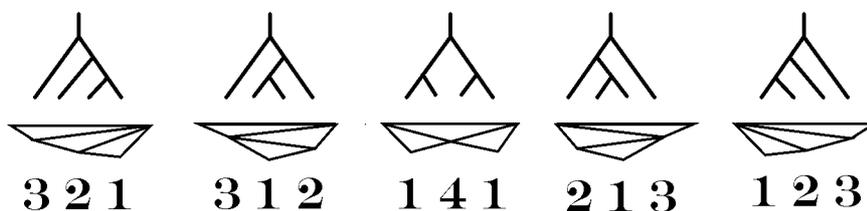


Fig. 8: The possible hierarchies on five events as trees and triangulations

The Stasheff polytope is the convex hull of these points. The dimension of the polytope is one less than the number of coordinates because the points are constrained to lie on the plane $\sum x_i = n(n + 1)/2$ (in dimension n). In musical examples, the numbers define relative weights (e.g., metrical accent) for the corresponding objects. The edges in the geometry represent elementary “flip” operations that move exactly one internal edge in the triangulation. A *leftward* flip replaces an upward-slanted edge with a downward-slanted one; a *rightward* flip does the opposite. Figure 9 gives the 2-dimensional polytope, with arrows pointing in the direction of leftward flips; as Loday (2007) observes, they establish an important partial ordering on the hierarchies.

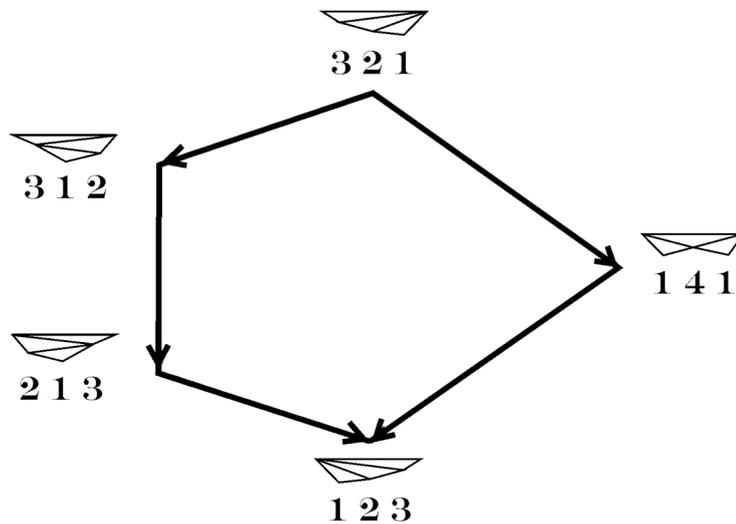


Fig. 9: The two-dimensional Stasheff polytope.

It is possible to define operations that relate rhythmic and melodic sequences whose hierarchies share an edge in the polytope (i.e., are related by a flip). Figure 10 presents a series of melodies that traverse the two-dimensional polytope starting at (1 4 1) and moving counterclockwise around the perimeter until arriving back at (1 4 1) with simultaneous rhythmic and tonal operations. Figure 11 shows the Stasheff polytope in three dimensions. Figure 12 gives an (arbitrary) sample traversal that circles the three-dimensional polytope through a series of flips in coordinated rhythmic-melodic structures.

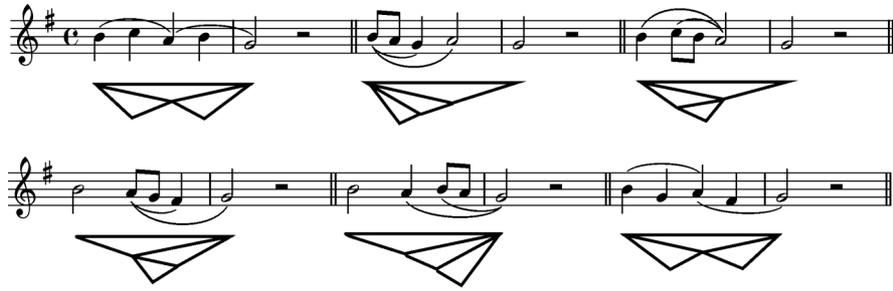


Fig. 10: A melodic traversal of the two-dimensional Stasheff polytope.

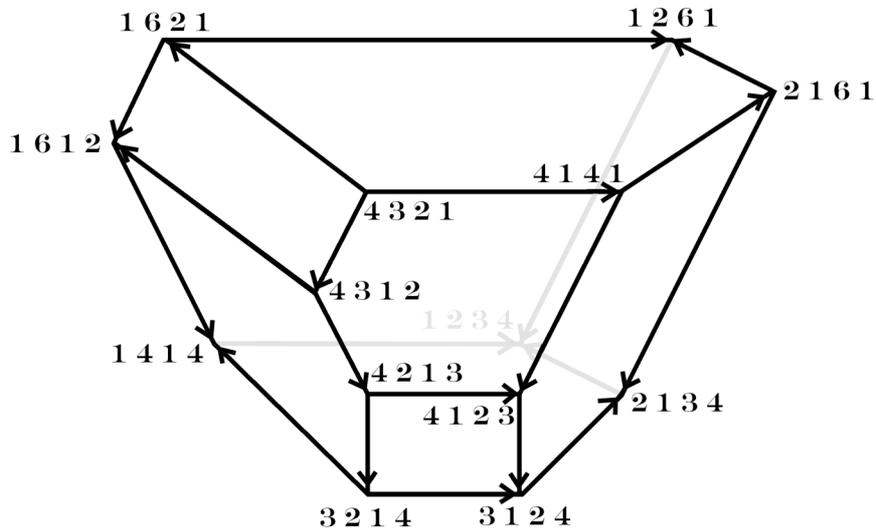


Fig. 11: The three-dimensional Stasheff polytope.

The Stasheff polytope structures the relationships not only between binary hierarchies, but between any hierarchy representable by a plane tree. All types of structure described here admit of mixed binary-ternary hierarchies, which are necessary for dealing with, e.g., triple and compound meter. The cells of the polytope correspond to the plane trees with $n + 1$ leaves, with higher-dimensional cells corresponding to trees with fewer internal nodes. The binary trees, with a maximum of internal nodes, correspond to the 0-cells (points) of the polytope. Mixed binary/ternary trees correspond to higher dimensional cells (edges, faces, etc.) depending on the number of ternary branchings they include.

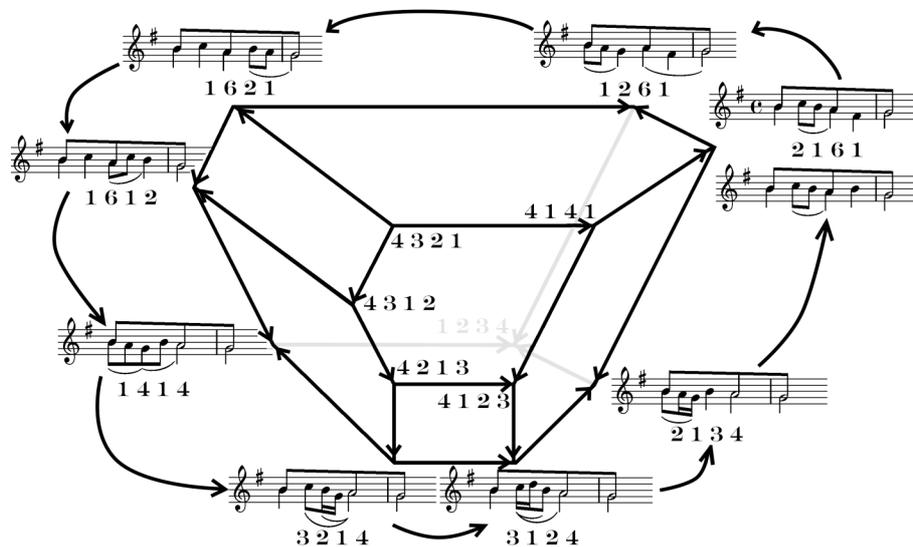


Fig. 12: A melodic traversal of the three-dimensional Stasheff polytope.

4 Relating Hierarchies on Different Musical Parameters

4.1 Conflict between Melodic and Metric Structures

The generalization of the concept of hierarchy to different musical parameters means that not only can one geometrically relate different hierarchies on a given parameter, but also hierarchies existing simultaneously on different parameters. Figure 13 shows the rhythmic operations of Figure 10 on a sequence with an unchanging melodic structure, resulting in various kinds of conflict between simultaneous rhythmic and melodic structure.

Schenker used such rhythmic operations on occasion to add emphasis to his tonal-hierarchical analyses of musical passages. Figure 14 reproduces a portion of his analysis of the theme from the C minor fugue of Bach's *Well-Tempered Clavier*. (It excludes the upper part of the compound melody and normalizes syncopations). The background of the theme is a descending line G–F–E \flat . Schenker gradually adds new elements in coordinated rhythmic and melodic structure, and then transforms rhythmic structure to misalign the two hierarchies. Thus, rhythm helps to clarify the structure of the melody at each introduction of a new element, while progressive transformations

bring it gradually in line with the actual rhythm of the fugue theme. The end result is a combination of hierarchies far apart in the five-dimensional Stasheff polytope—a heavily left-weighted tonal structure against a right-weighted rhythm.

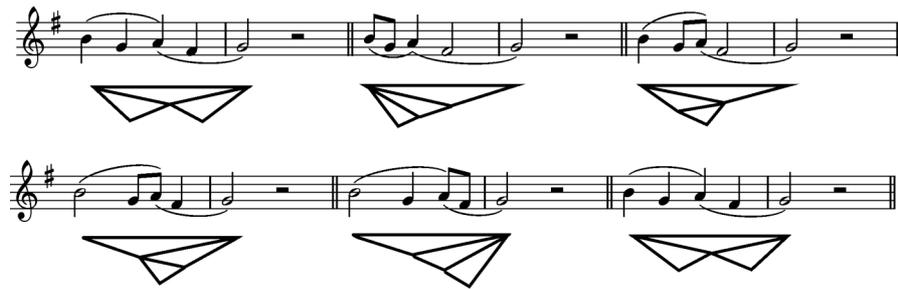


Fig. 13: Rhythmic transformations against a constant melodic structure.

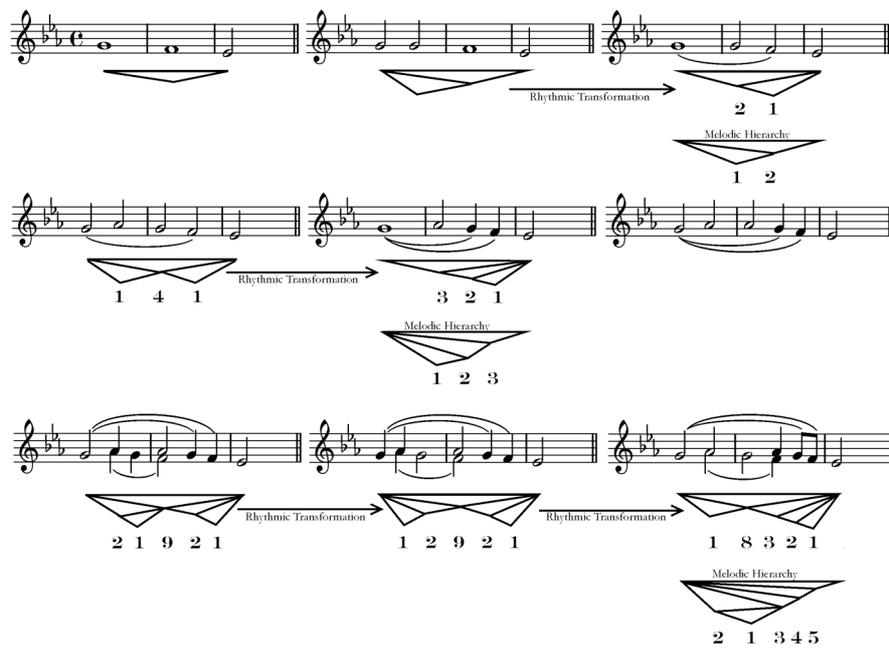


Fig. 14: Rhythmic transformations in Schenker's analysis of the C minor fugue subject.

4.2 Melodic Structures in Counterpoint

Geometrical relationships between hierarchies can also describe aspects of tonal counterpoint. Counterpointing melodies with melodic and rhythmic structures in perfect coordination produce successions of consonant thirds and sixths, as in the first counterpoint of Figure 15. Transforming the melodic and rhythmic structures in one voice against the other voice creates oblique dissonances. Adding conflict between melodic and rhythmic structures *within* one voice to the conflict between voices produces accented dissonances. For example, Figure 16 transforms the rhythm of the upper voice in the fourth counterpoint of Figure 15.



Fig. 15: Transformations of melodic/rhythmic hierarchy in counterpoint with a constant melody.

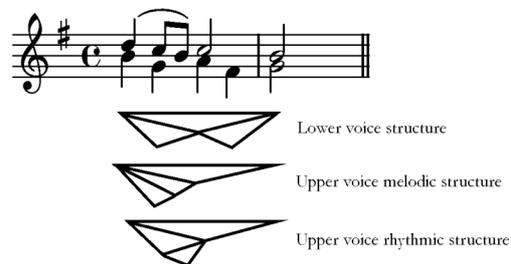


Fig. 16: Accented dissonance as a relationship of three hierarchies.

4.3 Relationships between Melodic and Harmonic Structure

Melodic structures also relate to harmonic structures through the properties of their hierarchies. Melodic and harmonic structures, however, relate through the contraction of their hierarchies, and therefore invoke the relationships between Stasheff polytopes in different dimensions. The fifths-sequence figure distinguishes harmonic hierarchies from melodic ones, so the contraction of thirds from below the fifths sequences of a

harmonic hierarchy creates an associated melodic hierarchy. Figure 17 shows this relationship; notice that the contraction here preserves the stepwise passing motions of the harmonic structure and introduces some new stepwise relationships, such as the step from leading tone to tonic. The contracted edges appear as vertical intervals in the musical notation of Figure 17.

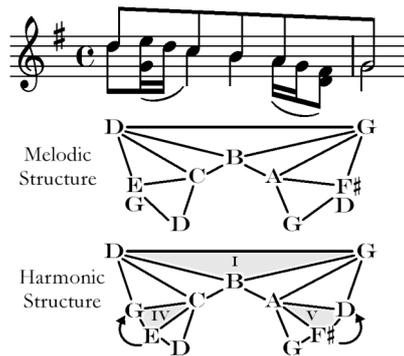


Fig. 17: Melodic structure as a contraction of harmonic structure.

In general, there might be no such contractions of a harmonic structure that do not annihilate parts of it. In such an instance, a complete representation of the harmonic structure requires multiple melodic structures in counterpoint, each a contraction of a common harmonic structure. Figure 18 shows a sequence from the $G\sharp$ minor prelude of the *WTC I* (mm. 19–21). The imitative counterpoint of the two upper voices consists of two different contractions of the underlying harmonic structure, and the two voices together express the complete structure of the harmonic sequence.

Each contraction in Figure 18 requires two vertex deletions. Geometrically, a vertex deletion defines an n -simplex in the n -dimensional Stasheff polytope (consisting of all the hierarchies reducing to a common structure with the deletion of one foreground vertex). The intersections of these n -simplexes reflect the corresponding simplicial structure on the $(n - 1)$ -polytope. Two successive vertex deletions therefore define an $(n - 1)$ -simplex of intersecting n -simplexes. Two melodic structures in counterpoint define a precise harmonic structure when the uncontracted melodic structures correspond to two (simplexes of simplexes of . . .) simplexes in some Stasheff polytope that intersect in precisely one point. For instance, the first half of the harmonic structure in Figure 18 (from $G\sharp$ to $F\sharp$) corresponds to a point on the 4-dimensional Stasheff polytope. The deletions of this structure for each melodic voice, ($G\sharp-F\sharp-E-C\sharp-F\sharp$ and $G\sharp-C\sharp-B-A\sharp-F\sharp$) each correspond to 3-simplexes (tetrahedrons) of intersecting 4-simplexes in the 4-dimensional polytope. The two sets of sixteen points encompassed by each of these

intersects in one point, which corresponds to the overall harmonic structure of the passage.

The figure displays two musical passages in G# minor, each with its corresponding chord diagrams. The top passage is the upper voice, and the bottom passage is the lower voice. Both passages are in 8/8 time and feature a complex melodic structure with many accidentals.

Upper voice melodic structure: G# - E - F# - D# - E. The chord diagrams show a sequence of chords: G# (I), E (II), F# (III), D# (IV), and E (V). The harmonic structure diagram shows a contraction of the harmonic structure with chords: G# (I), F# (II), E (III), D# (IV), and E (V).

Lower voice melodic structure: B# - G# - A# - B - A# - G# - E. The chord diagrams show a sequence of chords: B# (VI), G# (V), A# (IV), B (III), A# (II), G# (I), and E (V). The harmonic structure diagram shows a contraction of the harmonic structure with chords: G# (I), F# (II), E (III), D# (IV), and E (V).

Fig. 18: Two melodies in counterpoint (from the G# minor prelude of Bach's *WTC I*) as differing contractions of a harmonic structure.

5 Conclusion

The geometry of hierarchy is fascinating by virtue of its depth of mathematical structure alone. Indeed, we have only scratched the surface of that mathematical structure here. Given the myriad ways that music is hierarchically organized this geometry illuminates it from many angles, showing the nature of relationships between different ways of hierarchically organizing rhythms or tonal patterns, conflict between the simultaneous rhythmic and tonal organization of a melody or multiple melodies in counterpoint, and the relationships between different types of hierarchical

organization that apply to tonal patterns. All of these relationships not only demonstrate the importance of hierarchy in tonal music, but also give the lie to the idea, implicit in Schenkerian analysis, that musical hierarchy is a unitary phenomenon. Indeed, a single hierarchy is too simple an object to represent the complex interactions that bring life to music in the tonal idiom.

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