Periodicity-Based Descriptions of Rhythms and Steve Reich's Rhythmic Style

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In his music from the 1970s through the 1990s, Steve Reich formulated a strongly coherent style that employed pattern repetition and rhythmic cycles in a consistent way from piece to piece. Much of Reich's compositional attention was clearly directed at the aesthetics of the rhythmic texture of the piece, so while there are strong common threads though the pieces—the use of cyclic rhythms, unvarying pulse, and rhythmic canons, for instance—there is also a variety in the kinds of rhythms he uses. In particular, Reich uses a variety of different kinds of rhythmic cycles, including ones that correspond to conventional sorts of regular meters (cycles of 8, 12, 16, 24, or 32 units) and irregular meters (such as cycles of 19, 22, or 33), as well as shorter cycles of one or two measures or longer ones of multiple measures, sometimes of mixed lengths. Nonetheless, all of these rhythms have an immediately recognizable similarity in style, a style which evolves out of Reich's process pieces of the 1960s.

We can pinpoint the essential stylistic characteristics shared by all of these rhythms by using the discrete Fourier transform (DFT). This method has been previously used by multiple authors to describe pitch-class sets.[1] It has also been used for rhythms.[2] The application to rhythms assumes a regular rhythmic cycle, and therefore is well-suited to Reich's cyclic rhythms. It is precisely analogous to the DFT on pitch-class sets when the rhythmic cycle consists of twelve units, AKA "beat classes" or "time classes," analogous to the twelve pitch classes. However, the application to rhythm works for cycles of any length.

The DFT can be understood as a way of representing a rhythm in terms of periodicities. Conventional concepts of meter are based on periodicities that exactly divide the complete cycle (the measure or hypermeasure). The DFT gives a complete description of a rhythm in terms of periodicity by considering, in addition to these kinds of conventionally metrical periodicity, periodicities that divide the cycle into non-integer multiples of the basic pulse. The main distinguishing feature of these kinds of periodicities is that their prototypes are maximally even but not perfectly even, while conventionally metrical periodicities have perfectly isochronous prototypes.[3] The special stylistic trait of Reich's

[1] Lewin 1959, 2001, 2007; Quinn 2006–7; Amiot 2007, 2016; Yust 2015a–b, 2016, Bernardes et al. 2016.

[2] Amiot (2011, 2016); Chiu (2018); Milne, Bulger, and Herff (2017); Milne, Bulger, Herff, and Sethares (2015); Sethares 2007; Yust (2017b, 2020).

[3] In this sense it is similar to a recent proposal by Guerra (2019), with the difference being the privileged status of maximally even divisions.

rhythms, observable across a number of different kinds examples, is their emphasis upon the kinds of periodicities with non-isochronous prototypes.

A previous paper (Yust 2021) focuses on Reich's rhythms that involve strictly repeating cycles of twelve or sixteen timepoints. While such cycles are very common in this music, there are a number of pieces that deviate from this template in one of two ways. First, Reich wrote a number of pieces that experiment with less regular rhythmic cycles. The examples discussed here include pieces based on 19-element (*Electric Counterpoint*, second movement), 20-element (*Eight Lines*), and 22-element cycles (*Sextet*, second and fourth movements). To draw connections between rhythms in these different kinds of cycles and describe the whole of this musical output as partaking of a highly consistent rhythmic language, it is necessary to adopt *time-continuous* rhythmic concepts, in contrast to the usual *time-discrete* concepts of traditional metrical theory and beat-class set theory. The DFT (in spite of its name) is conceptually time-continuous, and thus serves as a theoretical bridge between the two.

In the first section of this paper I present the DFT as a periodicity-based description of a rhythm, and by virtue of that a sort of generalization of traditional metrical theory which is also based on periodicity in a more restricted sense. Using the DFT we can describe cyclic rhythms with *spectra* that show their underlying periodicities, and I show how we can compare spectra across different kinds of rhythmic cycles. I then use these tools first to discuss the formation of Reich's rhythmic style in the early phase pieces, and the transitional pieces, *Clapping Music* and *Drumming*, and the germinal role of rhythmic combinatoriality in this period. I then show how the *Clapping Music* rhythm is foundational to many later pieces, including *Music for 18 Musicians*, *New York Counterpoint*, *Electric Counterpoint,* and *Vermont Counterpoint*. Finally, I consider irregular rhythmic cycles from *Eight Lines, Sextet, Electric Counterpoint,* and *Desert Music*, and compare spectra across different cycle lengths to reveal commonalities between all of these.

DFT and a time-continuous concept of periodicity

Fourier transforms decompose a periodic *signal* (generally understood as a function over time that repeats at some fixed time interval) into a sum of periodic components, each of which is a simple sinusoidal function, whose period divides the larger period of the signal evenly. The *discrete* Fourier transform, the method used in this article, does this with a discrete signal, which can be expressed as a vector.

For the purposes of this article, the signal is a rhythm. As an illustration, consider the rhythm for Reich's *Clapping Music*, reproduced in Figure 1, which I refer to as Reich's "signature rhythm."[4] This is understood as repeating indefinitely, making it a periodic signal.

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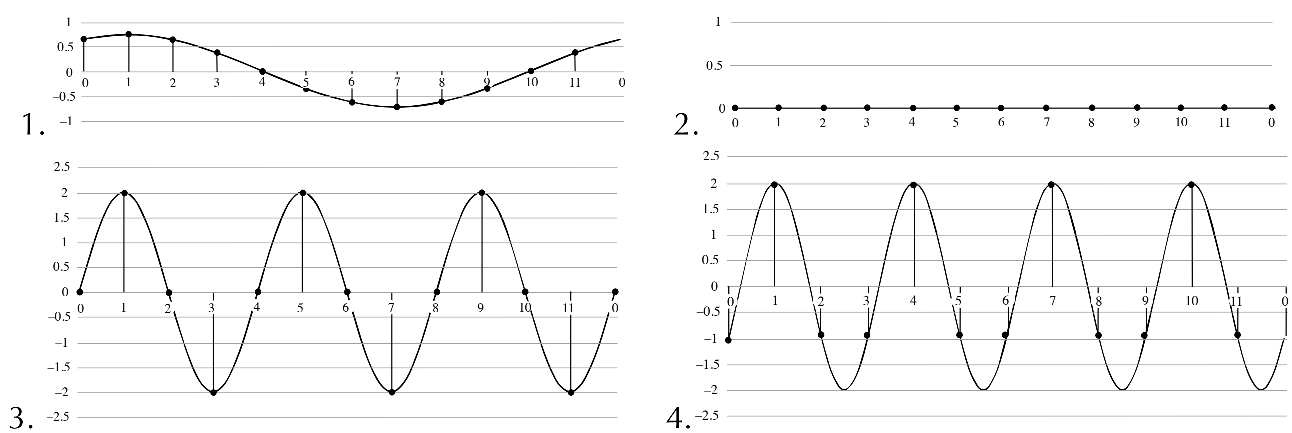
[4] This rhythm and its significance for Reich is also discussed at length in Yust 2021.

Figure 1: Reich’s signature rhythm

The vector for the signature rhythm is (1,1,1,0,1,1,0,1,0,1,1,0). Each place in the vector corresponds to a timepoint in the cycle. If a note is present at a given timepoint, we put a 1 at that place in the vector, and otherwise we put a 0. Notice that it would also be possible to give a more differentiated weighting to a rhythm. A simple example (which will be important below) would be making superposition of two rhythms by adding their vectors. The resulting vector will have 0s, 1s, and 2s, indicating how many onsets between the two rhythms are present at a given timepoint.

The DFT transforms these n-place vectors into a set of n sinusoidal components. Figure 2 gives the DFT of the signature rhythm as an example. The sinusoidal components divide the overall cycle into $1 – n$ subcycles. Each component is sampled at the n timepoints, whose values add up to the original vector. Hence, this transformation shows which simple periodic functions are a best fit to the rhythm. In the case of the signature rhythm, they are numbers 5 and 7, with 3, 4, 8, and 9 also being significant. Each component can be expressed with two parameters. One gives the height of the sinusoid, which we call the *magnitude*, and the other indicates where the sinusoid peaks in the cycle, which we call the *phase*. Together, each set of parameters gives a Fourier *coefficient*, indexed by k, the number of times its period divides the overall period. If the overall period is n, the *frequency* of the component is k/n. The period of the component is just the reciprocal of this, n/k (in units of the basic pulse).



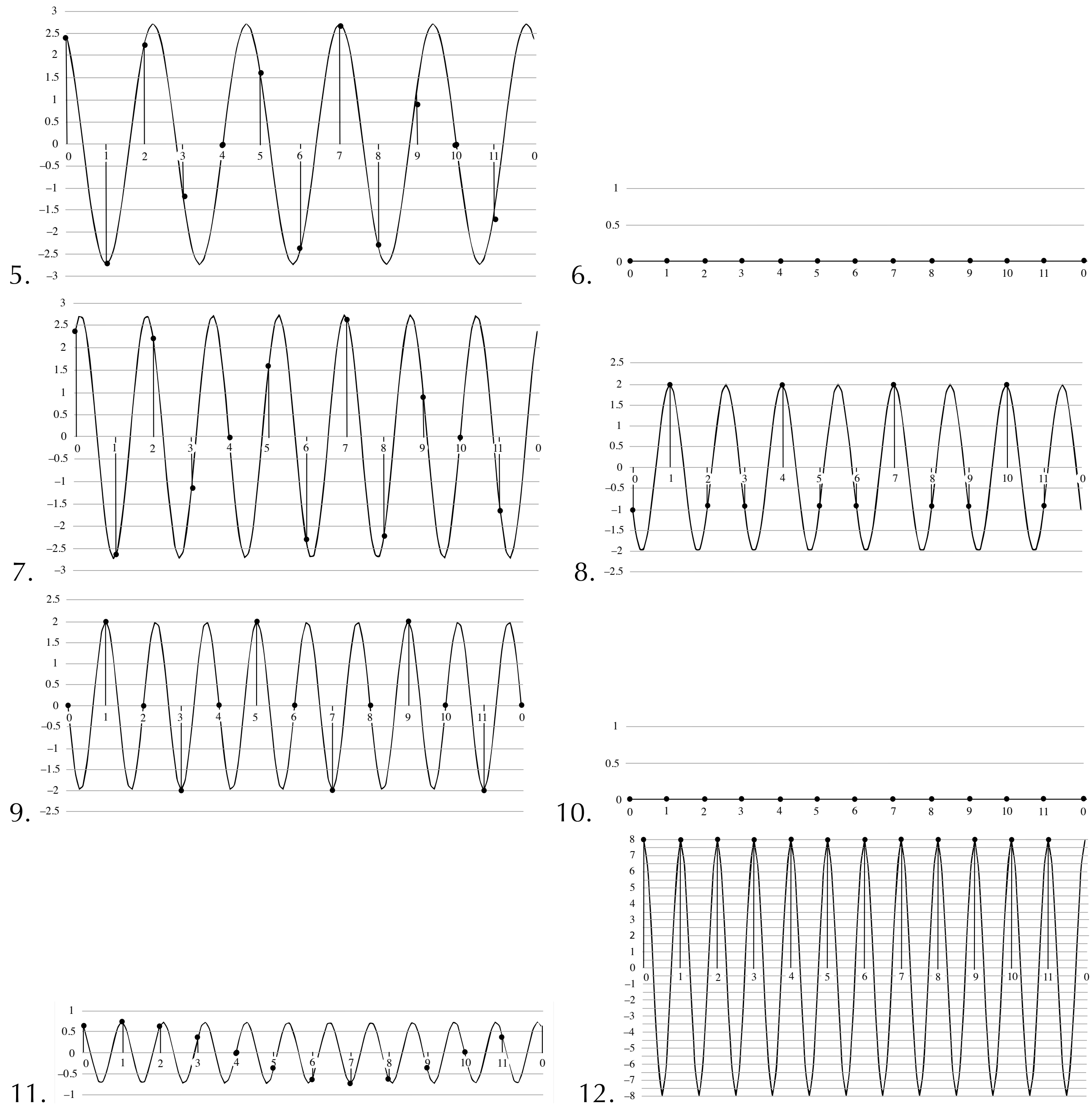


Figure 2: The complete DFT of the signature rhythm in Figure 1(a)

The first observation to make about the DFT is that it has a basic symmetry. The *k*th component is indistinguishable from the (*n* – *k*)th component, once it is discretely sampled. This equivalence is known as *aliasing*: in general, when we discretely sample a periodic signal by *n*, any frequency given by adding or subtracting a number *k* from any multiple of *n* is equivalent. This makes sense, because discretely sampling reduces the information in the signal to a finite amount, and therefore only a corresponding number of frequencies remain distinguishable. In the example of the signature rhythm (Figure 2) the rhythm is discretely sampled as 12 numbers. Coefficients 1–5 require two parameters (a magnitude and a phase), while coefficient 6 requires just a positive or negative magnitude (since a shift in phase reduces the value at each point by the same amount). By the aliasing principle, the twelfth component is equivalent to a *zeroth* coefficient, which is simply a constant, indicating the number of onsets in the rhythm. This is also expressible as a single magnitude. Therefore the DFT converts the twelve numbers of the sampled signal to another set of twelve numbers, seven magnitudes and five phase values.

We can make this information-theoretic explanation of the DFT more precise: it is a *reversible* transformation, meaning that a set of DFT coefficients corresponds to exactly one rhythmic signal (with the weights of the timepoints freely varying as real numbers). This is what the word "transform" refers to in this context: it means that, when we look at the DFT coefficients, we are looking at the same object (the rhythm) from a different perspective, essentially measuring it on a different set of axes. The music theoretic significance of this is that by using the DFT we can express a cyclic rhythm in terms of purely periodic information.

The idea of a vector as a discrete "sampling" of the time continuum comes from engineering applications of the DFT, where sampling does result in a loss of information and is usually seen as a computational expedient. In our application to notated rhythms, the sampling is already performed by the process of notation; in a sense it is the composer who performs the sampling by using notation, rather than the analyst. The aliasing of frequencies is therefore the result of a compositional decision: by choosing a twelve-element cycle for his signature rhythm, Reich makes it impossible to distinguish periodicities that divide the measure in 7 parts from those that divide the measure into 5 parts. They correspond to the same rhythmic *quality*, to use language coined by Ian Quinn (2006–7) for the pitch-class analogue of the DFT, already co-opted for the rhythmic case in Yust 2021.

Applications of the DFT to the description of pitch-class sets, beginning from Quinn's (2006–7) adaptation of Lewin (1959, 2001), use the same mathematical method with a more unusual definition of the "signal space," where the initial vector is a *pitch-class vector*, a discrete octave-periodic function over the pitch continuum. The definition of the signal space in the rhythmic application, as time, matches the more familiar engineering applications of the DFT. However, given the long history of transference between these two spaces in music theory, through the pitch-class/beat-class analogy,[5] Lewin's and Quinn's application of the DFT to pitch-class space is not intuitively so far-fetched. The analogy is especially strong in case like Figure 1, where the rhythmic cycle has twelve elements.

[5] Initiated by Babbitt (1962) and drawn upon by, among others, Pressing (1983), Rahn (1983, 1996), Cohn (1992b), Roeder (2003), Milne, Bulger, and Herff (2015), Milne et al (2017), and Yust (2020).

We might say, then, following Pressing (1983) and Rahn (1996), that Reich's signature rhythm is similar to a *diatonic rhythm*, the beat-class analog of a diatonic scale, but with one additional onset. The subset rhythm (1,0,1,0,1,1,0,1,0,1,1,0) is the analog of an F major scale, to which we add a C♯ to get Reich's rhythm. The diatonic rhythm, which is a well-known timeline common across sub-Saharan Africa,[6] is special as a *maximally even* rhythm. Quinn (2006–7) and Amiot (2007) have shown how maximally even sets serve as prototypes of DFT coefficients. For this reason, Amiot (2016, 2017b) and Yust (2015a, 2016) refer to the 5th coefficient for pitch-class sets as "diatonicity." The prominence of the $k = 5$ periodicity in Reich's signature rhythm (Figure 2) is a kind of rhythmic diatonicity, indicating a similarity to the diatonic rhythm, a fact of historical importance that will be further explored below.

One important music theoretic question about the DFT is how it relates to concepts of meter, which are also fundamentally about periodicity. Among the milestones in the history of the theory of meter, two of the most important and oft-cited are a key passage from J.P. Kirnberger's *Der Kunst des reinen Satzes in der Musik* of 1776, and an illustration from Viktor Zukerkandl's *Sound and Symbol* of 1956. The Kirnberger passage ([1776] 1982, 381–385) starts with a set of equally spaced pulses, which then become meter by regular groupings of different kinds. This basic *time-discrete* strategy of conceptualizing meter is now the norm. Zukerkandl (1956, 187) offered a different metaphor, picturing "metric waves" of different periodicities that gradually ebb and flow with the different metrical levels. As Justin London (2012, 64) has noted, Zukerkandl's idea, though it remained loosely metaphorical in his own writing, anticipated the theory of driven oscillators that has become the standard neuroscientific explanation of meter (Large and Palmer 2002, Large and Snyder 2009). It also closely resembles the sinusoidal functions of Fourier transforms. Zuckerkandl's metaphor and the modern neuroscientific theory differ in a crucial mathematical sense from Kirnberger's explanation, as *time-continuous* descriptions.

Although the DFT is *discrete*, that is only because of the time-discrete nature of notated rhythms. It is understood as a discretization of a continuous transform (the Fourier transform more generally), and therefore belongs conceptually with time-continuous theories like Zuckerkandl's and Large's. The time continuous nature of the theory shows up in one crucial difference from traditional metrical theory, which only considers subdivisions of a metrical cycle with integer periodicities. In the 12-cycle of Reich's signature rhythm, we have metrical concepts associated with $k = 2$ (the beats of a compound $\frac{6}{4}$ meter), $k = 3$ (the beats of a $\frac{3}{2}$ meter), $k = 4$ (the beats of a $\frac{12}{8}$ meter), and $k = 6$ (the subdivisions of a $\frac{6}{4}$ or $\frac{3}{2}$ meter), but none associated with $k = 5$.[7] This makes the principal quality of Reich's rhythm invisible, in a sense, to traditional metrical theory. This quality, however, is essential to a complete periodic description of a 12-cycle rhythm, in

[6] Jones 1959; Pressing 1983; Rahn 1983, 1996; Agawu 2003
[7] I discuss the role of integer periodicities in traditional notions of meter and the metrical status of non-integer periodicities also in Yust 2021.

the sense of a reversible transformation, like the DFT. To recognize it, however, we have to consider the rhythm as a possible approximation of a kind of periodicity that is impossible to perfectly match in a 12-cycle rhythm. That idea of temporal approximation requires a continuous metric.

For a given value of *n*, we can think of the presence of a periodicity *k* in terms of *prototypes*, or rhythms that maximize the value that *k*. As Quinn (2006–7) and Amiot (2007, 2016) have shown, the prototypes are maximally even rhythms, or subsets of maximally even rhythms. For instance, Figure 3 shows prototypes of $k = 6$, $k = 5$ or 7, and $k = 4$ or 8, for $n = 12$. Using Clough and Douthett's notation these are ME(12, 6), ME(12, 7), and ME(12, 8). The first is perfectly even, i.e. perfectly isochronous. The third is not perfectly isochronous, but is translationally symmetrical at the dotted quarter. The second, because 5 and 7 share no factors with 12, does not have any precise translational symmetry, although it has near symmetry at a translation by 5 or 7 eighth notes. This is the "diatonic rhythm." Reich bases a number of his pieces on the diatonic rhythm (the finale of the *Sextet* is one good example), which he first encountered in his study of Ghanaian and other West and Central African traditions (Scherzinger 2018, 2019). London (2012) observes that a concept of non-isochronous meter is necessary to account for the full range of musical practice and uses maximal evenness as a well-formedness constraint for non-isochronous meters.

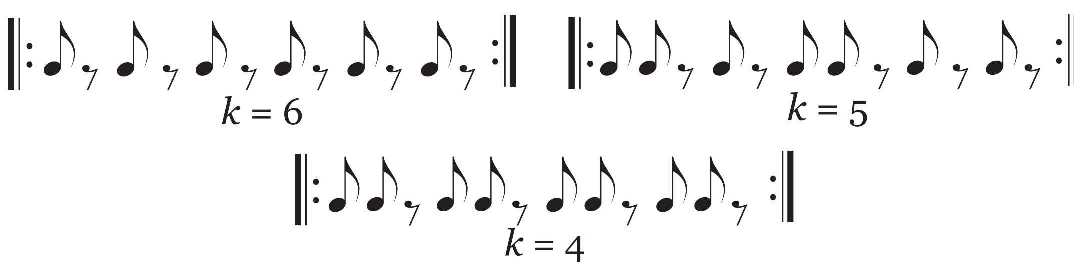


Figure 3: Prototype rhythms (maximally even rhythms) for $k = 4$–6

Reich's signature rhythm in Figure 1 is similar to the ME(12, 7) rhythm, but significantly, it is not exactly the same. This is a general feature of many of the cyclic rhythms Reich uses in works from the 1970s and 1980s; they are close to maximally even rhythms without being equivalent to them. Since the DFT, loosely speaking, compares a given rhythm to all the possible maximally even subsets and supersets, the similarity of Reich's rhythms to maximally even ones will show up as prominent periodicities in the DFT. In the last section of the paper, I will compare a number of these to reveal additional commonalities.

As a further consequence of the time-continuous conceptual underpinnings of the DFT is one particularly critical to the project of this article, the comparison of rhythms across different kinds of rhythmic cycles. If we regard the coefficients of each DFT as indices of different frequencies (or periodicities), then this kind of comparison can rely on continuous metrics by comparing coefficients across rhythms with similar frequencies.

For example, the signature rhythm, reproduced as Figure 4(a), is the basis of the first part of *Music for Pieces of Wood*, composed by Reich contemporaneously with *Clapping Music*. The second part of *Music for Pieces of Wood*, Reich uses the 4/4 rhythm in Figure 4(b), which appears to be a truncation of the 6/4 rhythm, in which the first two beats have been removed. This description, while initially satisfying, is inadequate upon closer inspection. There are a number of ways one could truncate the 6/4 rhythm to get a 4/4 rhythm. For instance, Figure 4(c) truncates the 6/4 rhythm by removing the last two beats. This rhythm does not seem to capture the quality of the 6/4 rhythm as well as the one that Reich uses (Fig. 4b).

Figure 4: The rhythms of first two parts of *Music for Pieces of Wood* (a–b), and a different truncation of the first rhythm (c)

It is hard to compare across different kinds of cycles using traditional metrical theory or beat-class set theory. There is no reason, for instance, to compare the timepoints in a 4/4 measure to the last four beats of 6/4 measure, as opposed to the first four, or any four consecutive beats. The DFT does allow for such comparisons, however. Our primary tool for doing so will be the *rhythmic spectrum*.

A *spectrum* summarizes the DFT by showing just the *magnitudes* of each of its coefficients.[8] Figure 5(a) shows the full spectrum of the signature rhythm of Figures 1–2 and 3(a) as a line graph on the DFT coefficients from 0 to $n - 1$, following the practice of Amiot (2016). Two points mentioned above are evident here: First, the value at $k = 0$ is equal to the cardinality of the rhythm. Second, the graph has a mirror symmetry around $k = n/2$, as a result of the aliasing principle, which says that if a rhythm has an eighth note pulse, periodicities shorter than a quarter note (2× the pulse) are indistinguishable from complementary ones longer than a quarter note. Therefore, we can see the significant information by showing just the spectrum from 1 to $n/2$, and that will be the usual practice throughout this article. The spectrum orders the coefficients as *frequencies*, k/n. They can be thought of, equivalently, as periodicities, n/k, as shown in the alternate labeling of the x-axis in Figure 5(a). While the signature rhythm features some periodicities at integer

[8] The use of spectra is therefore similar to Quinn's (2006–7) idea of quality space in the pitch-class domain, which also eliminates phase information from consideration.

multiples of the basic pulse, at the half-note and dotted quarter, its most prominent periodicity is at $2^2/_5$ eighth notes, a division of the 6/4 measure by five (i.e., a frequency of 5/12). Or, equivalently by aliasing, at $1^5/_7$ eighth-notes, or a frequency of 7/12.

The spectra for the 4/4 rhythms in Figures 4(b) and (c) can be compared to the spectrum for the signature rhythm by overlaying the spectra on the same frequency/periodicity axis, as Figures 5(b)–(c) do. The rhythm that Reich uses, as Figure 5(b) shows, has a peak at 3/8, very close to the peak of the signature rhythm at 5/12. The similarity of the 6/4 and 4/4 rhythms that Reich uses in *Music for Pieces of Wood* is a similarity of their basic periodicities: $2^2/_3$ is a nearest approximation among even divisions of 8 to $2^2/_5$. In addition, the irregularity of the rhythms, which is associated with the fact that their principal periodicities are not integer multiples of the eighth-note grid, is itself a similarity. The hypothetical rhythm that Reich rejects, as Figure 5(c) shows, peaks at a half-note periodicity that evenly divides the cycle, and is not close to the signature rhythm periodicity of $2^2/_5$.

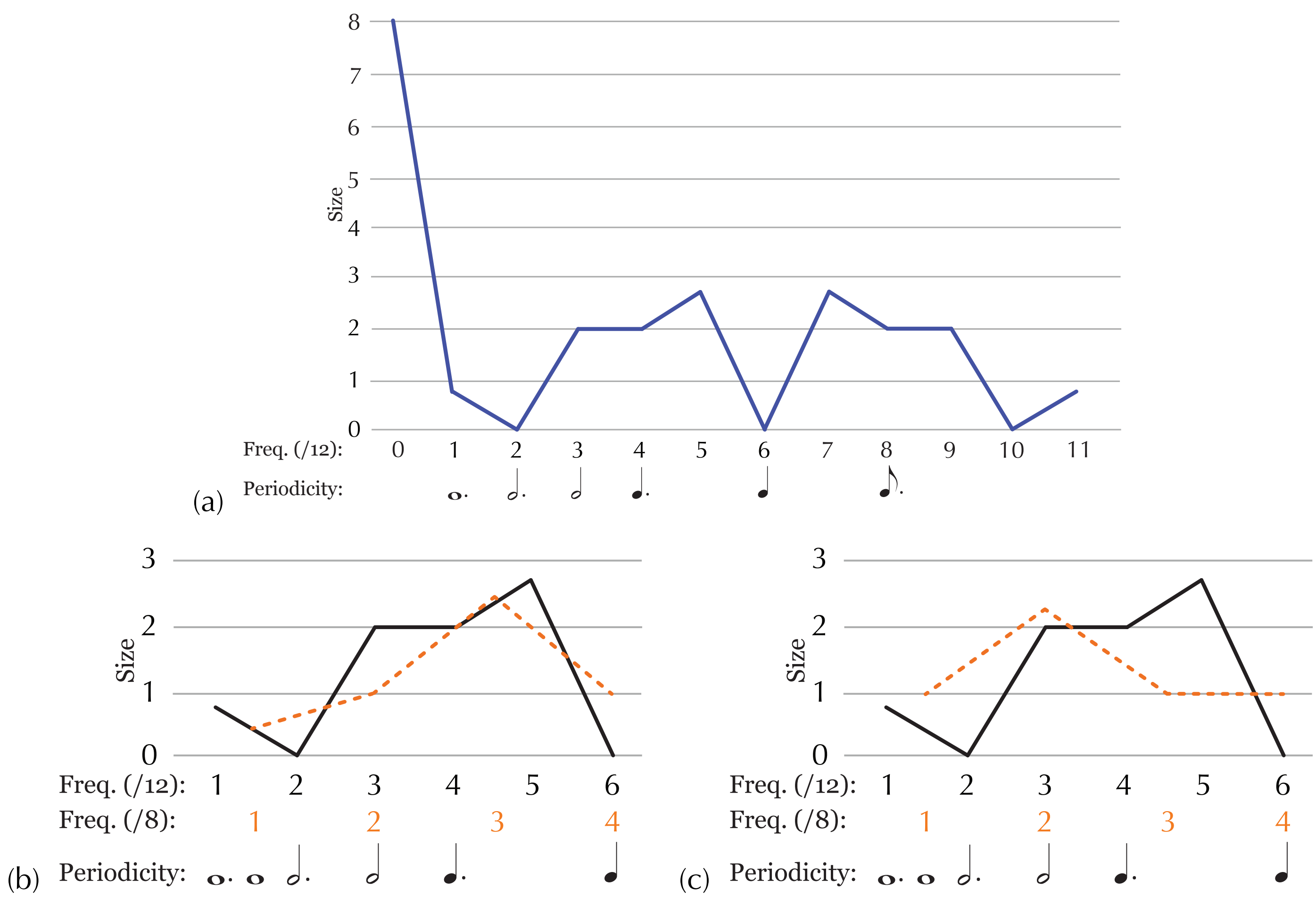


Figure 5: The spectrum of the rhythm in Figure 4(a) compared to that of Figure 4(b) and (c)

Rhythmic spectra are incomplete representations of a rhythm. They are independent of rotation of the rhythm with respect to the downbeat, and retrograde, corresponding to the transposition and inversional invariance of spectra in the pitch-class domain (Amiot 2016, Yust 2015a). The spectrum may be understood as the "purely intervallic" aspect of the rhythm. For this reason, they are usually the most important first consideration in characterizing a rhythm. The information missing from the spectrum consists of the *phases* of the coefficients, their orientation with respect to a downbeat. Rotation of the rhythm (beat-class transposition) shifts all of the phase values, but does not change the spectrum. Spectra excluding the zeroth coefficient are also invariant with respect to complementation.

A consideration of phases can be important after identifying significant components, usually ones that are prominent in the rhythmic spectra. In application of the DFT to pitch-class sets, for instance, we find that coefficients 3 and 5 are prominent for important tonal sets (such as triads and scales), and therefore tonal collections can be related by comparing the phases of their 3rd and 5th coefficients (Yust 2015, 2017a). When limiting consideration to a single periodicity, the phase and magnitude of the corresponding coefficient can be represented in a single Cartesian space.[9] For instance, consider the three rhythms of Figure 4 in the space for the half-note periodicity (3/12 or 2/8). This is shown in Figure 6. The half-note cycle is represented by the unit circle in this space; each time point within the cycle corresponding to a position on the circle, starting from 12-o'clock and proceeding clockwise. Because it is a half-note in length, there are four eighth note positions within it, numbered from 0 to 3. Each coefficient is a point in the space, where the distance from the origin is the size of the coefficient, and the angular distance from 0 is the phase. These points can be determined by addition of the unit vectors corresponding to each timepoint in the rhythm, as described by Quinn (2006–7).

[9] This is the complex plane, with the real part of the coefficient on the y-axis and the imaginary part on the x-axis (contrary to the mathematical convention which puts the real part on the x-axis). I will also invert the imaginary axis (positive values to the left) so that time moves in a clockwise direction.

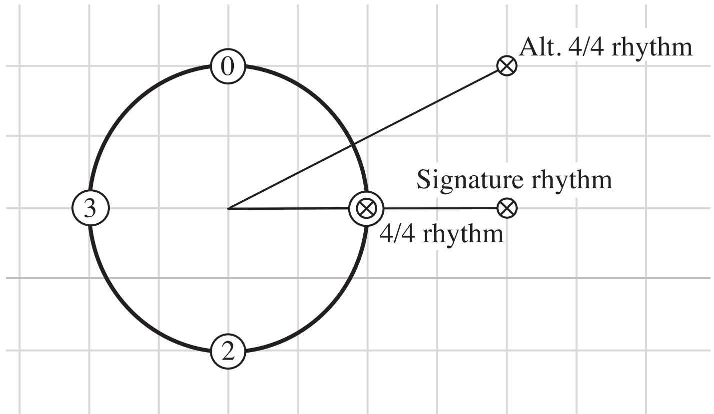


Figure 6: The rhythms of Fig. 4 in the Cartesian space for the 𝅗𝅥 periodicity

Meaningful comparisons across cycle lengths in this kind of Cartesian space can only happen when there is an exact shared periodicity between them. For the rhythms in Figure 6, we may note that the 4/4 rhythm Reich uses in *Music for Pieces of Wood* matches the 6/4 rhythm precisely in phase, even though it is smaller in size. The 4/4 rhythm that he does not use is similar in size, but closer to the downbeat in phase, making it a more metrical and less syncopated rhythm in a meter with a half-note layer.

The story of Reich's development of a distinctive rhythmic style involves the melding of his interest in African rhythms and other non-European rhythmic languages with the gradual transition from the technique of phasing in the 1960s to that of cyclic canons in the 1980s, noted by Robert Schwartz (1990) and Twila Bakker (2019). The use of rhythmic spectra and the accompanying periodicity-based theory of rhythm helps to explain how the mechanics of these processes evolve with gradually more complex rhythms at each stage of this development, as the next section demonstrates. Following this, I discuss the crucial moment in the early 1970s when Reich formulates his signature rhythm, the basis of *Clapping Music*, and show how this leads to the exploration of similar rhythms in rhythmic cycles of varying lengths in *Eight Lines, Sextet*, *Desert Music*, and *Electric Counterpoint*. The last section of this paper uses DFT spectra to explain the similarity of all of these rhythms.

Combinatoriality in the Early Phase Pieces

The development of Reich's rhythmic language begins with his encounter with West African drumming in the early 1960s, initially through the medium of A.M. Jones' (1959) transcriptions. As Martin Scherzinger [2018] points out, ideas about African rhythm initially inspired the phasing process that defined the first stage of his career, starting with the mechanical phasing of the early tape pieces, *It's Gonna Rain* and *Come Out* of 1964 and *Melodica*, to phasing as a performance technique in *Piano Phase* and *Reed Phase* in 1966–67, followed by more sophisticated phase-based pieces *Violin Phase* and *Phase*

Patterns, and finally to the reconceptualization of phasing as canon in *Drumming* (1971) and *Clapping Music* (1972). Over the course of these pieces he tried out a number of methods of rhythmic construction, progressing from regular, combinatorial, and generated rhythms to the touchstone signature rhythm which was to set the agenda for the next two decades.

Reich's early interest in rhythmic combinatoriality is evident in *Melodica*, the first of the phase pieces to be composed with a notated melody (as opposed to pre-existing recordings), shown in Figure 7. In the first part of the piece, this pattern, played on the two stereo channels, gradually goes out of phase with itself by an eighth-note. When this is complete, the combination of the two channels is duplicated and then gradually phase shifts by a quarter-note until all of the Bs coincide in octaves.

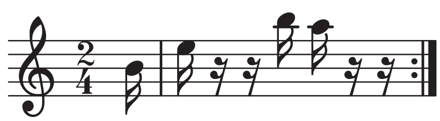


Figure 7: The basic material for *Melodica*

The *Melodica* rhythm is written so that when the first stage of the phasing process is complete, after one channel has shifted ahead of the other by an eighth note, the melody interlocks with itself to cover the entire eight-element universe with no overlap. Its essential property is thus *combinatoriality*, the rhythmic analog of transpositional combinatoriality of pitch-class sets: the rhythm, when "transposed" or translated (shifted ahead) by an eighth-note, combines with itself to cover all eight time points without overlap. This rhythm is also translationally symmetrical: it is equivalent to itself under translation by a quarter note. This is a necessary condition for the rhythmic combinatoriality to occur at the short distance of an eighth-note (so that the phase process can proceed slowly without taking too long). The rhythm of Figure 7 is thus a unique solution (combinatorial under eighth-note translation as opposed to sixteenth-note or quarter-note translation).

Piano Phase exists in two versions (see Suzuki 1991, Chapman 2019). Figure 8 shows the first version. Alternating stems indicate the alternation between two overlapping hands. The resultant pattern is made up of two simple patterns with the same rhythm, combinatorial at the sixteenth-note. Besides the separation of hands, the melodic patterns also distinguish the interleaved parts, with the left hand having a basic arched melodic contour repeating only at the measure, while the right hand repeats every half measure. This creates a multi-leveled hierarchy to the phasing process: the left hand and right hand chords combine then separate over each eighth note of phasing; the right hand pattern goes in and out of phase over a quarter note, and the left hand only comes back into phase over the complete half note of phase shift.

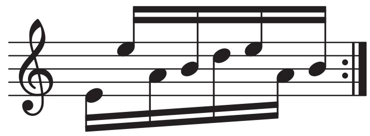


Figure 8: The basic pattern for an earlier version of *Piano Phase*

The final version of *Piano Phase* expands the rhythmic cycle one more time to the twelve-element universe of sixteenth notes in a 6/8 meter, using the pattern shown in Figure 9. The phasing process still has three levels, but instead of all levels being duple, one level involves a factor of three.

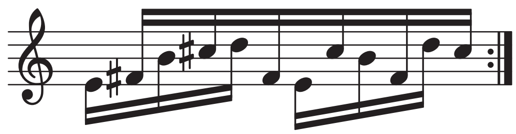


Figure 9: The basic material of *Piano Phase*

The evolution of these early phase pieces thus reveals an experimentation with simple forms of rhythmic combinatoriality. Richard Cohn (1992b) has demonstrated the role of more complex combinatorial patterns in subsequent pieces, *Violin Phase* and *Phase Patterns*. In these pieces the combinatorial rhythms do not have translational symmetries, which requires that the interval of translation is the half-measure. Figure 10 shows the basic material of *Phase Patterns*. Again, the combinatorial rhythm is divided between hands, here playing the lower and upper voices of a single chord. The four-element rhythm fills the entire eight-element universe under translation by four eighth notes.

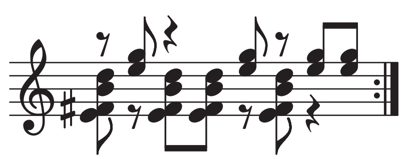


Figure 10: *Phase Patterns,* m. 1

The combinatorial property of these rhythms is directly related to their rhythmic spectra. The essential principle is the *convolution theorem* of Fourier theory. This property, which says that Fourier transforms change convolution into multiplication, is critical for applications of the DFT to rhythm and pitch class sets, as explained by Amiot (2011,

2016) and Yust (2016).[10] A *convolution* of rhythms is a translational combination of them. In other words, to convolve two rhythms repeat one rhythm, translating it to start at the position of each element in the other rhythm, and add all of these together. For instance, a convolution of the *Phase Patterns* rhythm, (1,0,1,1,0,1,0,0), with (1,1,0,0,0,0,0,0) will add two copies of the first rhythm, one starting on the downbeat, and one starting an eighth-note later:

$$(1\ 0\ 1\ 1\ 0\ 1\ 0\ 0) \circ (1\ 1\ 0\ 0\ 0\ 0\ 0\ 0) =$$

$$\begin{array}{r} (1\ 0\ 1\ 1\ 0\ 1\ 0\ 0) \\ \underline{+\ (0\ 1\ 0\ 1\ 1\ 0\ 1\ 0)} \\ =\ (1\ 1\ 1\ 2\ 1\ 1\ 1\ 0) \end{array}$$

Convolution is thus the same thing as rhythmic canon, where one rhythm in the convolution is the subject of the canon, and the other is the arrangement of the starting points of each canonic voice. The convolution theorem says that we can get the spectrum of the combination by multiplying the spectra of these two rhythms. For instance, the complete spectrum of the phase patterns rhythm (rounded to two decimal places) is

$$(4,\ 1.08,\ 0,\ 2.61,\ 0,\ 2.61,\ 0,\ 1.08)$$

While the spectrum of the (01) dyad is

$$(2,\ 1.85,\ 1.41,\ 0.77,\ 0,\ 0.77,\ 1.41,\ 1.85)$$

And multiplying these gives the spectrum of the resultant:

$$(8,\ 2,\ 0,\ 2,\ 0,\ 2,\ 0,\ 2)$$

In a combinatorial rhythm, there is some canon that gives a completely flat resultant, meaning it has zeros everywhere in its spectrum except at $k = 0$. For the *Phase Patterns* rhythm this occurs at (1,0,0,0,1,0,0,0), whose spectrum is (2, 0, 2, 0, 2, 0, 2, 0), with zeros in all the places where the rhythm itself has non-zero values (except the zeroth). I will refer to the rhythm of canonic entries as the “canonic configuration” or “configuration rhythm.”

We can use this theorem to classify all of the combinatorial rhythms of the 8-element universe. Let us consider just those rhythms combinatorial in two parts, which we can do by looking at all possible dyad configuration rhythms. There are only four distinct dyadic spectra, which are given in Figure 11. Clearly it is hard to get combinatoriality for intervals 1, 2, and 3. For 1 and 3, the rhythm must have zeros in coefficients 1–3, which means it

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[10] The convolution theorem, for example, is essential to the discovery and classification of tiling canons (Amiot 2011, 2016), the relationship of the DFT on pitch-class sets to interval content and Lewin’s (2001) interval function (Amiot 2016, Yust 2016). Auto-correlation, used for meter finding by White (2019), is also a form of convolution.

must maximize coefficient 4. This only occurs with the prototype of coefficient 4, 2-2-2-2, which is the combinatorial rhythm for the first version of *Piano Phase*. (Note: throughout the paper I will indicate rhythms with a successive-interval notation like this.) For interval 2, the rhythm must have zeros everywhere except component 2. This, similarly, can only occur if the rhythm is symmetrical at the half-measure (to get zeros in 1 and 3) and balanced between odd and even elements (to get a zero at 4). The unique solution up to translation is the rhythm of *Melodica*, 1-3-1-3.

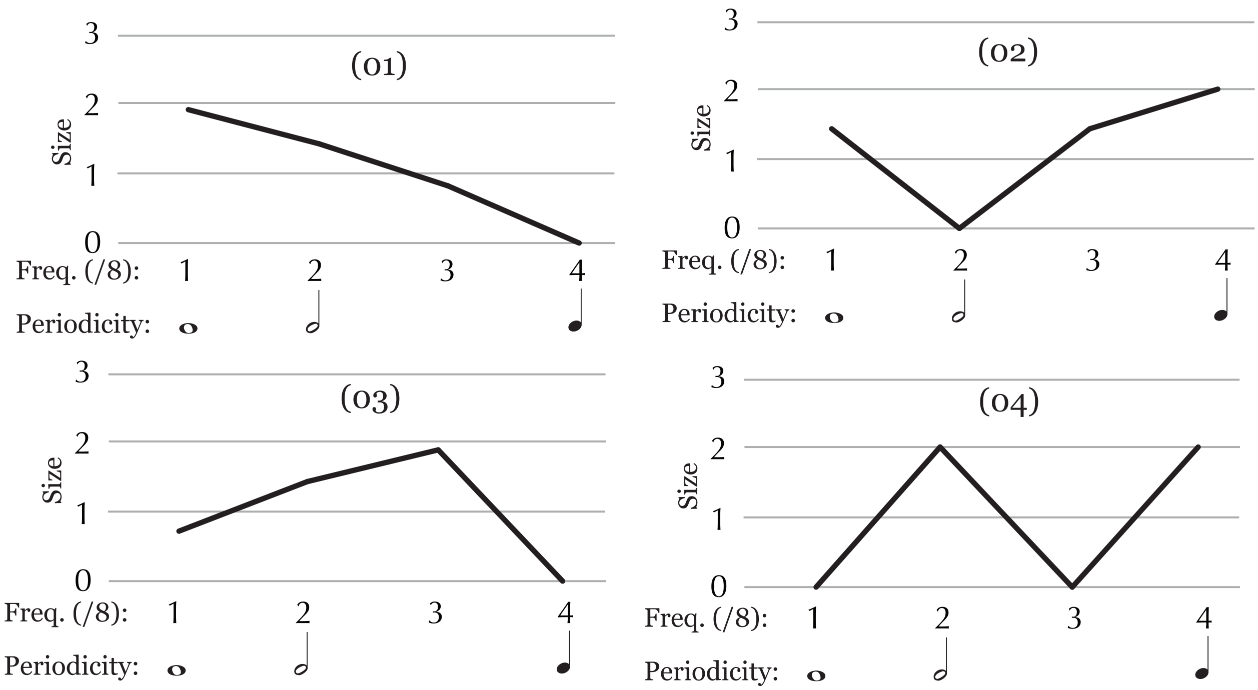

Figure 11: The spectra of all dyads in the 8-element universe

Interval 4 has more potential for combinatoriality because it has zeros for all of the odd-numbered coefficients, so the rhythm only needs to have zeros for the even ones. That is, it needs to be balanced at the half-measure and at the quarter-measure. It turns out that the only way to do this is to maximize coefficient 1 or 3 with a *generated* rhythm, a rhythm constructed by reiterating a single interval. A rhythm generated by interval 1 will maximize coefficient 1, the periodicity corresponding to the full cycle (because it will bunch onsets up as much as possible). Interval 3 is the closest approximation to the periodicity 8/3 (≅ 2.67), so generating a rhythm with this interval maximizes coefficient 3. When the cardinalities reach exactly half the size of the cycle (4), these odd-interval generated rhythms have zeros on all the even coefficients.[11] Although there is little musical

[11] Pressing (1983) and Toussaint (2013) discuss a number of interesting generated rhythms, and Cohn (2016) discusses a particular set of generated rhythms common in popular

interest in rhythms generated by 1, the ones generated by 3, and maximizing coefficient 3, are of particular musical value. The three-note rhythm generated by 3 is the familiar *tresillo* rhythm and the five-note one is its complement, the *cinquillo*, both of which are maximally even rhythms common in many popular and folk musics. The four-note 3-generated rhythm, 3-2-1-2, is a subset of the *cinquillo* and a superset of the *tresillo*. This is the rhythm Reich uses for *Phase Patterns*; its spectrum is given in Figure 12(a). All of the possible two-part combinatorial rhythms of the 8-element universe, except the very imbalanced 1-1-1-5 rhythm, are thus featured in one of these early phase pieces.

The effect of the phasing process can be visualized in a Cartesian space. Figure 12(b) plots the phasing process in the space for the third coefficient, the principal periodicity of the *Phase Patterns* rhythm. A rotation of the rhythm by an eighth-note (indicated by T_1) always changes the phase by the same amount without changing the magnitude. After four iterations of the process, the phasing rhythm is exactly opposite the original, which is the point of combinatoriality where the two vectors cancel one another out.

Violin Phase is an interesting development in Reich's use of combinatoriality where the property occurs for a *three*-part canon, but the underlying principle (maximization of a Fourier component through use of a generated rhythm) is similar to that of *Phase Patterns*. The idea that the piece is based upon, Figure 13(a), fills most of the 12-element cycle with onsets, but the rhythm of primary interest is the four-element rhythm made by the double-stops with the open E string, Figure 13(b). Figure 13(c) gives the spectrum of this rhythm and the spectrum of interval 4, which is the point where the first phasing process stops, and the tape picks up the canonic voice from the live violin. The rhythm is generated by interval 5, maximizing coefficient 5 and having zeros in the third and sixth coefficients. The canonic interval maximizes precisely these coefficients, the third and sixth, but it is 1 rather than zero on the other coefficients. Therefore the combination is not flat, but has exactly the same spectrum as the original rhythm. It is, in fact, the complement rhythm. To get a flat rhythm we need a three-part canon, with entries at 4-4-4. The spectrum of this rhythm is also shown in Figure 12(c), and this three-part canon is the ultimate goal of the second phasing process of *Violin Phase*.

All of these points about *Phase Patterns* and *Violin Phase* and combinatoriality have already been made by Cohn (1992b) using a more direct algebraic method. The reformulation of Cohn's observations using Fourier analysis may be worthwhile for its mathematical elegance alone, but more important for present purposes is that it will help connect combinatoriality to the development of Reich's style below. Combinatorial rhythms have to have zeros on a number of Fourier coefficients, which implies that they also have to maximize others, and will tend therefore to be prototypes of certain rhythmic qualities. In particular, they are often generated rhythms, which always maximize a given component (and are the *unique* way to maximize that component if it is co-prime to its

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music. On the mathematics of generated sets and the DFT, see Amiot 2007, 2011, 2016, Yust 2015a.

universe). The most interesting qualities from this perspective are those that are co-prime to their universe and are therefore typified by less symmetrical rhythms.

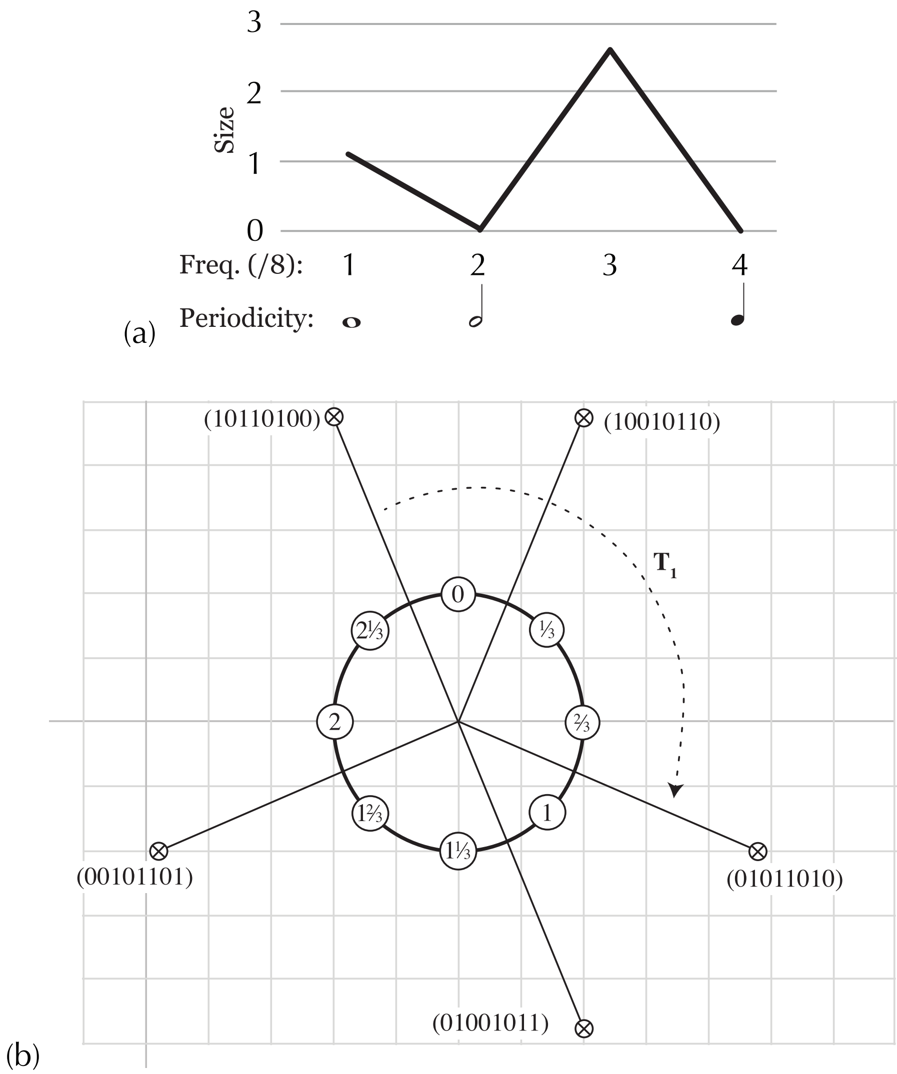

Figure 12: (a) The spectrum of the basic rhythm of *Phase Patterns* (see Fig. 10) (b) the phase space for coefficient 3 and different rotations of the rhythm in this space.

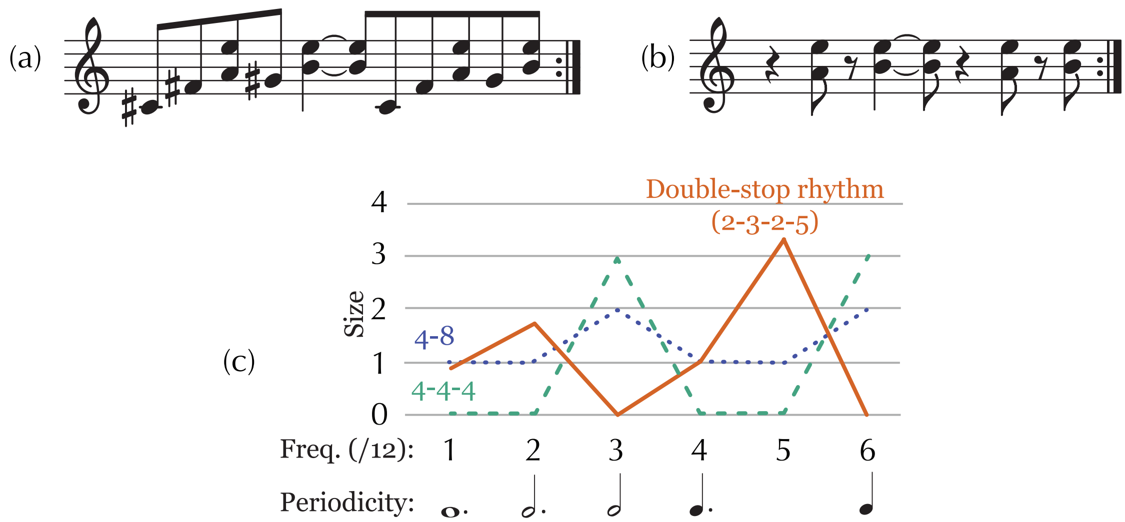

Figure 13: The basic pattern of *Violin Phase* (a), the double-stops of the pattern, (b), and spectra of the double stop rhythm and two canons that serve as the goals of phasing processes, (c)

These are also the rhythms that Reich encountered in his studies of traditional African and Balinese music, that are not characteristic of the European classical tradition. According to Scherzinger (2019), *Violin Phase* is based on a traditional Ghanaian pattern. The diatonic rhythm, which appears often in Reich's works of the 1970s and 1980s (such as the last movement of *Sextet*), and is a prototype of 5/12, is common in various West and Central African traditions. Tenzer (2019) observes that the generated rhythms of *Phase Patterns* and *Six Pianos* are typical of Balinese *kotekan* (interlocking patterns) that Reich studied in the early 1970s. Therefore, there are two possible explanations of how Reich's rhythmic language develops across these early phase pieces: one explanation is that he enriched the phasing idea with more complex rhythms learned through study of African and Balinese music. The other is that he arrived at generated rhythms by working through the logical conclusions of his early interest in how combinatorial rhythms give shape to the phasing process. Both explanations are undoubtedly partially true, giving credence to Reich's own way of explaining the influence of African music on him: "the influence of my trip to Africa on my composition was more in the nature of encouragement than change of direction [. . .] my trip there basically confirmed the direction I was already going in" (2002, 149), "the effect of my visit was basically *confirmation*" (2002, 106, Reich's emphasis). Because generated rhythms are prevalent in these repertoires, and generated rhythms were precisely the ones that satisfy the combinatoriality conditions, it is impossible to disentangle the two explanations. Regardless, since the idea of phasing itself was sparked by reading Jones (1959), as mentioned at the beginning of this section, the inspiration from African music is foundational, one way or the other. Reich implied as much, saying "the influence of African music on my music really happened much earlier, in 1963 and '64" (2002, 106).

Drumming* and *Clapping Music

The next stage in the development of Reich's rhythmic style leads to the turning point that marks the end of the phase-piece period of his career and into the multi-movement and large ensemble works of the 1970s and 1980s. Starting with *Clapping Music*, a single distinctive, rhythm, the signature rhythm, becomes ubiquitous in his music, and he leaves combinatoriality behind. What he learned from the experiments with combinatoriality in the phase pieces, however, particularly the properties of complex rhythmic qualities like 5/12, is crucial to all of this later music.

Reich's first feature-length large ensemble piece, *Drumming*, is transitional in this story. It is also based on a single rhythm (remarkably for a piece that lasts about an hour without pause!), shown in Figure 14(a). Again, by means of stemming (alternation of hands) Reich divides the rhythm into two parts. Now, however, the two parts are not related by translation. The spectra in Figure 14(b) show that they do both have zeros on all odd numbered components, and the right hand is a prototype of component 4. An interesting feature is that the sum of these rhythms has exactly the same spectrum as the left hand by itself (the right hand's only non-zero component, 4, is oblique in phase to the left hand), because the left hand relates to the complement of the total rhythm by translation.

(a)

(b)

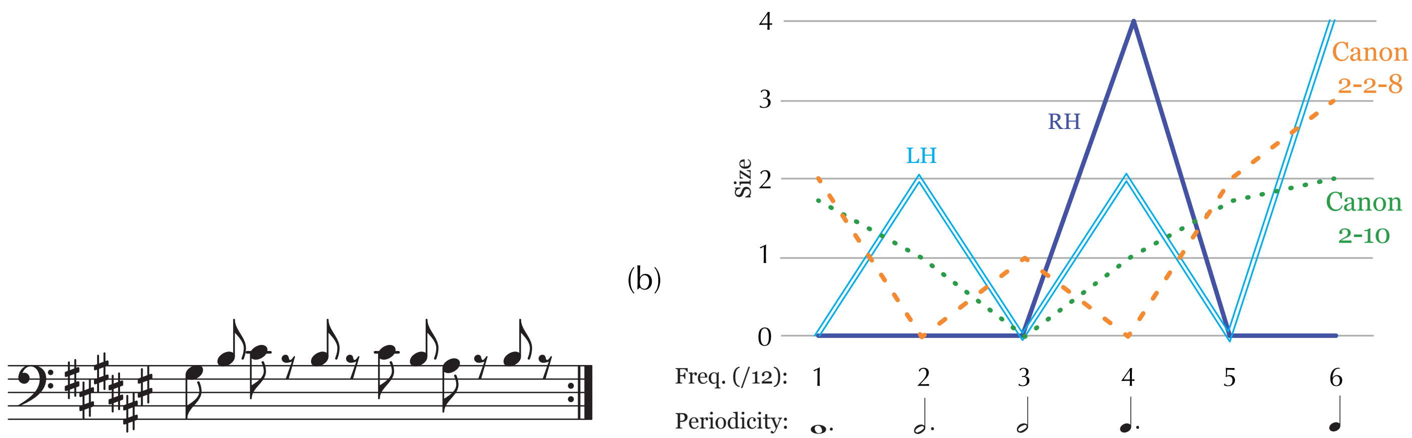

Figure 14: (a) The basic pattern of *Drumming*, (b) spectra of the left hand and right hand rhythms and canons

Reich's phasing process in *Drumming* occurs in three parts (like *Violin Phase*) and prioritizes the two- and three-part canonic configurations 2-10 and 2-2-8, whose spectra are shown in Figure 14(b).[12] The two-part configuration preserves the distinctive dotted quarter periodicity of the right hand while reinforcing coefficient 6 (the quarter-note beat)

[12] See Duker 2013 for a more complete analysis of *Drumming*.

of the left hand (as well as preserving coefficient 2). The three-part configuration has zeros for coefficients 2 and 4, and so also totally flattens out the right hand, realizing the combinatoriality of that part. In the process, though, it further reinforces coefficient 6, so that the quarter-note beat of the right hand takes over. Indeed, this is evident in the resultant patterns played over these canons. Here, then, we have something like a partial combinatoriality: if combinatoriality is understood as a canon that flattens the entire rhythmic spectrum, then we might understand a canon that results in zeros in all but one element of the spectrum to be a kind of incomplete combinatoriality. This connects *Drumming*, in which a phasing process terminates at a resultant that has zeros everywhere except at the quarter-note periodicity, to the earlier phase pieces, and also to *Clapping Music* and *Music for Pieces of Wood* where, as we will see, the goal canons result in zeros everywhere but the dotted-quarter-note periodicity.

The signature rhythm of Figure 15(a) is the other key component of Reich's stylistic transition in the 1970s, along with the shift from the concept of phasing to that of canon, already evident in the privileging of certain canonic configurations in the phasing processes of *Drumming*. In *Clapping Music*, where it first appears, the rhythm lacks any kind of differentiation between attacks, such as the differentiation produced by the alternation of hands in the *Drumming* rhythm. The same is true of *Music for Pieces of Wood*, also based on the signature rhythm. In later pieces, *Music for 18 Musicians*, *Desert Music*, and *Sextet*, melodic settings of the rhythm suggest different parsings on the basis of register and contour. One possible parsing appears throughout the marimbas in movement IIIa of *Music for 18 Musicians* and midway through the first movement of *Sextet,* as shown in Figure 15(b–c). Here a relationship to the *Drumming* rhythm becomes evident: the alternation of register splits the rhythm into an upper part which is a prototype of coefficient 4 (perfectly even), and a lower part which has the rhythm 2-3-4-3. This is precisely the complement of the basic rhythm, so a translation of the lower part by 6 can combine with the complete rhythm to precisely cover all twelve onset positions.

Later pieces like *Music for 18 Musicians* do not use this canonic interval of 6, but it is the basic canon for *Music for Pieces of Wood*, which suggests that Reich may have had this parsing in mind when first formulating the rhythm. In *Music for Pieces of Wood* the 2-3-4-3 subsets of the two canonic parts interlock to cover eight onsets, while the four isochronous onsets overlap making a 4/12 prototype resultant rhythm, a cross-rhythm against the six isochronous onsets of Clave 1. We can see this by multiplying spectra in Figure 14(c): the signature rhythm has zeros at 2 and 6 while the canonic interval of 6 has zeros on all odd-numbered components, leaving 4 as the only non-zero component. As Keith Potter (2011) shows in his discussion of Reich's sketches, Reich's initial work on *Clapping Music* and *Music for Pieces of Wood* proceeded simultaneously, and, throughout the process, combinatoriality (in the form of hocketing or interlocking patterns) was a foremost consideration, as was the dotted-quarter periodicity, explicitly realized in the early sketch in which the signature rhythm first appears. The importance of canon at interval 6 makes it all the more interesting that this canonic interval is virtually non-existent in the many uses of the signature rhythm in the following decades.

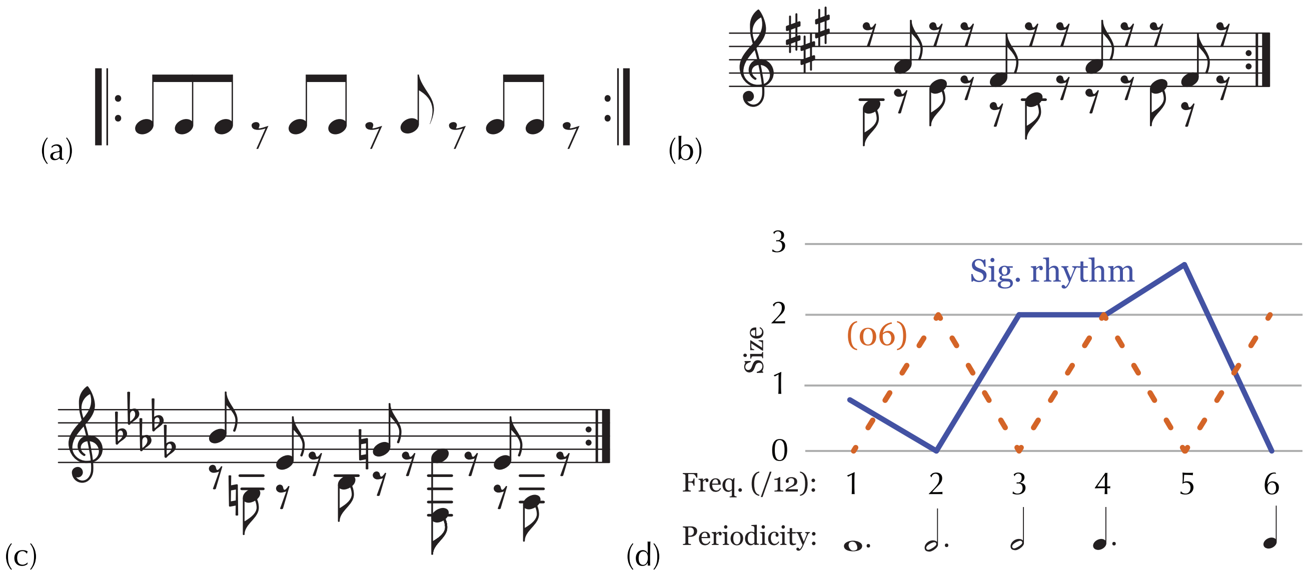

Figure 15: Reich's signature rhythm (a), as it appears in *Music for 18 Musicians* pt. IIIa (marimba), with stemming added to show registral segmentation (b), and *Sextet* Reh. 59 ff. (marimba), also with alternate stemming added (c), the spectrum of the signature rhythm with the spectrum of interval 6 (orange), the canon in the first part of *Music for Pieces of Wood* (d)

The "partial combinatoriality" in *Music for Pieces of Wood* illustrates that the signature rhythm, which would define the next phase of his career, preserves a trace of Reich's earlier concern with rhythmic combinatoriality, even as combinatoriality is abandoned as an explicit feature in later pieces. In the next piece to use the signature rhythm, *Music for Eighteen Musicians*, the melodic contours and separation between hands in piano parts more often emphasize a different component of its spectrum, 5/12. Figure 16(a) is an example from movement I that illustrates the most common subdivision in the work. The subset made by the high points in the melodic contour is a prototype of 5/12: 2-2-3-2-3. Figure 16(b), from the fourth movement, shows a similar segmentation that isolates a different 2-2-3-2-3 subset in the upper register. In both cases, the lower register rhythm is 3-4-5 or 4-3-5. As the spectra in Figure 16(c) show, the maximally even 2-2-3-2-3 isolates the coefficient-5 quality of the signature rhythm, while the 3-4-5 subset has a similar spectrum to the overall rhythm, but favoring the 3rd coefficient. Thus, in the first four movements of *Music for Eighteen Musicians*, where canons are absent, the focus is instead on eliciting rhythmic qualities, particularly the 5/12 qualities of the signature rhythm, through segmentations.

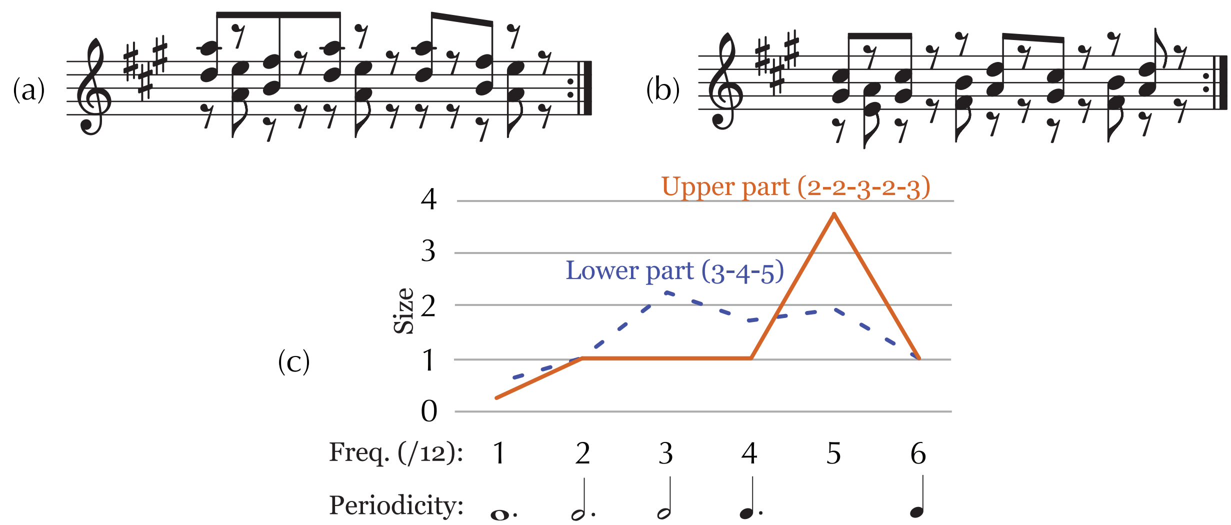

Figure 16: *Music for 18 Musicians* m. 109 ff., clarinet and voices (alternate stemming added) (a), and m. 321 ff., clarinet and voices (alternate stemming added) (b), and spectra for the 2-2-3-2-3 and 3-4-5 subsets of the signature rhythm (c)

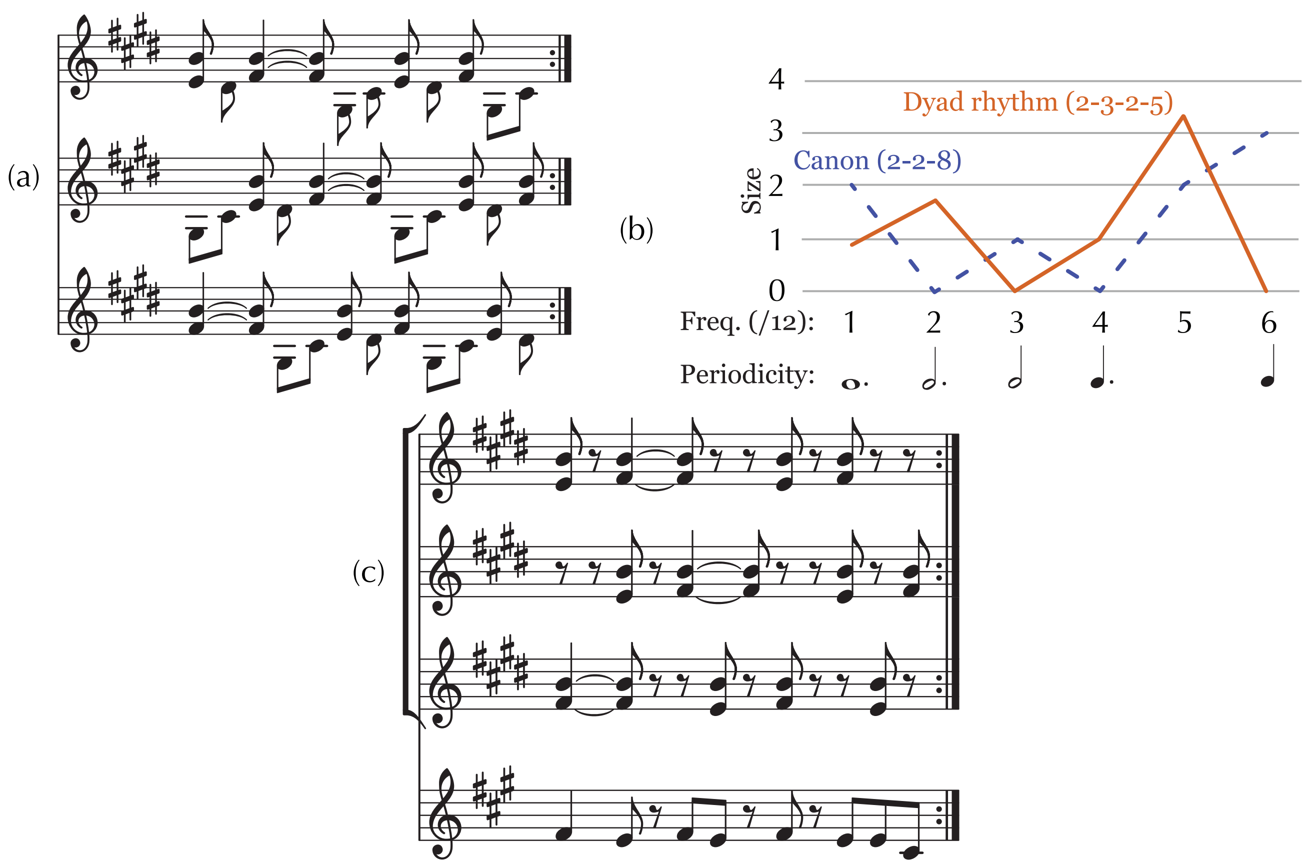

Figure 17: (a) Canon from *Music for 18 Musicians* V (pianos, m. 372 ff.), (b) Spectra for the rhythm of the dyads (2-3-2-5) and the canon configuration (2-2-8), (c) The canon on dyads only, and the resultant rhythm appearing at m. 382 ff. (marimbas, piano, voices)

This interest in coefficient 5 is underscored by what happens in movements V and VI, the central movements of the work. Canons finally appear in movement V but they use the *Violin Phase* rhythm, not the signature rhythm that is used throughout the first four movements. Recall that Reich originally devised this rhythm such that the four double stops would be combinatorial in a three-part canon at the half note. The double-stop rhythm, 2-3-2-5, is generated by a 5♪ interval, maximizing component 5, so that in convolution with the component-3 prototype 4-4-4, a completely flat rhythm results. In *Music for 18 Musicians* he does essentially the opposite, making a three-part canon with the pattern 2-2-8, which emphasizes coefficient 5 and cancels out most other coefficients, shown in Figure 17(a–b). The resultant rhythm of the dyads is thus a generated rhythm that maximizes coefficient 5, close to the prototype diatonic rhythm. Towards the end of movement V, Reich extracts this resultant rhythm in the marimba part shown in Figure 17(c), which becomes the basis of movement VI, in alternation with the diatonic rhythm (removing the second-to-last onset). Movement VII then returns to the signature rhythm of movement I, completing a process that amounts to an elaborate compositional essay on 5/12 rhythmic quality.

These kinds of canons, which reinforce rather than flatten the principal periodicities of the rhythm, become the norm in subsequent pieces. As an example, Figure 18 shows two four-part canons on the signature rhythm from the third movement of *Electric Counterpoint*. The first canon (in 3/2) has entries spaced 2-3-4-3, the complement of the rhythm itself, meaning that the canonic configuration has exactly the same spectrum as the rhythm, serving to exaggerate its quality-profile. The contour of the line, on the other hand, emphasizes the perfectly even component-4 prototype subset, 3-3-3-3. The reason for this is revealed at the time signature change to 12/8. There is no change here in the basic pulse, nor does the rhythm change, or the contour of the canonic line. Only one part in the canon shifts, the second staff, by one eighth note. The result of this shift, however, is to throw the weight of the canon configuration rhythm, now 2-4-3-3, to component 4, resulting in a palpable feeling of metric shift. The perceptual rabbit-duck effect of hearing the same rhythm and melody from different metrical perspectives, particularly with the minimal one eighth-note "voice leading" on the canon configuration, is musically effective, and the central thesis of the whole movement. Reich does something similar with the diatonic rhythm in the finale of *Sextet*, and another closely related rhythm in the finale of *New York Counterpoint*. (See Roeder's [2003] analysis of the latter.)

Even when the signature rhythm does not seem to be present, it often lurks just below the surface. The first and third movements of *New York Counterpoint* are based on canons in 12-cycles of eighth notes whose rhythms are closely related to the signature rhythm. The interior second movement seems to be a respite from it. The tempo shifts by a factor of two, and the basic melody, shown in Figure 19, occupies two measures of 3/4 with a sixteenth note rhythmic unit (equal in tempo to the eighth notes of the framing movements), making a 24-cycle. The spectrum of this rhythm (Figure 19c) resembles that of the signature rhythm from 1–6, then flattens out at approximately 1 from 7–11.

Figure 18: Canons on the signature rhythm in the third movement of *Electric Counterpoint* (mm. 116–117)

We can see why this happens by splitting the rhythm into one consisting of just the onsets that fall on the eighth-note grid of the meter, and those off the grid. Both of these rhythms are 12-cycle rhythms on an eighth-note grid. Considering them on a sixteenth-note grid, where every other onset position has a zero, is an instance of "oversampling," meaning that the rhythm is represented with a higher sampling rate (here, sixteenths instead of eighths) than necessary. The result of an oversampling (see Amiot 2016 or 2017a) is a repetition in the spectrum. Here, the complete spectra of the 12-cycle rhythms, with their mirror symmetry (aliasing), appear in coefficients 0–11 of the oversampled spectra, and are repeated in coefficients 12–23. The on-beat rhythm (Figure 19c) is, in fact, just a rotation of the signature rhythm, and has the same spectrum, which can be seen in Figure 18(c). The off-beat rhythm, 4-3-5, has only three elements and has a similar spectrum to the signature rhythm, with peaks at 3 and 5.

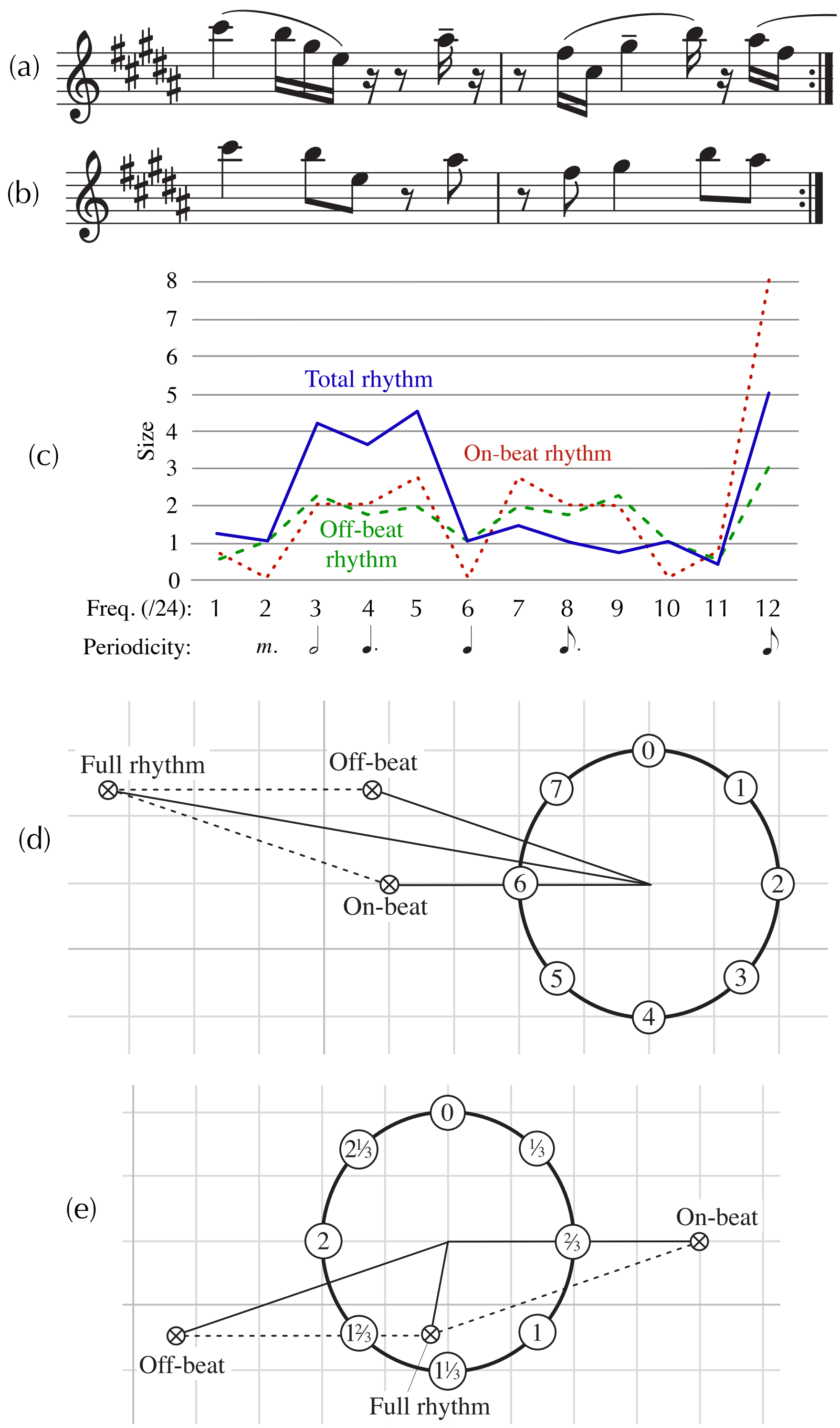

Figure 19: The subject of *New York Counterpoint* II, (a), reduced to onsets on the eighth-note grid, (b), and the spectrum of the whole rhythm and the rhythm split into onsets on and off the eighth-note grid, (c). Cartesian plots for coefficients 3 and 9 of the three rhythms are given in (d) and (e).

When the two rhythms add together they reinforce one another at coefficients 1–5 and weaken one another at 7–11. This is the result of phase relations between them. Notice that the elements of the off-beat rhythm always fill in an eighth-note span of the on-beat rhythm. Therefore the off-beat rhythm maps onto a subset of the on-beat rhythm when shifted a sixteenth-note in either direction. As a representative case, Figures 19(d) and (e) show the consequences of this with Cartesian plots of coefficients 3 and 9. The first of these is the same space as Figure 6 above. Because the on-beat rhythm is the signature rhythm shifted back by a quarter note, half of a cycle in this space, it is the same distance from the origin but the opposite angle. In the space of coefficient 3, the intervals ±1 represent small phase differences (transposition by ±1 is a 45º rotation around the origin), whereas they are relatively large (135º rotation) in the space for coefficient 9. Therefore, the off-beat rhythm, which is similar to transpositions of the on-beat rhythm by one sixteenth in either direction, is close in phase to the on-beat rhythm in coefficient 3, and distant in coefficient 9. When these are added together, then, they reinforce one another in coefficient 3, but largely cancel one another out in coefficient 9. This reasoning can be extended to all mod-12 complementary pairs of coefficients: the translations by a sixteenth are smaller for *k* closer to 0, and larger for *k* closer to 12. Therefore Reich's method of constructing this rhythm suppresses the higher-frequency coefficients (7–11), a reproduces the shape of the signature-rhythm spectrum in the lower-frequency coefficients, 1–6.

While Reich's earlier pieces, such as *Violin Phase* and *Six Pianos*, as well as *Music for 18 Musicians*, single-mindedly quarry all the possible manifestations of a single rhythmic quality, 5/12 or 3/8 in these pieces, the sustained interest in the signature rhythm responds to the multivalent potential of its richer rhythmic spectrum, with prominent components at 3, 4, and 5. The example of *Electric Counterpoint* above (Figure 18) illustrates the play between these in its shifting meters. Another example, *Vermont Counterpoint*, the first of the *Counterpoint* series, begins by building a three-part canon on the melody in Figure 20. Its rhythm is closely related to the signature rhythm, but omitting one onset throws its weight towards coefficient 4 and away from 3, as the spectrum shows. The piece begins by building a canon with entries related by dotted quarters, 3-3-6, further emphasizing component 4 so that the piece begins with a gradual shift from the initial 3/2 meter to a 12/8 feel, purely by means of canon-building.

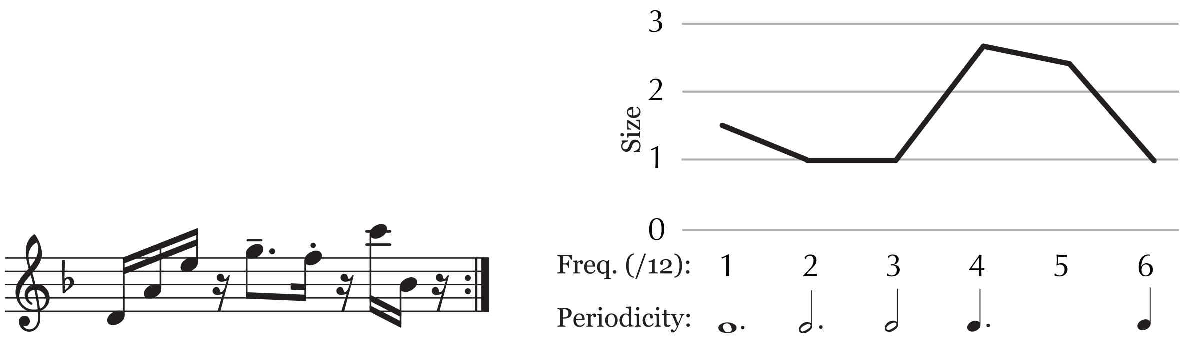

Figure 20: The initial melodic of *Vermont Counterpoint* (a), and its spectrum (b)

Other cycles: *Eight Lines, Sextet, Electric Counterpoint*, and *Desert Music*

The signature rhythm, along with the methods of cyclic canon, resultant patterns, and graduated pattern building that Reich formulated over the course of writing his phase pieces, was the wellspring of a rhythmic language that Reich would continue to cultivate for three decades. The first major fruit of this style is *Music for 18 Musicians*, with its large-scale continuous multi-movement plan, one of Reich's most identifiable works and pivotal in his career. This work, as we have seen, is based almost entirely upon the signature rhythm. Hence, despite the differentiation of instrumental textures, techniques, and harmony that shape the work, it remains "minimalist" in its devotion to an unvarying pulse and the 12-element rhythmic universe. Though an attachment to the 12-element universe of the signature rhythm would persist for many years, Reich remained interested in more irregular metrical cycles as well, and the influence of the signature rhythm can be seen there also. His 1979 *Octet*, revised with changes of instrumentation as *Eight Lines*, is a notable example, a piece using many of the large-ensemble techniques he developed in the 1970s, but with 10- and 20-element rhythmic cycles.

Eight Lines is based in its entirety upon a series of canons on the rhythm and melodic contour in Figure 21(a), or (only at the very beginning of the piece) on a repetition of the first measure of the pattern. Because the two halves of the pattern are very similar, there are two ways to go about analyzing it, by thinking of it as two closely related rhythms in 10-element cycles, as in Figure 21(b), or as a single 20-cycle rhythm, as in Figure 21(c). The two approaches arrive at the same conclusions by different means. The second of the two 10-cycle rhythms is maximally even, a prototype of component 4. This component (unlike any in the 12-element universe) is neither co-prime to nor a divisor of the cycle (10). The ME(10, 4) and ME(10, 6) rhythms therefore have a translational symmetry (at 5) but are not isochronous. The rhythm in the second measure of Figure 21(a) is ME(10, 6), and its contour emphasizes the complementary ME(10, 4) rhythm (♩. ♩ ♩. ♩).

The first measure of Figure 21(a) is similar to the second, but adds an onset (in the third position) and changes the contour to outline a ♩. ♩. ♩ ♩ rhythm. The resulting spectrum in Figure 21(b) still has a prominent component 4, but raises the floor so that the odd components all have magnitude 1 rather than zero. Figure 22 reduces the rhythm to the one outlined by the melodic contour and gives it spectrum. The variation of the maximally even pattern in the first measure has the effect of introducing a non-zero ♩ periodicity (frequency 10/20 or 5/10), while maintaining the principal ♩‿♪ periodicity (8/20 or 4/10). The ♩ periodicity is important in that it matches the beats of the notated meter, and is also the basic pulse for augmented rhythms in the string parts, discussed further in the next section.

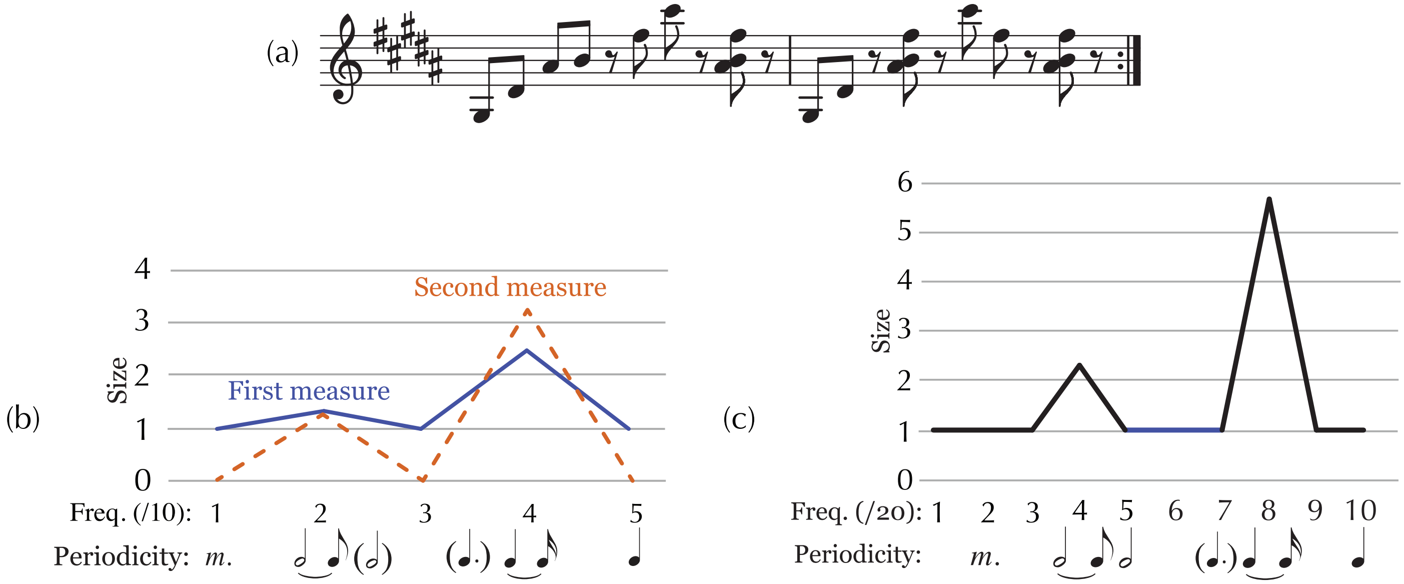

Figure 21: Ostinato for *Eight Lines* (a), the spectrum of each measure in a 10-element universe (b), and the spectrum of the entire rhythm in a 20-element universe

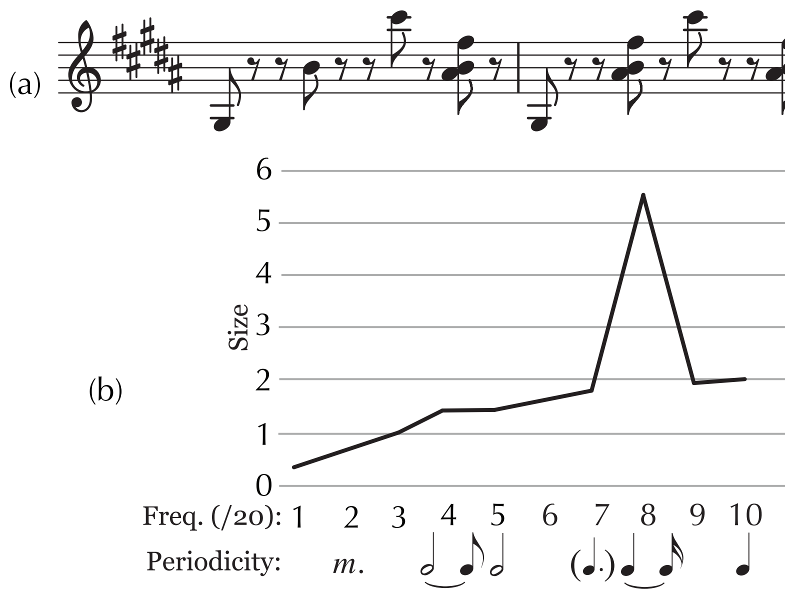

Figure 22: Reduction of the *Eight Lines* ostinato (a), and its spectrum (b)

Reich's use of the 20-cycle in *Eight Lines* is a logical continuation of his exploration of the 5/12 quality of the signature rhythm in *Music for 18 Musicians*. The generator for prototypes of this quality (such as the diatonic rhythm) is the interval 5. In general, a generator for *k/n* occurs at the interval closest to a multiple of the periodicity *n/k* (Clough and Douthett 1991, Amiot 2007). For $n = 12$, $k = 5$, this is $2(12/5) = 4\,{}^{4}/_{5}$, which is close to 5. Imagine then switching the role of this interval, 5♪, from generator to fixed cycle by scaling the basic periodicity up slightly from 12/5 ($2\,{}^{2}/_{5}$♪) to 5/2 ($2\,{}^{1}/_{2}$♪). This produces a frequency of 2/5, equivalent to 4/10 or 8/20, whose prototype 2-3 rhythm serves as the basic material of *Eight Lines*. In other words, the rhythmic palette of *Eight Lines* can be viewed as a simplification of *Music for 18 Musicians*, achieved by finding a nearest approximation to the 5/12 quality with a smaller denominator, 2/5. By the same logic, we could also imagine a piece based on 3/7 quality (with prototype 2-2-3 rhythms). The difference is that 7/3 ($2\,{}^{1}/_{3}$) is smaller, closer to 2, than 12/5 ($2\,{}^{2}/_{5}$), whereas 5/2 ($2\,{}^{1}/_{2}$) is larger. There are deeper consequences to these observations to be unpacked in the next section.

Eight Lines is a single-movement six-part arch form built out of two- and three-part canons on the melodic contour of Figure 21(a), with two-part canons in the outer sections (1 and 6) and three-part canons in the others. The three-part canons of sections 2 and 5 have configuration rhythms 2-4-14 and 4-2-14 and the interior parts (2 and 3) have configuration rhythms generated by interval 2: 2-2-16. Other differences between the canons involve phase only, so there are only three spectra to consider, given in Figure 23. The coefficients of primary interest here are number 8, the principal 𝅘𝅥‿𝅘𝅥𝅯 periodicity of the canonic rhythm; 10, the 𝅘𝅥 beat; and 4, a secondary 𝅗𝅥‿𝅘𝅥𝅮 periodicity. The two-part configuration on interval 4 weakens the main 𝅘𝅥‿𝅘𝅥𝅯 periodicity (8) while reinforcing the 𝅗𝅥‿𝅘𝅥𝅮 and 𝅘𝅥 periodicities. The three-part 2-4-14 configuration does the same, even more strongly. The 2-2-16 configuration, by contrast, adds emphasis to the anti-metrical 8/20 periodicity, so that in the central part of the piece, the polymetric potential of the $2\,{}^{1}/_{2}$♪ periodicity of the basic rhythm against the quarter-note periodicity of the meter comes into focus. At the same time, it de-emphasizes the half-measure periodicity of component 4, the significance of which is that it is present in a rhythm that the strings first play in part 1 (rehearsals 0–3), returning at the beginnings of part 3 (rehearsals 16–18), part 4 (rehearsals 32–35), and part 6 (rehearsals 65–66), and more strongly in a development of that rhythmic idea in rehearsals 19–31 of part 3 and 50–63 of part 5. The strings play on a quarter-note grid throughout.[13] With the quarter note as a unit, the 5♪ periodicity of coefficient 4 can instead be understood as a 4/10 frequency, equivalent to the 8/20 of the main rhythm in augmentation. We will refer back to this multi-leveled rhythmic design in the next section.

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[13] Not counting two passages in parts 2 and 4 (rehearsals 9–13 and 40–44) where the lower strings play a resultant pattern from the canon.

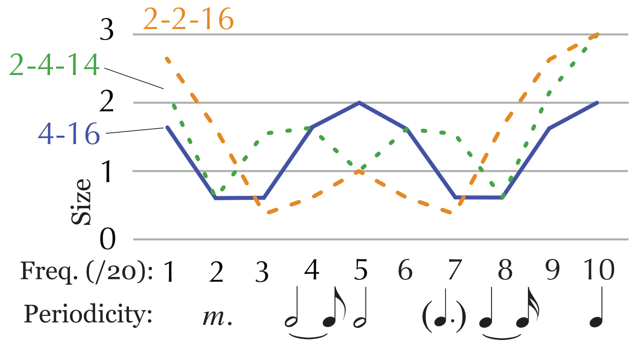


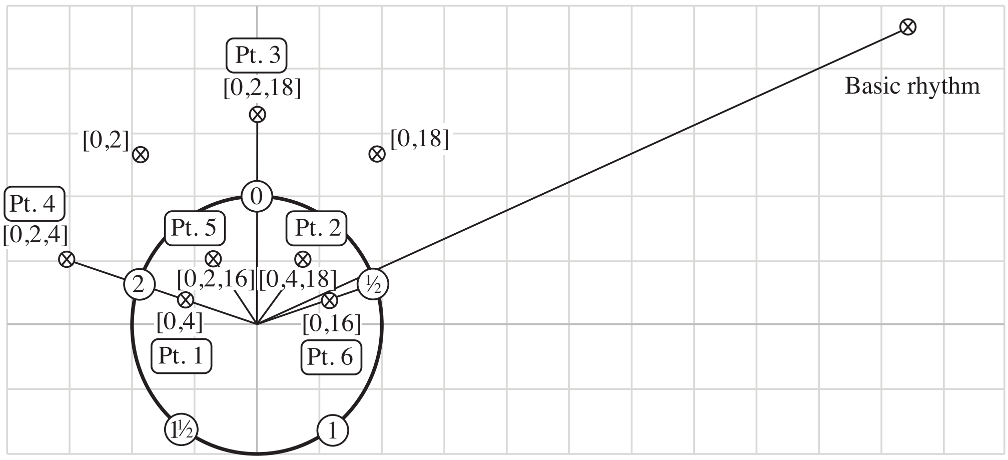
Figure 23: Spectra for the canonic configurations in *Eight Lines*

Each of the canon configuration rhythms occur in two phase orientations in the piece, which also express the arch form of the piece. The configurations are:

Pt. 1 (r.0–3): [0,4]
Pt. 2 (r.4–14): [0,4,18];
Pt. 3 (r.16–28): [0,2,18];
Pt. 4 (r.31–43): [0,2,4];
Pt. 5 (r.47–57): [0,2,16];
Pt. 6 (r.68–74): [0,16]

Corresponding canons of the symmetrical arch form relate by retrograde around timepoint 0 (parts 1–6 and 2–5) or 1 (parts 3–4). Figure 24 plots the canon configurations in two Cartesian spaces, for coefficients 8 and 4. The basic rhythm is behind the beat by about $^1/_5$ of a period (a 𝅘𝅥𝅯 for coefficient 8 and an ♪ for coefficient 4) in both cases. According to the convolution theorem, while multiplying their magnitudes, convolution of two rhythms *adds their phases*. The [0,4] canon of part 1 therefore brings the phase of the basic rhythm into alignment with the downbeat, perfectly for coefficient 8 and approximately for coefficient 4. Coefficient 8 follows a regular pattern going out of alignment in part 2, getting larger and still out of alignment in part 3, back into alignment at part 4, then weakening and going gradually out of alignment in parts 5 and 6. Coefficient 4 is weak in parts 3 and 4, then comes back to strength nearly a full ♩ behind the downbeat in part 5, remaining behind the downbeat in part 6. Overall then, as the arch design shifts focus to one anti-metrical periodicity, 𝅗𝅥‿♪, then a more extreme one, ♩‿𝅘𝅥𝅯, and back, there is also a directional process of phase shift of these periodicities—particularly the first, 𝅗𝅥‿♪, which remains strong in the last two sections—out of sync with the downbeat.

(a)



(b)

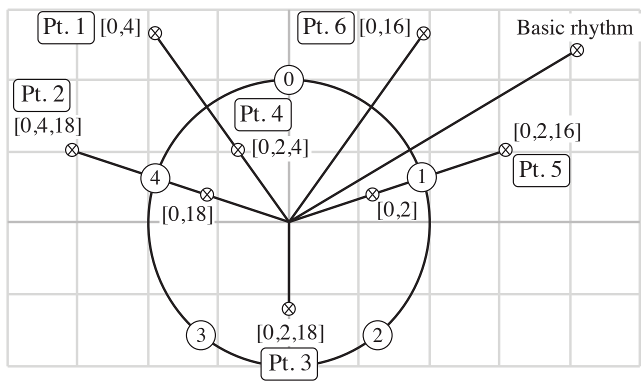


Figure 24: Rhythm and canons in Cartesian spaces for the 8th (a) and 4th (b) coefficients

In instrumental works of the 1980s, Reich begins to use interior movements as opportunities to explore more complex rhythmic cycles, while outer movements take advantage of the 12-cycle and the play between the different possible even and uneven divisions of it. *Sextet* is a prime example, where Reich exploits the possibilities of a 22-element cycle in the second and fourth movements.

Reich first used alternating meters of 6/4 and 5/4, creating a cycle of 22 eighth-notes, in *Variations for Winds, Strings and Keyboards* of 1979, the next piece he wrote after *Octet*. From the sketches reproduced in Potter (2017), we can see that Reich arrived at this metrical scheme by expanding a repeating 5/4 meter: He began with the simple pattern of Figure 25 made up of continuous eighth notes and an unvarying up-down contour. In the final composition, he introduces rhythmic interest by expanding alternately the fourth and fifth eighth note of the original pattern to a quarter note, making an alternating cycle of two 11-eighth-note patterns. The quarter-note beat of the meter and the simple alternating melodic contour now interact with the odd 11-cycle of the pattern, which goes in and out of phase across each two-measure cycle. He uses a similar idea for the first movement of his later orchestral work, *The Four Sections*, but notated instead in a cycle of meters 4/4–2/4–3/4–2/4.

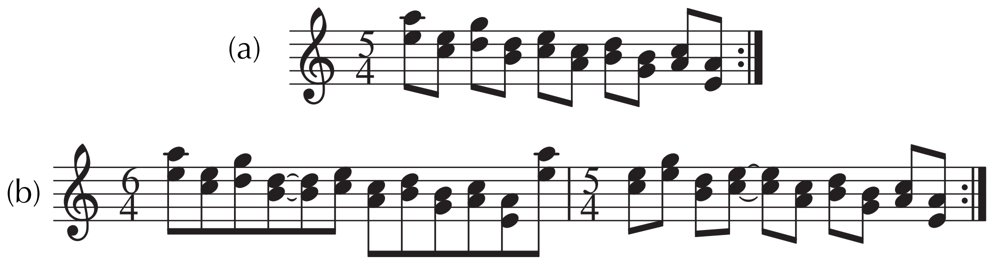


Figure 25: The initial ostinato for *Variations*, as it appears in an early sketch (a), and in the final version (b)

The interior movements of the *Sextet* use the 22-cycle in a manner that reflects the rhythmic techniques developed in connection with the signature rhythm. The second movement is based on the melody in Figure 26(a), which we can look at in three ways: the total 18-onset rhythm, the contour- and articulation-based reduction in Figure 26(b), and the accompanying bass drum rhythm. The spectra of these are given in Figure 26(c). A 9/22 frequency is evident in the rhythm itself, as well as the reduction. The bass drum emphasizes only on-beat notes (on the quarter-note grid of the meter), so it is effectively an 11-cycle rhythm. The eighth-note grid "oversamples" this rhythm by a factor of 2, so that component 11 (half of 22) has a value equal to the cardinality (like component 0), and from 1–10 the spectrum is retrograde symmetrical. Besides emphasizing the quarter-note beat, the bass drum also has peaks at 4 and its complement at 7. The basic rhythm echoes the first of these (4/22) while the reduced rhythm of Figure 26(b) echoes the other (7/22).

The movement alternates between two two-part canons on this melody, at intervals 3 and 4. Figure 27 compares the spectra for these to the reduced melodic rhythm. The first canon, at interval 4, reinforces the quarter-note beat in the bass drums (coefficient 11), as well as, to a lesser extent, the secondary coefficients 4 and 7 (periodicities of $5^1/_2$♪ and $3^1/_7$♪), while it de-emphasizes the main coefficient 9 ($2^4/_9$♪) of the basic rhythm. The second canon, at interval 3, instead drops the ♩ periodicity (coefficient 11) to zero, and reinforces coefficient 9 and, even more strongly, 7.

This texture returns in the fourth movement of *Sextet* with the addition of a new pattern in the vibraphones, Figure 28(a). This new pattern is closer to a 9/22 prototype, the ME(22, 13) rhythm, than the rhythms of the second movement, but it is not quite that. It could be understood as the combination of ME(22, 9) and ME(22, 7) rhythms intersecting in exactly one onset (position number 5), which happens to be the only sustained chord of the pattern, as shown in Figure 28(a). The spectrum of this rhythm, Figure 28(b), has a clear principal peak at 9 and secondary peak at 7. The main quarter-note canonic interval (2-20) reinforces the principal component 9 and suppresses the secondary component 7, while the alternative dotted-quarter canonic interval (3-19) shifts focus to component 7.

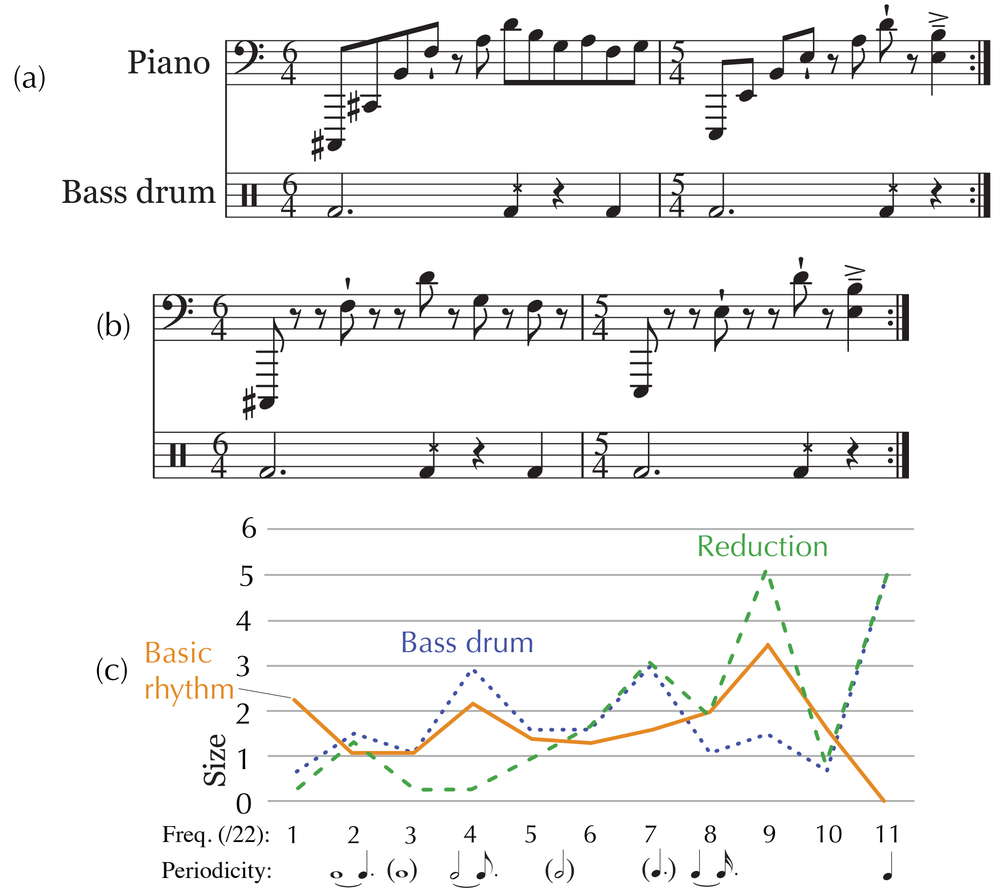


Figure 26: The ostinato of *Sextet* ii (a), a contour-based reduction of it (b), and the spectra of these (c)

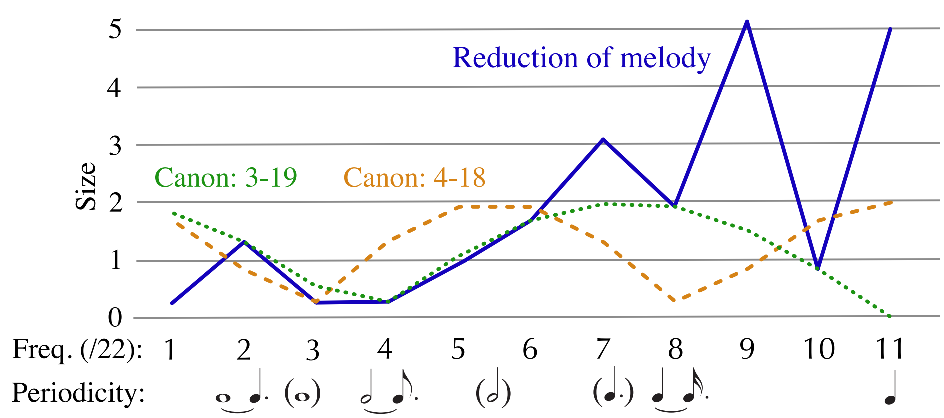


Figure 27: Spectra of the canons and reduced melody in *Sextet* ii

This vibraphone pattern is very similar to the signature rhythm—in fact, the first (6/4) measure is the signature rhythm shifted back a quarter-note, while the second (5/4) measure is a truncation of it (with the sustained onset on beats 3–4 removed). This similarity is reflected in a similarity of spectra, illustrated in Figure 29. The peak at 9/22 ($2^{4}/_{9}$♩♪) is very close to the peak at 5/12 ($2^{2}/_{5}$♩♪), and the peak at 7/22 ($3^{1}/_{7}$♩♪) is very close to 4/12 (3♩♪). We will further generalize this finding shortly, and endeavor to explain how it relates to Reich's rhythmic language.

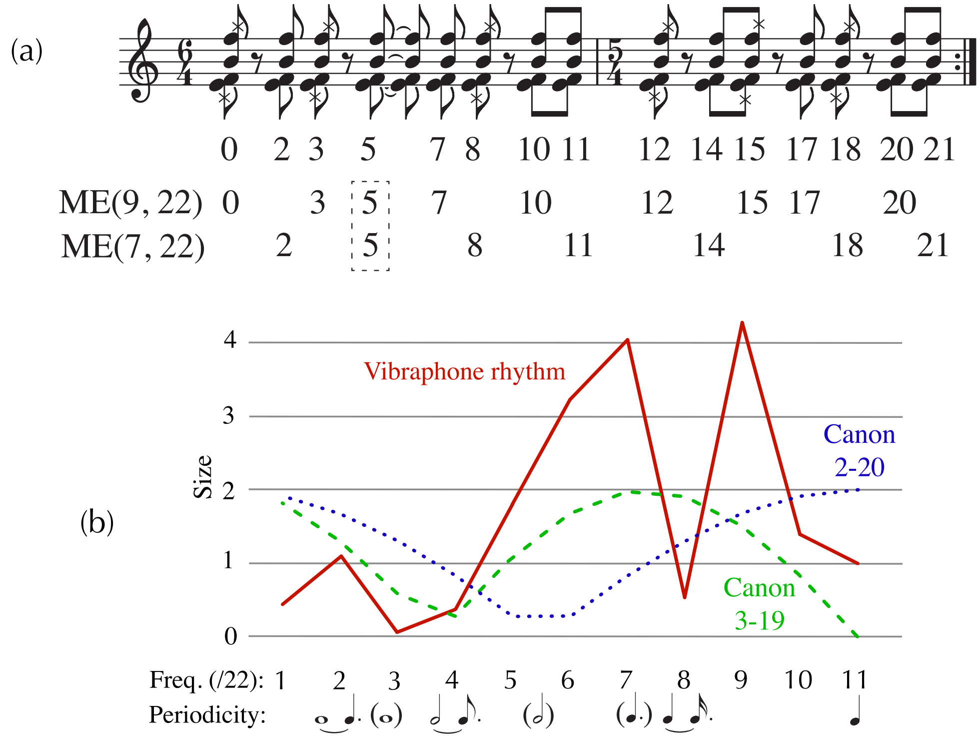


Figure 28: The vibraphone part from *Sextet* IV, (a), and
its spectrum with those of the two canonic configurations, (b)

We also find an interesting irregular rhythmic cycle in the interior movement of *Electric Counterpoint*. The basic melodic idea of the movement is given in Figure 30(a). The cycle here is a length of 19 eighth-notes, divided up into measures of 3/4, 5/8, and 4/4. Since there is only one isolated sixteenth-note in the melody, the sixteenth-note level is not particularly interesting, so Figure 30(b) shows the spectrum, analyzed on the 38-cycle of sixteenth notes, from coefficients 1–9 only, to compare it with the 19-cycle spectrum of the canon. (Recall that oversampling results in repetition in the spectrum, so the almost-oversampled rhythm of Figure 30a has an approximate symmetry from 1–9 to 10–18.) The spectrum has a distinct principal peak at 7, which indicates that it is almost (though not exactly) a ME(19, 12) rhythm. Specifically, if the sixteenth-note embellishment were removed and onset following it moved ahead by an eighth note, the result would be ME(19, 12). The rhythm of canon entries for the movement, 1-2-16, reinforces the $2^{5}/_{7}$♩♪ periodicity of coefficient 7, as its spectrum in Figure 30(b) shows.

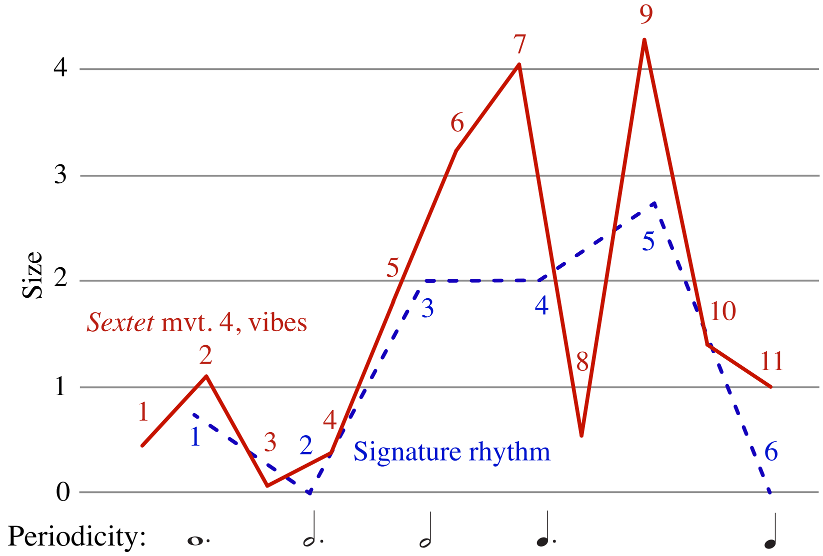


Figure 29: The spectra of the signature rhythm and vibraphone rhythm from *Sextet* IV, proportionally adjusted to compare similar periodicities.

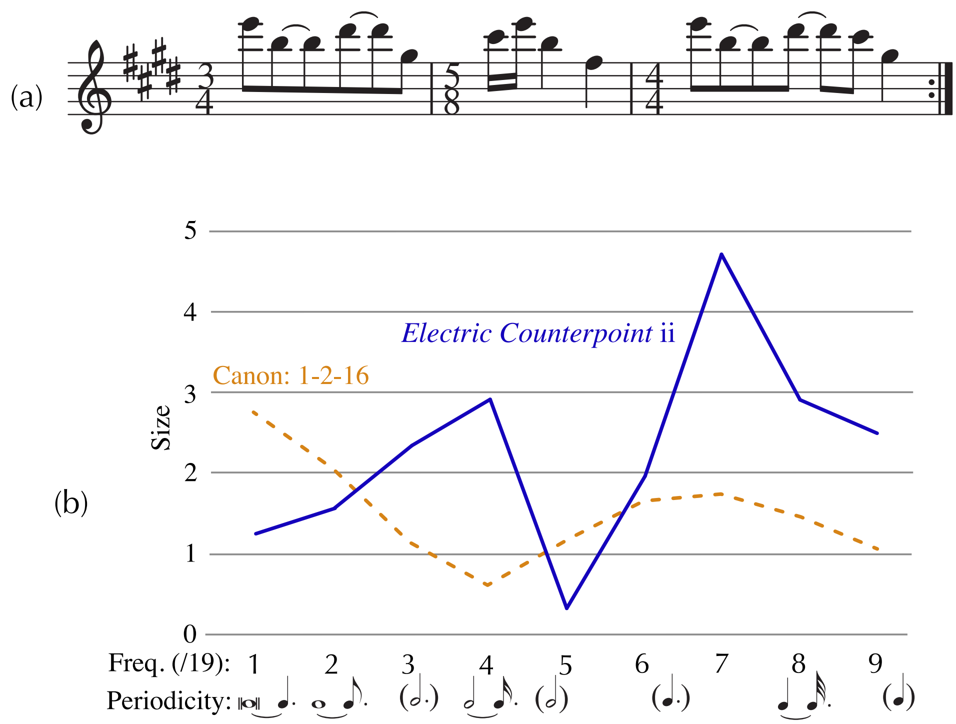


Figure 30: The basic melody of *Electric Counterpoint* ii, (a), and its spectrum with that of the canon entries, (b)

A rhythm-pitch analogy is tempting here: the ME(19, 12) pattern also gives a chromatic scale in 19-tone equal tuning, an equal-tuning approximation of mean-tone temperament. It seems unlikely, though, that Reich had such an abstract analogy in mind. The coincidence can instead be explained by common motivating principles in the two modalities: approximation to the mean-tone fifth in pitch, which explains both the ME(12, 7) diatonic scale and the ME(19, 12) chromatic scale, and a similar approximation of a desired periodicity ratio in the rhythmic case. Exactly what this desired frequency ratio is requires further investigation. That is the task of the next section.

Reich constructed more complex rhythmic cycles in the central movement of his large-scale orchestral work of 1983, *The Desert Music.* One of these is the canon on “It is a principle of music to repeat the theme,” given in Figure 31(a). An interesting feature of this canon is that Reich uses an irregular series of meters to notate it, but because the entire cycle ultimately amounts to 32 eighth notes in length, it could have been notated in 2/4 or 4/4. Horlacher (2000–2001) points this out and argues that such a duple metrical hearing might emerge over the course of the repetition of the canon, framing the irregular rhythm as syncopation against a regular meter rather than a simple articulation of a continually shifting meter, as notated. From the theory and tools we have developed thus far, however, traditional concepts of meter, which virtually force us to consider a duple interpretation (since only powers of two divide 32) look excessively limiting. We can instead ask a somewhat different question: what periodic divisions of the cycle, not limited to integer multiples of the eighth note, are more or less strongly expressed by this rhythm? Figure 31 shows the spectrum of the rhythm for this canon, for which the most prominent periodicities are 𝅝 (coefficient 4) and $2^{6}/_{13}$♪ (coefficient 13). The first of these suggests that there might be something to Horlacher’s rebarring in 2/4, though at the two-measure level, not at the one-measure level (the 𝅗𝅥 periodicity is almost completely absent). However, noting that Reich uses changing meters primarily to accurately set the linguistic stress pattern of the text, the rhythm of stressed syllables deserves consideration as well. Its spectrum also appears in Figure 31. In the stressed-syllable rhythm, $2^{6}/_{13}$♪ clearly dominates while 𝅗𝅥 is weakened. Thus, while a 4/4 or 8/8 barring works for the basic rhythm, it does so at cross purposes to the linguistic stress. Another important consideration is the canon. The rhythm of entries for the three-voice canon is 3-2-27, which neutralizes coefficient 4, and strongly reinforces 13.

The next large canon of the movement is shown in Figure 32(a). The cycle is only slightly longer, 33♪. Although the rhythm is quite different from the “It is a principle” canon, one notable similarity is the peak at coefficient 13 in the spectrum (Figure 32b). There are also similar subsidiary peaks at the ♩. periodicity (11), similar to coefficient 10/32 ($3^{1}/_{5}$♪ periodicity) in the previous canon, and 4 (𝅗𝅥‿𝅘𝅥𝅰 vs. 𝅗𝅥) and 7 ($4^{5}/_{7}$♪ vs. $4^{4}/_{7}$♪). The main difference is that the “difficult” canon also has a high value at 1, which reflects the fact that the rhythmic activity is concentrated in one part of the cycle (towards the end), making it more imbalanced overall. The spectrum for the rhythm of canonic entries, 3-2-28, is also given in Figure 32(b), and is similar to the previous canon. It primarily reinforces the principle ♩. periodicity, and neutralizes the 𝅗𝅥‿𝅘𝅥𝅱 frequency that otherwise is present in the basic rhythm.

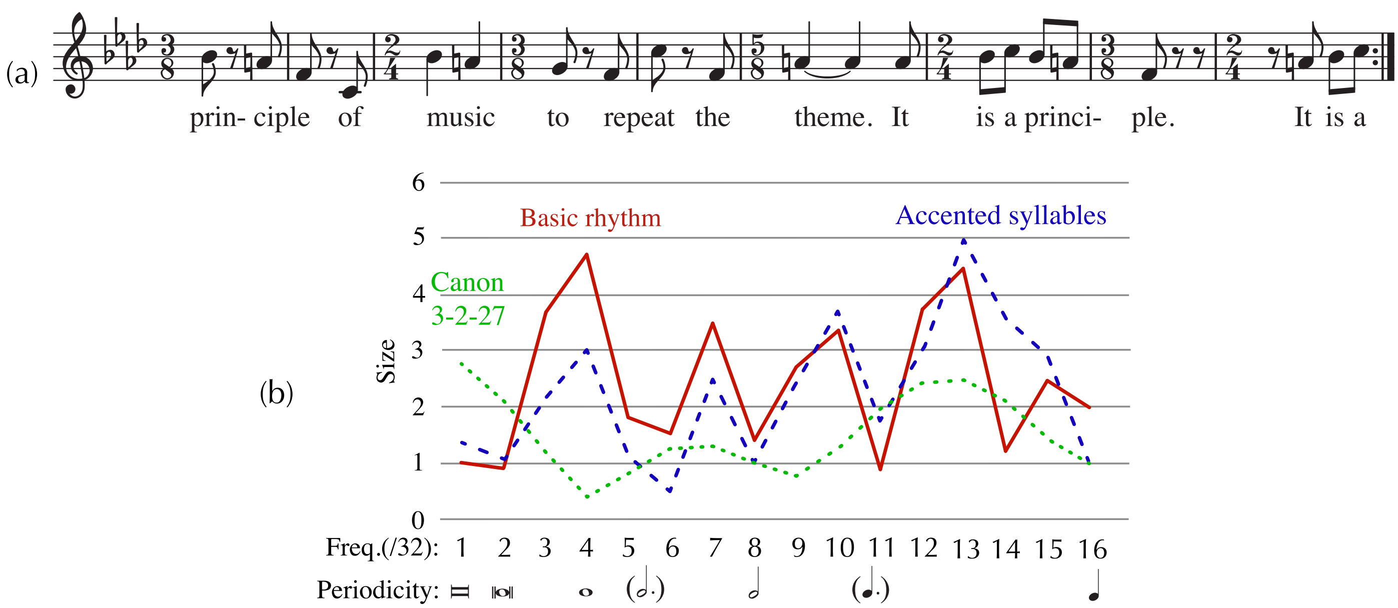


Figure 31: The subject of the "It is a principle" canon from Reich's *The Desert Music*, reh. 163, (a), and the spectra of the basic rhythm, accented syllables, and canon entries, (b)

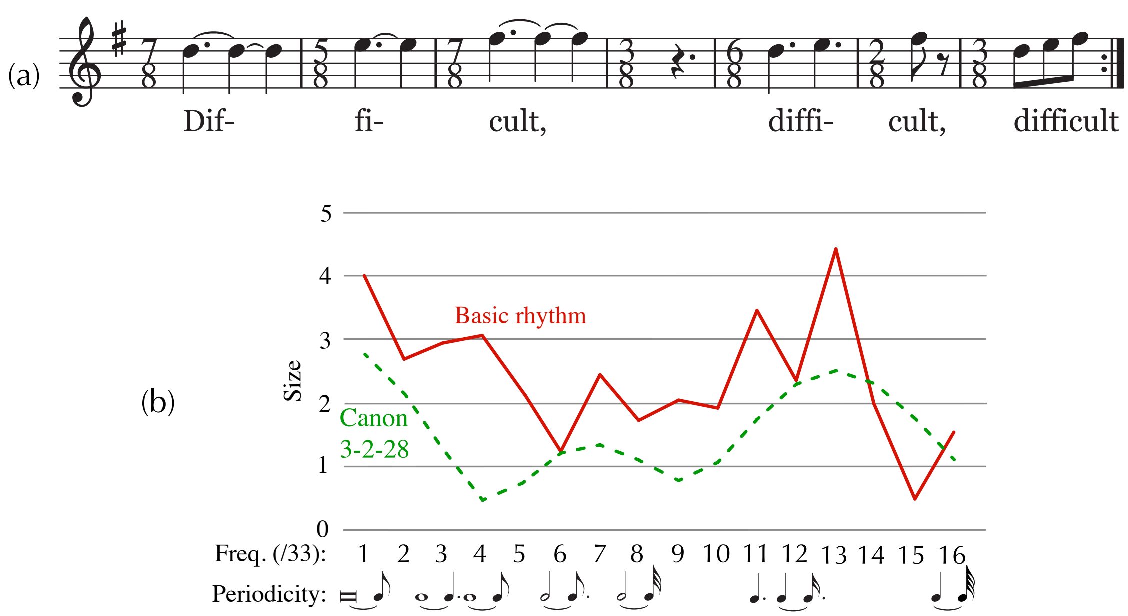


Figure 32: Melody for the "Difficult" canon from *The Desert Music* reh. 184, (a), and its spectrum and that of the canon entries, (b)

A Coherent Rhythmic Style: Comparing Spectra across Cycles

We have seen a number of rhythms in many different kinds of cycles, and now can consider what common features they all have, and how they define a coherent rhythmic style. When initially considering the rhythms used in *Music for Pieces of Wood*, the signature rhythm and the *cinquillo* rhythm (which is also the basic rhythm of *Six Pianos*), we were able to compare these by proportionally adjusting spectra and relating nearby periodicities. We used the same method to show the resemblance of the vibraphone rhythm of *Sextet* IV to the signature rhythm. This can be extended to rhythms from all of the different kinds of cycles we encountered thus far. Figure 33 overlays proportionally adjusted spectra for • the basic rhythm of *Six Pianos* (8-cycle), • the signature rhythm (12-cycle), • the basic rhythm of *Electric Counterpoint* ii (19-cycle), • the ostinato of *Eight Lines* (20-cycle), • the vibraphone ostinato of *Sextet* IV (22-cycle), • the rhythm of the "It is a principle" canon from *The Desert Music* (32-cycle), and • the rhythm of the "Difficult" canon from *The Desert Music* (33-cycle). Each spectrum has a principal peak in a very similar location, near the 5/12 periodicity of the signature rhythm. This means that there is an approximately fixed ratio relating the eighth-note pulse to the main periodicity that divides the different-length cycles.

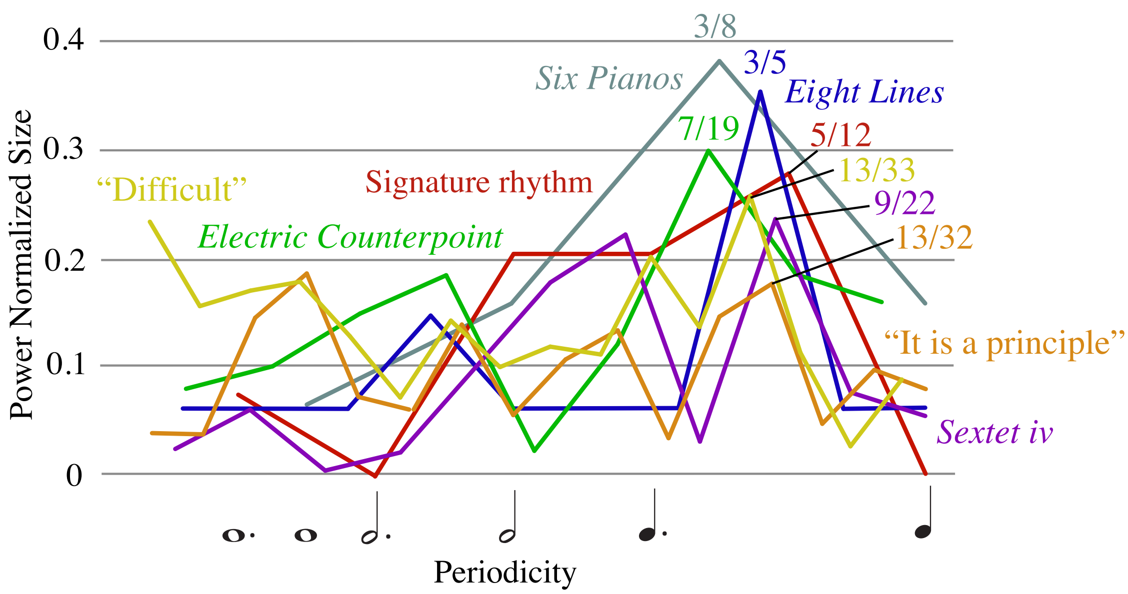


Figure 33: Spectra for rhythms from *Six Pianos, Clapping Music, Eight Lines, Sextet iv, Electric Counterpoint,* and *The Desert Music*, proportionally scaled. Values are normalized by power (the sum of squared magnitudes).

Recall that spectra are symmetrical around *n*/2, so a peak at *k*/*n* can equivalently be understood as a peak at (*n* – *k*)/*n*. Since the basic rhythms usually fill the *n*-cycle with more than *n*/2 onsets, they tend to resemble the ME(*n*, *n* – *k*) rhythm more than the complementary ME(*n, k*) one (with *k* < *n*/2). For instance, Reich's signature rhythm is more like the diatonic rhythm, ME(12, 7) than the complementary pentatonic rhythm, ME(12, 5). If we therefore compare the larger of the complementary frequency ratios for each of these rhythms—5/8, 12/20 (= 3/5), 7/12, 12/19, 13/22, 19/32, and 20/33—they range from 0.58 to 0.63, which means they are all fairly close to the reciprocal of the golden ratio (1/φ = 0.618). Or, equivalently, the periodicities, 5/3, 8/5, 12/7, 19/12, 22/13, 32/19, 33/20, are approximations of the golden ratio (φ = 1.618). Specifically, the frequency ratios are always the closest irreducible approximation of 1/φ for the given denominator (treating the *Eight Lines* ratio as 3/5). Before declaring this a mystical revelation of sacred geometry, though, let us consider what the musical interest of frequencies in this range might be.

Like the golden ratio, the ratios for Reich's rhythms avoid the simplest fractions, specifically having principal periodicities well below 2, but above $1\frac{1}{2}$. These simple fractions are significant as the simple metrical frequencies: a period of 2 corresponds to a basic duple grouping of the pulse, and $1\frac{1}{2}$ (or 3) corresponds to a basic triple grouping. The golden ratio is an optimization of rational-number avoidance, and therefore rational approximations of it are the ones that avoid confusion with the simplest ratios (ones with small denominators) as much as possible. This is often used to explain the frequent occurrence of approximations to φ in phyllotaxis, or the arrangement of leaves around a stem, and other biologic structures.[14] The ratio 1/φ is very far from the simplest ratios, 1/1 or 2/1, and also avoids the next simplest ratios, 1/2 and 2/3, by being in the middle of the range between them, somewhat closer to 2/3. Reich's rhythms amplify this kind of frequency ratio for two reasons: they are based on patterns that fill the measure with a relatively evenly distributed mixture of quarters and ♩ ♪ rhythms, and they do so in an irregular fashion. That is, they express a periodicity somewhere between a quarter note and dotted quarter, while avoiding exact translational symmetries (except at the overall cycle). And by choosing a relatively even number of quarters and dotted quarters (♩ ♪ rhythms), slightly favoring dotted quarters, the result is a frequency ratio close to 1/φ.

According to this explanation, we might think of Reich's rhythms as approximations of the diatonic rhythm. Its frequency ratio, 7/12, is the exact halfway point between 1/2 and 2/3. All of the ratios from Reich's rhythms are best approximations to 7/12 except for 12/19 (7/19 from *Electric Counterpoint* ii) and 20/33 (13/33 from the "Difficult" canon), but all of them are also larger than 7/12 (farther from 1/2). Therefore, an accurate characterization of these ratios is that they are the next values above 7/12 for the given denominator, so that they are between 1/2 and 2/3, but closer to 2/3 than to 1/2. As the ratio gets closer to 1/2, the maximally even rhythm has longer strings of straight quarter notes, so "closer to 2/3 than 1/2" means having a maximally even prototype with no more than 3 consecutive

[14] See, e.g., Adam 2009. The idea of "maximum irrationality" can be made mathematically precise, as can be found this or a host of other sources.

quarter notes. This explanation is more satisfying biographically, since, we know that Reich's signature rhythm was derived from the diatonic rhythm, the African bell pattern (see above), and it is easy to see him generalizing basic features of the diatonic rhythm (as an irregular well-distributed mixture of ♩ and ♩ ♪ rhythms) while reworking the rhythm in larger and more irregular cycles. However, the idea of rational-avoidance more generally, as a feature of the golden ratio, remains of value as a possible explanation of why this particular aspect of the diatonic rhythm was so fascinating to Reich and resilient, dominating decades of his composition output.[15]

Another feature of these rhythms is that, while they are all similar to maximally even rhythms, and reflect that similarity in a single dominant peak in the Fourier spectrum, only one of them (the 8-cycle rhythm of *Six Pianos*) is actually maximally even or generated, so none of the others completely maximize their principal component. They all have some secondary spectral peak. Except in the 8-cycle, where there are few alternatives, Reich pointedly avoids these maximally even rhythms, which certainly were available to him.

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[15] Another aspect of the golden ratio of possible significance here is a derivation from the type of approximate augmentation that Reich uses in *Tehillim* and other pieces. As a simple example, consider an approximate augmentation of the ME(5,3) rhythm ||: ♩ ♩ ♪ :|| which replaces eighths with quarters and quarters with dotted quarters. This produces the ME(8,3) rhythm ||: ♩. ♩. ♩ :||. The principal periodicities of the first rhythm are $2\frac{1}{2}$♪ (5/2), $1\frac{2}{3}$♪ (5/3), and the full cycle, 5♪, and those of the second rhythm are $2\frac{2}{3}$♪ (8/3), $1\frac{3}{5}$♪ (8/5), and the cycle (8♪). Under an exact augmentation of the first rhythm by 8/5 (mapping the cycle-lengths exactly), the 5/3 periodicity would become an 8/3 periodicity. This is the sense in which the second rhythm is an approximate augmentation of the first; essentially, the coefficient numbers for a spectral peak in the two rhythms match, the larger of the two complementary coefficient numbers in the first (3), is the smaller of the two in the second. Since the coefficient number corresponds to a division of the cycle, both cycles are divided in a similar way by their principal periodicity (into 3). A second feature of interest, though, is that if we compare the spectral peaks as periodicities (multiples of the pulse rather than as divisions of the cycle), they are a good approximation: $2\frac{1}{2}$♪ is close to $2\frac{2}{3}$♪, and $1\frac{2}{3}$♪ is close to $1\frac{3}{5}$♪. Notice that because this is stated in terms of spectral peaks, it does not depend on the two rhythms matching in cardinality; all of this would also be true for the ME(8,5) rhythm ||: ♩ ♪ ♩ ♪ ♩ :|| which embeds the ME(5,3) pattern in multiple ways, and also is a quality-preserving elaboration of its augmentation, the ME(8,3) rhythm. Generalizing this, let the first rhythm have a cycle length a and principal coefficients $a - b$ and b. To match the coefficient number of a principal peak (b) and the original rhythm's cycle (a) as principle peaks of the augmented rhythm, let this rhythm have principal coefficients b and a and cycle length $a + b$. The principal periodicities are then $a/(a - b)$ and a/b for the first rhythm and $(a + b)/b$ and $(a + b)/a$ in the second. To get perfect equivalence in the periodicities, we would have $a/b = (a + b)/a$, which is satisfied by $a/b = \varphi$. Since φ is irrational, this situation can only be approximated in notatable rhythms.

He was well aware of the 7/12 maximally even diatonic rhythm, for instance, but he clearly favored the non-maximally-even signature rhythm and derivatives of it. We saw in the previous section that the secondary ♩. periodicity of the signature rhythm is important in Reich's usage, where the centerpiece of many compositions becomes the musical play between the different possible periodicities and resulting metrical perspectives that can be brought out of a single rhythm.

Therefore, in addition to this principle periodicity, the secondary periodicities of these rhythms are also essential to their particular musical value. Figure 34 divides up the spectra from Figure 33 to show the periodicity ranges where these secondary peaks occur. Four rhythms have a secondary peak at or just below the frequency for a ♩. periodicity, and four have peaks at or above the frequency for 𝅗𝅥‿♪ periodicity. The two rhythms from *The Desert Music* also have peaks around 𝅝 (8♪).

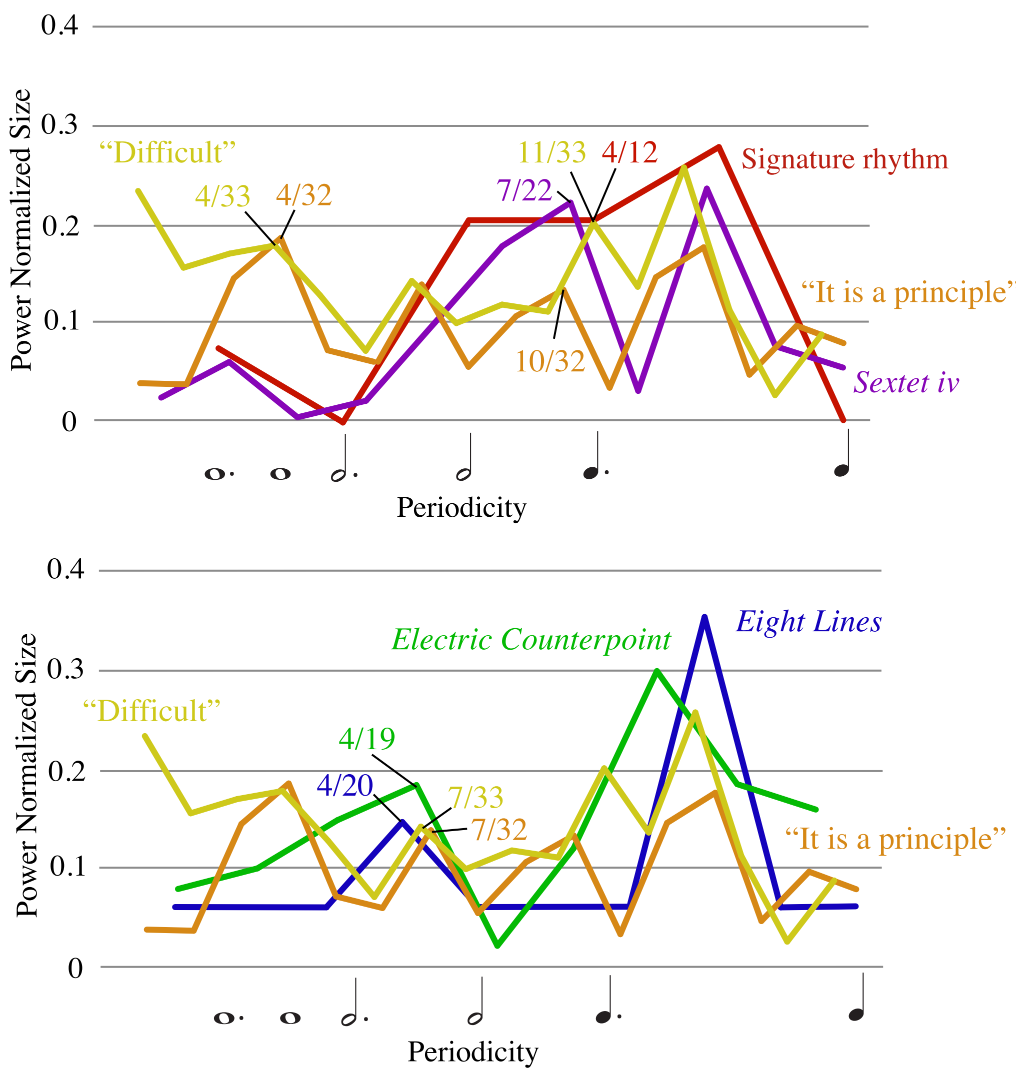

Figure 34: Secondary peaks in the spectra from Figure 33

The peaks near ♩. suggests a derivation from the important secondary ♩. periodicity of the signature rhythm. The analysis of the second movement of *Sextet* above has already given a hint as to musical significance of the periodicity in this region. The vibraphone rhythm in the fourth movement (the rhythm represented in Figures 33–34) accompanies a canonic melody in the pianos that is common to both the second and fourth movements, shown in Figure 26. This melody is coupled with a simpler bass drum rhythm, which only strikes at timepoints on the quarter-note grid. The bass drum rhythm, therefore, could be described in an 11-cycle of quarter notes, rather than the 22-cycle of eighth notes required to describe the piano (and vibraphone) rhythms. It therefore has the characteristic symmetry of an oversampled rhythm (the mirror symmetry caused by aliasing appearing between 0 and *n*/2). The peak at coefficient 7 in this 11-cycle rhythm is one of the golden ratio approximations. Since the pulse unit for this 11-cycle is a quarter-note, the periodicity of coefficient is at $1^{4}/_{7}$♩, equal to $3^{1}/_{7}$♪. When oversampled, it becomes a peak at 7/22 with the same periodicity ($3^{1}/_{7}$♪) since the pulse unit is now an eighth. The peak at 7 in the 11-cycle rhythm also has a complementary peak (by aliasing) at 4, a periodicity of $2^{3}/_{4}$♩ or $5^{1}/_{2}$♪, which also becomes a coefficient 4 peak in the oversampled 22-cycle rhythm, with the same periodicity ($5^{1}/_{2}$♪). Referring back to Figure 26, we note that these two 22-cycle periodicities appear in the basic rhythm of the pianos (coefficient 4) and its reduction (coefficient 7). The latter (7/22) is also present in spectrum of the vibraphone rhythm in Figures 26 and 34.

This secondary periodicity can therefore be explained as a replication of the golden ratio approximation augmented by a factor of 2, and illustrates another important feature of Reich's rhythmic style: his metrical contexts are usually duple at the lowest level, with a clear quarter-note beat. This quarter-note beat gives the irregular periodicities something to pull and tug against. His emphasis on the ♩. periodicity of the signature rhythm (in *Music for Pieces of Wood*, *Music for 18 Musicians,* and *Vermont Counterpoint*, for instance), as opposed to the equally salient 𝅗𝅥 frequency, can be seen in this light: although the ♩. periodicity evenly divides the 12-cycle, it does not line up with quarter-note grid. In *Music for Pieces of Wood*, Clave 1 articulates the quarter-note grid throughout, so when the canon on the signature rhythm reduces to a pure ♩.-periodicity rhythm, the result is a cross-rhythm with Clave 1.

Eight Lines provides an example of how Reich enforces and uses the quarter-note rhythmic layer. Figure 35(a) presents the basic material at rehearsal 17 with the canon at the quarter note in the pianos and a longer-breathed series of chords in the strings, excluding only the elaborations in the flute. The strings' rhythm, on a quarter-note unit, is 2-2-3-3 (rotated to 3-2-2-3), which is the same rhythm, in augmentation, as is outlined in the first measure of the pianos (rotated to 3-3-2-2) with the basic unit of the eighth-note. The spectrum of this rhythm, Figure 35(b), has the same "golden" frequency ratio of 4/10. Since the basic unit is a quarter here, this corresponds to a $2^{1}/_{2}$♩ periodicity, equal to a 5♪ periodicity, which corresponds to coefficient 4 of the 20♪ cycle. This periodicity is reflected as a secondary spectral peak in the canonic rhythm of the pianos (Figure 21c), which can therefore be explained by the interaction with augmented rhythms.

Figure 35: *Eight Lines* reh. 17, pianos and strings, (a), and the spectrum of the strings' rhythm, (b)

Use of augmentation, like in *Eight Lines,* was a persistent element of Reich's works for larger ensembles in the 1970s, going back to *Music for Mallet Instruments, Voices, and Percussion* of 1973. (See analyses in Suzuki 1991, 495–499 and Schwartz 1981–2.) The augmentation of Reich's characteristic rhythmic irregularities to the quarter-note level explains secondary periodicities of the signature rhythm (12-cycle), the *Eight Lines* rhythm (20-cycle), and the *Sextet* second and fourth movement rhythms (22-cycle). If we consider Reich's preferred periodicities to be around $1^5/_7$♪, or, equivalently (by aliasing), $2^2/_5$♪, the diatonic periodicities, then we would expect preferred secondary frequency ratios to be approximately twice that, $3^3/_7$ or $4^4/_5$. This is true of the secondary peak of the *Eight Lines* rhythm at 5♪ (approximating $4^4/_5$) and of the *Sextet* rhythms at $3^1/_7$♪ (approximating $3^3/_7$).

These secondary peaks also occur, however, in pieces with odd cycles. The 19-cycle rhythm for *Electric Counterpoint* has a secondary peak at $4^3/_4$♪, which is a good approximation of $4^4/_5$. The explanation from *Eight Lines* and *Sextet* does not immediately transfer to this example: the *Electric Counterpoint* cycle does not divide up into an integer number of quarter notes. If the listener groups the eighth notes in pairs, she will hear not a 19-cycle, but a 38-cycle, with the rhythm repeated twice, exchanging the on-beat and off-beat patterns each time.[16] Indeed, this seems a likely way of hearing the rhythm, and Reich encourages it: the rhythm has frequent consecutive quarter notes, and the parallelism between the 3/4 and 4/4 measures reinforces their shared quarter-note beat. Also, the solo guitar entries for the third canonic part, the one whose quarter-note beat is offset from the

[16] This is a generalization of Cohn's (1992a) "switchback" principle.

others, are *forte* instead of *mezzo forte*, and begin on two quarter notes. Assuming this 19-quarter-note grid, then, the rhythm reduced to on-beat attacks is just the original rhythm multiplied by 2 modulo 19. Multiplying a rhythm this way rearranges its coefficient numbers by the same modular multiplication, so the 4/19 subsidiary peak in the original rhythm becomes an 8/19 subsidiary peak in the quarter-note rhythm. This is in the range of the "golden" frequency ratios.

The explanation of this rhythm from *Electric Counterpoint* highlights a nuance in how we have thus far explained the rules of the preferred periodicities: at the eighth-note level, the use of a $1^{7}/_{12}$ (coefficient 12) instead of a $1^{8}/_{11}$ (coefficient 11) periodicity that better approximates $1^{5}/_{7}$ suggests a tendency to avoid periodicities too close to 2. This makes good musical sense if one of the purposes of irregular periodicities is the tension they create with a quarter-note beat. The rule for the subsidiary peaks, however, seems to be simply to approximate $3^{3}/_{7}$ or $4^{4}/_{5}$, without necessarily avoiding 4 (= 𝅗𝅥), because, presumably, tension with a half-note beat is not a primary function of these periodicities. In the case of the 19-cycle, the result is a different ratio when transferring the subsidiary peak to the quarter-note level (8/19) than is preferred at the eighth-note level (7/19). The difference has the added effect of avoiding a simple linear relationship between the two frequency ratios.

We can give a similar explanation of the canons from *The Desert Music*. The "It is a principle canon" has a main peak at a $2^{6}/_{13}$ periodicity and subsidiary peaks at $3^{1}/_{5}$, $4^{4}/_{7}$, and 8 (coefficients 13, 10, 7, and 4). The $4^{4}/_{7}$ periodicity is a nearest approximation to $4^{4}/_{5}$. The $3^{1}/_{5}$ periodicity is nearest to $3^{3}/_{7}$ from above. This does not account for the peak at periodicity 8, however, which is actually larger than the one at $2^{6}/_{13}$ (recall that the latter is made more salient by the accent pattern of the text). We can give a possible explanation of this peak by dividing the pattern up into two quarter-note rhythms, as we did for *New York Counterpoint* ii in Figure 19. Due to the irregular metric notation and canons in odd multiples of eighth notes, it is not obvious which quarter-note grid counts as on-beat, so let us refer to the two rhythms as the "even" and "odd" subsets. The spectra for these are given in Figure 36, revealing an interesting feature of the rhythm. If we take each target periodicity suggested by our analysis of Reich's rhythms thus far: (a) $2^{2}/_{5}$, (b) $3^{3}/_{7}$, and (c) $4^{4}/_{5}$; the coefficients available in the 32-cycle just above and below each target value are (a) 12–13, (b) 9–10, and (c) 6–7. In each case, *both* frequencies are present, one localized to the even subset and the other to the odd subset. The even subset coincidentally favors the even components (6, 10, and 12) and the odd-beat rhythm the odd ones (7, 9, and 13). By symmetry constraints, peaks also occur at 3 and 4 to match the peaks at 12 and 13. For each of these pairs of adjacent coefficients (3–4, 6–7, 9–10, and 12–13), one is dominant in the overall rhythm (4, 7, 10, and 13) such that the symmetry of the quarter-note rhythms is "smudged." An explanation for the peak at coefficient 4, then, is that it is the approximate complement of the principal peak at coefficient 13 when constrained to the quarter-note grid.

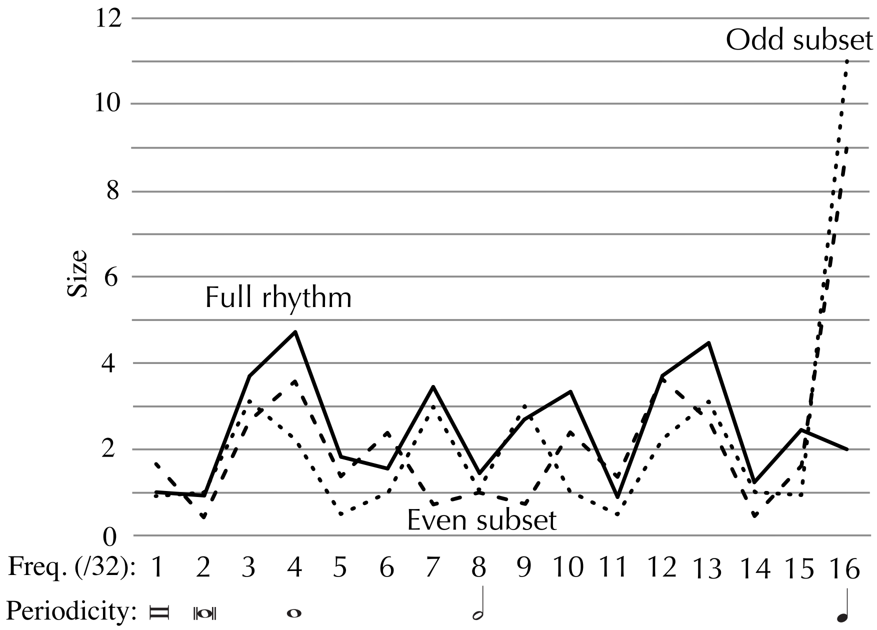

Figure 36: Spectra of the "It is a principle" canon: the full rhythm and split into odd and even eighth-note subsets

The 33-cycle of the "Difficult" canon is similar to the 19-cycle of *Electric Counterpoint* in that the 33-cycle has a close approximation to $1^{5}/_{7}$ from above ($1^{14}/_{19}$), but the principal frequency ratio is $1^{13}/_{20}$, which is further from $1^{5}/_{7}$ but below it. The subsidiary peak at $4^{5}/_{7}$ is a nearest approximation to $4^{4}/_{5}$. There is a larger subsidiary peak at ♩., similar to the signature rhythm. This is not a best approximation of $3^{3}/_{7}$ in the 33-cycle ($3^{3}/_{10}$ is closer), but avoids another simple linear relationship between peaks (coefficient 10 to 20). Finally, the coefficient 4 peak approximately complements the principal peak in a similar way that the coefficient 4 peak does in the "It is a principle" canon.

Conclusion

The aspect of Reich's music that perhaps most immediately impresses itself upon the listener is his devotion to the pulse. Long works like *Music for 18 Musicians* proceed from start to end with an unrelenting eighth-note pulse, never varied in tempo or subdivided. This music might therefore seem like the ideal candidate for discrete methods of analyzing rhythm. However, the rhythmic patterns Reich builds upon this scaffolding defy such methods, because their common source is periodicity that deliberate seeks the cracks between simple integer groupings of the pulse. Analysis with rhythmic spectra reveals a strong stylistic consistency in Reich's rhythms, and a web of interdependencies between their underlying periodicities and other persistent aspects of Reich's style, cyclic canons and augmentation.

The use of rhythmic spectra to trace the development and isolate the stylistic traits of Reich's rhythms illustrates the efficacy of this method for understanding rhythm and generalizing away the inflexibility of traditional notions of meter. The potential for future application of this theory to other repertoires and musical problems is hopefully apparent from many of the explanatory tasks assigned to it here. Among these are • expanding discrete concepts of periodicity used in traditional explanations of meter to concepts that allow for non-integer periodicities, • recognition of the significance of maximally even rhythmic patterns as prototypes of non-integer frequency ratios, • application of mathematical results relating to the Fourier transform to musical problems (such as classifying combinatorial rhythms or characterizing rhythmic canons using the convolution theorem), and • comparing rhythmic spectra across different cycle lengths by comparing nearby periodicities.

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