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INTRODUCTION: FORMALIZATION AND SCHENKERIAN ANALYSIS

Schenkerian analysis is not so much a theory or an analytical method as it is a symbolic language for expressing intuitions about tonal music. The people who speak this language have many different theories about tonal music, and many different approaches to analyzing it. After all, the state of music theory would be very sorry indeed if the structure of tonal music were a problem solved by somebody a hundred years ago and since then the primary distinction of music theory instructors was that they knew all about that guy and whole-heartedly agreed with him. In truth, a music theorist must not only be able to speak the language of Schenkerian analysis, but also have insights about music to express in this language. Schenker's contribution was not to develop a theory of music—understanding "theory of music" in the sense of "quantum theory" or "evolutionary theory" as a set of verifiable statements about music—, nor to develop a method of analysis—understanding "method" as a replicable process for making analytical decisions. Rather, he developed a novel language, both in his terminology and symbols, that is especially well suited to the expression of ways of hearing music.

It is easy to run into a lot of confusion about this. For instance, it is easy to think that Schenker's *Ursatz* is a theoretical assertion that all the experts have come to agree upon, much as biologists have come to agree upon Darwin's theory of evolution. Or, to repeat a brief summary of Schenkerian theory I once heard from a knowledgeable musician: "Schenker said that all music basically boils down to I-V-I." If this were the case, it would be quite remarkable that in a hundred years no one has thought to empirically test the idea that music boils down to I-V-I. This would be like trying to verify that predicates must always follow subjects because wherever one finds a thing it can always be said to be doing something. Wherever one finds a piece of tonal music, it can be said to have a beginning, middle, and ending.

For instance, David Beach (1977) says of the *Urlinie* and *Ursatz* that "Schenker arrived at them, or more precisely discovered their existence, after years of searching for the fundamental and natural laws of tonality" (280).[1] Beach's statement implies that the *Ursatz* is something that objectively exists in music apart from any person's theoretical formulation of it. For this to be true there ought to be some formulation of the idea of an *Ursatz* that makes it possible to say, for a particular piece of music, that it is not in fact derived from an *Ursatz.* (That is, the existence of the *Ursatz* and *Urlinie* ought to be falsifiable). However, neither Schenker nor Beach has ever proposed such a formulation: in fact, according to the conventions of Schenkerian analysis, any piece of music that includes some sort of cadence in the tonic key can be (and will be) derived from the *Ursatz* in a Schenkerian analysis.[2] This is not a flaw of Schenker's theory because Schenker didn't propose it as a scientific theory; it only becomes a flaw if one wishes to make a scientific theory out of it (to prove, perhaps, that the *Ursatz* was something that Schenker discovered, not something he just made up).

Actually, the *Ursatz* is not a theoretical assertion but a linguistic convention. Not only that, it is quite an important convention because it enables one to speak meaningfully about such things as head tones, structural dominants, and structural endings. The rebel theorist is free to test such propositions as "every piece of tonal music has a V chord with its fifth in the upper voice," but she should not be surprised if falsifying this proposition by finding a piece of tonal music that has no such V chord fails to stop people from talking about structural dominants. The rebel theorist will hear responses such as, "actually, the fifth in the structural dominant is an implied tone here,"

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[1] This is not an isolated claim of Schenker's "discovery" of the *Ursatz*: in fact these claims follow the lead of Schenker himself, who says in *Meisterwerk II*, "alles Religionsempfinden, alle Philosophie, Wissenschaft drängt zur kürzeften Formel, ein ähnlicher Trieb ließ mich auch das Tonstück nur aus dem Kern des Ursatzes als der ersten Auskomponierung des Grundklangs (Tonalität) begreifen; ich habe die Urlinie erschaut, nicht erreichnet!" Schenker uses deliberately biblical (or perhaps Husserlian?) language here ("ich habe die Urlinie *erschaut*"). He certainly doesn't mean to imply *discovery* in the scientific sense.

[2] Schenker himself claimed that certain composers, such as Wagner and Bruckner, did not compose from the *Ursatz*. However, this doesn't mean that one could not interpret their work in terms of an *Ursatz* with a certain amount of effort, only that these composers didn't hear their own work through the *Ursatz* and therefore such an interpretation wouldn't reveal the artistic agency of the composer.

or "the structural dominant in this piece is actually not a V chord." The problem is not that the term "structural dominant" is ill-defined; rather, it is not defined in terms of the elements of music—as a V chord, a metrically emphasized chord, or anything of the sort—but normatively: "from the top down," as that musical event that represents the penultimate event in the *Ursatz*. To be sure, there are many conventions about what musical elements tend to make up a structural dominant, but any of them may be broken; what it "basically boils down to" is that when I say "X is the structural dominant of this piece of music," I am asserting something about how I hear the music, that I hear X as the musical event that fundamentally prepares the conclusion of the piece.

Fields of mathematics are also languages, not theories. It would not be an especially worthwhile expenditure of one's time to test whether, in all cases, putting nine things together with seven other things always results in sixteen things. Trying to convince algebraists that, actually in some cases it is possible to put nine things together with seven things and get four things, would be like trying to convince Schenkerians that, actually there are some pieces of tonal music that don't have structural endings. Neither of these claims would strike the listener as verifiable claims about the world so much as pleas that we should change the rules of the language game—as in, "sometimes, it is better to adopt the convention that 9 + 7 = 4 when adding things like pitch classes," or "when talking about this particular genre of music it better expresses my hearing to have a rule that says some pieces have endings whereas others do not." To elaborate on the latter case, it is certainly possible to give a standard Schenkerian analysis of the Schumann song "Im Wunderschönen Monat Mai" and to discuss "the structural ending of 'Im Wunderschönen Monat Mai.'" However, it would also be reasonable for someone to say "It better expresses my hearing of this song to say that it lacks an ending" and to develop an analytical method that resembles Schenkerian analysis but revises the standard *Ursätze* in order to give a meaningful sense to "lacking an ending." Yet, this would in no way "disprove" Schenkerian "theory"; the most it could do would be to become such a popular way of speaking about music that it supplanted the older Schenkerian terminology.

This being said, it would be irresponsible to paint over the vast differences between a field of mathematics and Schenkerian analysis by saying that when you get right down to it, they both are languages. There is good reason why we distinguish between languages that are mathematical and those that are not. A mathematical language is one that is valuable for its rigor and precision, not for its ability to communicate. To demand the same precision of a natural language would strangle the language's function as a medium for formulating novel ideas. It would be impossible for mathematicians to carry on their work only expressing themselves in mathematical languages: the work of a mathematician is not "speaking mathematics," although that is something a mathematician can do. The work of a mathematician is creating, modifying, and describing mathematical languages, and that work is done with the aid of some (necessarily) non-mathematical language; call it a "meta-language" if you will.

On the other hand, there is nothing wrong, in principle, with musicians carrying on a discourse about music purely in the language of Schenkerian symbols. Natural languages such as English or German are useful as supplements to the language of Schenkerian symbols, only because the Schenkerian language is relatively simple and confining in its scope, but they aren't necessary as a "meta-language" although they can serve this function also. That is, the work of Schenkerian theory is not only to describe and modify the Schenkerian language, but also substantively to make statements about music in the language. In this sense the Schenkerian language is more like natural languages than mathematical ones.[3]

It is essential to keep all this in mind when embarking on the project of "formalizing Schenkerian analysis," because the question, "what is the point of doing this?" inevitably arises. Of course, the reason one hears this question so often is probably

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[3] John Rahn (1989a) makes a related point in explicitly Wittgensteinian terms: "Theories are language-like, and using formal theories is a language-game. The game played by music theorists emphasizes communication, not segregation or prediction. One of the reasons that Schenker's theory is so popular is its ability to support discourse among analysts, so that significantly differing perceptions of the structure of a piece can be articulated precisely." I concur with Rahn's fundamental philosophical stance, although I question the precision of the Schenkerian language as it presently stands. This entire paper, in fact, can be read as a revelation of the imprecision of Schenkerian language, showing the countless ways one might interpret concepts as basic to Schenkerism as prolongation.

that some people find the task of formalizing musical analysis more fun than playing chess while others find it more dull and laborious than doing their taxes, but it's fair to expect a ready explanation of what a formalization is supposed to do and what it isn't supposed to do. "Formalizing" something means incorporating it into a mathematical language. In the case of Schenkerian analysis, the formalization is not intended to supplant the Schenkerian language; indeed, it is not possible to formalize every imaginable statement one could make in this language. For instance, we will be especially concerned below with statements saying that one thing prolongs another; these are a highly important class of statements in the Schenkerian language and we can and should formalize them. However, this says nothing about statements about, for instance, voice exchange. These could also be formalized—perhaps they should—, and the formal model of prolongation probably would inform the process of formalizing voice exchange. Yet, this is a separate matter, and there is no end to such separate matters.

To be sure, often it is the formalizers who are to blame for the bad reputation of formal and mathematical theories because they conflate the process of formalization with one of its functions, that of generating testable hypotheses. Thus, mathematical theories have a reputation for being prescriptive. It is true that formal theory is necessarily prescriptive of terminology, but it is never in principle prescriptive of analysis. That is, formal theory sets limits on how one can use certain terms, because such limits make it possible for speaker and listener to achieve a fuller understanding of one another's insights. However, a formal theory should never tell us that there is only one way to hear a particular piece of music, only that the analyst with unusual insights should express them with language that has not already been claimed for other purposes.

Presenting formalization as prescriptive falsely separates analysts into camps like warring nations who are unified in accepting national identities that are, in reality, completely arbitrary. When the "other side" accepts the mistaken idea that formalization is inherently prescriptive of analysis, they become needlessly mystical about "musical truth." For instance, consider the following comment of Carl Schachter in his article "Either/Or," in which he passingly maligns the idea of a "theory of reduction:"

> I strongly doubt that such methods or theories can be made to work, for I believe that the understanding of detail begins with an intuitive grasp of large structure, however imperfect or incomplete, a process that is ultimately resistant to rigorous formalization. (167)

Schachter seems to present here two paths to musical truth: intuition and formalization, and decides in favor of intuition, believing that formalization leads to a labyrinth of dead-ends. The result is a quite a nebulous recommendation to the reader desiring a better understanding of music: improve your musical intuition—what is, in fact, a mystical smoke screen on the model of Schenker's own "if you're not one of the geniuses, you just aren't ever going to get it."

This is not really what Schachter means, though. The observation he makes in this article is that many analytical conundrums must be solved in novel or unanticipated ways, through a comprehensive understanding of the music. The problem is that Schachter accepts the false premises of the authors—I would guess that Lerdahl and Jackendoff, whose theory I will discuss in part two, are foremost among them—who he is reacting against: that the ultimate goal of formal theory is to replace analytical insight with mechanical processes. Schachter is quite right to point out that this is impossible, since the formal theory cannot hope to anticipate every analytically relevant insight one might have about a piece of music. However, the dichotomy between intuition and formalism is a false one: without intuition, there is no formalism; formalism is built upon intuition. And yet, without the means of expression provided by formalism, intuition inevitably evaporates into the Brownian motion of misunderstanding. Intuition and formalization are not two things that one must choose between; they are two essential components to the process of constructing a theory of music.

Although we can't expect, and indeed shouldn't hope, to eradicate all imprecise speech by replacing it with mathematical speech, it is irresponsible to not use precise language when it is possible. Consider, for instance, a physicist who is describing the motion of a ball thrown in the air. It would inappropriate for her to say, "the ball moves in a kind of arch-like shape" when she could say, "the trajectory of the ball traces a parabola." The latter statement, in the context of a world with Cartesian geometry, is fairly precise: the listener knows how to check for himself whether it is true. The first statement is vague: the listener doesn't know exactly what counts as arch-like and what doesn't. In a world without Cartesian geometry, in which an "arch" was something one walks under on their way to the garden and no one had really thought of describing a shape as "arch-like," the first statement might be appropriate. But if this started a trend of people talking about arch-shapes, eventually the community of ballisticians would do well to make up their minds about what exactly "arch-like" is supposed to mean, or else develop a set of terms to replace "arch-like" (semicircular, parabolic, hyperbolic, et c.).

Furthermore, neither of these statements about the motion of a ball is mathematical; a ball is not a mathematical object. However, the second statement invokes a mathematical language by using the geometrical term "parabola." The value of having a language of Cartesian geometry is not, ultimately, to make statements in the language itself (except insofar as this is an amusing mental exercise), but to lend precision to terms that can be used in a broader linguistic context.

Similarly, the more music theorists talk about prolongation, the more they will disagree about particular statements regarding prolongation. Such arguments will be valuable if they address the question of how exactly we should define prolongation and related concepts—that is, if they address the question of how best to construct a formal theory of prolongation. If they fail to do this, the debate will be endless and futile, like fans of opposing football teams heckling one another from either end of the stadium, because it will be a debate over statements that have been insufficiently defined.

Furthermore, "formalization" doesn't necessarily have to look like mathematics. Thoughtful people who cringe at the word probably engage in formalization all the time; they just prefer to leave the top of the mathematical toolbox latched or use a handsaw rather than install an electrical system to power the table saw. While it certainly is possible to take formalism much further than what the situation at hand calls for, the reader who takes some time to familiarize herself with a certain amount of mathematics will make the effort well worth the while by reaping a comprehensive understanding of what is at stake in submitting music-analytic concepts to formal description.

To illustrate the need of formalization in the case of the idea of prolongation, consider the following comment of William Benjamin from his article "Models of underlying tonal structure." He argues in this part of his article that in tonal music one will frequently find four consecutive events, call them W, X, Y, Z in that order, where Y is a "prolongation" of W by reiteration while X is a "prolongation" of Z by anticipation. In his example Y is a reiteration of a cadential dominant chord, W, with an intervening cadential 6/4, X, resolving to a tonic chord, Z. Most theories of prolongation would make such "overlapping" prolongations impossible, requiring the cadential 6/4 to be part of the prolonged cadential dominant rather than an anticipation of the resolution. According to Benjamin, the cadential 6/4 is also a prolongation of the tonic resolution by anticipation. He goes on to say,

> That prolongations, and particularly high-level prolongations, routinely overlap in tonal music may seem self-evident to many readers. This would make it seem all the more remarkable that virtually the whole of our recent theoretical tradition asserts that they do not. (44-5)

Obviously there's a problem here. Benjamin uses the word prolongation as if it meant something quite definite and unambiguous and sees the problem as being that the whole of our recent theoretical tradition has simply ignored the self-evident truths of the music it has been analyzing and the prolongations contained therein. Of course, this is preposterous: the problem is that, while most people understand the word "prolongation" in such a way that the idea of overlapping prolongations is an oxymoron, Benjamin understands the word in a much more general way. What is going on here is not that Benjamin has discovered something happening in music that no one had ever noticed before, but that he disagrees over the way that words should be used to describe what is happening in the music. If he had presented the problem in this way, he could have given more appropriate arguments in favor of his case—e. g. explaining how the word prolongation would be more useful under his definition—and avoided the misplaced condescension of the passage quoted above.

Furthermore, a brief consideration of the matter makes it quite clear, I think, that Benjamin's and other's extension of the term prolongation to allow overlapping relationships actually causes a great deal of confusion and would thus be undesirable. On the one hand, prolongations would only overlap in particular and localized circumstances: saying that prolongations can overlap arbitrarily would tremendously water down and otherwise substantially alter the meaning of the term. Yet, to take away such a nicely straightforward and definite aspect of the term for such a relatively insignificant phenomenon would needlessly complicate its meaning.

Furthermore, the proper resolution to the "problem" is not very difficult to see: Benjamin uses the phrase "prolongation by anticipation" to describe something that isn't really a structural tonal relation. There's no reason that we can't notice such an anticipation without calling it a kind of prolongation: indeed, in the case of the cadential 6/4 it is nothing like what we usually refer to as a prolongational relationship, because the relationship between the cadential 6/4 and the tonic resolution is one that doesn't make sense unless it is mediated by the intervening resolution to the cadential dominant. This kind of anticipation is more like motivic relationships in music. For example, in the finale to Brahms' first symphony, the introduction anticipates the second theme of the movement, yet nothing could abuse terminology more than calling the introduction a prolongation of the second theme by anticipation. Such a usage would make bread pudding out of the noble grilled-cheese sandwich that is Schenkerian analysis.

A formalization of prolongation thus serves many functions. Above all, it focuses the discussion of prolongational issues by separating out claims about prolongation in music from those that address the usage of the word prolongation, and it eliminates misunderstandings from discourse of the latter kind. It also separates prolongational claims from ones that engage some aspect of music that is not properly prolongational. Furthermore, it allows us to divorce the term "prolongation" from the historical person of Schenker, in that once the question becomes "how should *we* use the word prolongation?" issues about how Schenker himself used the term only become relevant insofar as Schenker's usage is preferable to some other being proposed. Finally, it allows us to confidently move "beyond Schenker." That is to say, it is only possible to distinguish extensions of Schenkerian theory from advances in music theory outside of the Schenkerian framework if we a definite sense of what the Schenkerian framework includes. Otherwise, debates over whether the Schenkerian model provides a more interesting or useful account of music than some other model will be inextricably tangled up with debates about music that properly belong within the Schenkerian framework.

Part 1: The MOP Model of Prolongation

The Concept(s) of Prolongation

The meaning of the term “prolongation” turns out to be a complicated subject meriting careful consideration. Therefore, before proceeding with any formal models of prolongation, it is necessary to examine the ways in which different authors use the term.

Putting one’s finger on the concept of prolongation is complicated for two reasons. The first reason is a lack of clarity about the abstractness of the subjects and objects of prolongation—that is, it is often ambiguous whether the things prolonging and being prolonged are particular musical events or more abstract theoretical constructs. The second is the history of the term (the main topic of this section): the concept of prolongation has metamorphosed extensively—by revision, reinterpretation, and misinterpretation—since Schenker first introduced it.

The primary source of ambiguity in concepts of prolongation is the common usage “to prolong a harmony.” Such a turn of phrase points to the abstract object of a “harmony,” which consists not in any particular configuration of notes but in a context in which one evaluates the notes of a musical passage—e. g. as “harmonic tones” or “non-harmonic tones.” This abstract concept of “harmony”—as I consider in more detail below—is akin to the Schenkerian *Stufe* (scale-degree). However, the word “harmony” also tends to take the more concrete meaning of a particular set of simultaneously sounded pitches—i. e. a chord. The ambiguity between these two senses of “harmony” is a source of confusion in the phrase “to prolong a harmony.”

This ambiguity may also extend to melodic pitches that appear in voice-leading graphs, especially those in background graphs. These melodic pitches are generally associated with particular foreground events, but can also accrue a sense of abstractness similar to that of “harmonies.” The analyst may then speak of the “prolongation of the initial tone” of the *Urlinie* and mean, at one moment, that the music is to be understood in the context of a melodically unresolved tonic harmony, and at the next moment, that the “initial tone” is a literal melodic event occurring (e. g.) in the first violins in measure 32.

If allowing such ambiguities may make a convenient cover for loose analytical reasoning, it certainly also presents an impediment to understanding the concept of prolongation. There is often a tension in Schenkerian analysis between the ideas expressed in language, which always entices us towards the more abstract and figurative, and the staff-notation symbolism central to Schenkerian dialogue, which is always moored to literal pitches and limited in its storehouse of conventional symbols. Throughout this paper I will focus on the relationships between literal pitch events, since one must ultimately construct any abstract tonal entities out of more concrete tonal material to give them a secure foundation of meaning. Abstract tonal entities are important and useful also, of course, but it's critical to always make their meaning clear and unambiguous.

There are two distinct and incompatible ways that the term prolongation may be used to relate concrete musical objects. Perhaps the more familiar usage is what I will call the *static sense* of prolongation. According to this convention, musical events themselves are the subjects and objects of prolongation. This is a common way to understand melodic prolongation, to view each less structural note as prolonging some more structural note. It is also possible to understand harmonic prolongation in the static sense—that is, to associate each harmony with a particular event, a chord, and to see each less structural chord as prolonging some more structural chord. However, the ambiguity of the term harmony discussed above sometimes makes it difficult to pinpoint examples of the static conception of harmonic prolongation.

A different understanding of prolongation—what I will call the *dynamic sense*—, views the *motion* between tonal events as prolonged by motions to other tonal events. That is, instead of saying that some event, X, prolongs some other more structural event, Y, according to the dynamic usage one says that an event X prolongs the motion from one more structural event, Y, to another more structural event, Z. Other ways one might say this are, for example, "the motion to X prolongs the progression from Y to Z," "X expands/prolongs the space between Y and Z," or "X delays the progression/resolution of Y to Z." The paradigmatic example of dynamic prolongation is the passing tone: to interpret a note as a passing tone is to identify it both as a stepwise progression from some note and to some other note.

In parts one and two of this paper, I'll crystallize these two usages into two distinct formal models, respectively, the *MOP model* (the subject of part one) and the *phrase-structure model* (the subject of part two). The brief tour through the history of "prolongation" that follows here first explores Schenker's own conception of prolongation and its transformation from his first uses of the term to his last writings, and then demonstrates the emergence of the dynamic sense of prolongation in the earliest English language interpretations of the Schenkerian approach and also how these early works opened the way for the static concept of prolongation.

If the evolution of language is supposed to proceed from the more literal to the more figurative, then the word prolongation is an odd duck, having charted the opposite course. Schenker's earliest employment of the word comes in a discussion in the first volume of *Kontrapunkt* of the possibility of an incomplete neighbor or passing tone in free composition, a license forbidden in strict counterpoint. In this case, the object of prolongation is not any musical entity but a basic law (*Urgesetz*) of dissonance treatment. Referring to the incomplete neighbor/passing tone figure he says,

> One sees, then, how one and the same basic phenomenon manifests itself in so many forms, yet without completely losing its identity in any of them! However much a given variant may conceal the basic form, it is still the latter alone that occasions and fructifies the new manifestation. But to reveal the basic form together with its variants, and thereby to uncover only prolongations of a fundamental law even where apparent contradictions hold sway—this alone is the task of counterpoint! (241)[4]

Thus, at its gestation the term prolongation reveals its central function in the task that was to consume the latter part of Schenker's published work, to show how the laws

[4] "Man sieht also, wie ein und dasselbe Urphänomen in so vielen Formen sich manifestiert und doch in keiner von ihnen sich ganz verliert! Will nun auch fürs erste die jeweilige Abwandlung noch so wenig den Urtypus erkennen lassen, gleichwohl ist es der letztere allein, der auch die neue Erscheinung zeitet und befruchtet. Gerade aber den Urtypus samt dessen Abwandlungen aufzuzeigen, und eben nur Prolongationen eines Urgesetzes zu enthüllen, auch dort, wo scheinbar Widersprüche gegen dieses zu Tage treten, ist allein Aufgabe des Kontrapunktes!" (315)

of strict counterpoint underlie the phenomena of free composition though they are sometimes hidden. A "prolongation," quite generally, is a rewriting of a strict law that preserves its spirit if not its letter, a movement from the more rigidly rule-governed to the less, not by breaking the rule but by following the rule in a freer way.[5] The purpose of the term for Schenker was to allow him to discuss the non-observance of a law while suggesting continuity from the law to the apparently contrary phenomena: thus "Prolongation des Gesetzes."[6]

Schenker uses the term prolongation throughout the second volume of *Kontrapunkt*, always in this figurative sense.[7] The thing being prolonged is always a law or procedure, never a particular musical passage or object. It's only in his subsequent analytical work that Schenker begins to employ the term prolongation more literally. The genesis of this new sense comes in *Tonwille 5*, in an otherwise modest analysis of number five of Bach's *Zwölf Kleine Präludien*. Schenker's analyses of the first five of the *Kleine Präludien* appear in *Tonwille* 4-5. In all of these he presents an *Urlinie-Tafeln*, a practice

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[5] As it turns out, "prolongation" is not the best translation of Schenker's "Prolongation" because of the temporal associations the term necessarily evokes in English. Thus the phrase "to prolong a law" sounds odd and inscrutable in English. In German the word is a borrowing from Latin associated primarily with commercial usages, such as extending a loan or renewing a contract. Thus, a better translation for "Prolongation" and "prolongieren" would be "extension" and "to extend." Thus "Prolongation eines Gesetzes" is an *extension* of a law. One must admit, however, that the somewhat poor translation of the term, which is now irrevocable in any case, has produced interesting results in English language Schenkerism. See also Alpern (2005), 51-3.

[6] Schenker finds frequent occasion in *Kontrapunkt* to air his dissatisfaction with the teaching of counterpoint that asserts a law only to admit that music is full of exceptions to the law. See in particular the author's introduction to *Kontrapunkt I*. See also Dubiel 1990.

[7] *Counterpoint II*, xviii-ix, 3, 4, 77, 119, 176, 179, 180, 192, 196, 213, 216, 228, 257, 271, 272 (*Kontrapunkt II*, xiv-xv, 3, 4, 77, 118, 171, 174, 176, 188, 192, 208, 211, 222, 248, 261, 262), and also in some of the section titles. The term also occurs in its figurative sense in *Counterpoint I*, 278 and 323 (*Kontrapunkt I*, 358-9 and 417); *Tonwille 2*, 53 (German ed., 4-5); *Beethoven, die Letzten Sonaten: op. 101*, 18; and *Tonwille* 5, 175 (German, 3). (The first three volumes of *Tonwille* were published between the publications of *Kontrapunkt I* and *II*). Note that the word "prolong" on p. 57 of *Counterpoint II* is not actually a translation of "prolongieren" but of "forttragenden," and similarly, "prolong its effectiveness" on 262 is Rothgeb's rendering of "fortwirken."

he uses throughout the analyses in *Tonwille* and *Meisterwerk*. These are illustrations on a single grand staff of the fundamental melodic line of a piece, indicated with large note-heads and scale-degree numbers above the system, accompanied by its elaborations and accompaniment in smaller noteheads. Though the *Urlinien* consist almost exclusively of stepwise motion (excepting transfer of register) they aren't *Urlinien* in the sense of Schenker's later theory, where only $\hat{3}$-$\hat{2}$-$\hat{1}$, $\hat{5}$-$\hat{4}$-$\hat{3}$-$\hat{2}$-$\hat{1}$, and $\hat{8}$-$\hat{7}$-$\hat{6}$-$\hat{5}$-$\hat{4}$-$\hat{3}$-$\hat{2}$-$\hat{1}$ qualify. For example, the *Urlinie* in Schenker's graph for number four of the *Kleine Präludien* (a scant 18 measure piece!) is $\hat{1}$-$\hat{2}$-$\hat{3}$-$\hat{4}$-$\hat{5}$-$\hat{4}$-$\hat{3}$-$\hat{2}$-$\hat{1}$-$\hat{2}$-$\hat{3}$-$\hat{4}$-$\hat{3}$-$\hat{2}$-$\hat{1}$.

In the analysis of the third of the preludes, however, Schenker finds an elegantly simple *Urlinie*: $\hat{5}$-$\hat{4}$-$\hat{3}$-$\hat{2}$, a mere fourth-progression. Schenker cautions the reader not to misinterpret this simplicity:

> The reader must be profoundly shaken when following the paths of imaginative power that coaxes out such a bold manifestation from such an intrinsically simple progression of the Urlinie and harmonies (shown in figure 1)—not in any way to disavow the simple as too simple, but indeed to confirm faith in its creative infinity though such diverse phenomena. (175)[8]

Schenker's figure 1 is reproduced in figure 1.1; it represents the first published example of Schenker illustrating the successive elaborations of the *Urlinie* in a series of vertically aligned voice-leading layers. His use of durational values in this early example is interesting: the apparent thirty-second-note runs on the lower system each represent a measure or two in duration in the music itself.

Schenker devised this illustration, as his comment explains, to show the reader that the simplicity of such an *Urlinie* may belie a fascinating and multifarious working-out even in the most background stages of elaboration. He obviously was fond of this method, as he immediately begins to apply it to subsequent analyses. He expands on the technique in the analysis of the fifth *Klein Präludium*, and gives these graphs the title of

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[8] "Mit tiefer Erschütterung muß der Leser in der nachstehenden Figur den Wegen der Einbildungskraft folgen, die aus einem an sich einfachen Vorgang in Urlinie und Stufen eine so kühne Erscheinung hervorlockt, nicht um das Einfache als zu einfach etwa zu verleugnen sondern um den Glauben an sein Zeugend-Ewiges noch durch die so weitfaltig." (*Tonwille 5*, 3)

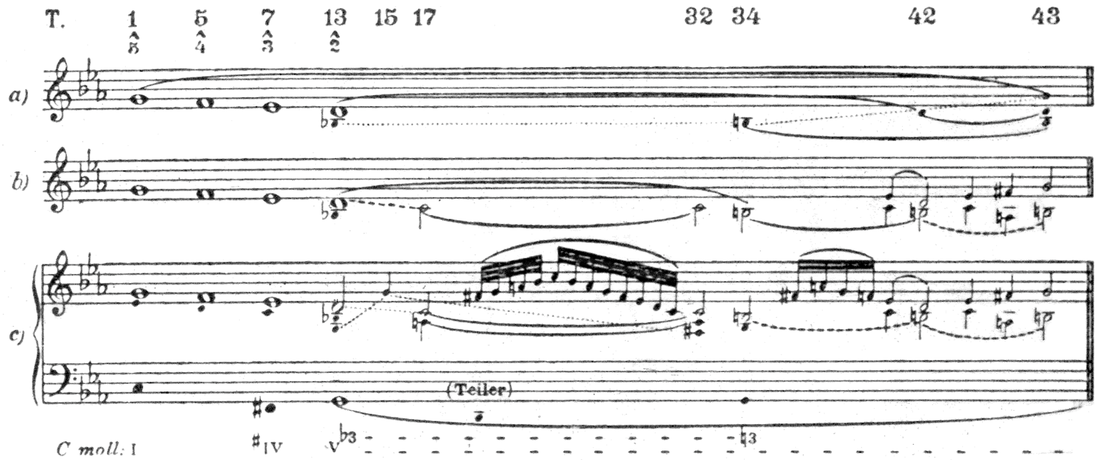

FIGURE 1.1: SCHENKER'S ANALYSIS *OF BACH'S* KLEIN PRÄLUDIUM *NO. 3*

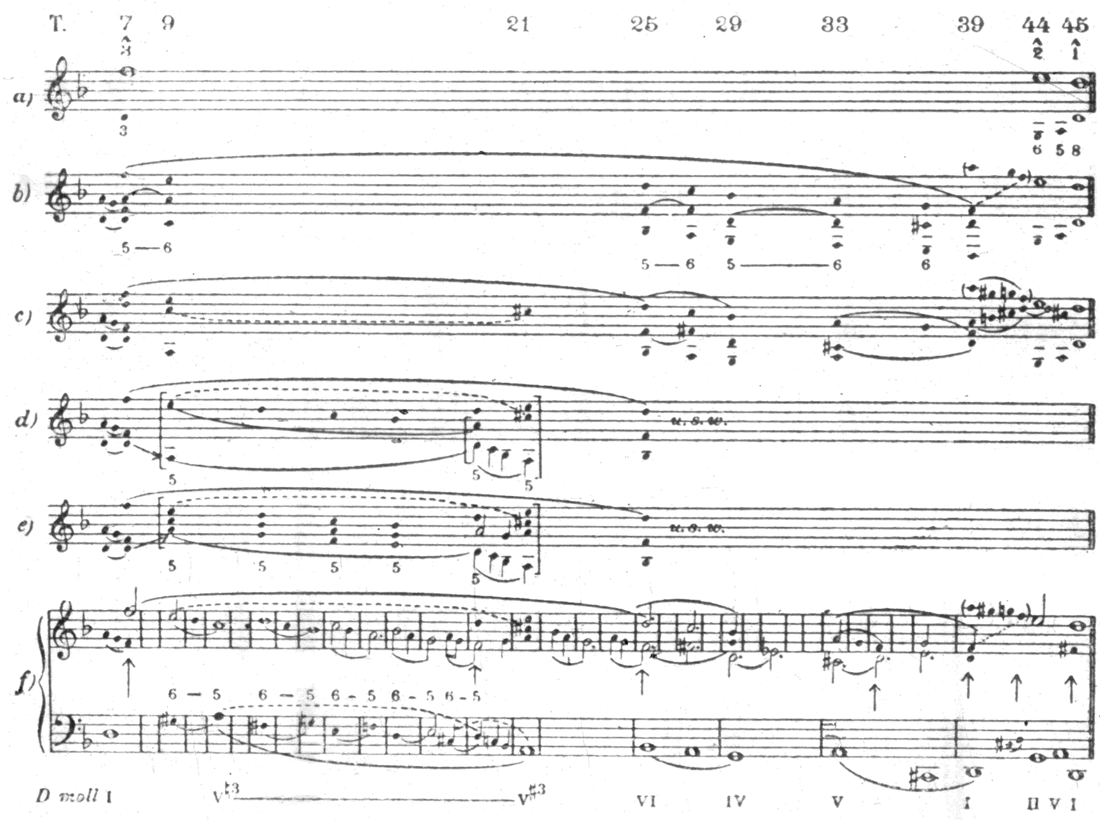

FIGURE 1.2: SCHENKER'S ANALYSIS OF BACH'S KLEIN PRÄLUDIUM *NO. 5*

Stimmführungsprolongationen (Voice-leading prolongations). (*Tonwille 5*, 8-9; English, 180) This example is reproduced in figure 1.2.

Schenker revisits the analysis of *Klein Präludium* no. 5 in the "Miscellanea" of *Tonwille 5*, as a means of illustrating "the *Urlinie* . . . as the source of voice-leading." (212)[9] It is clear that what he has in mind is that the *Stimmführungsprolongationen* reveal the operation of the laws of strict counterpoint in composition. He says, "in figure 1a," (figure 1.2a),

> the notes of the *Urlinie* can be seen in the two-voice *Ursatz*. One may already observe that this setting is somewhat freer than the voice-leading that would be formed in the setting of an actual *cantus firmus*—the material would not be enough for a *cantus firmus* setting—but in any case the purity in the progression of intervals is in accordance with the precepts of strict counterpoint.[10] (212-3)

He goes on to explain how b) and c) are prolongations of a), saying that

> Although within the octave descent [of prolongation b)] the voice-leading may also . . . comply with the demands of strict counterpoint, its principle validity remains the derivation from the fundamental voice-leading in figure 1a, which alone authenticates it as an octave descent.[11] (213)

In other words, even though b) itself essentially follows the laws of strict counterpoint rather than their prolongations, it still should be considered a prolongation because it is properly understood only in the context of the more basic strict counterpoint in a). Similarly, the voice leading of c) "is based on the insertion of chromatic notes, which are

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[9] "die Urlinie . . . als den Ursprung der Stimmführung." (45)

[10] "Bei a) der Figure. 1 sind die Urlinie-Töne zu sehen, in zweistimmigen Ursatz. Man darf schon diesen Satz als eine erste Freiheit gegenüber einer auf einen wirklichen *Cantus firmus* gestellen Stimmführing betrachten—für einien *C. f.*-Satz als wäre das hier gegebene Material zu klein—, jedenfalls aber entspricht die Reinheit in der Führung der Intervalle den Geboten des strengen Satz." (45)

[11] "Mag auch innerhalb der Octavsenkung die Stimmführung schon an sich . . . den Forderungen des strengen Satzes entsprechen, ihre Hauptgewähr aber bleibt die Herkunft von der grundlegenden Stimmführung bei a), die allein sie als eine Octavsenkung . . . beglaubigt." (45)

forbidden in strict counterpoint. . . . The justification for this voice-leading lies once again above all in its derivation from b) and a), even if it also has its own justification."[12]

This explains how Schenker arrived at his new use of the term "prolongation." In constructing series of voice-leading layers to show the connection between the *Urlinie* (the image of strict counterpoint in the composition) and the music itself, it occurred to him that the prolongations of the laws of counterpoint in free composition could be broken down and illustrated through this powerful technique. Thus, since they are generally arrived at through prolonged laws, the process of getting from one voice-leading layer to another is a "prolongation," an extension of the more basic voice leading.[13]

The *Stimmführungsprolongationen* quickly become a staple of Schenker's analytic procedure. He even adds one to the analysis of the second movement Beethoven's Fifth Symphony, which was conceived long before its publication in *Tonwille 5*.[14] This is the second figure of the analysis of the second movement (*Tonwille 5*, 33; English edition, 202). They elucidate only a small part—the first eight notes, occupying the first 15 measures—of the sprawling *Urlinie* he identifies in the movement. This is a general rule for *Stimmführungsprolongationen* in the *Tonwille* analyses: except in very short pieces, Schenker uses them to explain parts of the *Urlinie* that occupy only a fragment of the entire piece. In the continuation of the Fifth Symphony analysis in *Tonwille 6* (9-25), Schenker uses the technique in three different places, all for relatively short spans of music: the transition to the last movement in mm. 325-374 of the third movement (15), measures 72-132 of the fourth movement (25), and measures 281-312 of

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[12] "[Die Prolongation bei c)] beruht auf der Einschaltung der im strengen Satz noch verbotenen Chromen. . . . Die Rechtfertigung auch dieser Stimmführung liegt wieder vor allem in ihrer Herkunft von b) und c), wenn sie in sich auch eine eigene trägt." (45)

[13] Indeed, the idea of *Stimmführungprolongationen* is crucial in Schenker arriving at the canonical *Urlinie* and *Ursatz* of *Der Freie Satz*. As I have shown, the procedure originated in an analysis with an *Urlinie* that was too simple and therefore needed to be derived in stages to show precisely its connection to the music. Once the process was established, however, the obvious question must have presented itself: why not extend the process backwards from the more complex *Urlinien* to arrive at a simple canonical form of *Urlinie* that can integrate the entire piece?

[14] See the preface to the English edition of *Tonwille 1-5*.

the last movement (27). He also uses them for brief passages in five different places in his analysis of Beethoven's op. 57 Sonata (*Tonwille 7*, 3-33).

Further examples where Schenker uses the *stimmführungsprolongation* graphs for short sections of a piece are the short analysis of Beethoven's op. 127 String Quartet (*Tonwille 7*, 39-41), the analysis of the fugue of Brahms' *Variationen und Fuge über ein Thema von Händel* (*Tonwille 8-9*, 28-35), the Schubert *Impromptu* op. 90, no. 1 (*Tonwille 10*, 14-21), and Mendelssohn's *Venetianisches Gondellied* op. 30, no. 6 (*Tonwille 10*, 25-9). He uses the technique for entire short pieces in the analyses of "Erbarm es Gott" from Bach's *Matthäuspassion* (*Tonwille 7*, 34-8), the theme of Brahms' *Variationen und Fuge über ein Thema von Händel* (*Tonwille 8-9*, 3-5) and also many of the variations, the Mendelssohn *Lied ohne Worte* op. 67, no. 6 (*Tonwille 10*, 30-1), the Haydn *Österreichische Volkshymne* (*Tonwille 10*, 11-3), and the Schumann *Kinderszenen* op. 15, nos. 1 and 9 (*Tonwille 10*, 34-5, 36-9). In fact, in all of the analytical essays in *Tonwille* volume 6 and onward, only three do not use the *stimmführungsprolongation* technique, all of them very slight. In volume I of *Das Meisterwerk in der Musik*, the continuation of the *Tonwille* essays, Schenker begins to apply the method more boldly, analyzing nine pieces in their entirety with large *stimmführungsprolongation* graphs. And in *Meisterwerk II* he takes the procedure to its logical conclusion: an extensive *stimmführungsprolongation* analysis for each movement of Mozart's G minor symphony in its entirety.[15]

Thus, "prolongation" adopts a more concrete sense in Schenker's later writings, referring to the elaboration of a particular voice leading pattern, and this new sense for the most part supplants the earlier usage of "prolongation of a law." The term however retains its association with the connection between strict and free composition. For example, in a commentary on a letter of Beethoven's that plays on the term *Wechselnote*

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[15] Schenker sometimes refers to these graphs as simply *Stimmführungsschichten* (levels of voice-leading), or "Wandlungen der Stimmführung" or *Stimmführungsverwandlungen* ("transformations of voice-leading"). Some other interesting designations are *Stimmführungsvergangenheit* ("voice-leading history," *Tonwille 10*, 34) and "der weg vom Hinter- zum Vordergrund" ("the path from the background to the foreground," *Tonwille 10*, 36).

(literally "changing-note," i. e. *cambiata*), Schenker takes the opportunity to censure "the sort of musicians whose brains are always incapable of prolongation,"

> those who go through "the school of the changing-note" first with Fux, then with Albrechtsberger, yet are never capable of grasping the generality of the term and at the same time its unfolding in the particular, those who thus do not recognize the strict counterpoint in free composition. Yet there are even fewer who are able to produce prolongations that, seemingly detached from each law and unrestrained in their liberty, in truth are fulfillments of a fundamental law of strict composition.[16]

The latter, of course, is the exclusive property of the musical genius, and the purpose of music analysis is to hear these prolongations, so seemingly unrestrained at the surface, as fulfillments of the fundamental laws.

Although the more literal usage of "prolongation" becomes primary in Schenker's later writings, *Das Meisterwerk in der Musik* and *Der Freie Satz*, the word never adopts the yet more concrete sense that it takes on in North American Schenkerism. In *Freie Satz* Schenker uses it frequently to refer to prolongational techniques (arpeggiation, unfolding, octave coupling, reaching-over, motion from an inner voice, mixture, interruption, neighbor-note, register transfer, substitution, and linear progression). Otherwise, Schenker generally uses the term to relate entire voice-leading passages, not individual musical events such as a chords, harmonies, notes, pitches, et c.

Ernst Oster's translation of *Freie Satz* is quite misleading on this point. Most of the instances of the word "prolongation" in the English are not translations of the German "*Prolongation*" but free translations of *Auskomponierung*, *Verwandlung*, and other terms into the modern American usage of "prolongation," which is quite different than

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[16] "die Gattung der ewig unprolongierbaren Musikergehirne, die mit Fux, dann mit Albrechtsberger . . . 'die schule der Wechselnoten' durchgehn, niemals aber das Allgemeine des Begriffs und zugleich seine Ausfaltung ins Besondere zu erfassen vermögen, . . . diejenigen also, die im freien Satz den strengen nicht wiedererkennen, noch viel weniger jene Prolongationen schaffen können, die, losgelöst schienbar von jedem Gesetz und ungezügelt in ihrer Freiheit, in Wahrheit Erfüllung eines Grundgesetzes des strengen Satz sind" (*Tonwille 8-9*, 42, my translation).

Schenker's usage of "*Prolongation*."[17] One example of how misleading this can be — arbitrarily chosen from numerous possible examples—is Oster's translation in §74 of

> Es besteht . . . die Möglichkeit, auf das Sinken der Urlinie durch zwei Quintfälle des Basses einzuwirken, was sich später auch auf die aus <u>einem beliebigen Einzelklang gezogene Oberstimme</u> übertragen lässt, (67)

as

> It is possible to strengthen the descent of the fundamental line by two descending fifths in the bass; at later levels this procedure can be transferred to <u>an upper voice that prolongs a harmony on any scale-degree</u>. (33)

As my added underlines show, the phrase "*einem beliebigen Einzelklang gezogene Oberstimme*," which literally means simply "an upper voice drawn from any single chord," becomes "an upper voice that prolongs a harmony on any scale-degree," introducing the technical terms "prolongation" and "scale-degree" where Schenker doesn't actually use them, and furthermore giving the impression that Schenker's concept of prolongation includes the possibility of prolonging a particular harmony.

While Schenker generally uses "prolongation" in *Freie Satz* to refer to entire voice-leading graphs, either as a whole or split into individual voices, in a few instances we find him breaking down prolongations of the *Ursatz* into parts. In §65 he asserts the impossibility of "a prolongation of the descending arpeggiation V—I" ("*eine Prolongation der Abwärtsbrechung V—I*")[18] at the first level. He refers to this fact again in §86, saying, "because of the step of a second in the *Ursatz*, $\begin{smallmatrix}\hat{2}-\hat{1}\\ \mathrm{V}-\mathrm{I}\end{smallmatrix}$ provides no

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[17] For reference: Schenker uses some form of "*Prolongation*" in §§18, 26, 45, 48, 53, 62, 64, 65, 66, 70, 71, 73, 82, 86, 89, 117, 123, 127, 133, 138, 143, 149, 155, 157, 184, 185, 186, 189, 192, 204, 243, 257, 278, 280, 282, 283, 284, 286n, 308, 312, 313, 323 (also *erste Abschnitt* chapter 3). Oster translates a form of "*Auskomponierung*" to "prolongation" in §§32, 166, 170, 189, 206, 230, 247, 248, 249, 297, 301, 310, 311, 313, 320, and a form of "*Verwandlungen*" to "prolongation" in §§12, 30, 47, 49, 50, 51, 68, 83, 168, 169, 170. Other introductions of some form of "prolongation" by Oster that don't correspond to "*Prolongation*" in the original are in §§49 ("*Stimmführungschicht*"), 71 ("*Wandlungen*"), 74, 77 ("*Übertragung*"), 86 ("*Fassung*"), 99 and 101 ("*Durcharbeiten*"), 133, 194 ("*Stimmführungschicht*"), 204, 212 ("*Gliederung*"), 224, 247 ("*Diminution*"), 277 ("*Stimmführungschichten*"), 279 ("*Übertragung*"), and 324 ("*Dehnungen*").

occasion for any further rhythmic conflict. Only at later levels can possible prolongations of the $\hat{2}$ also provide opportunity for a special contrapuntal melodic development of the bass."[19] This is the only instance in *Freie Satz* where we find Schenker referring to a particular event in the voice-leading graph as an object of prolongation. However, "prolongations of the $\hat{2}$" here is apparently a shorthand for "prolongations of the progression from $\hat{2}$"—that is, the "step of a second" from the previous sentence—because later in the sentence he says that these prolongations "have their origin in the descending fifth $\frac{\hat{2}-\hat{1}}{\mathrm{V}-\mathrm{I}}$," and gives a reference to §189, which discusses "prolongations of V—I" ("*Prolongationen bei V—I*").[20] Schenker also mentions "prolonged versions of I—V" ("*prolongierten Fassungen von I—V*") in §186 and "prolongational forms of the ascending arpeggiation I—V" ("*Prolongationsformen der Aufwärtsbrechung*") in §189.[21]

These passages demonstrate a certain degree of precedent in Schenker's writing for the dynamic sense of prolongation, in that he applies the term to isolated two-element progressions from the *Ursatz*. However, Schenker only infrequently uses the term in such a specific way. Contrastingly, the first extended works on Schenkerian theory in the English language, Adele Katz's *Challenge to Musical Tradition* and Felix Salzer's *Structural Hearing*, use the word prolongation broadly and extensively both as a general concept and to explain details of analyses, investing a great amount of theoretical significance in it. Indeed, this focus on prolongation is appropriate given its association with the fundamental motivation behind Schenker's theory, the demonstration of the principles of strict counterpoint operating in free composition, but the meaning of the

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[18] *Free Composition*, 31; *Freie Satz*, 233 (Note the spelling error in the original).
[19] *Free Composition*, 36 (I have altered the translation slightly to make it more literal). "$\frac{\hat{2}-\hat{1}}{\mathrm{V}-\mathrm{I}}$ gibt wegen des secondschrittes der Urlinie keine Veranlassung zu einer weiteren rhythmischen Auseinandersetzung mehr. Erst in den späteren Schichten können etwaige Prolongationen der $\hat{2}$ zu einer besonderen kontrapunktisch-melodischen Entfaltung auch des Basses Veranlassung geben." (*Freie Satz*, 70).
[20] *Free Composition* p. 69; *Freie Satz* p. 113.
[21] *Free Composition* p. 69; *Freie Satz* p. 112-3.

word has already been altered significantly in these early introductions to the Schenkerian approach.

Although precise attribution is impossible, the evidence points to Salzer as the motivating force behind crucial aspects of the transformation of the term at its introduction into the language of North American music theory. Though *Challenge to Musical Tradition* (1945) was published seven years in advance of *Structural Hearing* (1952), much of the content of Katz's book was influenced by her discussions with Salzer, who arrived in New York in 1940 and was a student of Schenker's before the latter's death in 1935.[22] In the case of the new concept of prolongation we can infer Salzer's influence from the remarkable similarities between the accounts in *Challenge to Musical Tradition* and *Structural Hearing*, and the differences between these and Katz's concept of prolongation in her 1935 article "Heinrich Schenker's Method of Analysis," written before she met Salzer.

Katz and Salzer both introduce the concept of prolongation in their books with a critique of mainstream American harmonic analysis and suggest curing it by recognizing a "structure and prolongation" dichotomy. This critique of roman-numeral analysis as a motivation for the Schenkerian method was a means of presentation they inherited from their common teacher, Hans Weisse. Weisse used the same approach in his article, "The Music Teacher's Dilemma," a publication of a lecture he gave at a meeting of the Music Teacher's National Association in 1935.[23] It's quite likely that he used a similar polemic in his teaching.

The term "structural" in itself was not new: Katz used it in her 1935 article, but in a limited way. However the technical sense conferred on "structural" through the structure/prolongation dichotomy and the identification of chordal events as "structural" and "prolonging" first appears in *Challenge to Musical Tradition*.

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[22] Katz expresses this debt in her acknowledgements: "to Dr. Felix Salzer I am especially grateful for the warm and unflagging interest he has shown from the inception of this book through its final phases, and for his provocative point of view which evoked so many stimulating discussions of problems dealt with in this book." See also Berry 2002, 118.

[23] See Berry 2003, 124.

Katz's 1935 definition of prolongation, which she identifies as a central concept to the Schenkerian approach, draws on the "Elucidations" ("Erläuterungen") section of *Das Meisterwerk*.[24] She defines it as "the extension of the simple form of Horizontalization by filling in the Space." "Horizontalization," as indicated by its capitalization, is Katz's translation of Schenker's "Auskomponierung"[25] (though a somewhat specialized version of *Auskomponierung* in that it applies only to the triad and its tonal spaces, $\hat{1}$–$\hat{3}$, $\hat{3}$–$\hat{5}$, and $\hat{5}$–$\hat{8}$). In Katz's definition, "filling in the Space" (where the capitalized "Space" is a translation of "Tonraum," the tonal spaces of the triad) suggests that prolongation is adding passing motion in the horizontalized triad. Yet Katz includes the *Zug* (the composed-out triadic interval with passing motion) as a form of "Horizontalization." She doesn't include it in her list of prolongational techniques, although she does include transformation of the passing tone into a consonance. This,

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[24] *Masterwork I*, 112-4; *Meisterwerk* 201-6; *Masterwork II* 118-20; *Meisterwerk II* 193-8. Katz probably gave careful attention to this source because Schenker presented it as content from the not-yet-published *Freie Satz*. In it we find the comment, "dissonance is transformed into a consonance because only consonance, with its tonal spaces . . . can [give rise to] new passing-note progressions and freshly burgeoning melodies. This comes about through prolongations in ever-renewing layers of voice-leading, through diminution, through motive, through melody in the narrower sense." ("Die Dissonanz wird in eine Konsonanz verwandelt, weil . . . nur diese allein mit ihren Tonräumen . . . wieder zu neuen Durchgängen, zu neu sich zweigender Melodie führen kann. Dies geschieht nun durch Prolongationen in immer neuen Stimmführingschichten, durch Diminution, Motiv, Melodie im engeren Sinn.") This notion of transforming dissonance into consonance as the mechanism for continually expanding voice-leading progressions originates in *Kontrapunkt II* (xv, 172, 181-2; *Counterpoint II*, xviii-xix, 176, 185-6) and it is crucial to the connection between strict and free composition so fundamentally tied to Schenker's concept of prolongation. (See "Prolongations as Passing Events" below.) This notion takes a prominent place in the exposition of "Heinrich Schenker's Method of Analysis," but evidently only because of Katz's choice of source: in *Challenge to Musical Tradition* Katz abandons the idea, apparently not seeing it as not especially relevant to the concept of prolongation.

[25] The term actually comes from Weisse: he uses it in multiple places in his Mozart analysis in "The Music Teacher's Dilemma," though not in an explicitly technical way. The first published source of the translation of *Auskomponierung* to "horizontalization" is Victor Vaughn Lytle's 1931 polemic, which, though scant on actual music theory, represents the earliest publication in the English language to deal with some of the technical details of Schenker's method. See Berry 2003, 148-9.

and the fact that she introduces "Horizontalization" before prolongation, suggests that the concepts are exclusive. However she speaks of "Prolongation by Horizontalization" later in the article (referring to Schenker's analysis of the E major Sonata for Violin Solo), demonstrating a certain amount of confusion at this point in her assimilation of Schenker's concepts of prolongation and *Auskomponierung*.

This confusion here stems from the fact that Katz is missing a critical element of Schenker's concept of prolongation that lies beneath the "Erläuterungen" article: the relationship between strict counterpoint and free composition. Though this aspect of the concept is not absent from *Meisterwerk*, it is more clearly expressed in the second volume of *Kontrapunkt*. The linear progression (*Zug*) is indeed a form of *Auskomponierung*, however, it isn't necessarily a "prolongation." This is because the third and fourth progressions (the generalized passing-tone figure, the filling in of the tonal space of the triad) are themselves perfectly in accord with the laws of strict counterpoint, whereas a prolongation, for Schenker, indicates a relaxation of strict counterpoint in free composition.[26] A prolongation, in other words, is a passing motion

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[26] Schenker says in *Masterwork II*, "free composition, through prolongation, supplements the third- and fourth-progressions, taken from strict counterpoint, with fifth- and sixth-progressions" (10. I have simplified Rothgeb's translation). ("Der freie Satz fügt den vom strengen Satz übernommenen Zügen im Terz und Quartraum prolongierend nun auch noch Quint- und Sextzüge," *Meisterwerk II*, 26). In *Kontrapunkt I* Schenker assigns the filling-in of a fourth to free composition, where either of the two intermediate diatonic notes might function as a passing event even though it must progress by third to one of the notes of the fourth. (248-9; *Counterpoint I*, 184-5) However, he also demonstrates a true fourth progression as an acceptable figure in third species. (298; 227) In *Kontrapunkt II* he more explicitly demonstrates the fourth progression as an element of three-voice third species counterpoint where both passing notes may be made dissonant, which is preferable to the more ambiguous situation of a consonant passing note: "it is precisely the dissonant nature of the middle tones that most fully promotes the concept of the fourth-space." (*Counterpoint II*, 73) ("Gerade . . . die dissonante Natur der mittlener Töne den Begriff des Quartraums am ehesten fördert," *Kontrapunkt II*, 118). He goes on: "the process of composing out that manifests itself in the sparse material of strict counterpoint thus undergoes, through development of the fourth space, an enrichment and intensification, even though it may still be far removed from the definiteness and precision of free composition." ("Die im kargen Material des strengen Satzes sich auswirkende Auskomponierung erfährt so durch die Entwicklung des Quartraumes eine

in disguise. Schenker's prolongational techniques are ways in which a composer may disguise the passing motion—by transferring into a different register, into a different voice, and so forth. Therefore, the third progression of the *Ursatz* in particular is not a prolongation.

Katz is correct that the transformation of a passing tone into a consonance is a prolongation, because it is disallowed in strict counterpoint where the *cantus firmus* (against which the passing tone is dissonant) is present. (In free composition the "*cantus firmus*" is imaginary, taking the form of a progression of *Stufe*).[27] She must have been acquainted with the essay in *Meisterwerk II* on the *Urlinie*[28] that includes a section entitled "the dissonance is always a passing event, it is never a chord." (9)[29] In this section Schenker writes,

> The characteristics that the dissonant passing tone acquired at its birth in the second species of two-voiced strict counterpoint remain with it also in the third species in a two-voice setting, and in the second and third species of settings of three and more- voices. . . . Even in the combined species certain prolongations of the dissonant passing tone may rest only on the fact that in them the horizontal tension above all is emphasized, even to the point of permitting a dissonance to be transformed into a consonance without relinquishing the inner nature of a passing tone. . . .

Bereicherung und Steigerung, mag dieser auch, wie oben zu sehen ist, von der Unbedingtheit und Bestimmtheit des freien Satzes noch weit entfernt sein.")

[27] See *Kontrapunkt II*, 259-62; *Counterpoint II*, 269-71.

[28] Besides the "Erläuterungen" article at the end of *Meisterwerk I* and *II* (201-6 and 193-8), Katz quotes analyses from *Meisterwerk I*, 75-98 (the Bach Sonata for Violin Solo) and *Meisterwerk II*, 55-98 (the C minor Prelude from WTC I). She also uses an analysis (the C major prelude of WTC I) from the *Fünf Urlinie Tafeln*. I have not been able to identify the source of Katz's apparent translation of Schenker calling the *Ursatz* the "perfect realization of tonality (the life of one and the same tone throughout the work) expressed through the Horizontalization of the tonic triad in two voices." It is possible that she perhaps had acquired unpublished advance material of *Freie Satz* through Hans Weisse.

[29] "Die dissonanz ist immer ein Durchgang, niemals ein Zusammenklang." (24-40) Rothgeb renders *dissonanz* as "dissonant interval." Note that the German term "Durchgang" can take the technical sense of "passing tone" as it does in second species counterpoint, but can also indicate more generally any passing event, such as a contrapuntal chord. Thus, Schenker is claiming that the passing tone of second species counterpoint is the model for all dissonance.

> The consonance that comes about this way to substitute for a dissonance is then further employed by free composition to sprout additional linear progressions; . . . but all this freedom to delude, to create tensions, is drawn only from the law of the dissonant passing note! (10)[30]

This passage makes a clear distinction between the passing tone itself, and the prolongations of the passing tone—i. e. figures derived from a prolongation of the law of the dissonant passing tone—, which here is its transformation into a consonance.

Katz is thus also correct in identifying prolongations as kinds of passing motions, ways of "filling in" the tonal space. However, she is not able to explain why the passing tone of the *Ursatz* is a "Horizontalization" rather than a prolongation of a horizontalization, because she doesn't invoke the relationship between strict and free counterpoint that motivates Schenker's theory of prolongation.

In *Challenge to Musical Tradition*, Katz clears up all this conceptual confusion by translating Schenker's concept of *Auskomponierung* of a harmony as "prolongation of a chord." This is convenient in a number of ways: first, it resolves the problematic relationship between horizontalization and prolongation, making horizontalization unambiguously a type of prolongation. Furthermore, it means that all expansions of musical material from one voice-leading level to the next count as prolongations. Also, Katz can now use the term horizontalization more freely: it no longer functions as a translation of a particular concept of Schenker's. In any case "Horizontalization" was never sufficient as a translation of *Auskomponierung*, because while "composing-out" usually applies to harmonies, Schenker also sometimes speaks of composing-out notes or melodic ideas.

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[30] "Die Merkmale, die der dissonante Durchgang bei seiner Geburt in der 2. Gattung des zweistimmigen strengen Satzes empfangen hat, bleiben ihm auch in der 3. Gattung des zweistimmung Satzes, in der 2. und 3. Gattung des drei- und mehrstimmigen Satzes erhalten. . . . Schon stehen in den Mischungsgattungen gewisse Prolongationen des dissonanten Durchgangs nur darauf, daß sich in ihnen vor allem die horizontale Spannung betont, bis zu dem Grade sogar, daß sie die Verwandlung der Dissonanz in eine Konsonanz gestattet, ohne das Wesen eines Durchgangs aufzuheben. . . .

Die Konsonanz, die so an Stelle einer Dissonanz tritt, benutzt der freie Satz dann wieder zur Spaltung in weitere Auskomponierungszüge . . . , doch wird alle diese Freiheit zu täuschen, zu spannen, allein aus dem Gesetz des dissonanten Durchgangs bezogen!" (25)

As a result, a new concept of “prolongation of a chord” emerges in this work.[31] Katz introduces prolongation with the idea of “widening the motion within a single chord through the use of prolongations.” (15) The following passage fully explains this idea:

> Thus, when we speak of motion within a chord, the reader will understand: (1) that the chord has been horizontalized; (2) that the arpeggiated interval forms a space-outlining motion; (3) that the passing chords within this space are of a contrapuntal and prolonging nature, and (4) that the motion as a whole constitutes a prolongation of a single horizontalized chord. (16)

Thus Katz’s notion of prolongation is explicitly dynamic. The prolongation of a chord is the expansion of a motion arpeggiating the chord. The notion also includes motion between different chords:

> In some instances the contrapuntal chords expand a single arpeggiated chord, while in others they prolong the space between two different chords of a harmonic progression. (16)

These concepts of prolongation of chord progressions extend readily to melodic prolongation in the voices that make up the progression (see Katz’s chorale analysis on pages 21-2).

Salzer uses the term prolongation in the dynamic sense also. He introduces his first discussion of prolongation (in part I of *Structural Hearing*, intended as an exegesis of basic Schenkerian ideas) with a description of music an expression of directed motion. (11) He explains this by means of a distinction between structural and prolonging chords and describes prolonging chords as “filling the space between” structural chords, a “means of passing” from one structural chord to another, and as “prolonging the motion” between the structural chords. (12-3)

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[31] There is only one isolated instance of a usage like this in Schenker’s major publications. This is in his analysis of the Mendelssohn *Lieder ohne Worts* op. 67, no. 6. Schenker points out “a very artful prolongation of the simple V^7” (“einer sehr kunstvollen Prolongation der einfachen V^7”) in his voice-leading graph. (*Tonwille 10*, 31) In all other instances, he reserves such a usage for the term *Auskomponierung*.

The idea of "chord prolongation" is quite prominent in Salzer's *Structural Hearing*, suggesting that he may have been the source of this translation of Schenker's *Auskomponierung* of a harmony (see above). Salzer introduces the idea thus,

> Contrapuntal chords do not only appear *between* two members of a harmonic progression; very often they move *within* a single harmony or chord. In such cases the function of these contrapuntal chords is to prolong and elaborate that single harmony or chord. (16)

Clearly the potential for confusion between the abstract and literal senses of "harmony" or "chord" is great here, but Salzer makes it clear that he intends the dynamic sense of prolongation:

> The role of the chords *within* the horizontalized and thus prolonged . . . chord is that of passing chords, not between two different harmonies but within the horizontalized intervals of a single harmony. (16)

Note also his adoption of the Katz/Weisse terminology of "horizontalization."

Katz's and Salzer's focus on chords in their introductory texts to the Schenkerian approach was momentous for the history of North American Schenkerism. In Schenker's work, the notion of *Stufe* (exposited first in *Harmonielehre* and expanded in *Kontrapunkt*) precedes most of his other most original and influential ideas. The *Stufe* concept, as I pointed out above, situates the principles of harmony in an imaginary *cantus firmus* that accompanies the composition, and thereby eliminates the necessity for discussing harmony in terms of literal chords. As a result, Schenker never needs to define any of his ideas in terms of chords. Katz's and Salzer's presentation of the Schenkerian approach in terms of chords in chorales has a subtle but profound effect: it locates harmonies in literal events, the structural chords, instead of the coordination of several contrapuntal factors suggesting a particular *Stufe* to the ear. This opens the door to the ambiguous notion of harmony that I mentioned at the beginning of this section. When Katz speaks of the "prolongation of a harmony," she means motion within the intervals of a harmony presented melodically. However, if a "harmony" is also a chord, the phrase "prolongation of a harmony" can also be used in the static sense.

By now it should be clear that the lines that define the static and dynamic senses of prolongation become easily blurred. We will see in subsequent sections of this paper (especially in Part 2) however that the distinction is crucial in the formalization of prolongation. Indeed, that is the reason for examining it so carefully here, and—looking at it from a different vantage point—the service of articulating a formal model is that it brings such distinctions, those that are significant but easily obscured, into relief.

A crucial aspect of the static/dynamic difference is that to conceive a musical event as a dynamic prolongation is to understand that event equally in terms of the structural event that it departs from and the structural event that it leads to, just as describing a note as a passing tone implies something both about its preparation and its resolution. Static prolongation conceives of the prolonging event as *either* departing from some more structural event *or* anticipating the more structural event.

One problem that arises immediately with the concept of dynamic prolongation is the instance of a prolonging event that occurs at the beginning of a piece. Katz faces this problem in her explanation of the initial ascent, which she identifies as a prolongation of the structural line. She says

> It may seem a contradiction to call these tones [those of the initial ascent] prolongations of structural top voice when they precede the tone . . . on which the structural descent begins. However, the ascending motion . . . forestalls the entrance of [the initial tone of the fundamental line] and thus expands the top-voice motion as a whole. Because of this expanding function, the tones which comprise the ascending motion are prolonging tones. (18-9)

Interestingly, Katz falls back on two Schenkerian concepts here. Most explicitly, she invokes Schenker's use of the term *Prolongation* (in *Meisterwerk* and *Freie Satz*) where it refers to the expansion of an entire voice-leading graph. She also describes the prolongation as a forestalling, invoking Schenker's concept of *Aufhaltung* as articulated in *Freie Satz*. Here Schenker describes many of the prolongational techniques as a kind of "delaying" (*Aufhaltung*), including initial ascent (§124) as well as prolongations of the *Baßbrechung* (§70), interruption (§90), and the neighbor-note (§§109-11).

The same problem presents itself to Salzer in *Structural Hearing*. Figure 1.3 (Salzer's example 5) shows a melody from Schumann's *Album for the Young* and Salzer's analysis of it. This melody, though its analysis is straightforward, presents a difficulty for description in terms of dynamic prolongation. The thirds are composed-out harmonic intervals, but the more structural notes of each third in this case *follow* the less structural. Salzer, using "prolongation" in its Schenkerian sense, says "the ascending thirds are the prolongations of a melodic structure directed downwards from G to C." (42) [32] This is fine and perfectly accurate: Schenker himself probably would have considered this a sufficient description of the situation. But Salzer, writing an introductory and pedagogical text, must explain in more detail exactly *how* these ascending thirds prolong this progression. Here the dynamic sense of prolongation fails us, because there is nothing preceding the initial C to initiate a motion to E that one can say the C prolongs. Consequently Salzer temporarily abandons the dynamic sense of prolongation and says, "we now realize that prolongations may also precede the structural tones."[33] Salzer's next example shows the retention of a note over the majority of a short melodic phrase, and summarizing the analysis he says, "for the greatest part of its course this melody does not move from one structural point to another, but around one structural tone." (43)

FIGURE 1.3: SALZER'S ANALYSIS OF A MELODY FROM SCHUMANN'S ALBUM FOR THE YOUNG

The *Introduction to Schenkerian Analysis* textbook of Forte and Gilbert, published 30 years after *Structural Hearing*, remarkably echoes this discussion in its

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[32] It's interesting that Salzer and Katz both fall back on the more general, neutral Schenkerian usage of "prolongation" when the analytical situation presents a difficulty for the dynamic sense of the term.
[33] Note that this departure from the dynamic sense of prolongation occurs not in Part I of *Structural Hearing*, Salzer's introduction to the concepts of Schenker's analytical approach, but in Part II, "the pedagogic and systematic approach to structural hearing." (xvii)

definition of melodic prolongation. Forte and Gilbert claim that there are three different types of melodic prolongation "motion *from* a given note," "motion *to* a given note," and "motion *about* a given note." (144) Thus, in the most widely used introductory text to the Schenkerian approach, the static sense of melodic prolongation has essentially replaced the dynamic sense of the term in the earliest extended English language works on the subject.

The narrative here, the biography of the concept of prolongation, is clear enough: the original Schenkerian concept of prolongation becomes volatile through the severing of the concept from its mother, the relationship between strict counterpoint and free composition, and its expansion through the translation of Schenker's *Auskomponierung* as prolongation. Then the new Schenkerians add the catalyst, a need for more careful systematization of the analytical procedure for introducing the concept to the new audience of English-speaking music theorists and students of music theory. The concept first adopts the dynamic sense but this state proves unstable, leading to a combustion that eventually leaves us with the equilibrium of static prolongation. In the following two sections I will try to restore stability to the concept of dynamic prolongation, to help prolongation retain its "free energy," in preparation for its formalization in the subsequent sections of this part of the paper.

Prolongations as Passing Events

In the previous section ("The Concept(s) of Prolongation") we saw that Schenker's own use of the term "prolongation" is more limited than either the dynamic or static usages. The dynamic sense of prolongation, however, as I will argue here, most closely reflects the original intent of the term in Schenker's writings. Of course, "original intent" itself shouldn't hold any currency in music theory; formal modeling of Schenkerian analysis is not constitutional law. If I cast my own lot in the end with dynamic prolongation it is not because of its historical priority but because I find that it leads to a more compelling theoretic construct. The reader, of course, is free to make up her own mind about this.

In this section I will attempt to restore to "prolongation" the critical aspect that is missing in the understanding of the term in North American music theory: prolongation as the link between strict and free composition. At the end of this survey of Schenker's theory of the strict/free counterpoint connection we will see that Schenker's model of prolongation is the dissonant passing tone of second species counterpoint. It is because the dissonant passing tone admits of an accurate description in terms of dynamic but not static prolongation that I claim dynamic to be the more Schenkerian of the two senses of prolongation. However, I engage this aspect of Schenker's theory here not only for this historical interest, but because his ideas of prolongation are themselves compelling and help to motivate the formal model that I pursue in the remainder of this part of the paper.

As we have already seen in the last section, the strict/free counterpoint relationship is essential to an understanding of its prolongation's place in Schenker's theory; it's the reason for Schenker's original introduction of the term and remains with it throughout its life in Schenker's writings, despite a considerable evolution in its usage. Therefore the logical place to look for an explanation of prolongation is in the writings in which Schenker tries to spell out the connection between strict and free composition in detail. Unfortunately this project of Schenker's was somewhat cut short. Throughout his completion of the second volume of *Kontrapunkt*, Schenker intended to compose an entire third volume that would address this subject.[34] However, this plan was set back by the questions of music analysis that increasingly occupied him, and *Freie Satz* was eventually to focus more on analysis than on the more theoretical content of the planned third volume of *Kontrapunkt*.[35]

The primary published source we have on which to base our understanding of prolongation then is the last part of *Kontrapunkt*, "Bridges to Free Composition" ("Übergänge zum Freien Satz"). Here Schenker identifies the phenomenon of the "passing event in multiple voices" (*mehrstimmig Durchgang*) as the crucial aspect of the bridge from strict to free composition. (*Kontrapunkt II*, 171-3; *Counterpoint II*, 175-7) The *mehrstimmig Durchgang* is a harmonized passing tone. That is, a passing tone

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[34] See Hedi Siegel, "When *Freie Satz* was part of *Kontrapunkt*."
[35] See William Drabkin's review of *Counterpoint*.

dissonant with the *cantus firmus* accompanied by and consonant with one or more notes in other voices (that may or may not be passing tones themselves), so that if the *cantus* is removed only consonance remains. When leaping motion in the other voices accompanies the dissonance, Schenker calls this a "leaping passing tone" (*springend Durchgang*), and describes the situation thus:

> Although the progression of both voices adheres most precisely to the principles of strict counterpoint, the difference between the two simultaneously operating laws nevertheless causes a conflict, which has the result that the dissonant nature of the passing tone cancels the consonant effect of the leaping interval. That it is precisely the dissonant passing tone which prevails in this situation probably rests on the fact that . . . such a passing tone confirms and extends the harmony of the downbeat in far greater measure—that is, it preserves the harmonic unity of the bar much more decisively—than does a consonant half-note. Thus one might even say that by its superior influence, this dissonant passing tone in a way ensnares the consonant leap into the realm of its own dissonance, so that in such a situation it may appear by no means inappropriate to speak of *horizontalization of the leap*—that is, of the leap as, again, only a passing tone—and to speak even of a "*leaping passing tone.*" (181-2)[36]

Because the leaping passing tone must be consonant with the passing tone proper as well as the *cantus*, if the *cantus* is removed, or "elided," the result is a transformation of the dissonant passing tone into a consonance. In free composition, the function of the *cantus* is taken over by *Stufen* as Schenker explains at the end of the "Bridges to Free Composition" chapter (*Counterpoint II*, 269-71, *Kontrapunkt II*, 259-62); this elision of

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[36] "Entspricht dann zwar die Führung jenes wie dieses dem strengen Satze aufs genaueste, so zeitigt die Verschiedenheit der sich gleichzeitig auswirkenden Gesetze dennoch einen Widerstreit, der nun in der Weise ausgetragen wird, daß die dissonante Natur des Durchganges die konsonante Wirkung des springend Intervalles aufhebt. Daß hiebei gerade der dissonante Durchgang siegt, beruht wohl darauf, daß durch einen solchen ja . . . die Harmonie des Niederstreichs in viel stärkerem Maße bestätigt und forgesetzt, das heißt, die harmonische Einheit des Taktes viel entschiedener verbürgt wird, als durch eine konsonante Halbe. Man darf daher auch sagen, daß der dissonante Durchgang durch seinen überlegenen Einfluß den konsonierenden Sprung gleichsam in die eigene Dissonanz mitreißt, weshalb in einer solchen Lage von einer *Horizontalisierung des Sprunges*, das heißt vom Sprung als weider nur von einem Durchgang, und zwar von einem "*springenden Durchgang*" zu sprechen durchaus nicht unangebracht erscheinen mag." (177)

the *cantus* makes such illusory consonances possible in free composition. (The previous section, "The Concept(s) of Prolongation," also discusses this aspect of the *Stufe* concept.)

Schenker promises, in the "Bridges to Free Composition" chapter, to show the "great fecundity" of this technique of fulfilling "*dissonant concepts by means of illusory consonances*" in *Freie Satz*. (*Counterpoint II*, 186; *Kontrapunkt II*, 182) In fact he apparently considered this to be the most important aspect of the work-in-progress, as it makes up the majority of the theoretical content of the "Erläuterungen" section Schenker published repeatedly in *Tonwille* and *Meisterwerk* as a preview of the content of *Freie Satz* (quoted above, in "The Concept(s) of Prolongation"). Nor was Schenker to disavow this theoretic standpoint by the time he had assembled *Freie Satz* for publication. He writes there,

> The *Ursatz* exhibits the first transformation of the primal dissonant *Urlinie* tone into a consonance. . . . This principle continues through all levels of the middleground, creating more and new levels which present new possibilities of transformation for dissonant passing tones . . . until the foreground, with its greatest freedom, shows voice-leading which is not recognizable as passing motion without the interpretation of relationships in the middleground and background.[37]

This passage illustrates that Schenker conceived voice-leading prolongations, even in his last theoretical work, as ultimately derived from the law of the passing tone in strict counterpoint. The recursive aspect of prolongations is a consequence of the principle of transformation of a dissonance into a consonance, which is why this principle and the

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[37] "Schon im *Ursatz* zeigt sich die erste *Verwndlung* eines ursprünglich *dissonanten Urlinie-Tones* in eine *Konsonanz*. . . . Durch alle Schichten des Mittelgrundes pflanzt sich dieses Gesetz fort, wodurch sich immer neue Schichten bilden mit neuen Verwandlungsmöglichkeiten für dissonante Durchgänge . . . bis die Vordergrund in seiner äußersten Freiheit Stimmführungen bringt, die ohne Deutung der Zusammenhänge in Mittel- und Hintergrund als Durchgang nicht zu erkennen sind." *Freie Satz* (103). I have altered Oster's translation (*Free Composition* p. 61) to make it more literal.

"leaping passing tone" that instantiates it are fundamental to understanding the transition from strict to free composition.[38]

In Schenker's later analytical work, this theoretic stance is reflected in his preoccupation with identifying *Züge*, or linear progressions, in the works he considers. In the "Further Considerations on the *Urlinie*" essays of *Meisterwerk I* and *II*, he identifies linear progressions as essential to the understanding of voice leading; he says "Translation from vertical to horizontal is effected by means of linear progressions [*Züge*]. . . . Anyone who has not heard music as linear progressions of this kind has not heard it at all!" (*Masterwork I*, 107)[39] (He repeats this declaration in the *Meisterwerk II* essay as well for added emphasis. (11)) Schenker makes it clear in these essays that the linear progression is the image of strict counterpoint in free composition. First, since both of the outer voices express linear progressions, he says (in the *Meisterwerk I* essay) that they must be thought of as a "prolonged form" ("prolongiert Form") of the outer voice setting of strict counterpoint, as "a setting of a treble and an inner voice above a conceptual lower voice, which carries the fundamental, or scale-degree, notes." (105)[40] This reiterates the idea of elision from *Kontrapunkt II* (explained above), which is crucial to the manifestation of passing motion in free composition. Schenker also writes in this essay,

> Access to the third-progression is given already by the initial stages of counterpoint in the second species of two- and three-voice writing . . . ; there, the unitary cantus-firmus note at once guarantees the perceived unity of the third progression as well. But even in free composition, the unity of the third progression is not cancelled merely by the fact that the prolongation occasionally turns the middle note of the third-progression—the dissonant

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[38] For a reference to more examples of Schenker describing various dissonance-formations as prolongations of the passing-tone figure, see Dubiel 1990.

[39] "Die Auswicklung bewegt sich in Zügen. . . . Wer Musik nicht in solchen Zügen gehört hat, hat sie überhaupt nie gehört!" (*Meisterwerk I*, 192) Rothgeb renders "die Auswicklung," "unfolding," freely as "translation from vertical to horizontal."

[40] "der Satz einer Ober- und Mittelstimme über einer gedachten Unterstimme, die die Grund- oder Stufentöne führt." (188)

> passing note—into a consonance (see 'Elucidations'). And thus it is also with the perceived unity of the fourth-, fifth-, and sixth-progressions. (107)[41]

He pursues this further in the "Further Consideration on the Urlinie" essay of *Meisterwerk II*. I have already quoted in the previous section ("Concept(s) of Prolongation") a lengthy passage from this essay in which Schenker traces the development of the *Durchgang* from second species in two voices through third species, three-, four-voice, and mixed species counterpoint to free composition, where "all this freedom to delude, to create tensions, [through linear progressions] is drawn only from the law of the dissonant passing-tone!" (10) Schenker quotes a passage from *Kontrapunkt II* concerning the leaping passing tone in this discussion. He also declares, at the beginning of the section,

> A linear progression always presupposes a passing note: there can be no linear progression without a passing tone, no passing note without a linear progression. Therefore, it is only by means of the linear progression—by means of the passing note—that it is possible to achieve coherence, to achieve synthesis of the whole! (9)[42]

And at the beginning of the essay, where he characterizes the linear progression as a "perceived tension" ("geistige Spannung") he states,

> This tension alone engenders musical coherence. In other words, *the linear progression is the sole vehicle of coherence, of synthesis*. (1)[43]

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[41] "Den Terzzug offenbaren schon die ersten Schritte des Kontrapunktes in der zweiten Gattung des zwei- und dreistimmigen Satzes . . . ; zugleich ist dort durch den einen *cantus firmus*-Ton auch die geistige Einheit dieses Zuges gesichert. Aber auch im freien Satze wird die den mittleren Ton des Terzzuges, den dissonierenden Durchgang unter Umständen konsonierend macht, siehe 'Erl.'. Und so ist es auch mit der geistigen Einheit der Quart-, Quint- und Sextzugs." (192) I have changed "free counterpoint" and "conceptual" in Rothgeb's translation to "free composition" and "perceived."

[42] "Der Auskomponierungszug setzt immer einen Durchgang voraus: kein Auskomponierungszug ohne Durchgang, kein Durchgang ohne Auskomponierungszug. Und also auch: nur durch den Auskomponierungszug, durch den Durchgang geht es zum Zusammenhang, zur Synthese des Ganzen!" (24).

[43] "Diese Spannung allein schafft den musikalischen Zusammenhanges, das heist: *der Auskomponierungszug ist der alleinige Träger des Zusammenhanges, der Synthese*." (11)

Schenker could hardly make this point less equivocally, that all compositional elaborations, all prolongations, must be understood as having their origins in the law of the dissonant passing tone of second species counterpoint. He repeats the point in *Freie Satz* also,

> Whatever the goal may be, the qualities inherent in the linear progression of the *Urlinie* and in the linear progressions at the first level remain the same at the later levels: a linear progression is, above all else, the principle means of creating content in passing motions—that is, of creating melodic content.[44]

It's worth considering more closely here the idea of the linear progression as a psychologically perceived tension span, the subject of the first section of the "Further considerations on the *Urlinie*" essay in *Meisterwerk II.* Schenker explains that this tension comes from the retention of the initial tone of the linear progression, because "the primary note [*Kopfton des Zuges*] is to be retained until the point at which the concluding tone appears." (1)[45] This recalls a discussion from *Kontrapunkt II*:

> Alongside all of the corporeality (which is always to be understood as independent) of the intervals available in strict counterpoint, the first appearance of the dissonant passing tone produces a curious intrusion of the imaginary: it consists in the covert retention, by the ear, of the consonant point of departure that accompanies the dissonant passing tone on its journey through the third-space. It is as though the dissonance would always carry along with it the impression of its consonant origin, and thus we comprehend in the deepest sense the stipulation of strict counterpoint, which demands of the dissonant passing tone that it always proceed by the step of a second and always in the same direction.
>
> The implications of this effect are of great importance: we recognize in the dissonant passing tone the most dependable—indeed the only—vehicle of melodic content. (57-8)[46]

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[44] "Wie immer das Ziel aber sei: die dem Urlinie-Zug und den Zügen der ersten Schicht anhaftenden Eigenschaften bleiben dieselben. Daher bedeutet auch in den späteren Schichten ein Zug vor allem das Hauptmittel einer Inhaltsbeschaffung in Durchgängen, das ist der Beschaffung eines melodischen Inhalts." (118-9) I have altered Oster's translation (73) to reflect Schenker's use of "Urlinie-Zug" rather than simply "Urlinie."
[45] "ist doch der Kopfton des Zuges so lang fortzutragen, bis der Endton erscheint." (11)
[46] "Bei aller stets als unabhängig zu verstehenden Körperlichkeit der im strengen Satze möglichen Intervalle enthüllt sich somit bei der Urerscheinung des dissonanten

He reiterates this point also in *Freie Satz*:

> In all linear progressions, whether descending or ascending, the principle of the primary tone holds: coherence is achieved through the mental retention of the primary tone. Each previous level vouches for the succeeding one, thus guaranteeing the indivisibility at the later levels so that unity prevails in the foreground as well as in the background.[47]

I'll invoke Schenker's ideas about mental retention and tension spans in the development of the MOP model of prolongation below.

Thus, it's clear that Schenker considered the dissonant passing tone to be the model of prolongation in general. The passing tone, furthermore, fits the dynamic sense of prolongation: it is an expansion of a motion from one note to another. Therefore, if we adopt Schenker's view that prolongations are applied extensions of the law of the dissonant passing tone, we should understand "prolongation" in the dynamic sense.

The static sense of melodic prolongation, on the other hand, fails to adequately describe the passing tone, because in the passing tone the dissonant event must relate to both the preceding and following melodic events. Relating the dissonant event to only a single, preceding or following, melodic event would be inadequate. To describe the situation in terms of static prolongation, then, one would have to say that the passing tone prolongs an *interval* and that interval is a particular event. That the passing tone prolongs an interval is perfectly accurate; yet that interval is melodic, so considering it as an event

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Durchganges gleichwohl schon ein seltsamer Einschlag von Vorgestelltem: er besteht in der geheimnisvoll wirkenden Erinnerung an den konsonanten Ausgangspunkt, die den dissonanten Durchgang auf seinem Weg durch den Terzraum begleitet. Es ist, als würde die Dissonanz auch den Einschlag der Ausgangskonsonanz stets mit sich führen, und man begreift so aus tiefstem Grunde die Vorschrift des strengen Satzes, die von dissonanten Durchgang fordert, daß er durchaus nur im Sekondschritt und durchaus nur in derselben Richtung fortgehe.

Die Tragweite dieser Wirkung ist höchst bedeutsam:

Wir erkennen im dissonanten Durchgang den verläßlichsten, ja einzigen Träger des Melodischen überhaupt." (59)

[47] "In allen Zügen, fallenden wie steigenden, wirkt sich das *Gesetz des Kopftones* aus: Das Forttragen des Kopftones erwirkt den Zusammenhang. Jede rückliegende Schicht bürgt für die nachfolgende also auch für die Unteilbarkeit und Einheit der Züge in den späteren Schichten, so daß einheit im Vordergrund wie im Hintergrunde waltet." (119) I've altered Oster's translation (79) to make it read more smoothly.

itself would use the term "event" in quite a different sense than it is used in the case of the passing tone itself. This mixing of insoluble meanings of "event" makes formalization of this idea—and, consequently, a consistent understanding of it—difficult if not impossible. Therefore there appears to be no satisfactory way to model Schenker's concept of prolongation using the term in its static sense.

Some Conceptual Problems in Theories of Prolongation

Before embarking on explicit formalization, I'd like to touch on some problems that arise in the theories of prolongation discussed above. I'll deal with three such problems here. The first is one I have not yet mentioned because it transcends the dynamic/static distinction and therefore doesn't appear to have played a significant role in the historical interaction between the two concepts. This is the problematic notion of voice in Schenker's writings. The later two conundrums are those we saw emerging with the systematic application of the idea of dynamic prolongation in the first extended English language works on Schenkerian analysis. The first of these, which we saw rearing its head in both Katz's and Salzer's books (see "The Concept(s) of Prolongation" above), is how to characterize a prolonging event dynamically when it occurs at the beginning or end of a piece—that is, when no more structural event either precedes or follows it. The last problem, a related one, is how to characterize incomplete progressions dynamically.

The problem of voices is specific to Schenker's own conception of prolongation. As we have seen in the previous two sections, Schenker's concept of prolongation emerged out of his attempts to show the laws of strict counterpoint operating in free composition, and the idea eventually became closely tied to his analytical procedure of illustrating voice-leading strata. Thus, the idea of a "voice" is essential to Schenkerian prolongation. However, Schenker's idea of "horizontalization," or "unfolding" (*Auswicklung*) introduces confusion into the notion of a voice.

In the previous section, we found that linear progressions (*Züge*) are the model of prolongation in Schenker's theory. Schenker portrays linear progressions as unfoldings of harmonic intervals represented by the initial and final tones of the progression. He therefore views the initial and final tones of the progression as belonging to different voices. Yet, this seems to contradict the characterization of the linear progression as a kind of passing motion: by its definition (in strict counterpoint), the passing motion must occur within a single voice. If the intermediate notes of the linear progression constitute a voice-leading motion, we are left wondering exactly what voice they belong to.

Schenker's cavalier neglect of such obvious theoretic quandaries issuing from his assertions should encourage us to sympathize with his early interpreters, and shows why Schenker's ideas were never to maintain their integrity entirely as they assimilated into the broader world of tonal theory. In fact it's immediately apparent that the only possible way to formulate a consistent sense of "unfolding" is to view voices as relative with respect to voice-leading strata. That is, the set of voices is distinct at each prolongational level, and an event participates in a different voice at each level in which it occurs.

Fortunately, the unfolding concept is not essential to the prolongation concept. Therefore my approach in the formal modeling will be simply to ignore unfolding for the purpose of defining prolongational structures and to address separately the question of how to represent unfoldings. See "Refinements of the MOP Model" below.

The next problem I would like to touch on here is that of describing on the dynamic model the circumstance of a prolonging event that has no more structural event either preceding or following it. Katz's solution to this problem in the case of an initial prolonging event—discussed above in "The Concept(s) of Prolongation"—was to appeal to Schenker's idea of a prolongation as a delaying or forestalling (*Aufhaltung*). Thus, if a prolongation of a motion from event X to event Y is a delaying of the progression from X to Y, then we can similarly regard the delay of the entry of an initial structural event as a kind of prolonging.

To make this more precise, consider the circumstance of an autonomous section of a larger piece, such as the trio of a minuet and trio, that begins with a relatively non-structural event. This event delays the first structural event of the trio, so that in the context of the entire movement seen as an integrated piece, it delays the progression of the structural conclusion of the minuet to the first structural event of the trio. However, because the minuet itself has a structural conclusion, the initiation of the trio is not necessitated by anything that precedes it. It is the initiation of the trio itself—i. e. the fact that the trio begins—, then, that brings about the necessity of the trio's first structural event, the event to which the initial prolonging event progresses.

In a sense, then the initial prolonging event of the trio prolongs the *initiation* of the trio to its first structural event. In other words, we can see the initiation of the trio itself as an event, and say that an event "prolongs a motion from the initiation of the piece" when it is an initial event at some structural level. Exactly which structural level is determined by the nature of the goal of this prolonged motion. In particular, if that event is the most structural of the entire piece, then it prolongs the motion of the initiation of the piece to its termination—where "the termination of the piece" is an event analogous to "the initiation of the piece."

When the trio is contextualized in the minuet and trio, the structural conclusion of the minuet adopts the place of the "initiation" event. Therefore, the "initiation" and "termination" events can be seen as stand-ins for the context of a musical passage. That is, when we analyze an autonomous passage independently of the larger piece of which it is a part, the initiation and termination events take the place of the context created by the entire piece. It is the autonomy of the passage that makes this semantically possible: the initiation of motions at all levels within the passage are not necessitated by events lying outside of the passage, so the initiation and termination events can take the place of this context without affecting the analysis of the passage itself. One way to think of the initiation and termination events for a complete piece, then, is to consider that the piece has the potential to be integrated into a yet larger context (even if it is not) and these formal events act as place-holders for that context.

Thus, in the formal model for dynamic prolongation below, every analysis has initiation and termination events as its most structural events. Saying that an event X prolongs the motion from the initiation event to another event Y is equivalent to saying that X is initial at some structural level or that it delays the entry of Y, and similarly for the termination event. If an event X prolongs the motion from the initiation to the termination, then it is the most structural event of the piece. That is, there is some highest level of reduction at which X, initiation, and termination are the only events.

Finally, let's consider the problem of incomplete progressions. It is this problem that perhaps persuaded Salzer to reject Katz's semantic resolution of the initial/terminal prolongation problem and provisionally adopt a static usage of the term prolongation.

In particular, consider the problem in terms of the melodic example that motivated Salzer's deliberation on it (shown in figure 1.3 and discussed in "The Concept(s) of Prolongation" above). This is reproduced again in figure 1.4 for reference. While the problem of the initial E of the melody being a prolonging event could be solved by saying that it functions as a delay of that event, this impairs the representation of the obvious parallelism of the passage shown in Salzer's analysis because it seems to represent the prolonging function of the initial E in a slightly different way than that of the D of the first measure and the C of the second measure.[48] Furthermore, though it's true that the D of the first measure, for example, prolongs the progression of G to F, it relates to F in a more direct way than it relates to G.

FIGURE 1.4: SALZER'S ANALYSIS OF A MELODY FROM SCHUMANN'S ALBUM FOR THE YOUNG

Such relationships must be understood on the dynamic model of prolongation as incomplete progressions. This is a generalization of the idea of an incomplete neighbor. That is, it's a motion that must be understood as having an elided origin or goal. This fits well with the Schenkerian idea of prolongation, which views the passing tone as the basic

[48] Salzer actually takes this from measures 4-8 of the *Stückchen* of Schumann's *Album for the Young*, but it's nearly the same as measures 0-4 and I will simply refer to it as if it is the beginning of the piece.

model of prolongation, and the neighbor note as the simplest transformation of it. Passing and neighboring tone figures include both an origin and goal. The incomplete neighbor figure, being less fundamental, is a neighboring or passing motion whose origin or goal has been left out.

In Salzer's example, the incomplete progressions are arpeggiations from F to D and E to C. That is, the G of measure 1 descends by step to F, and F is prolonged by arpeggiation to D. However, the F preceding D is missing; we must infer it from the F that follows. One way to understand this might be to view the prolonging D as a motion from an imaginary event. Such a way of thinking cannot take us very far though, because the literal motion it tries to mitigate, the motion from G to D in this case, may itself be prolonged. In the Schumann melody the G-D is in fact prolonged—by an incomplete neighboring motion. In other cases, such a motion, at the "loose end" of an incomplete progression, might be prolonged by a more basic passing motion. Therefore, the G and D in question, and other similarly juxtaposed events, cannot be completely unrelated; they constitute, in some sense, a motion.

A more successful solution is to assert that the elided event leaves an empty space, a vacuum, if you will, that is filled by the nearest structural event. Thus, in Salzer's example, the G fulfills the function left vacant by the elided F of the arpeggiation to D; it serves as the origin of this arpeggiating motion. In other words, the G event substitutes for F as the head of this prolongation. Thus, it is true that "D prolongs the motion from G to F," but one could also say more precisely that "D prolongs the motion from G to F as an incomplete arpeggiation to F," or "D prolongs the motion from G (as a substitute for an elided F) to F," emphasizing the asymmetrical nature of the prolongation.

Recognition of the asymmetry of dynamic prolongations in Salzer's example solves the parallelism problem. In particular, according to my explanation of initial prolonging events above, the E of the anacrusis is a prolongation of the motion from the initiation of the melody to the first structural event, the G of measure 1. This is an incomplete progression, as it must always be for such initial prolongations since the initiation event is always a stand-in for a musical context. In this case the initiation event substitutes for a G event in this prolongation, defining the note E of the anacrusis as prolongation of G by arpeggiation. The situation is exactly the same for D in measure 1 and C in measure 2, restoring the sense of parallelism to the passage. In fact, the parallelism is now enriched: we see that the prolongation of D in measures 3-4 is in fact the same arpeggiation observed in the previous prolongations but now in complete form.

I will invoke this discussion in development of the formal model below. As the model takes shape, it will be possible to see more clearly why I have resolved these semantic problems in just the way I have, although the less explicitly formal explanations here are self-sufficient.

Graphs and Digraphs as Analytical Models

Existing formal models of prolongation, which I'll discuss in part two, formalize what I've called the static sense of melodic and/or harmonic prolongation. (See "The Concept(s) of Prolongation" above). In this section, I'll develop a contrasting formal model for dynamic prolongation that I call the MOP model (for reasons that will soon become clear). It will be useful to first have some basic graph theory terminology on the table right away. I will explain the necessary terminology now, but save the more mathematical definitions for part four of the paper.

"MOP" stands for "maximal outerplanar graph," which is a particular kind of *undirected graph*. A graph is a set of objects, called *vertices*, and a set of relationships between the objects, called *edges*. (The terms "graph" and "vertices" are interchangeable with "networks" and "nodes" respectively, although "networks" are usually directed).

In our music-analytic application, the set of objects, the vertices, will be events in a piece of music. Of course, the term "events" is general enough to allow for many interpretations. For now, we will focus on the simple and important case where each event is a note in a melody, and the graph is a prolongational analysis of that melody. However, it will be worth keeping in mind other possible interpretations of the term events: an event can be a simultaneity of notes, for instance, or it can be a harmony. The reader can probably imagine all kinds of other interpretations of the term "events." However, it's important that an event is something that can be located temporally in a particular piece of music. It is indeed possible also to use graphs to represent relationships between abstract musical objects that linger in the musical background like platonic ideals—for example "the pitch-class G" in general rather than "this particular G"—but I will not regard these as "events" and not address this particular analytical application of graphs.

An edge in a graph always relates two vertices. We say that these vertices are the *endpoints* of the edge, and they are *incident* upon the edge. In addition, if two vertices share an edge in a graph, we say that they are *adjacent* in that graph. (Because "adjacent" has this technical meaning, I will avoid using it otherwise).

In an undirected graph, I define the edge set as a subset of the set of all vertex pairs. This way of defining the edge set has a few important implications. First, it's impossible for an edge to relate a vertex to itself. (In graph theory terminology the graph has no *loops*). Second, there can be no more than one edge between any two vertices. (The graph has no *multiple edges*). And finally, the edges are undirected. That is, if u and v are vertices of the graph, there is no difference between an edge "from u to v" and "from v to u." Unless I indicate otherwise, a "graph" is always an undirected graph.

These features distinguish an undirected graph from one that might be more familiar to some readers, the directed graph, or *digraph*. This construct is essentially the same as what David Lewin calls a "node-arrow system" in *Generalized Musical Intervals and Transformations*. (See the next section, "David Lewin's Node-Arrow Systems"). In fact, Lewin's analytical application of the node-arrow system is quite similar to the application I have suggested for graphs, in spite of the fact that I have chosen to revert to the mathematically more standard terminology.

An edge of a digraph is a two-member list of elements of the vertex set. Therefore, an edge can go from a vertex to itself, there can be two edges between any particular pair of vertices, and an edge has an inherent direction.

The distinctions between graphs and digraphs have important semantic implications. In a MOP defined as an undirected graph, the edges represent inherently undirectional relationships between musical events. These relationships are undirectional simply because of their generality: an edge in the MOP indicates that two events bear a direct relationship to one another. The edges also can (more usefully) represent more specific relationships that are directional in nature; for example, an edge from u to v can represent "v is a prolongation of an interval involving u,"[49] "v is a more foreground event than u," "u is the initial and v is the final event of a prolongational span," or "u directly precedes v in some reduction of the music." All of these imply an orientation to the edges of the MOP, making it into a digraph. However, while the first and second produce the same orientations and the third and fourth also produce the same orientations (ones that always point from an earlier to a later event), the orientations of the first two are different from those of the second two. Eventually it will be best to think of the MOP analysis as a sort of doubly oriented graph (and, in fact, my representations of MOP analyses always reflect this in the horizontal and vertical placement of vertices). These orientations will be demonstrated in the second part (in the sections "Combinatorial Comparison of MOPs and Binary Phrase-Structure Trees" and "Comparisons of MOPs and Phrase-Structure Trees via Backgroundness Partial Orderings") For the time being, however, it is simpler to develop MOPs as undirected graphs and save the orienting of the graph for later.

David Lewin's Node-Arrow Systems

The only difference between Lewin's node-arrow system and the general concept of a digraph is that in *Generalized Musical Intervals and Transformations* Lewin stipulates that a node-arrow system must have loops on all of its nodes. It is not entirely clear why Lewin adds this condition to the definition of a node-arrow system, especially considering that it was not included in the earlier formulation of a node-arrow system in "Transformational Techniques in Atonal and Other Music Theories." Also, Lewin adds this to the definition with the qualification, "for present purposes," suggesting that he only put it there to simplify some later definition or proof.

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[49] Note that I often use the term "interval" to mean a *particular* melodic interval. That is, I don't take the term "interval" to imply harmonic or vertical, nor do I use it only in the abstract sense of a general distance between pitches.

I suspect that the reason for Lewin's addition of the loop condition was to reduce the possible types of homomorphism between node-arrow systems. With the loop condition, an onto mapping of arrows to arrows implies an onto mapping of nodes to nodes (because if there's a node x in the second node-arrow system with no corresponding node in the first, then there's a loop in the second system, (x, x), that cannot have a corresponding arrow in the first system). Without it, there could be a node x in the second system that participates in no arrows, so that an onto mapping of arrows is possible even if there is no node in the first system corresponding to x. More importantly, with the loop condition an injective mapping of arrows implies an injective mapping of nodes—by contrapositive, if $x \to z$ and $y \to z$ then $(x, x) \to (z, z)$ and $(y, y) \to (z, z)$. Therefore Lewin doesn't need to worry about a special kind of homomorphism where a non-injective mapping of nodes produces an injective mapping of arrows. (Note also that simply eliminating loops entirely doesn't accomplish this).

The loop condition has interesting repercussions when the semigroup of the transformational graph has no identity element. Consider, for example, the set of multiplicative pitch-class transformations $\{M_0, M_3, M_4, M_6, M_8, M_9\}$ acting on pitch-classes. This set is closed under composition despite the exclusion of the identity M_1, so it is indeed makes a semigroup under composition. Without the loop condition on node-arrow systems, there is nothing especially wrong with the graph in figure 1.5, although its rightmost node is forced to have the contents 0. However, given the loop condition, this graph is impossible, because there is no possible label for the loop on the node in the middle. If we expand the semigroup to include an identity M_1, then the graph in figure 1.5 is ambiguous: the loop on the leftmost node can be labeled by either M_3 or M_1 and the loop on the rightmost node can be labeled by M_8 or M_1. These labels are not trivial: for

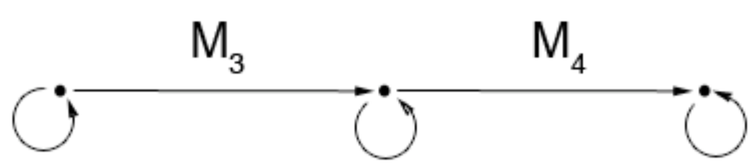

FIGURE 1.5: AN IMPOSSIBLE NODE-ARROW SYSTEM

instance, if M_1 is the label on the leftmost node then this node can have any pitch-class as its contents, but if M_3 is its label it can only contain pitch-classes 0, 3, 6, or 9.

To give a fuller comprehension of the situations that arise with Lewin transformation networks, I offer to the reader figure 1.6 as an exercise. The semigroup for this transformational graph is the set of integers {0, 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28} acting on the integers 0-29 by multiplication modulo 30, under the operation of composition. (This set of integers is just the integers from 0-29 that are not coprime to 30, excluding 1). There are three parts to the exercise: first, I claim that there is only one possible labeling of the loops for each node. What are they? Second, I claim that there are only fifteen possible ways to add contents to this graph: what are they?

The third part of the exercise is to give a music theoretic interpretation of the graph. I will do this part myself: first, consider the isomorphism of $\mathbf{Z}_5 \times \mathbf{Z}_3 \times \mathbf{Z}_2 \rightarrow \mathbf{Z}_{30}$ given by (a, b, c) → (6a + 10b + 15c) mod 30. This changes the labels of figure 1.6 to those of figure 1.7. I leave it to the reader to convince herself that this is an isomorphism under the multiplicative operation, defined on the direct product group as
(a, b, c) • (d, e, f) = (ad, be, cf).[50]

The contents of the nodes are now ordered triples to which I give the following interpretation: each object refers to a note of a jig in a pentatonic scale, (G, A, B, D, E) ≡ (0, 1, 2, 3, 4), written in 6/8. The first number in the triple gives the pitch-class of the note while the second gives its eighth-note metrical value: 0 for on-beat, 1 for an eighth-note after the beat, and 2 for two eighth-notes after the beat, and the third gives its beat value: 0 for the first half of the measure and 1 for the second. As a multiplicative transformation, a zero in the first place turns all notes into G, a one retains the pitch, a four inverts the scale around G (exchanging A with E, and B with D), and two and three permute the pitch-classes A, B, D, and E cyclically and leave G alone. In the second place, a zero puts every note on the beat, one is the identity, and two exchanges the first

[50] Or the reader could just trust me on this. If some of the mathematical jargon here is unfamiliar, a good reference is the first chapter of David Dummit and Richard Foote, *Abstract Algebra*.

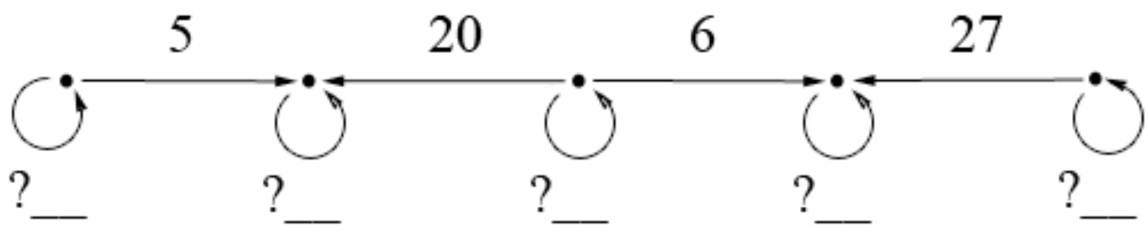

FIGURE 1.6: A NODE-ARROW SYSTEM WITH ONLY ONE POSSIBLE LABELING FOR ITS LOOPS

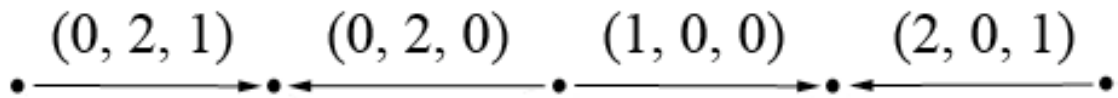

FIGURE 1.7: A NODE-ARROW SYSTEM ISOMORPHIC TO THAT OF FIGURE 1.6

and second offbeat eighth-notes. In the third place, one is the identity and zero moves all notes to the first half of the measure.

An example analysis using the network of figure 1.7 is given in figure 1.8. This is the first two measures of a traditional Irish jig called Christie Barry's #2. As I have formulated the transformational graph, the contents of figure 1.8 are actually "illegal," because three of the nodes have contents that are non-trivially transformed by the semigroup element on the loop for that node. Therefore, we have to imagine the graph

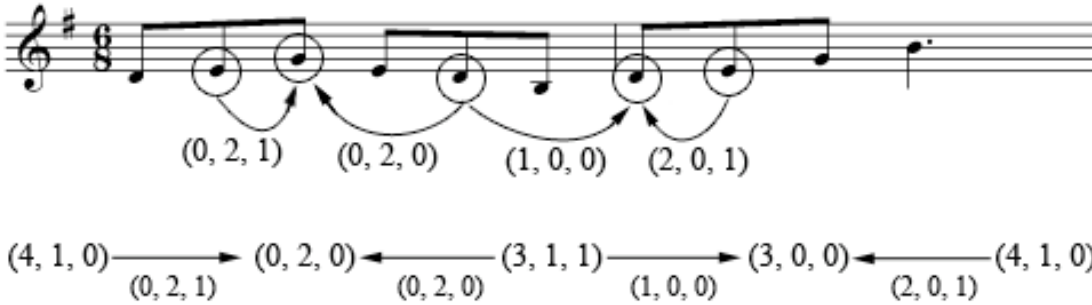

FIGURE 1.8: AN ANALYTICAL APPLICATION OF THE NODE-ARROW SYSTEM OF FIGURE 1.7 TO THE FIRST TWO MEASURES OF "CHRISTIE BARRY'S JIG #2"

either without Lewin's loop condition on node-arrow systems or with a semigroup that includes the identity. (Note that the choice between these doesn't appreciably change the semantics of the analysis).

I mainly intend this analysis as an illustration that the transformational graph of figure 1.7 can in fact reflect musical intuitions, not that the particular intuitions of figure 1.8 are especially interesting. However, it is interesting in any case to see how such transformational graphs work. Because the semigroup is restricted to elements of $\mathbf{Z}_5 \times \mathbf{Z}_3 \times \mathbf{Z}_2$ that include at least one zero (excepting, perhaps, the identity element), every arrow reflects some kind of "normalization," either metrical or tonal. We can think of an arrow pointing forwards in time as a stabilization and an arrow pointing backwards as a destabilization. The first E going to G shows a tonal stabilization accompanying a metrical transformation at the eighth-note level. The G then destabilizes in pitch to D and metrically by moving to the second half of the measure. This D then stabilizes metrically, moving to the first beat of the measure, and so forth. I think the reader can imagine how such semigroup transformations could be quite useful in reflecting intuitions about tonal music in a way that group operations fall short.[51] (A funny

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[51] For the reader interested in pursuing these matters further, I offer the following suggestions. A simple representation of pitch-class in a tonal context is given by $\mathbf{Z}_3 \times \mathbf{Z}_3$ where (0, 0) refers to the tonic note, (1, 0) to the tonic third, and (2, 0) to the tonic fifth. The second place can then represent stepwise displacements of these: 1 for upward and 2 for downward. The reader should work out how the elements of this set act on such diatonic pitch-class representations as multiplicative operations. Various elaborations of this simple semigroup are possible by simply extending the direct product: for example $\mathbf{Z}_3 \times (\mathbf{Z}_3 \times \mathbf{Z}_3)$ might be useful with the elements of the first $\mathbf{Z}_3$ representing tonic, dominant, and subdominant harmonies. Or, we could use $\mathbf{Z}_3 \times (\mathbf{Z}_3 \times \mathbf{Z}_2)$ with the two $\mathbf{Z}_3$'s interpreted as in the simple case, and the $\mathbf{Z}_2$ indicating diatonic neighbors for 0 and chromatic neighbors for 1. (These all have interesting semantic repercussions when they give multiple representations to the same pitch-class).

We could combine such multiplicative semigroups with additive groups if we extend Lewin's transformational graphs to "algebraic graphs" by replacing the semigroup of the definition of a transformational graph with a ring. (For definitions of and information about rings, see Dummit and Foote's *Abstract Algebra*, chapter 7). However, this recommendation comes with a warning that devising a ring that reflects intuitions about pitch relationships in tonal music is not quite as straightforward as it may seem!

consequence of the combination of pitch-class and metrical transformations is that it is impossible for a transformation to simultaneously destabilize a note metrically and stabilize it tonally, or vice versa).

This example illustrates a few points about transformational networks. First, transformational networks using semigroups that are not groups have potential for musically interesting analytical applications. Second, the only reason to label a loop of a transformational graph with a non-identity element is to restrict the contents of the node (where there is presumably some general analytical reason for doing so). Third, there is generally no especially compelling semantic reason to exclude the identity from the semigroup of a transformational network. In the example, the exclusion of the identity transformation from the semigroup means, for one thing, that every note analyzed either has to be a G, on a beat, or in the first half of the measure (making the D on the fifth eighth of the first measure of the jig impossible contents for any node of the network). Beyond that, every note analyzed must have a zero in at least one place in common with the labels on every arrow it participates in. (This is the problem with the E's on the second eighths of the first two measures as node contents). It's difficult to think of a reason why one would want to impose such a restriction as a general rule.

Thus, these odd repercussions of the loop condition on node-arrow systems when the semigroup of the transformational graph lacks an identity element certainly do not constitute a reason for its inclusion. If Lewin had left out the condition, there would be no obvious analytic reason for adding a loop labeled with the identity transformation to any node of a transformational graph. While there may be a reason for adding a loop labeled by a non-identity element in some cases where the semigroup of the graph is not a group, when the loop condition on node-arrow systems forces every node to have a loop, labeling one of these loops with a non-identity element serves the same semantic function.

A more fundamental problem that arises with Lewin's use of directed graphs is the lack of flexibility in orienting them. As I suggested above, the advantage of developing a model of prolongation with undirected graphs is that it is easier to add orientations to an undirected graph than it is to change the orientations of a digraph. In music theory applications, the orientations of edges have a great deal of semantic importance. However, in Lewin's system, especially in its application to tonal music, a great amount of difficulty comes about from the fact that there are multiple ways that the orientation of an edge can be meaningful. For example, an arrow labeled DOM from x to y properly means "x is a dominant of y." This is different from the meaning of an arrow labeled SUBDOM from y to x, which means "y is a subdominant of x," even though x and y can often have the same contents in either case. If x, y, and z are nodes with the contents C major, G major, and C major respectively, then it is impossible to have arrows from x to y to z reflecting the order of events and modeling the intuition that x-y-z represents a I-V-I progression in C major. This is because the only choice of label for x → y is SUBDOM, which means something quite different. The problem is that when an arrow is used to ascribe tonicity to the event at its head, it cannot independently reflect the order of events (unless "more tonic" events always follow "less tonic" events, which obviously is not always the case).

Lewin makes various attempts at incorporating temporal information into transformational networks in section 9.7.6 of *Generalized Musical Intervals and Transformations*. I find none of these especially satisfying next to the comparatively simple solution of allowing multiple orientations of the same undirected graph, one of which can represent temporal information while the other represents relative structural weight.

Maximal Outerplanar Graphs

So much for the "graph" part of "maximal outerplanar graph," but what about the "maximal outerplanar" part? Given a set of musical events, there are many different ways of building a graph out of them. Saying that an analysis of prolongation consists in any way of relating events pairwise wouldn't give much of a useful meaning to the term prolongation. Therefore, for a graph to represent the prolongational relationships between events, there must be a more particular definition of what qualifies as a graph of prolongational relationships. It seems to me that the most intuitively appealing and useful such definition identifies prolongational relationships between events with a particular class of graph, which is the maximal outerplanar graph.

Unfortunately, defining this class of graphs as “maximal outerplanar” is not the most musically enlightening way to define them. The last two sections of the paper describe a number of other ways of circumscribing the class of MOPs that tell us something musically relevant about the graphs. I have chosen MOP as the general term for the class because it is mathematically the most obvious way of defining it and consequently is the name that this type of graph goes by in the existing literature of graph theory.

It will be easiest to understand the meaning of “maximal outerplanar” in the context of an example, so I beg the reader’s patience concerning its definition for the moment as I construct an example.

The simplest and also most important type of prolongational relationship is that between notes of a melody. For this reason, a fugue subject will provide an ideal example for development of the theory, since it is relatively short, monophonic, and plays an important role in establishing the tonality of a piece. In the third section of the paper, I’ll show how to expand the use of graphs as representations of prolongation beyond single melodies, giving them a more general analytical applicability.

Consider the subject of the C major fugue from book II of the WTC, shown in figure 1.9. Below the music is an analysis taken from William Renwick’s insightful book, *Analyzing Fugue*. (116) I will construct a MOP that interprets Renwick’s analysis in terms of dynamic prolongation. Of course, the analysis leaves it up to the reader to figure out what to do with the many notes that it excludes: this is not difficult to do, but to simplify matters I will deal with just the reduced melody in reconstructing the analysis as a MOP.

As another preliminary note, there is one obvious aspect of the analysis that I will ignore: that is the C, the last eighth-note of the first measure, which Renwick presents as part of an independent line. This note is very important: it immediately orients the listener to the tonality of C major. Also, Renwick’s analysis of this note, I think all will agree, is a good one: this C gives a fleeting aspect of compound melody to the subject,

Figure 1.9: The subject of Bach's C major fugue (WTC 2) and Renwick's analysis

serving more of a harmonic than a linear function. However, because the C is part of a different voice according to the analysis, it doesn't properly participate in the prolongational relationships of the main line considered by itself (although it does help us to hear those prolongational relationships).

As I said above, the vertex set of the graph consists of all events of the passage being analyzed. I'll call these events by their pitch-class names (since there is no danger of confusion concerning the register of notes), and pre-index them according to what measure they occur in (to distinguish the various G's, E's and F's). The sequence of events in the analysis, then, is ${}_1$G, ${}_2$A, ${}_2$G, ${}_3$F, ${}_3$E, ${}_4$D, ${}_4$G, ${}_4$F, ${}_5$E. We can write an edge of the graph by simply concatenating the names of the edges—e. g., ${}_1$G-${}_5$E might be an edge. (The dash here is only to make it easy to read.) If we think of the graph as undirected then ${}_5$E-${}_1$G (for example) indicates the same edge as ${}_1$G-${}_5$E. However, I will generally write the edge names as if the graph is directed by temporal precedence. (So ${}_1$G-${}_5$E could be an edge but ${}_5$E-${}_1$G cannot). The graph will also include vertices to represent the initiation and termination events (described in "Some Conceptual Problems in Theories of Prolongation" above), which I will denote $\bigcirc i$ and $\bigcirc t$.

The edge set of the graph includes all event pairs that define the boundaries of a prolongational span. By "prolongational span," I mean something analogous to the span of a linear progression in Schenker's theory—i. e., where a melodic analysis identifies a linear progression, the initial and final notes of the progression should make an edge representing that linear progression. The edge set of a MOP generalizes this: a prolongational span could indicate the retention of a note (e. g., over a neighboring motion), a motion filled in by arpeggiation, or an incomplete motion where one of the events involved conceptually substitutes for a missing event (—see "Some Conceptual Problems in Theories of Prolongation" above).

A more precise way of indicating the meaning of an edge in the graph is to appeal to Schenker's idea of retention of the head-tone. (See "Prolongations as Passing Events" above). There is an edge between any two events that are in *direct succession* in the analysis, where that direct succession could either be literal or could occur through the mental retention of the initial note over a particular prolongational span. (In other words, a prolongational span is a period of time over which a single event is mentally retained, either because the event is a note that sounds over that span of time or because it initiates a motion whose continuation is delayed over that period of time, while other events may intervene). In other words, two events that make up an edge are consecutive in some reduction of the passage. Thus, in a melodic analysis, notes that are connected by a slur or beam in Schenkerian notation will be connected by an edge in the MOP representation of the analysis.

The initiation and termination events always form a prolongational span, ○*i*-○*t*. Semantically, this edge, ○*i*-○*t*, indicates that the passage under analysis constitutes a complete and coherent whole.

The edge set also will contain a set of edges that represent trivial prolongational spans, where a trivial prolongational span is one between two events that are literally consecutive in the sequence of events. I call them "trivial" because there is nothing in the sequence of events that could be said to be "prolonging" these motions. In our analysis, these edges are ${}_1$G-${}_2$A, ${}_2$A-${}_2$G, ${}_2$G-${}_3$F, ${}_3$F-${}_3$E, ${}_3$E-${}_4$D, ${}_4$D-${}_4$G, ${}_4$G-${}_4$F, and ${}_4$F-${}_5$E. As it turns out, all of these event-pairs are in fact prolonged in the music, since we are analyzing a reduction of the subject itself. However, these trivial relationships are necessary even if we are analyzing the musical surface, because the prolongational relationship is a generalization of the idea of the motion of one event to another governing a particular span of time in the music, which is literal in the case of trivial prolongations.

Furthermore, the more background prolongational relationships are built out of the more foreground ones, beginning with the trivial ones. Also among the trivial relationships we should include $\bigcirc i$-G_1 and ${}_5E$-$\bigcirc t$, which say that ${}_1G$ is the first note and ${}_5E$ the last note of the passage.

Figure 1.10 shows the edges so far included in the graph's edge set (where an edge is represented by a line connecting the incident vertices of the edge). This type of graph is called a *cycle-graph*, for obvious reasons (I will save the more formal definition for later). Note that the positioning of the vertices in the drawing is completely arbitrary as far as the graph (as an undirected graph) is concerned. Yet, I have drawn them so that horizontal position corresponds to melodic order and no two lines in the drawing cross one another. A graph that can be drawn without intersecting lines in this way is called *planar*. (Not to be confused with the use of the term "plane" in "plane tree," a term I will use in part two).

Cycle-graphs are the basis of MOP representations of musical phrases, understanding the term "phrase" here very generally to mean a sequence of events that can be heard as consecutive and as a whole make up a single coherent motion. Thus, the cycle-graph consists of a series of edges relating consecutive events, plus one edge relating the initial event to the final event representing the complete prolongational span of the sequence. So far, then, our graph asserts that the passage under analysis makes up a complete phrase, but fails to identify any internal relationships between these events.

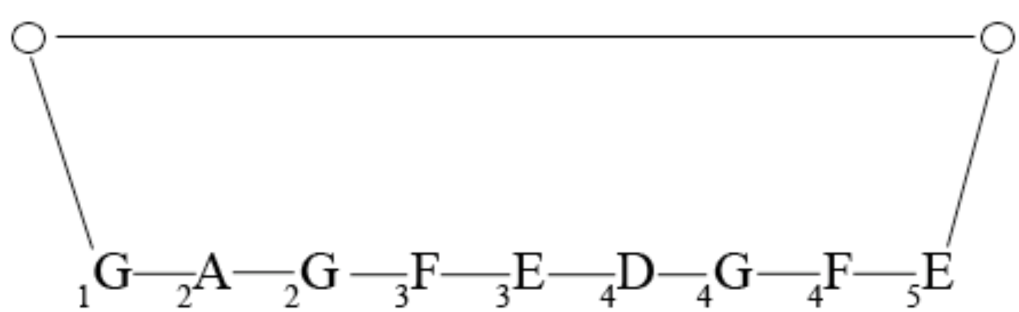

FIGURE 1.10: A CYCLE REPRESENTING THE REDUCED MELODY OF THE FUGUE SUBJECT

Yet, there is one important thing missing from the cycle-graph of figure 1.10 as a representation of a musical phrase: as a mathematical object, none of the edges of the cycle are distinguishable from any others. Thus, the cycle-graph by itself doesn't tell us which edge represents the prolongational span and which represents an interval between consecutive events. Therefore, one edge of the graph must be distinguished as the edge of the prolongational span. I will call this the *root edge*. In this case, the root-edge of the cycle-graph is ○*i*-○*t* (as it always is, by convention, for a cycle that includes these vertices). Furthermore, we must have some way of telling which is the initial and which is the final event on the root edge (without reference to the names on the vertices). This is accomplished by assigning an orientation to the root-edge pointing from the initial to the final event. Thus, the MOP representation of a musical phrase is an *oriented-edge rooted cycle-subgraph* of the MOP.

Fortunately, the definition of a *cycle* allows us to avoid the somewhat cumbersome construction of an oriented-edge rooted cycle-subgraph. The usual definition of a cycle follows naturally from the idea of a *paths*.[52] Imagine that you're traveling from one musical event to another in the fugue subject in such a way that each two consecutive events you visit share an edge in figure 1.10. If you never visit the same vertex twice, the sequence of vertices that results is called a *path*. For instance, $_1$G-○*i*-○*t*-$_5$E-$_4$F-$_4$G-$_4$D is a path in figure 1.10. By writing out the names of the events on the path in order, we can see both the vertices and the edges included in the path. For instance, the edges of $_1$G-○*i*-○*t*-$_5$E-$_4$F-$_4$G-$_4$D are $_1$G-○*i*, ○*i*-○*t*, ○*t*-$_5$E, $_5$E-$_4$F, $_4$F-$_4$G, and $_4$G-$_4$D.

A *cycle* is a path that includes an edge between its initial and final vertices. One cycle of figure 1.10 is given by the sequence ○*i*, $_1$G, $_2$A, $_2$G, $_3$F, $_3$E, $_4$D, $_4$G, $_4$F, $_4$E, ○*t*. To distinguish this notationally from the path ○*i*-$_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_4$E-○*t*, I will indicate the final edge with a closed bracket: ○*i*-$_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_4$E-○*t*]. A cycle has the same number of vertices and edges, unlike a path whose number of edges is one fewer than the number of vertices. Also, every cycle-graph of n vertices actually

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[52] See Brandstädt, Le, and Spinrad 1999 (2).

includes 2n different cycles, because you can in principle start and end on any of the n events and there are two possible directions to travel around the cycle.

Although this definition of cycle is standard, most of the mathematical literature fails to draw such a careful distinction between "cycles" and "cycle-graphs" or "cycle-subgraphs." In the case of the MOP model of prolongation, the semantic importance of the distinction makes it worth the terminological nit-picking. A cycle of a graph technically designates not only a cycle-subgraph, but more specifically an oriented-edge rooted cycle-subgraph, because the edge from the initial to the final vertex of the cycle can be interpreted as an oriented root edge.

The edges in the graph of figure 1.10 tell us only the order of events in the passage; they don't say anything about the analysis. To complete the graph, we'll add edges to represent each prolongational span in the analysis. It is important to recognize that the goal here is simply to represent the shape of the music as implied by a particular analysis (in this case, the analysis of figure 1.9). The reader may be accustomed to thinking of formalizations as prescriptive of analysis, as attempts to "prove" that a certain analysis is the correct one. But this sort of prescriptiveness is not at all an inherent property of formal modeling. In the present case, I don't think figure 1.9 gives the only, or even the best, possible analysis of the fugue subject. However, I would like to show that some of its most basic properties can be represented in a mathematical construction. This construction is quite general: in its essential form it doesn't even make any necessary reference to the nature of the events (their pitch, metrical placement, et c.) that label the vertices of the graph.

I'll show two ways to approach the completion of the MOP analysis of the fugue subject: a top-down ("synthetic") and a bottom-up ("analytic") method.

The top-down method proceeds as follows: first we find the most background event of the passage, the one that can represent the passage in a wider context. Of Renwick's beamed notes (in figure 1.9), the note G will function as a head-tone for the entire piece, so we choose ${}_1$G. This is then attached to the root edge of the basic cycle shown in figure 1.10 (○*i*-○*t*) by adding the edges ○*i*-${}_1$G and ${}_1$G-○*t*, as shown in the first stage of figure 1.11. (○*i*-${}_1$G is already in the cycle of figure 1.10). These edges say that

$_1$G is both an initial and final note of the passage (that is, it may remain after all other events of the passage have been reduced out), and the triangle that is formed, ○*i*-$_1$G-○*t*], indicates a prolongation; in this case $_1$G prolongs the motion from initiation to termination. (This is all redundant information, of course).

Of the new prolongational spans created by choosing $_1$G as the prolongation of ○*i*-○*t*, ○*i*-$_1$G is trivial (meaning that there are no events preceding $_1$G), so we look for an event to prolong $_1$G-○*t*. In other words, we're looking for a note that's terminal and completes a motion from $_1$G. Such a motion is shown by the beams in figure 1.9, and is completed by $_5$E. Therefore $_1$G-$_5$E is an edge representing that third-progression, $_5$E-○*t* (already in the cycle of trivial prolongations) indicates that $_5$E is terminal, and the triangle $_1$G-$_5$E-○*t*] represents the prolongation of $_1$G-○*t* by $_5$E—that is, $_5$E delays the termination of a phrase dominated by the event $_1$G. This is shown in the second stage of figure 1.11.

Again, of the new prolongational spans created, only one, $_1$G-$_5$E is non-trivial. The event that most directly prolongs this motion is the passing tone $_3$F, as shown by the beams in Renwick's analysis. So we add the edges $_1$G-$_3$F and $_3$F-$_5$E to the graph, as shown in the third stage of figure 1.11. Because both $_1$G-$_3$F and $_3$F-$_5$E are non-trivial, this creates two smaller cycles that, along with the triangle $_1$G-$_3$F-$_5$E] (which represents the prolongation of $_1$G-$_5$E by $_3$F that we have just identified), split up the larger cycle $_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_4$E]. These are $_1$G-$_2$A-$_2$G-$_3$F] and $_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_5$E].

These smaller cycles can be considered as melodic phrases, just as the overall event sequence. Thus, we analyze them in the same way. Renwick shows with his stems that he considers $_2$A to be the most direct prolongation of $_1$G-$_3$F. It isn't clear from figure 1.9 whether $_2$A is simply an incomplete upper neighbor to $_1$G or a subdominant arpeggiation to $_3$F, but this distinction is inconsequential to the resulting structure of the MOP. The ambiguity of the prolongation of $_3$F-$_5$E is more problematic, because, since Renwick stems both $_4$D and $_4$G it's unclear which of these is the more fundamental prolongation of $_3$F-$_5$E. The choice of $_4$G would mean that $_4$D would have to be explained in terms of an arpeggiation, either from $_3$F or to $_4$G. Since Renwick indicates only tonic harmony in effect here, perhaps allowing $_4$D to appear as a lower neighbor to $_5$E, as in the

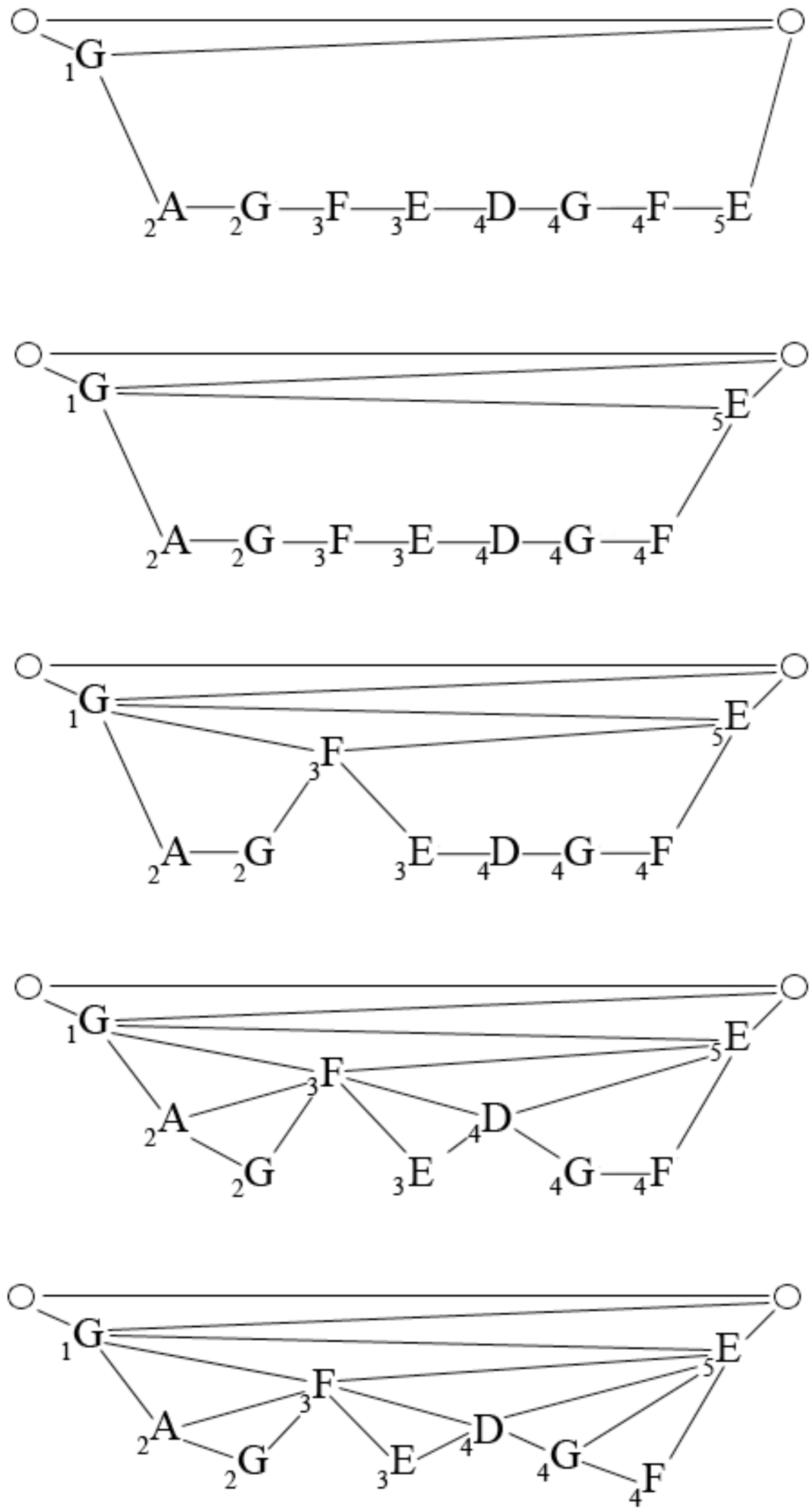

FIGURE 1.11: THE TOP-DOWN CONSTRUCTION OF A MOP

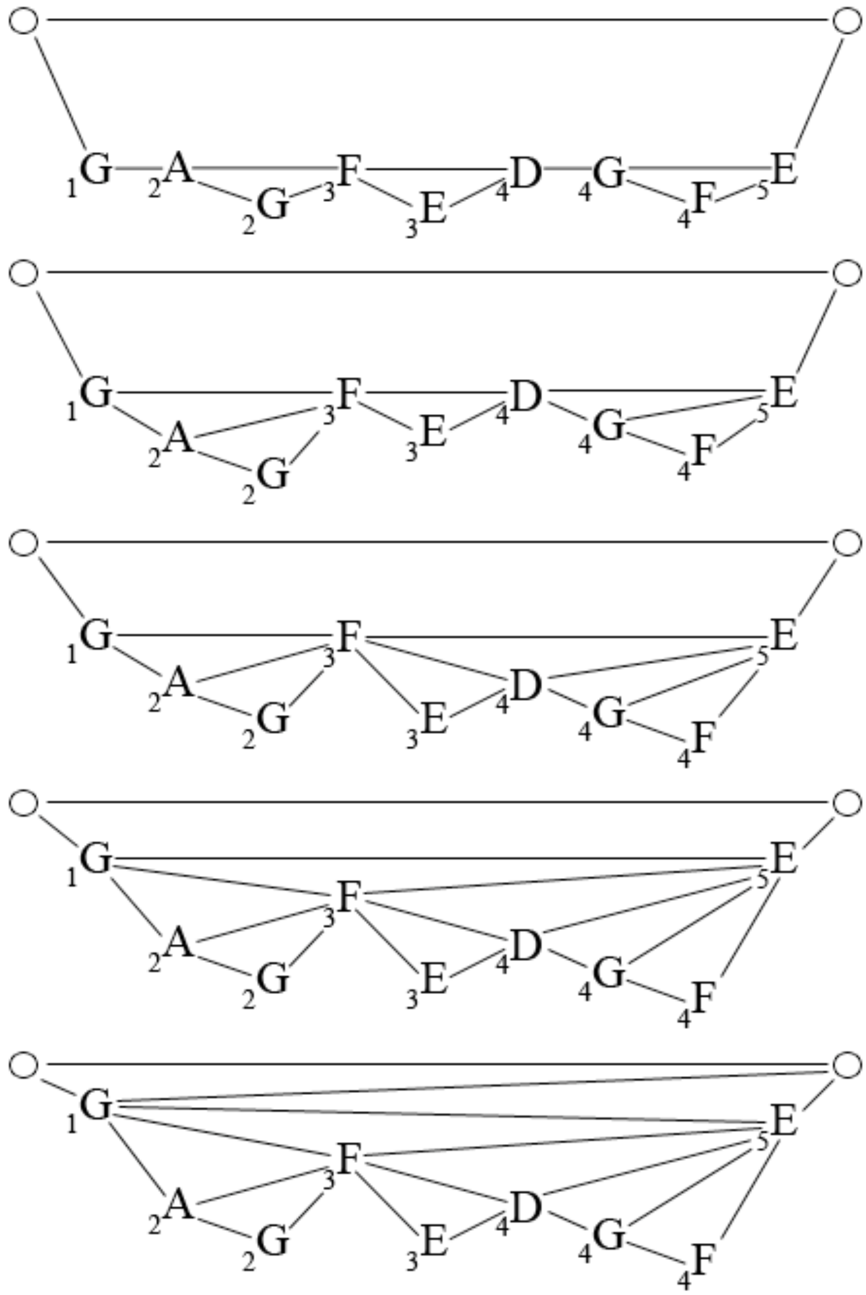

FIGURE 1.12: THE BOTTOM-UP CONSTRUCTION OF A MOP

fourth graph of figure 1.11, is a better choice. I'll discuss this further in the next section ("Notational Enrichments of MOPs").

Because we've represented the prolongations $_1$G-$_2$A-$_3$F] and $_3$F-$_4$D-$_5$E] as incomplete progressions in this analysis, the note $_3$F—according to the discussion in "Some Conceptual Problems in Theories of Prolongation" above—substitutes for $_1$G as the destination of $_2$A, and $_3$F substitutes for E as the origin of $_4$D. The edges $_2$A-$_3$F and $_3$F-$_4$D represent the prolongational spans created by these substitutions (see the fourth graph of figure 1.11). In the next section we'll supplement the notational system of MOPs to reflect the special nature of these prolongational spans, but for now our primary concern is simply which edges to include in the graph.

The edge $_1$G-$_2$A that shows $_2$A as a neighbor to $_1$G is trivial, and the edge $_4$D-$_5$E creates a new cycle, $_4$D-$_4$G-$_4$F-$_5$E]. The analysis is completed by finding the event that directly prolongs $_4$D-$_5$E to fill in this last "hole" in the graph. In this case the slur from $_4$G to $_5$E that denotes the third-progression gives the edge $_4$G-$_5$E to complete the graph.

Figure 1.12 shows the bottom-up analytical procedure, which starts from the edges representing trivial prolongations in figure 1.10. First we find the events that Renwick depicts as most foreground; these are $_2$G, $_3$E, and $_4$F, shown as passing tones by Renwick's slurs. These can be reduced out by adding edges between the events adjacent to them on the cycle: $_2$A-$_3$F, $_3$F-$_4$D, and $_4$G-$_5$E—that is, the notes connected by slurs in the analysis. This leaves the cycle ○*i*-$_1$G-$_2$A-$_3$F-$_4$D-$_4$G-$_5$E-○*t*], a reduction of the fugue subject, as shown in the first graph of figure 1.12.

Then the process of elimination continues: $_2$A is subsumed by the beamed stepwise progression of $_1$G to $_3$F, and $_4$G by the stepwise progression from $_4$D and $_5$E. The second graph of figure 1.12 includes the new edges, $_1$G-$_3$F and $_4$D-$_5$E, leaving the smaller hole of ○*i*-$_1$G-$_3$F-$_4$D-$_5$E-○*t*]. The edge $_3$F-$_5$E, shown by beams in Renwick's analysis, further reduces this, as in the third graph of figure 1.12. The final steps, then, are similar to the first steps of the top-down construction, and result in the same MOP.

Recalling my explanation of the term planar above, the drawing of the completed MOP in figure 1.13 illustrates the planarity of the MOP (because none of its lines cross). This is not all, though: the drawing of figure 1.13 also presents the MOP as an *outerplanar* graph. This means that there's a way to draw the graph so that, not only do no two lines intersect, but also if we imagine that the edges of the graph enclosing a region in the plane, all of the vertices are on the perimeter of this region.

Now, imagine that we add another edge to the MOP—say $_2$G-$_3$E, as in figure 1.14. This graph is no longer outerplanar: the note $_3$F is surrounded by the cycle $_1$G-$_2$A-$_2$G-$_3$E-$_4$D-$_5$E], and there is no way to draw the graph on a plane without crossing edges and avoid this. In fact, if we add any new edge to the graph of figure 1.13, the result will be a graph that isn't outerplanar. This is what *maximal* outerplanar means: the graph is outerplanar and there is no way to add a new edge and still have an outerplanar graph.

As I said above, this characterization, while mathematically interesting, doesn't say much about how the graph makes a good model of dynamic prolongation. However, a small modification of it will prove more enlightening.

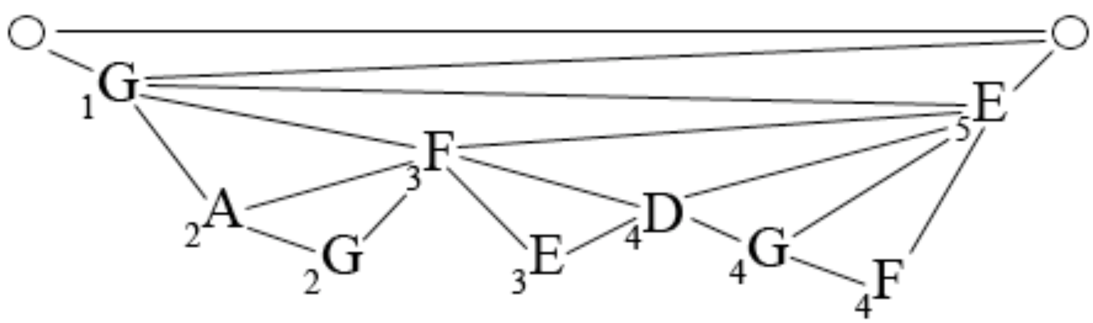

FIGURE 1.13: A MOP ANALYSIS OF THE FUGUE SUBJECT

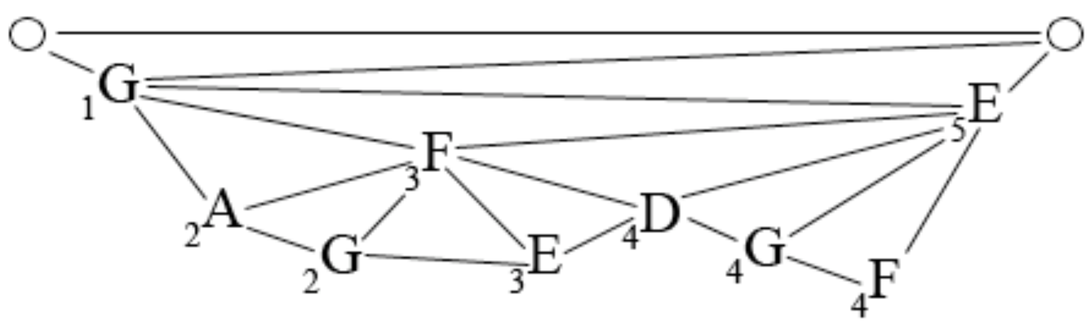

FIGURE 1.14: AN EDGE ADDED TO A MOP MAKES IT NO LONGER OUTERPLANAR

Notice that there are many ways to pick cycles out of the graph in figure 1.13 that correspond to a melodic phrases. Some cycles pick out coherent parts of the passage, such as ${}_1$G-${}_2$A-${}_2$G-${}_3$F], ${}_3$F-${}_3$E-${}_4$D], and ${}_4$D-${}_4$G-${}_4$F-${}_5$E]. Some include ○*i*-○*t* and represent reductions of the passage—as does the cycle ○*i*-${}_1$G-${}_2$A-${}_3$F-${}_4$D-${}_5$E-○*t*]. Still others may pick out coherent parts of some such reduction, such as ${}_3$F-${}_4$D-${}_4$G-${}_5$E]. There's always one cycle that includes every event in the melody. This is called the *Hamiltonian* cycle.

A *Hamiltonian* cycle, in the terminology of graph theory, is one that includes every vertex in the graph. In a MOP, such as figure 1.13, there is always exactly one Hamiltonian cycle-subgraph and it is the one that includes every edge between consecutive events and ○*i*-○*t*. By specifying ○*i*-○*t* as an oriented root-edge we identify precisely the Hamiltonian cycle that represents the entire sequence of events as a musical phrase. In figure 1.13, for instance, this is ○*i*-${}_1$G-${}_2$A-${}_2$G-${}_3$F-${}_3$E-${}_4$D-${}_4$G-${}_4$F-${}_5$E-○*t*].

Say that C is any cycle of a MOP, and it has n vertices. This cycle corresponds to some musical phrase. If we delete all the events of the MOP that are not part of C (and, of course, all the edges they participate in), what we ought to be left with is a prolongational analysis of the musical phrase defined by C. To make the notion of dynamic prolongation we've been pursuing thus far precise, I'll define two conditions on what counts as a prolongational analysis of C.

First, if C has more than 3 vertices, there must be at least one edge between two vertices of C other than the edges of C itself. Such an edge is called a *chord*. A chordless cycle of four or more vertices in a graph is called a *hole*. Semantically, a hole tells us that the phrase defined by the cycle makes up a prolongational span, but it neglects to relate the notes within that span. In the construction of the fugue subject we made a point of filling all such holes, because the holes leave the analysis incomplete: they fail to distinguish between possible ways of relating events within the chordless cycle. Below (in the section "Maximality and Chordality") I will entertain the possibility of leaving holes in an analysis, but for now let's stipulate that the analysis should be complete. Such a graph, one with no holes, is called a *chordal* graph.

Second, if C has more than one chord, these chords can't *cross*. That is, if a, b, c, and d are four events occurring in that order in the cycle, then ac and bd would be crossing chords. They are called crossing because there's no way of drawing them inside the cycle without having the lines cross. The reason we put this condition on a prolongational analysis is that crossing chords create the unintelligible situation where b prolongs the interval a-c while c prolongs the interval b-d.

This is precisely the restriction that William Benjamin and others would like to eliminate from the definition of the word prolongation. I think I have already sufficiently argued this case in the introduction, but it's worth adding the observation here that Benjamin's idea of prolongation relies heavily on the static sense of the term. According to Benjamin's usage, prolongation is simply a relationship between two events: one prolongs the other. And, in addition, the temporal relationship between events is not particularly important: the event c can prolong a without that fact necessarily affecting an event b that occurs between them in sequence. I find this quite counterintuitive, especially considering the temporal implications of the word "prolong."

In contrast, according to the dynamic sense I pursue here, "prolonging" is a relationship that an event can hold to a time span that includes that event, a time span that is defined by the events that delineate its boundaries (and not, for instance, by a clock or a meter). For instance, if ${}_3$F is a passing tone between ${}_1$G and ${}_5$E, then ${}_3$F prolongs the motion from ${}_1$G to ${}_5$E. One could say that ${}_3$F bears a relationship of prolonging to both ${}_1$G and ${}_5$E individually in some sense, but ${}_3$F cannot prolong ${}_1$G unless ${}_1$G is going somewhere and ${}_3$F provides a stop along the journey. Otherwise ${}_1$G and ${}_3$F are just two events, one after the other.

These three conditions give one way to completely define a MOP: it is a crosschord-free chordal Hamiltonian graph. Given Hamiltonicity, I think it is not difficult to see how the crosschord-free condition replaces outerplanarity, and the chordal condition replaces maximality in the definition of a MOP. I will prove this rigorously in part five of the paper. For now, it is important only to recognize how in the context of a particular musical interpretation of the mathematical object of a graph, the conditions that circumscribe the class of graphs suitable as a representation of prolongation help to remove much of the ambiguity about the meaning of the word. Furthermore, debates about my usage of the term prolongation can be directed at particular aspects of the model. (For example, if one disagrees about the necessity of completeness in a prolongational model, they can modify the chordality condition that helps to define a MOP).

Refinements of the MOP Model

In the previous section ("The MOP Model of Prolongation") I pointed out that the model developed there is restricted to relatively basic information about the prolongational analysis of a sequence of events. In particular, while the analyst is concerned with the particular nature of the events being related in the analysis, the mathematical construction is otherwise blind to them: it regards them as just a sequence of so many events. This has the advantage of freedom and simplicity: it offers the analyst many possible prolongational structures—some of them plausible, many of them junk—and allows for a concise description of the mathematical object that represents the analysis. The disadvantage, of course, other than having to sift out the junk analyses, is that certain musically important distinctions are lost: for instance, the fugue subject analysis in figure 1.13 doesn't tell us whether $_2$A-$_3$F is heard as a prolonged interval of the subdominant, or as the loose-end of an incomplete neighbor progression (as I described it above).

To remedy these flaws I propose a few notational conventions that make the drawings of MOPs more expressive. I won't rigorously formalize these notational conventions here, but it certainly is possible to do so, and also to use them to fence off some of the junk analyses. I'll develop this notation here in the context of melodic prolongation that I've already set up, and extend it later (in part 3) to contrapuntal prolongation. Also, at the end of this section I'll briefly address the problem of unfoldings that I brought up in "Some Conceptual Problems in Theories of Prolongation."

The notational convention here consists in drawing edges in four different possible ways to indicate different types of prolongational spans. First, ordinary lines will be reserved for the most fundamental types of melodic prolongation, passing and neighboring motion. Such edges, thus, will always be incident on stepwise-related notes.

Arpeggiations, on the other hand, are indicated by bold lines, whereas the retention or octave transfer of a pitch is shown with a bold line crossed by two slashes.

The fourth and last type of edge is drawn with a broken line, and denotes a relationship created by substitution—in other words, recalling the discussion from "Some Conceptual Problems in Theories of Prolongation" above, the "loose end" of an incomplete progression. To illustrate, let's return to the melodic analysis of Salzer's (shown in figures 1.3 and 1.4) that motivated that earlier discussion. Figure 1.15 gives a MOP representation of this analysis with the enriched notation.

Consider, for example, the motion from $_1$G to $_2$F in measures 1-2. This motion is prolonged by an incomplete arpeggiation from F to D, so a bold line for $_1$D-$_2$F indicates the arpeggiation while a broken line for $_1$G-$_1$D shows that this edge results from a substitution of $_1$G for the origin of the arpeggiation. This edge, $_1$G-$_1$D, is itself prolonged by an incomplete motion, here an upper neighbor from G. The ordinary line for $_1$G-$_1$A shows the neighbor relationship while the dotted line for $_1$A-$_1$D indicates that the resolution of the upper neighbor is elided resulting in the juxtaposition of $_1$A and $_1$D.

The edges involving the initiation and termination vertices are also considered incomplete progressions, since these "formal" events substitute for the possible ways of musically contextualizing the given passage. So, in figure 1.15, $_0$E-$_1$G and $_1$G-$_4$C are incomplete arpeggiations. This example also includes an example of retention in

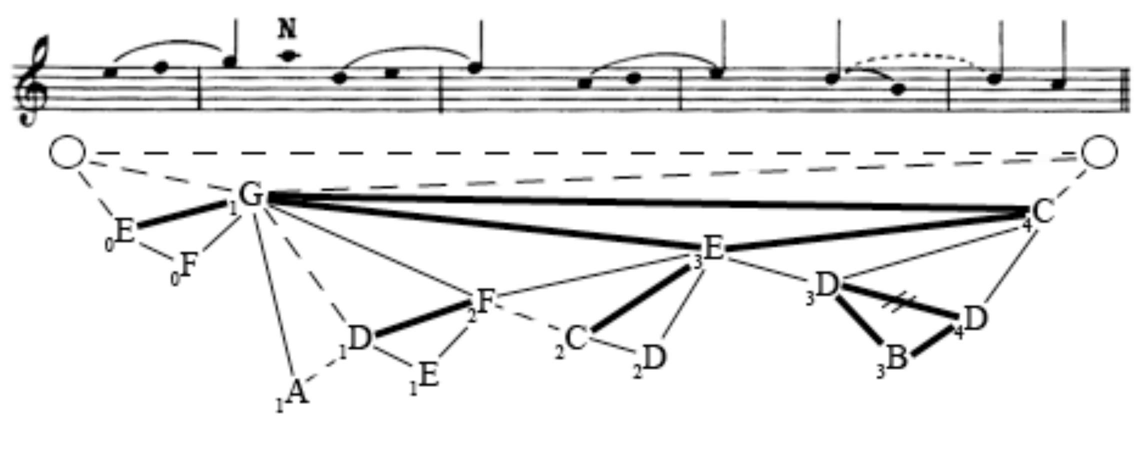

FIGURE 1.15: A MOP REPRESENTATION OF SALZER'S ALBUM FOR THE YOUNG *ANALYSIS*

measures 3-4 with the edge $_3D$-$_4D$, and a fifth-progression made up of a passing arpeggiation $_1G$-$_3E$-$_4C$ and the passing-tone progressions $_1G$-$_2F$-$_3E$ and $_3E$-$_3D$-$_4C$.

Figure 1.16 uses this notation to add content to the MOP representation of the fugue subject analysis in figure 1.13. Excluding the formal prolongations, there are three incomplete progressions in this analysis, the incomplete neighboring motions $_1G$-$_2A$ and $_4D$-$_5E$ and the incomplete arpeggiation $_4G$-$_5E$. Passing tones prolong the loose-end spans $_2A$-$_3F$ and $_3F$-$_4D$, illustrating why such spans must be included in the graph in spite of the fact that they aren't considered to be intervals of harmonic significance.

The two arpeggiations of this analysis, both of them from G to E, reflect the choice to avoid implying any harmony other that C major in the analysis (by making $_4D$ an incomplete lower neighbor to $_5E$). In fact, this is not really an accurate representation of Renwick's conception of the fugue subject, as I will show momentarily. First, let's explore some other ways of hearing the subject through dynamic prolongation.
Figure 1.17 gives the same MOP, but shows the motion $_2A$-$_3F$ as a subdominant arpeggiation rather than a substitution. The possibility of such a prolongation is interesting: $_2A$ is still in a sense an incomplete upper neighbor to $_1G$, but this doesn't seem to preclude its also being an incomplete arpeggiation to $_3F$, rather than a mere juxtaposition with $_3F$. What has actually happened is that by attaching two different types of incomplete progression (neighboring and arpeggiating) to one another, we obtain a new sort of composite complete progression: arpeggiation to (or from) a neighbor.

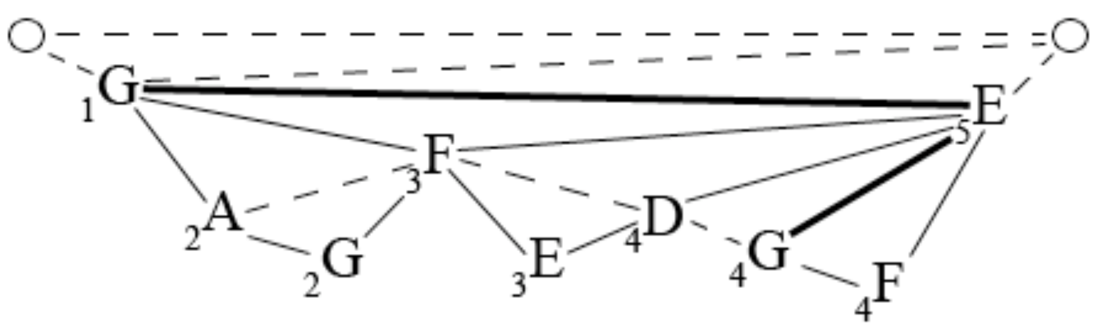

FIGURE 1.16: A NOTATIONALLY ENRICHED VERSION OF RENWICK'S FUGUE SUBJECT ANALYSIS

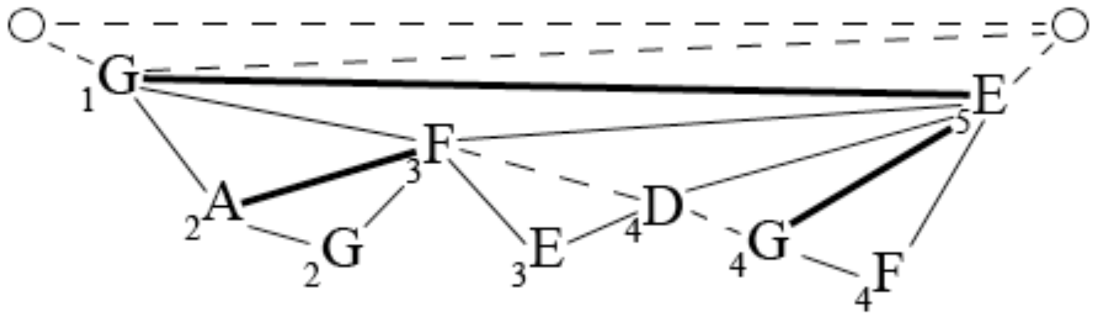

FIGURE 1.17: A SLIGHTLY DIFFERENT READING OF RENWICK'S ANALYSIS

The reader will probably have noticed already an interesting fact about this notational system: the labels on the vertices themselves basically determine whether an edge is drawn as a thin line or double-slashed line. Stepwise progressions will always receive a thin line, while retentions and octave-transfers will always receive a slashed line.[53] The thick lines and broken lines are left, in this game of musical chairs, to fight over the leaping motions. Thus, as far as actual content goes (beyond making the drawings of MOPs more expressive in appearance), the choices made in applying this notational system to a melodic MOP analysis are whether the leaping motions are actual arpeggiations or not.

While it's no skin off the back of the formal model to allow for such distinctions, they begin to appear often arbitrary and fictitious upon closer examination. The analysis of Renwick's in figure 1.9 comes from a larger analysis of the exposition of the fugue, and the ambiguities we observed in it are simply a result of its abbreviation in service of a wider analytical picture. Figure 1.18 shows his more detailed analysis of the subject. Here we see that Renwick hears the subject as not merely a composing-out of I, but implying an entire progression, I-IV-V^7-I. If one holds a very rigid view on "structure and prolongation," one might see this as a contradiction: are those IV and V chords structural or aren't they? Yet, it's difficult to maintain such an absolutist stance: in fact

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[53] One might dispute whether a stepwise progression might sometimes constitute an arpeggiation (i. e. of a seventh chord). Schenker's notion that the seventh of a chord always be heard as a passing motion from the root argues against this possibility, but there is presently no pressing formal reason to exclude it.

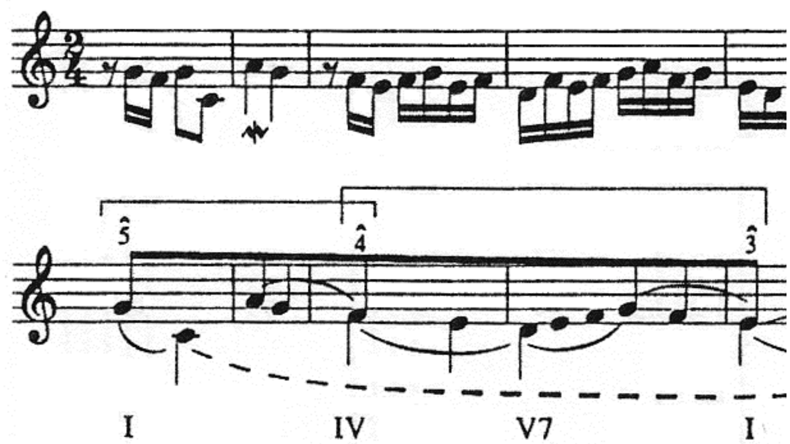

FIGURE 1.18: RENWICK'S MORE DETAILED ANALYSIS OF THE FUGUE SUBJECT

the implied IV and V are relatively structural in the very narrow context of an analysis of the subject by itself, but not very structural in the context of the entire exposition.

If we thus acknowledge the relativity of "structure and prolongation," and also adhere to the Schenkerian idea that harmonic progression is not an independent reality but arises out of contrapuntal patterns (—see the discussions of Schenker's *Stufen* in "The Concept(s) of Prolongation" and "Prolongations as Passing Events" above—), it calls into question the idea of distinguishing between "harmonic" and "non-harmonic" leaps. It's simpler to say: a leap by itself is nothing more than a leap and any leap has the potential and tendency to suggest an arpeggiation, but it's the extent of the leap, the backgroundness of the prolongational span it defines, that gives it power to trump other leaps in the defining the harmonic progression that we hear. It is natural, therefore, that the detail of the harmonic patterns we observe depends proportionately on the distance from which we take our vantage of the music. Nevertheless, the distinction between incomplete neighbors and arpeggiating neighbors is useful and I will retain it.

While we have Renwick's analysis of the fugue subject in figure 1.18 on the table, let's look again to find the source of ambiguity about the prolongation of the step $_3$F-$_5$E. In Renwick's more detailed analysis here we can see that he is unclear about this in figure 1.9 because in fact he hears *both* $_4$D progressing stepwise to $_5$E and $_3$F-$_5$E being prolonged by $_4$G as a sort of interruption. Perhaps at this point, William Benjamin will say “Aha!,” because to incorporate both of these into a melodic analysis would create crossing prolongations. Unfortunately for Benjamin, Renwick seems to share the view that such melodic prolongations are incomprehensible, because he finds it necessary, in order to show these two prolongations, to separate the notes of the subject conceptually into two voices, making $_4$D a passing tone from $_1$C to $_5$E in the lower voice so that the motion to $_4$G can occur simultaneously in a higher voice that expresses the main progression G-F-E.

In “Some Conceptual Problems in Theories of Prolongation” above, I pointed out that Schenker’s notion of unfolding introduces problems into the important contrapuntal idea of a voice. In particular, to fully adopt the idea that progressions can express unfoldings, it’s necessary to accept that the number of voices and content of voices can change depending on the level at which one views the music.

Here’s a possible formal way of dealing with this situation: I’ve already developed a model in which a particular sequence of notes, once fixed and assigned to a single voice, can be given a prolongational structure. Such a structure can also apply to a sequence of simultaneities, seen as a sort of first species counterpoint of consonances. (I’ll explore this idea in more depth in part three of the paper.) For instance, consider figure 1.19 as a reduction of Renwick’s analysis of figure 1.18 to a simple two-voice counterpoint of consonances over an imaginary *cantus firmus* of *Stufen*. Here, each event is a dyad, expressed as an ordered list of notes with the lower voice in the first place. The notation from melodic MOPs can be loosely extended to such an analysis: the progression ($_1$C, $_1$G)($_5$E, $_5$E) is an arpeggiation in both voices, the progression ($_3$F, $_2$A)($_3$F, $_3$F) as an arpeggiation of the upper voice over a stationary lower voice, and ($_3$F, $_3$F)($_4$D, $_3$F) similarly an arpeggiation in the lower voice against a stationary upper voice, so these all are drawn with thick lines. The other prolongational spans have stepwise motion in at least one voice, so (following Schenker’s idea of “ensnaring the leap”—see “Prolongations as Passing Events” above) they’re drawn with ordinary lines.

Of course, the voices individually should have the same prolongational structure as the composite, though with equivalent notes contracted into a single vertex. Thus, the graph of figure 1.20 gives the basic structure of the upper voice. Now consider the

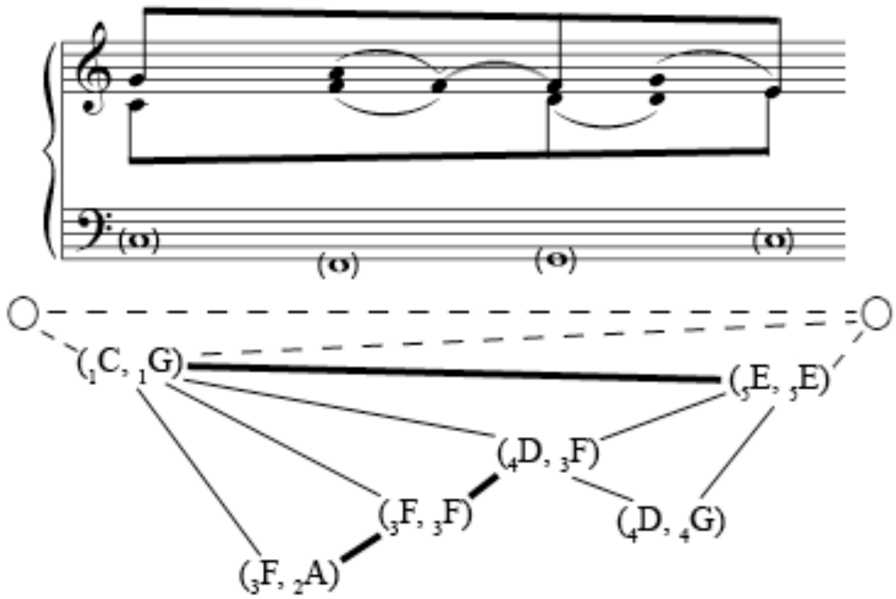

FIGURE 1.19: A "FOLDED" VERSION OF RENWICK'S ANALYSIS

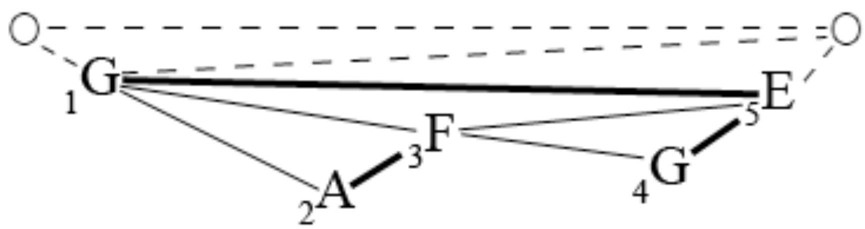

FIGURE 1.20: THE UPPER VOICE ISOLATED FROM THE ANALYSIS OF FIGURE 1.19

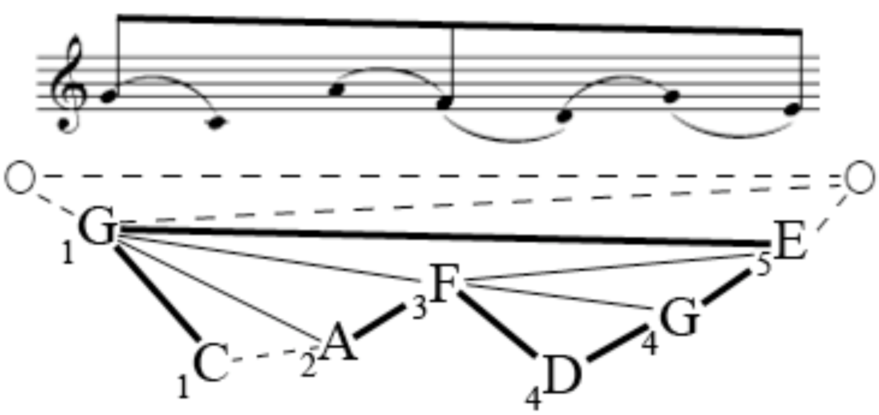

FIGURE 1.21: THE UNFOLDED ARPEGGIATIONS OF RENWICK'S ANALYSIS

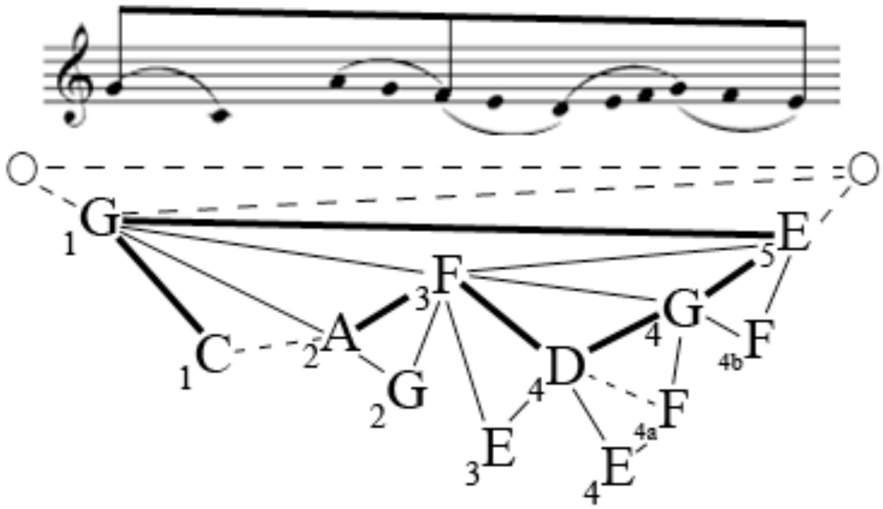

FIGURE 1.22: THE UNFOLDED VERSION OF RENWICK'S ANALYSIS WITH PASSING TONES

transformation of the counterpoint of figure 1.19 in figure 1.21: all of the notes are incorporated into a single voice by unfolding the consonant intervals into arpeggiations. That is, the basic structure of the upper voice in figure 1.20 is retained, while the lower voice notes are added as prolonging arpeggiations that either precede or follow the note they arpeggiate to. In the case of $_1$C this results in an incomplete arpeggiation from $_1$G, while $_4$D is a complete arpeggiation since it accounts for the unfolding of both ($_4$D, $_3$F) and ($_4$D, $_4$G). This basic structure can then be filled-out with passing motion as in figure 1.22. (The fourth-progression from $_4$D to $_4$G in this analysis is discussed briefly in "Maximality and Chordality" below). We can regard Salzer's *Album for the Young* analysis (figure 1.15) as an unfolding in a similar fashion, as figure 1.23 illustrates.

Figure 1.22 is particularly interesting in that some of the events in the analysis appear to take on multiple harmonic meanings. For instance, $_3$F is the root of the subdominant in the arpeggiation $_2$A-$_3$F, but in the following arpeggiation $_3$F-$_4$D the same event appears as a seventh of the dominant. Similarly, $_4$G seems to take on the meaning of both tonic fifth (in $_4$G-$_5$E) and dominant root (in $_4$D-$_4$G). This calls to mind Schenker's concept of "nodal points" (*Knotenpunkten*). (See *Counterpoint II*, 58; *Kontrapunkt II*, 57-8). Schenker demonstrates the phenomenon in second species counterpoint where a note on the downbeat can complete a passing motion from a previous measure and

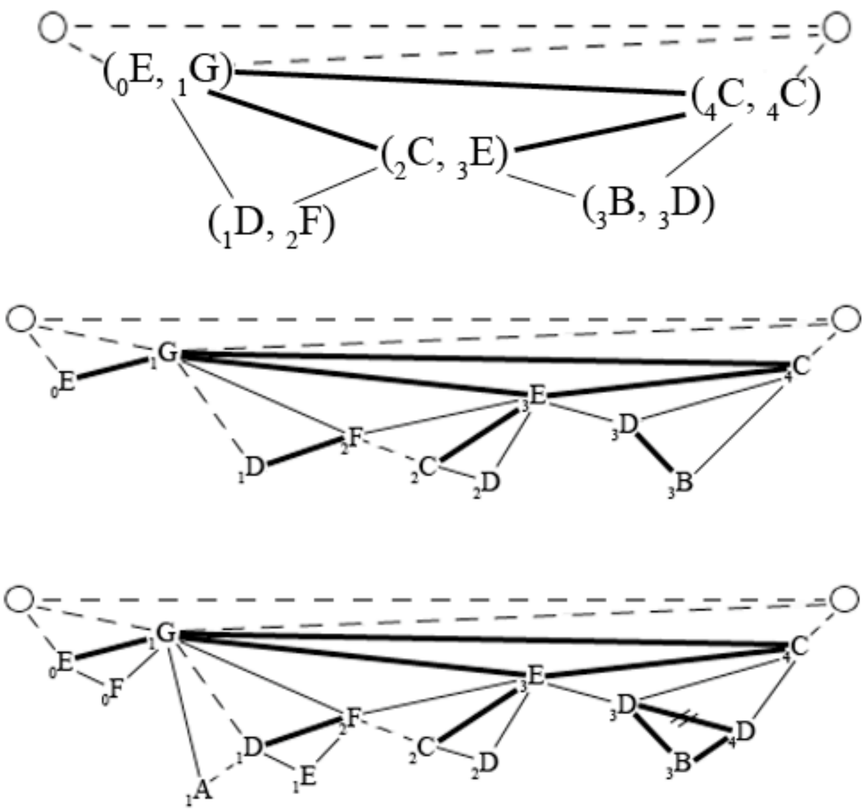

FIGURE 1.23: UNFOLDING TRANSFORMATIONS IN SALZER'S ALBUM FOR THE YOUNG *ANALYSIS*

initiate a different consonant leaping motion over a new *cantus firmus* note, so that as a completion of the preceding motion it has a different harmonic meaning than it does in the motion it initiates. This phenomenon is particularly interesting because it illustrates the fact that harmonic meaning is not a property of the melodic event itself; rather harmonic meaning is a property of the motions from one event to another. In the analysis of figure 1.22, $_3$F isn't exactly a subdominant event or a dominant event; rather, $_3$F completes a subdominant motion and initiates a dominant motion.

I won't rigorously formalize the model of unfolding offered here due to limitations of space and time, but this brief exposition demonstrates a couple of important points about prolongation. First, unfolding can be developed as a construct independent of the most basic aspects of prolongation. According to the formal model of unfolding I've suggested here, we can identify prolongational structures both for melodies and for counterpoints of melodies and view unfoldings as a transformations relating such prolongational structures. Second, it's impossible to go very far with a model of prolongation that doesn't account for multiple voices, since we tend to hear counterpoint and harmony even in the simplest of melodies. The discussion of unfolding here offers a small preview of the contrapuntal model of dynamic prolongation described in part three of this paper.

A Comparison of Analyses Using the MOP Model

After laying out an unambiguous model of at least the most essential characteristics of dynamic prolongation, having such a model proves to be an invaluable tool for analysis. In the introduction I quoted Carl Schachter's misgivings about a "theory of reduction," and while I certainly don't pretend to have formalized anyone's intuitive grasp of large structure, I would like to show how even the simple formal model of prolongation I have developed in the previous section greatly clarifies what is at stake in making the analytical "either/or" choices that Schachter discusses. (Schachter, 1990) I will revisit this analysis in the third part, at which point some added formal apparatus will allow a much more complete representation of Schachter's analytical insights.

Figure 1.24 shows a passage from the second movement of Haydn's Symphony No. 99 and Schachter's analysis below it. (170) Since we have only discussed the prolongation of melodic events so far, saving counterpoint for the third part, I will represent Schachter's analysis as a prolongational analysis of the main melodic line. This will be sufficient to demonstrate the points he makes in his discussion of the excerpt. Schachter presents two plausible conflicting analyses of this passage and decides in favor of the one shown in figure 1.24. Below Schachter's analysis are the harmonic progression implied by this analysis, and the one implied by the non-preferred analysis, which Schachter doesn't show in Schenkerian notation. The point of contention between the two analyses is in the interpretation of measures 7-12.

Schachter's idea of a "theory of reduction," as I pointed out in the introduction, is that of a prescriptive formal model, one that prescribes a particular analysis given a particular passage of music. Our model of prolongation, on the other hand, itself gives no

FIGURE 1.24: HAYDN SYMPHONY NO. 99, 2ND MOVEMENT, AND SCHACHTER'S ANALYSIS

preference to either of Schachter's two conflicting analyses. In fact, the model defines a finite (but huge) number of possible analyses, many of which are worthless, and makes no claim for any of them in particular. The musical observer is still indispensable, to filter out the worthless analyses and to deliberate on the relative merits of the remaining analyses, not to mention to evaluate the usefulness of the formal apparatus itself. Whether you agree with Schachter that a single correct or artistically truest structural analysis for most any musical passage exists but there's no practical limit to the musical expertise required to discover that analysis, and hence no mechanical way of deciding it, or you take the position that each analysis expresses a different hearing, some more plausible than others but none absolutely correct, the clarity of discourse that a formalized model makes possible—not it's potential to make analytical decisions for you—is indispensable to the process of evaluating an analysis.

My goal here, then, is to provide representations of Schachter's two analyses to illuminate the relative merits of each. Because I'm using the MOP model, I'll also be characterizing Schachter's analysis in terms of dynamic prolongation. While I make no presumptions on whether Schachter himself would agree that the dynamic sense of prolongation accurately reflects his own sense of the term, the discussion of his analysis in these terms shows that the language of dynamic prolongation is able to faithfully communicate the musical intuitions he illustrates in his analysis of the Haydn excerpt.

To use the MOP model as developed so far, we must first pin down a sequence of melodic events to analyze. Schachter's analysis boils measures 1-6 down to the events that operate at the level of the entire first 16-measure phrase. These are the initial tonic third B and the dominant fifth A of measure 4. This works well for the MOP model, since this reduces the first 6 measures to those events that bear some dynamic prolongational relationship to events in the music in question, measures 7-12. As for measures 7-12, a fairly complete list of melodic events, excluding only those that belong to an inner voice, is ${}_{7}$G-${}_{7}$B-${}_{8}$A-${}_{8}$G#-${}_{9}$A-${}_{9}$B-${}_{9}$C-${}_{9}$D-${}_{9}$E-${}_{10}$D-${}_{10}$C-${}_{10}$B-${}_{11}$A-${}_{11}$B-${}_{11}$C-${}_{12}$B (using the by-now-familiar notation where the index on each note refers to the measure number in which the event occurs).

Figure 1.25 shows the prolongational relationships that both analyses agree upon. These include the foreground prolongations by passing tone, ${}_{7}$B-${}_{8}$A-${}_{8}$G#, ${}_{9}$A-${}_{9}$B-${}_{9}$C, ${}_{9}$C-${}_{9}$D-${}_{9}$E, ${}_{10}$D-${}_{10}$C-${}_{10}$B, and ${}_{11}$A-${}_{11}$B-${}_{11}$C, the arpeggiation ${}_{8}$A-${}_{9}$C-${}_{9}$E, and ${}_{8}$G# as an arpeggiating neighbor, as well as the most background relationships, ${}_{1}$B as the main event of the passage, ${}_{7}$B and ${}_{12}$B as retentions of ${}_{1}$B, and ${}_{11}$C as a structural neighbor. This leaves a short sequence of six events where the analytic dispute lies: ${}_{7}$B-${}_{9}$E-${}_{10}$D-${}_{10}$B-${}_{11}$A-${}_{11}$C shown by the "gray area" in figure 1.25.

One thing that the graph of figure 1.25 doesn't include is the interruption that Schachter indicates parenthetically in his analysis. The reason for this is that Schachter's beaming and scale-degree indications contradict the assertion of a true interruption in Schenkerian terms. According to Schenker (see *Free Composition*, 36-7; *Freie Satz* 71-2), the *Urlinie* tone arrived at just before the interruption—$\hat{2}$, in the case of a third-progression *Urlinie*—is the structural passing event rather than a lower neighbor to the

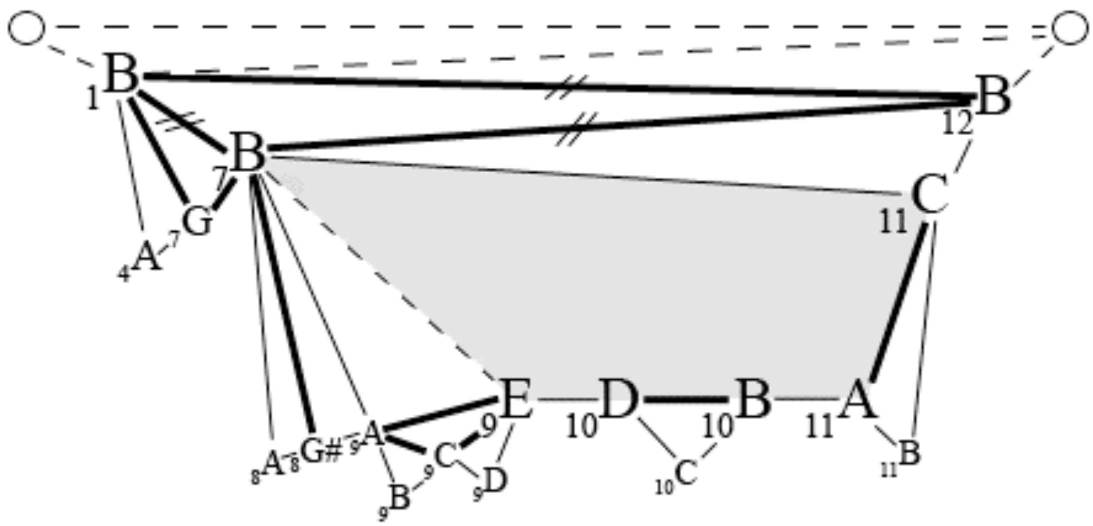

FIGURE 1.25: THE PROLONGATIONAL RELATIONSHIPS AGREED UPON BY SCHACHTER'S TWO ANALYSES

preceding *Urlinie* tone. Thus, to assert an interruption in the MOP analysis, $_{4}$A, rather than being a prolongation of $_{1}$B-$_{7}$B, would have to be a prolongation of the motion from $_{1}$B to a G that resolves the third-progression from $_{1}$B as well as a subsequent third-progression from $_{7}$B. This could potentially be the grace-note G in measure 12, though not according to Schachter's analysis of measures 7-12. Furthermore, there's no arrival at G in the remainder of the exposition, which pushes forward to a half cadence in measure 16 and ends on a tonicized D major in measure 34 (which is, of course, the true interruption in Schenkerian terms). Therefore Schachter puts the interruption symbol in parentheses, to say that the moment in measure 7 gives the "sense" of an interruption though not a true interruption, as in measure 34. Since the only plausible continuation of a third-progression through $_{4}$A is $_{7}$G, I have represented Schachter's analysis of these measures as a third-progression resulting in the retention of the initial tone B.

Figure 1.26 shows the two potential analyses of the gray area: first, Schachter's not-preferred option where $_{7}$B and $_{10}$B make up a prolongational span representing the tonic, $_{11}$A represents the "real" II^{6}, and $_{10}$E is an upper neighbor to the fifth of the tonic, $_{10}$D, and second, Schachter's preferred analysis, where $_{10}$D is passing from $_{11}$E of the "real II^{6}" to $_{11}$C, the seventh of the dominant.

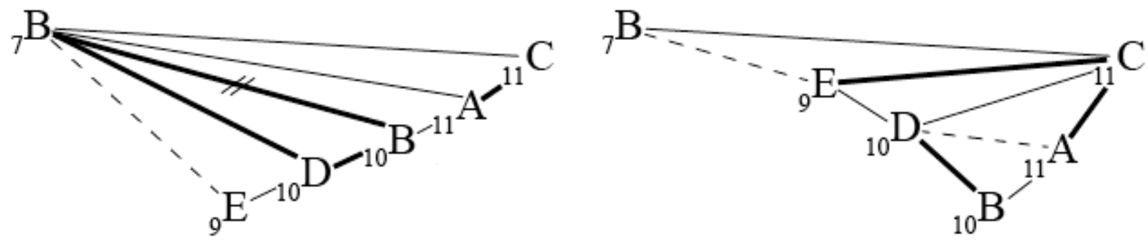

FIGURE 1.26: THE DISPUTED PROLONGATIONS IN SCHACHTER'S TWO ANALYSES

Thus, the mathematical properties of the formal model nicely allow us to narrow down the analytical dispute to a simple choice of how to place three non-crossing chords in a 6-cycle, even though the choice of analysis actually crucially affects the interpretation of three measures of music. It also shows us that the dispute is certainly over the prolongational interpretation of the passage and not something else. For instance, someone might dispute Schachter's claim that the G chord in measure 10 is not a "real" tonic, and argue that the progression is not II^6 (mm. 9-11)-V^7 (m. 11)-I (m. 12), as Schachter has it, but II^6 (m. 9)-VII^7 (m. 10)-I (m. 10)-II^6 (m. 11)-V^7 (m. 11)-I (m. 12). This person, however, is still free to agree with the second analysis of figure 1.26. If that person did accept the second analysis of figure 1.26, the dispute would not be over the prolongational interpretation of the passage but merely what the conditions are under which an analyst ought to call something a ("real") harmony. Certainly the prolongational interpretation would inform this dispute: for instance, Schachter would say that a I that prolongs a II^6 is not functioning as a tonic. In that case, the issue is not one of deciding upon a prolongational interpretation, but of the proper usage of the word "tonic" given a prolongational interpretation.

The complete MOP corresponding to Schachter's analysis is shown in figure 1.27. Only a couple of the decisions made in this representation of Schachter's analysis are not explicitly shown in the analysis Schachter gives in Schenkerian notation. First, in the passage $_7$B-$_8$A-$_8$G#-$_9$A-$_9$B-$_9$C-$_9$D-$_9$E, he shows the notes $_8$A, $_9$B, and $_9$D as passing tones by a slur and $_9$C as an arpeggiation in the supertonic chord. However, his analysis doesn't specify the choice of analysis for $_7$B-$_8$G#-$_9$A-$_9$E, whether to connect $_7$B to $_9$A or $_8$G# to $_9$E. Yet it is obvious that connecting $_7$B to $_9$A is correct, since a prolongation of

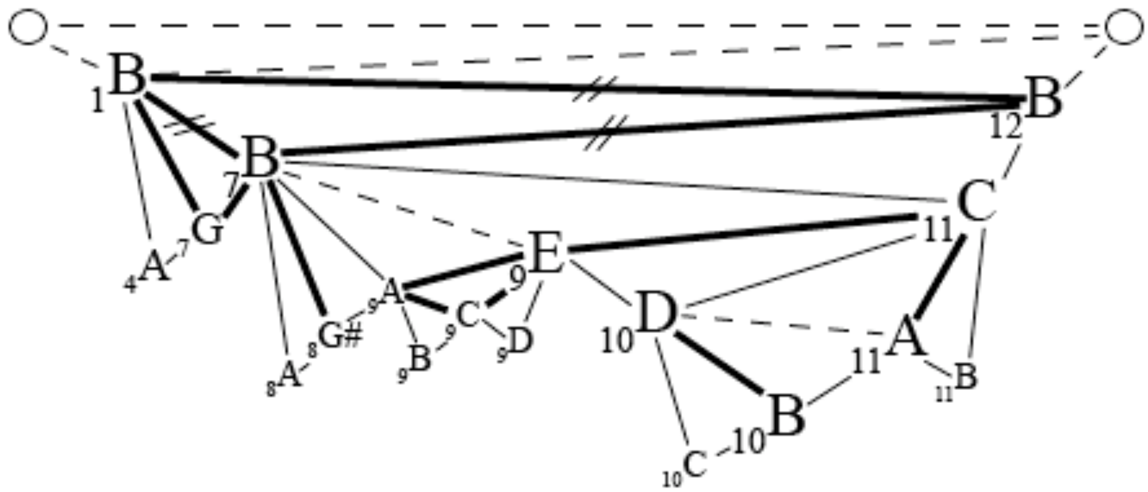

FIGURE 1.27: THE MOP REPRESENTATION OF SCHACHTER'S ANALYSIS OF THE UPPER VOICE

$_8$G#-$_9$E would have to be a arpeggiating motion over measure 9 which wouldn't fit with the hearing of a II^6 harmony in that measure.

Second, in the prolongation of $_{10}$D-$_{11}$C, Schachter's slurs show $_{10}$C and $_{11}$B as passing tones. This leaves the cycle $_{10}$D-$_{10}$B-$_{11}$A-$_{11}$C], where I chose to add the edge $_{10}$D-$_{11}$A rather than $_{10}$B-$_{11}$C. While Schachter's stemming shows the connection from $_{10}$D to $_{11}$C, the choice of $_{10}$D-$_{11}$A versus $_{10}$B-$_{11}$C is a choice between two pairs of events involving one stemmed and one unstemmed note. Yet the direction of the melodic line, which completes a descent at $_{11}$A as well as its strong metrical position and its coincidence with an important arrival in the bass suggest that $_{11}$A take on a more structural status here. Therefore I interpret $_{10}$D-$_{11}$A-$_{11}$C] as an incomplete arpeggiation. (I will address this question further in the next section, "Maximality and Chordality").

This shows two things: first, the Schenkerian symbols themselves do not require the analyst to produce an analysis that is unambiguous in terms of prolongation. Second, despite this fact, Schachter is very careful to leave no prolongational relationship ambiguous: in those places where the Schenkerian notation itself does not determine all of the prolongational relationships, they can be inferred from Schachter's harmonic analysis. This is not at all accidental: Schachter is careful to present an analysis that gives a complete account of measures 7-12, and the ease with which his analytical symbols translate into the form of a MOP supports the proposition that Schachter's idea of prolongation is consistent with the concept of dynamic prolongation as I have formalized it here.

The elements of a MOP analysis correspond closely to the slurs, stems, and beams of Schenkerian notation. Furthermore, the graphical presentation of a MOP has the advantage of showing very clearly what is required to make an analysis complete and what is not required. Despite this and other advantages, I don't intend to propose that my visual depictions of MOPs replace Schenkerian notation. Schenkerian notation has obvious advantages of its own: because it uses staff notation it is easy to see how it corresponds to the score and it readily invokes musical intuitions, and in addition the analyst can use it in combination with Schenkerian symbols that are not purely prolongational in nature. The graphical presentation of MOPs is more useful as a supplement to the usual Schenkerian notation than it is on its own.

Just to alleviate any confusion, however, I should point out that the formalization I have proposed is independent of any way of representing it with dots and lines. I offer these visualizations as an aid to understanding the nature of the mathematical object. Too often the music theorist tries to transplant a formal model without being careful to dig up the roots, the mathematical ideas that the visual paraphernalia strives to express, without which the formal model cannot grow and thrive. Though I'll admit to a fondness for the visual presentations of figures 1.9 and 1.14, my goal here is not to advance the cause of such drawings but rather the mathematical object of a maximal outerplanar graph as a representation of a prolongational analysis, a mathematical object that could be visually represented in numerous ways, including with noteheads and slurs on a staff.

Finally, before leaving the topic Schachter's article, let me address once again his general distaste for formal models. The formal model I have injected into his discussion of the Haydn slow movement is not concerned about "working." That is, it doesn't try to successfully reproduce Schachter's analysis through a mechanical procedure, or give a formal representation to anything as complex and multifaceted as Schachter's intuition about large structure. Schachter's choice of analysis, while it certainly implicitly takes into account all the resulting prolongational relationships shown in figure 1.27, is ultimately based on motivic factors that themselves have little to do with prolongation. While the MOP model certainly informs prolongational decisions by helping the analyst understand exactly what each decision entails, it doesn't prescribe a solution to any analytical dilemma. It simply enumerates the possible analytical decisions and gives a representation to the prolongational structures that result from each one. I hope that Schachter himself would agree that this tool helps music analysts do their job without putting any of them out of work.

Maximality and Chordality

The assertions about the nature of prolongation made by the MOP model divide up neatly into mutually independent parts. The underpinning of the entire model is the framework provided by the graph-theoretic interpretation of the concept of dynamic prolongation. In the section "Maximal Outerplanar Graphs" above I erected three walls upon that foundation intended to house a recognizable concept of prolongation. These three walls are represented by the graph properties of Hamiltonicity, outerplanarity or the property of being crosschord-free, and maximality or chordality.

Hamiltonicity represents the assumptions that a sequence of events has a definite order and comprises a prolongationally autonomous musical phrase. For an analysis of melodic prolongation, these are essential. In part three, we'll see that strictness about the ordering of events must be relaxed to some extent in representing music in multiple voices in counterpoint. The individual voices must still adhere to strict ordering, however, to avoid allowing the relaxation of Hamiltonicity in contrapuntal analysis to destroy the temporal nature of the analysis. Thus, the melodic MOP, for which Hamiltonicity is an essential condition, is a necessary precursor to the construction of a contrapuntal model of dynamic prolongation.

The property of outerplanarity or of being free of crossing chords is also essential to a unified and comprehensible prolongational analysis, as I have argued already in the introduction and in "Maximal Outerplanar Graphs." To give another angle on this, outerplanarity is crucial, in addition to Hamiltonicity, to assigning consistent time-orientations to prolongational spans. For instance, while a Hamiltonian outerplanar graph may only have one Hamiltonian cycle-subgraph, one that isn't outerplanar may in general have multiple. This means that it would be possible, given a non-outerplanar graph as a prolongational analysis, to infer multiple orderings to the events analyzed. More precisely: given any non-outerplanar Hamiltonian prolongational analysis on a fixed set of events (including an initiation and a termination event), for at least one edge of the graph it will be impossible to determine the order of the events in the prolongational span it represents. (See the description of confluence in part 4). Therefore, outerplanarity is essential to the notion that a prolongational span is a motion in time.

This leaves the property of maximality or chordality, the one property of a MOP whose necessity for a description of melodic prolongation is debatable. In "Maximal Outerplanar Graphs," I argued in favor of this condition on the grounds that a hole (a non-trivial chordless cycle) is an unanalyzed sequence of events in a prolongational span. Therefore a non-maximal Hamiltonian outerplanar graph (or "HOP") might be characterized as an incomplete prolongational analysis: it includes passages that could be further elucidated in terms of their prolongational structure but are not. Yet there may be cases where such "incompleteness" is desirable; the question is just how much incompleteness does it take before a graph is too "thin" to constitute a prolongational analysis.

Let me approach the issue from a somewhat different angle: the chordality condition means that all prolongations (of any number of events) can be built up out of an easily circumscribed set of possible elemental prolongation-types (given that the type of events participating in the analysis is well-defined and limited). For instance, in the model of melodic prolongation we've developed in this part the paper, there are seven basic types of elemental prolongations: passing/neighboring motion, arpeggiation, the arpeggiating neighbor, incomplete neighbor, incomplete arpeggiation, and repetition/anticipation or octave transfer. Each of these has a particular look in the graph-drawing notation developed in "Refinements of the MOP model" above, and each categories could also be further broken down by what type of motion they prolong, in some cases by the order in which different types of motion occur, and in other ways as well.[54] Yet, even this short list of six categories is sufficient to allows us to survey the possibilities and acknowledge for each that in general it represents a phenomenon that we would call prolongation. This is important because the fidelity of the model to our common-sense notion of prolongation is dependent on the comprehensibility of each of the possible building blocks.

Admitting the possibility of holes means admitting the possibility of prolongations that aren't constructed out of these simple building-blocks involving only three events. Furthermore, admitting the possibility of holes without restriction would mean making it impossible to survey the possible types of "atomic" prolongations, because then in principle any number of events could constitute an irreducible prolongational span. I think the surveyability of atomic prolongations is important to the evaluation and understanding of the model and its musical implications, however. Therefore, the admission of holes into the analytical model comes with a qualifier: each hole should be identified by type in such a way that one can evaluate whether it represents a musical phenomenon that constitutes a type of prolongation, and one, furthermore, that cannot or should not for whatever reason be broken down into smaller prolongational spans.

Two instances from the analyses above arguably provide examples of legitimate prolongational holes. Both are prolongational spans of four notes. (I see no compelling justification for larger holes). The first type is a fourth-progression. Usually, a fourth-progression can be broken down into a third progression and a step, but it might be desirable sometimes to leave the fourth-progression unanalyzed. For instance, in the somewhat more detailed analysis of the C major fugue subject presented in figure 1.22, the interval ${}_4$D-${}_4$G expands to the fourth progression ${}_4$D-${}_4$E-${}_{4a}$F-${}_4$G]. I broke this down into an incomplete neighbor ${}_{4a}$F-${}_4$G and third-progression ${}_4$D-${}_4$E-${}_{4a}$F]. Because of the

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[54] For completeness we should technically add to the list the "formal" prolongation indicated by two broken lines from the initiation and termination vertices, which designates a particular event as the most background of the passage. Also, of the six categories, all are determined by both prolonging edges except the last, which is characterized by only one of its two edges. We could break this last category down into four accordingly, but somehow it only seems to deserve one.

foregroundness of the progression, one might find such an analysis to be so much unwarranted splitting of hairs. The addition of the edge ${}_{4}$D-${}_{4a}$F, while not especially revelatory, in any case doesn't contradict the assertion of a fourth-progression. Thus the admission of unanalyzed fourth-progressions is certainly a possible, though not necessary, innovation in the model.

For another example, consider the one prolongational relationship that was ambiguous in Schachter's analysis, the analysis of ${}_{10}$D-${}_{10}$B-${}_{11}$A-${}_{11}$C] into the arpeggiating-neighbor progressions ${}_{10}$D-${}_{10}$B-${}_{11}$C] and ${}_{10}$B-${}_{11}$A-${}_{11}$C]. (See figure 1.27 and the discussion there). Here's a different description of the situation that may indeed best reflect Schachter's own understanding of it: the motion of ${}_{10}$D to ${}_{10}$B and from ${}_{11}$A to ${}_{11}$C each represent unfoldings of harmonic intervals. There is a significant stepwise relationship between the upper notes of these thirds, ${}_{10}$D and ${}_{11}$C, and—thinking in terms of the model of unfolding I offered in "Refinements of the MOP Model" above—the unfolding from ${}_{10}$D is projected forwards in time while the unfolding from ${}_{11}$C is projected backwards in time. (In Schenker's terminology, this is an *Untergreifen*). One might claim that this offers a complete account of such a prolongation, and that, indeed, prolongational relationships shouldn't be asserted between the inner voice and upper voice notes other than the unfolding relationships. Again, the assertion of an incomplete arpeggiation at ${}_{11}$A-${}_{11}$C is perfectly accurate and consistent with the *Untergreifen* account of this passage. Therefore, one might accept *Untergreifen* and *Übergreifen* as exceptions to chordality in the prolongational model, but one can also recognize *Untergreifen* and *Übergreifen* as phenomena while adhering to the strictly chordal model.

Thus, while reasonable exceptions to the chordality condition may be musically justified, they appear to be few and isolated and do not appear to be necessary exceptions. That is, the strict MOP model of prolongation that ignores such exceptions is not a terrible distortion of musical intuition. Therefore, while we'll keep the possibility of four-holes in melodic analyses in mind, in general I will assume that graphs of melodic prolongation are chordal.

PART 2: PHRASE-STRUCTURE MODELS OF PROLONGATION

The General Phrase-Structure Model of Prolongation

The idea of formalizing Schenkerian analysis is not new, of course. Three groups of authors stand out for proposing relatively rigorously formulated analytical models: John Rahn (1979), Stephen Smoliar (1980), and Fred Lerdahl and Ray Jackendoff (1983). While their formalizations tend to focus more on rules of derivation than the structure of the model itself, all of these authors draw upon concepts from linguistics to structure the analytical model. More specifically, they borrow Chomsky's idea of a phrase structure grammar to structure the representation of musical analysis. This is propitious for a comparison with the present approach, since a phrase-structure-grammatical analysis can be represented with an elaborated graph (called a rooted plane tree).

Chomsky uses the term "phrase structure grammar" to distinguish this type of grammar from two other general models, one weaker and the other stronger: the finite-state grammar and the transformational grammar. (Chomsky (1965)) All of these models can be thought of as "generative": i.e. as algorithms for generating a set of sentences, which is presumably the set of all grammatical sentences if the rules of the grammar are properly defined. However, it is not only important that the generative grammar produces a particular sentence: the way in which the grammar produces the sentence describes a grammatical structure for the sentence, and this structural description ought to have some sort of explanatory power to make the theory worthwhile. The same is true in the case of a "generative grammar" for music theory: the production of a piece of music by the grammar is actually a structural description of the music.

The finite-state grammar, while appealingly simple, is too weak to represent language because its rules can relate a word only to the words directly before it or after it, without taking into account the context of the word in the entire sentence. This is also true, speaking in somewhat vague generalities, of Schenkerian analysis: a finite-state grammar will fail to represent it because of its lack of sensitivity to the context of a note or phrase within a piece.

A phrase structure grammar, on the other hand, generates a sentence by beginning with a string *S* (which stands for "sentence") and successively rewriting the string until arriving at a terminal string. The designation "terminal string" serves to distinguish the sentences themselves from abstractions of sentences such as *NP* + *VP* (which means "noun phrase followed by a verb phrase"). The rules of rewriting are restricted to those that replace a single constituent of the string with one or more constituents. For instance, a rule might say $NP \rightarrow D + N$, which means, "You can replace a noun phrase with a determiner followed by a noun."

We can represent the structural description resulting from such a production with a directed graph in the following way: let the graph have a vertex corresponding to each constituent of each string in the derivation, and let there be an edge to from each constituent not in the terminal string to those derived from it by some rule. The result is called a phrase-structure tree. As an illustration, figure 2.1 shows a phrase structure tree for the sentence "Bach is a great composer." The rewriting rules represented in this tree are $S \rightarrow NP + VP$, $NP \rightarrow N$, $VP \rightarrow V + NP$, $NP \rightarrow D + N$, and $N \rightarrow A + N$. The tree in figure 2.1 is a digraph, as are all phrase-structure trees, but since the relative vertical position of vertices is sufficient to show the orientation of edges, the arrowheads are usually left out and it is understood that all edges are directed downwards.

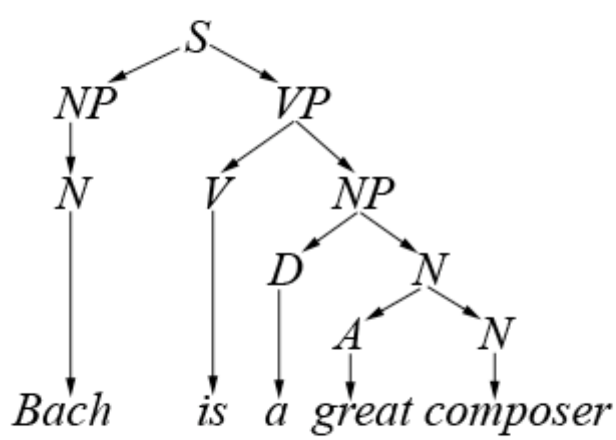

FIGURE 2.1: A GRAMMATICAL PHRASE-STRUCTURE TREE

Similarly, one can define a phrase-structure tree that analyzes a series of musical events. I will define a general model of such a phrase-structure tree to give an overall picture of how Rahn's, Lerdahl and Jackendoff's, and Smoliar's theories compare to MOPs as formal models of prolongation. While this gereral model is faithful to Lerdahl and Jackendoff, it departs from both Rahn and Smoliar in significant ways. There are two reasons for these departures: first, I would like to compare these models to the basic MOP model presented in the previous chapter, which applies only to a single sequence of events, as in a single melody. Both Rahn's and Smoliar's theories apply to a more complex contrapuntal organization of events, but for the purpose of comparison I have taken the liberty of formulating a theory that separates out the simple from the contrapuntal aspects, which I will deal with in the next part. Second, I am only interested in these theories insofar as they deal with prolongation.

In the phrase-structure model, the foreground events are the terminal vertices (the leaves) of the tree. Non-terminal vertices are labeled with one of the events below them. Therefore, in general every rule must be of one of the forms $x \rightarrow a_1 + a_2 + \ldots + a_m + x + b_1 + b_2 + \ldots + b_n$ or $x \rightarrow x + b_1 + \ldots + b_n$ or $x \rightarrow a_1 + \ldots + a_m + x$ or $x \rightarrow x$, where m and n may be any positive integers, and $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n$ are events such that in the music a_1 precedes a_2 precedes . . . precedes a_m precedes x precedes b_1 precedes . . . precedes b_n. Usually, however, rules of the form $x \rightarrow a + x$ and $x \rightarrow x + b$ are most common. The reason for this will become clearer below. The exact nature of these rules and their dependence on the nature of the musical objects involved is of particular interest to all of the authors mentioned. However, we are presently concerned only with the general form of representation given by this method of analysis.

An example of such an analysis is shown in figure 2.2, and gives a possible analysis of the reduced C major fugue subject similar to that of figure 1.16. This digraph is called a tree because there is one vertex, called the *root*, which has no vertices above it, and all other vertices have exactly one edge with a higher vertex. If v is a vertex of such a tree, the vertex adjacent to and above v is called the *parent* of v, the vertices adjacent

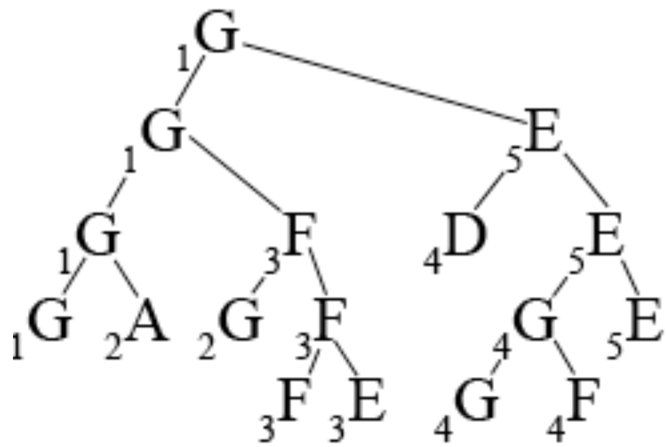

FIGURE 2.2: A BINARY PHRASE-STRUCTURE ANALYSIS OF THE FUGUE SUBJECT

to and below v are called *children* of v. If there is a directed path from a vertex u to a vertex v, then u is an *ancestor* of v. A vertex with no children is a *leaf*.[55]

This model is somewhat different than Smoliar's in that Smoliar's analyses are in the form of such a phrase-structure tree, but, in a monophonic analysis, all of the non-terminal vertices are labeled SEQ (for "sequence") rather than with one of the events below them. (I will discuss the other possible label, SIM, in the next part as a tool for contrapuntal analysis). Figure 2.3 shows an example of such an analysis (extracted as an analysis of the upper part from the analysis of Mozart K. 283 that Smoliar provides in "A Computer Aid for Schenkerian Analysis"). Obviously, such a tree simply gives a grouping of events rather than an analysis of prolongations, since it doesn't tell us what a sequence of events prolongs. This wouldn't be much of an analysis by itself; in fact, in Smoliar's theory, it is in the history of construction of the tree that most of the analysis itself lies.

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[55] The reader will no doubt notice that the analogy to genealogical trees implied by this terminology is misguided unless we're talking about a genealogy of amoebae. Interestingly, the terminology would be more appropriate for MOPs—where every child has two parents—except that it imputes a somewhat disturbing inverted morality to MOPs, since vertices in MOPs are only allowed to have children with their closest relatives and can only have one child with each of their multiple partners.

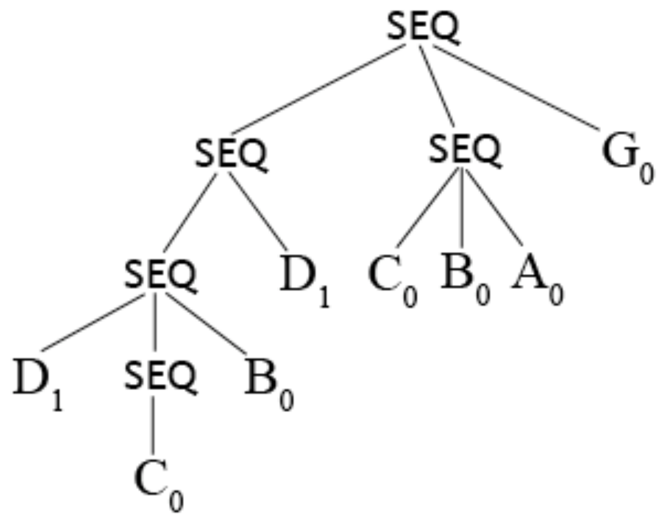

FIGURE 2.3: SMOLIAR'S ANALYSIS OF THE UPPER PART IN MOZART K. 283

This history of construction is a series of transformations that convert a trivial tree step by step into the final phrase-structure tree. Smoliar views these as transformations on the model of Chomsky's transformational grammars.

Transformational grammars, in Chomsky's theory, consist of two things: a set of terminal strings of a phrase-structure grammar, and a set of transformations that apply to those terminal strings and their constituent structure (where a constituent structure is the structure imposed on the string by its phrase-structure derivation). In other words, the construction of a sentence in a transformational grammar is a two-stage process: first, a series of phrase-structure rules is applied to the symbol *S* in one or more different ways to produce a terminal string with a constituent structure, or a set of multiple terminal strings with constituent structures. Then, a series of transformational rules derives a sentence from these terminal strings. In "Refinements of the MOP Model" in part one, I suggested a similar such transformational system to represent the Schenkerian concept of unfolding in the MOP model, where the MOPs represent prolongational structures on the "folded" and "unfolded" versions of the melody and a transformation between these two MOPs represents the unfolding (though I didn't rigorously formalize this process).

However, unlike Chomsky, Smoliar doesn't separate his operations into transformational and phrase-structure rules, although many of them can be represented as phrase-structure rules. This makes the task of interpreting the history of construction of the tree as an analysis somewhat difficult.

Figure 2.4 shows a history of construction for the tree of figure 2.3, showing the operations used at each step (but leaving out the node to which the operation applies, which is obvious from an inspection of the trees). The first two transformations are couched in Smoliar's URSATZ operation. In addition, the first and fourth transformations are actually compositions of two operations: PAR ("parallel") and RAUSKOMP ("reverse *Auskomponierung*"). The result of these two operations is a downward arpeggiation.[56] The other transformations represented are EXTEND, which copies a note, and PT, which adds passing tones between two notes. The operations PAR ∘ RAUSKOMP and EXTEND could be readily defined as phrase-structure rules, and thus imply event labels for the non-terminal vertices as shown in figure 2.5.

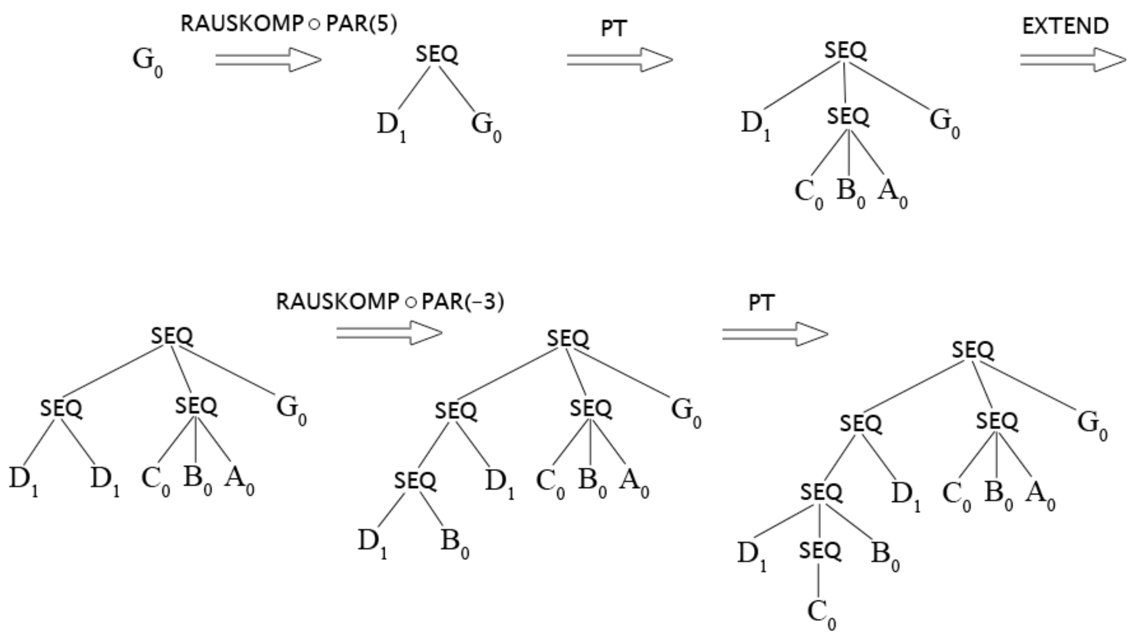

FIGURE 2.4: THE HISTORY OF CONSTRUCTION OF SMOLIAR'S ANALYSIS

[56] By composing these operations, I gloss over the fact here that there are intermediate steps with SIM vertices representing simultaneous events. This aspect of Smoliar's system reflects the idea that arpeggiations are composed-out simultaneities.

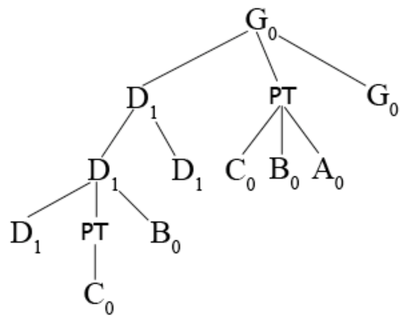

FIGURE 2.5: SMOLIAR'S ANALYSIS WITH ADDED NON-TERMINAL VERTEX LABELS

However, the operation PT doesn't work like a phrase-structure rule. Consider the second transformation: the operation adds a non-terminal vertex between the notes D_0 and G_0 to show that the children of these vertices have an equal prolongational dependence on both D_0 and G_0. Therefore, semantically PT is more like the top-down construction of a MOP demonstrated in part one. This is also true of Smoliar's lower auxiliary and upper auxiliary operations (LA and UA). Thus, Smoliar's model strives to represent linear prolongations (neighboring and passing motions) as per the dynamic sense of prolongation, where the prolonging event relates equally to the preceding and following events, while it represents arpeggiation and repetition in a more standard phrase-structure fashion, as a relationship between two foreground events with one being prolongationally prior to the other (—that is, one event arises earlier in the prolongational history than the other).

Unfortunately Smoliar's system is too limited in the types of melodic relationships it can express. For example, it's impossible to accurately represent Renwick's analysis of the C major fugue subject as I explained it in the section "Maximal Outerplanar Graphs" in part one. According to that description, an overall passing motion ${}_1$G-${}_3$F-${}_5$E is elaborated by incomplete neighbors ${}_2$A and ${}_4$D and passing tones ${}_2$G and ${}_3$E. Figure 2.6 derives the analysis of ${}_1$G-${}_2$A-${}_3$F-${}_4$D-${}_5$E in Smoliar's model (illustrating Smoliar's method for generating incomplete neighbors), and figure 2.7 gives non-terminal vertex labels for the resulting phrase-structure tree. It is impossible at this point to add passing tones between ${}_2$A and ${}_3$F or between ${}_3$F and ${}_4$D, because the PT

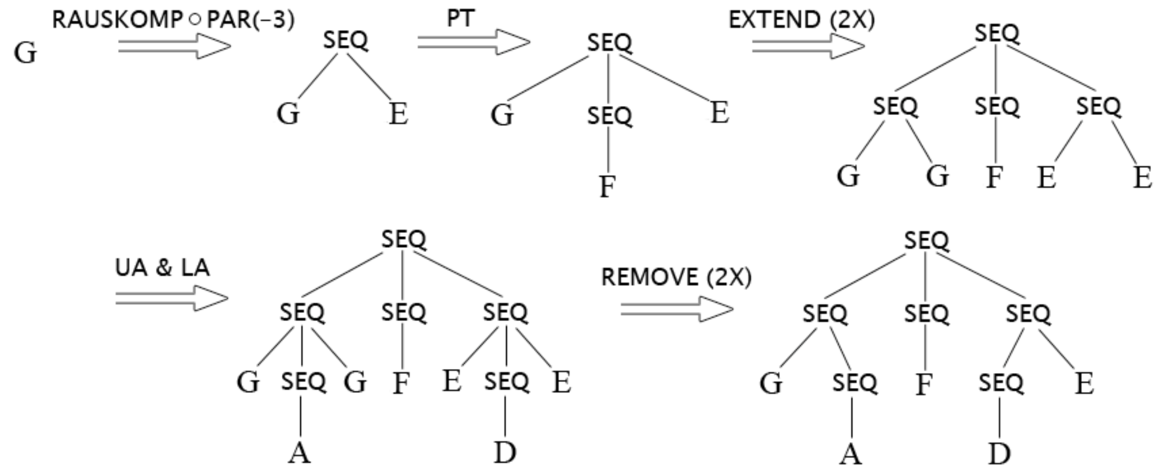

FIGURE 2.6: A PARTIAL ANALYSIS OF THE FUGUE SUBJECT IN SMOLIAR'S MODEL

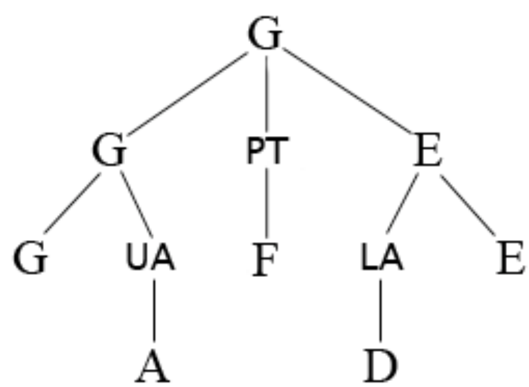

FIGURE 2.7: NON-TERMINAL VERTEX LABELS FOR THE ANALYSIS OF FIGURE 2.6

operation cannot apply to events such as these that aren't siblings in the tree, even if they are consecutive in the order of foreground events implied by the tree.

Another approach might be to adopt the more complete version of Renwick's analysis of the fugue subject that I gave in the discussion of unfolding in "Refinements of the MOP Model." Figure 2.8 presents a tree with non-terminal vertex labels to represent such an analysis. (I leave it up to the reader at this point to reconstruct the prolongational history that the non-terminal vertex labels imply.) Smoliar's system fares better in representing such an analysis, but it is still impossible, once we have derived $_4$D as an

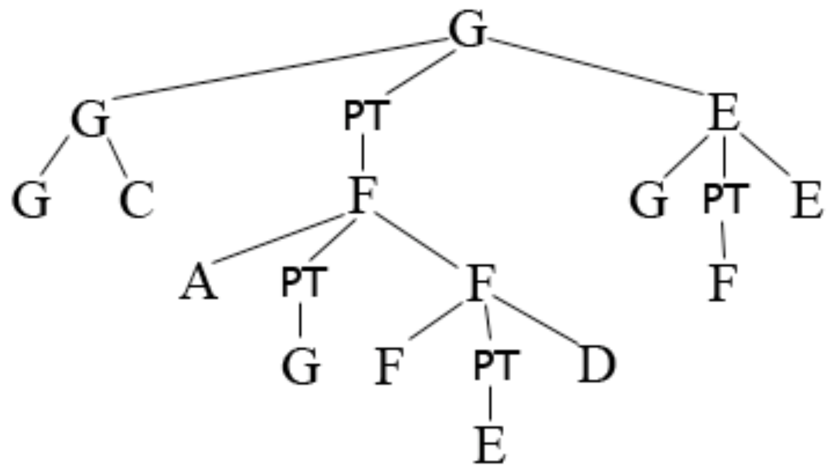

FIGURE 2.8: A DIFFERENT ANALYSIS OF THE FUGUE SUBJECT IN SMOLIAR'S MODEL

arpeggiation from $_3$F to assert the arpeggiation $_4$D-$_4$G or to fill this motion in with passing tones, as the MOP analysis of figure 1.22 does.

We can understand this more fully if we show the relationships of Smoliar's trees in the graph notation of MOPs. Figure 2.9 shows the sequence of events with slurs between notes with some horizontal harmonic relationship in the tree of figure 2.8 (above the note-names) and slurs between notes in some linear relationship below the note-names. The harmonic relationships are those between events that label some two vertices sharing an edge in the tree (—i. e., one is derived from the other through PAR and AUSKOMP or EXTEND operations in the prolongational history—), and the linear relationships are those between events labeling some two vertices that have an aunt/niece relationship in the tree (—one is derived from the other through PT, UA, or LA operations). Figure 2.10 rearranges these relationships visually in a MOP-like format with thick lines for arpeggiations and thin lines for linear motions. This illustrates that Smoliar's analyses include some prolongational building blocks similar to those of MOPs, the triangles that show passing motion within the arpeggiation of a third or neighboring motion from a repeated note, but unlike MOP analyses, these are linked together by sharing individual events rather than pairs of events.

Another difference between Smoliar's model and general phrase-structure model I've proposed is that the leaves of Smoliar's trees are, properly speaking, simply pitches

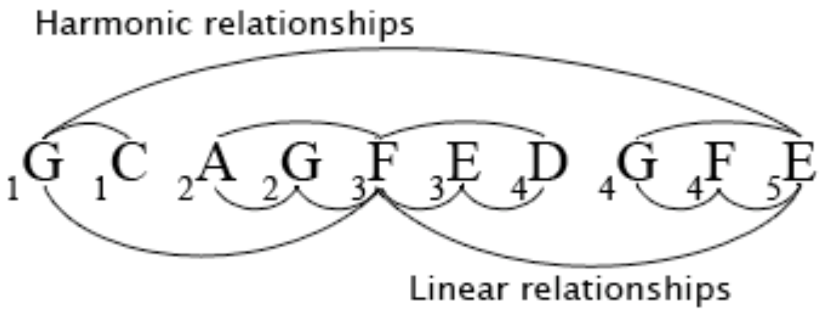

FIGURE 2.9: RELATIONSHIPS IMPLIED BY THE ANALYSIS OF FIGURE 2.8

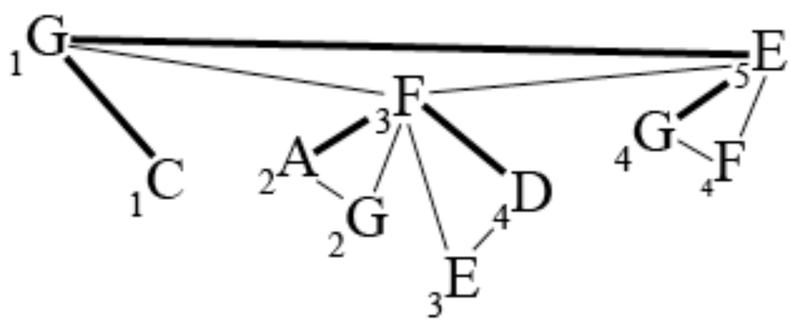

FIGURE 2.10: ANOTHER ARRANGEMENT OF THE RELATIONSHIP OF FIGURE 2.9

rather than events (which is why the note-names in figures 2.6, 2.7, and 2.8 are unindexed). Yet they still are meant to correspond to particular events in the music, so regarding them as events themselves is not a great distortion. This change in the definition of events only really affects Smoliar's EXTEND operation, which replaces a single pitch with two copies of the pitch in sequence. In the general phrase-structure model I propose here the copies have to refer to distinct musical events, one of which is also associated with the parent event, whereas Smoliar's analyses don't say that either of the identically pitched prolonging events is associated with the parent any more than the other.

The general phrase-structure model is also not equivalent to Rahn's model, but with some minor qualifications it is equivalent to a limited version of it. In Rahn's system, which I'll explore in more detail in part three, the collection of "events" is not a sequence of labels but a set of pitches, each with a time-point of initiation and release.

Thus, the raw material of Rahn's model is essentially a schematic representation of the pitch-time information of a score. Prolongations are then represented as transformations of this schematic score, and they come in two types: *neighbor prolongations* and *arp-prolongations*. The neighbor prolongations behave like phrase-structure rules. In an arp-prolongation, on the other hand, a sequence of pitches at one level becomes a simultaneity of pitches at a next-higher level. The phrase-structure model of this part won't include anything like these arp-prolongations. This is necessary because arp-prolongation introduces a contrapuntal element into the analysis, which is properly the subject of part three. In that part (in "The Representation of Counterpoint in Rahn's Model") I'll consider ways of incorporating arp-prolongations into a phrase-structure.

However, Rahn's system includes all the tools for constructing an analysis without arp-prolongations. If we define all the necessary chords as reference collections, then we can define arpeggiations as neighbors with respect to one of these chordal reference collections according to Rahn's definition VC. For instance, in figure 2.2, the necessary chordal reference collections are a tonic and subdominant chord in C major. According to this and the previous modification of Rahn's system, one can view every edge in figure 2.2 as representing an NC-prolongation relationship.

However, there is one complication in that Rahn's system requires the arp operation to deal with repeated notes because it's impossible to define two notes of the same pitch as neighbors using Rahn's definition VC. So to fit the general phrase-structure model Rahn's definition of a "neighbor" would need to be modified so that two notes with the same pitch can be neighbors, or we must add a "repeated note" operation.

Second, the events labeling non-terminal vertices in Rahn's theory differ in substance from the events below them. In particular, an event in Rahn's system includes a time-point of initiation and release, and the non-terminal event, while it adopts the pitch of one of its children, adopts the initiation point of the earliest child and the release point of the latest, so that it is not properly identical to either. This is an obstacle to the comparison of Rahn's model with the MOP model, since the MOP model applies to events as labels distinguished by their place in a sequence but lacking such specific durational characteristics.

Yet, in the simplified version of Rahn's system given by the general phrase-structure model, where events are defined as they are for MOPs, the initiation and release points are unambiguously implied by the labels and their position in the tree (because the labels refer to a particular piece of music, which includes all the necessary durational information). In this sense, switching the way the model labels events is trivial, since one can derive pitch/time-point event labels unambiguously from the referent event labels.

The converse of this is not quite true: the pitch/time-point labels do not unambiguously refer to an event in the music when a repetition of a note is reduced via the repeated-note operation we needed to add above. The referent labeling requires that one of these repeated notes be identified as the prolonged event for the time-span, whereas Rahn's method simply asserts that the pitch is prolonged, not a particular event with that pitch. (This is essentially the same situation that arises with respect to the EXTEND operation of Smoliar's model).

Thus, Rahn's model of prolongation in "Logic, Set Theory, Music Theory" is different than the general phrase-structure model I am discussing here, in that prolongation in Rahn's model is a relationship between sets of pitches that correspond to some time-span in the music but not necessarily any particular event within that time span, rather than a relationship between particular foreground musical events. However, the general phrase-structure model resembles Rahn's model closely enough that the comparisons between it and the MOP model also illuminate the differences between Rahn's model and the MOP model.

Comparison of Chomsky's Phrase-Structure Grammar to the General Phrase-Structure Model of Prolongation

There are some significant differences between the phrase-structure approach to grammar and the phrase-structure approach to musical analysis. In the linguistic model, the constituents of the terminal string, which are actual words, are distinguished from all other constituents, which are abstract grammatical labels. Furthermore, they are derived from these grammatical labels by rules that replace a single abstract object with a single concrete object and are of a different sort than those that replace grammatical labels with other grammatical labels. In the musical case, all objects and rules are of the same kind.

Thus the situation in language and music would be more analogous if a sentence consisted of a string of grammatical labels such as $N + V + D + A + N$. Or, better yet, if we could write a phrase structure grammar where all objects were actually words of the sentence itself, so that we had such rules as "Bach" → "Bach is", "is" → "is composer", and so forth. The closest we can come to a solution of this problem is to make careful distinctions between multiple vertices labeled by the same event when interpreting musical phrase-structure trees. For instance, the note $_5$E is a label on three different vertices in figure 2.2, each at a different level of abstraction, or reductional level; one vertex refers to a foreground note $_5$E, another refers to $_5$E as a reduction of the phrase $_4$G-$_4$F-$_5$E and another as a reduction of $_4$D-$_5$E (at a level "above" $_4$G and $_4$F).

The musical grammar of Allan Keiler (1979), a forerunner to Lerdahl and Jackendoff's theory, is closer to Chomsky's theory in this way. Keiler's theory of prolongation, however, is not Schenkerian in that its prolongations are not derived from the melodic motion of voices in counterpoint, but simply as relationships between roman numeral labels for chords. Yet this method does allow Keiler to differentiate between abstract labels such as "Tonic," "Dominant," "Dominant Prolongation," "Tonic Completion," and the less abstract terminal labels "I," "V," and so on.

Another significant difference between the linguistic and music-theoretic phrase structures is that in linguistics there is a notion of grammaticality that the phrase-structure grammar is meant to capture. In music, there are no distinctions to be drawn between "well-formed" and "ill-formed" musical phrases that compare to grammaticality both in definiteness and comprehensiveness. That is to say, while it may be possible to find sentences (in English, for example) where there is some question about their grammaticality, the vast majority of sentences are unambiguously grammatical or ungrammatical. Furthermore, the distinction applies to novel sentences, sentences that have never been spoken before or at least heard before by a particular listener. The first test of a model of grammar is that it is able to distinguish between obviously grammatical and obviously ungrammatical sentences.

In music, on the other hand, there are no such clear-cut distinctions between "correct" and "incorrect" musical phrases. Lerdahl and Jackendoff, in their attempt to make a *generative* theory of tonal music (i. e. a theory that distinguishes between tonal music and not-tonal music the way a generative grammar distinguishes between grammatical and non-grammatical sentences) appeal to stylistic norms as a correlate to grammaticality. However, they only show how the theory generates existing pieces of tonal music, not how it might exclude pieces of not-tonal music or generate novel pieces of tonal music. One must admit that it not even reasonable to hold this out as an eventual goal of the theory, since no matter how we draw the stylistic boundaries we think a generative theory will demarcate ("all tonal music," "the eighteenth-century keyboard sonata first movement," "the Chopin Nocturne," et c.) there is no obvious way to decide whether the theory has succeeded. If the criterion is the agreement of expert listeners, we cannot hope for the kind of broad unanimity one gets with judgments of grammaticality. If the criterion is circumstantial in nature, as in "that the piece is one composed by Chopin," then we are dealing with a finite category and "novel utterances" are impossible. If our criterion is theoretical in nature, such as "that the piece follows the rules of modal counterpoint," then we would simply be begging the question.

For this reason, authors such as Rahn (1989b) quite reasonably reject any such "generative" aspirations in their formal theories. Nor is this especially a problem: even in the case of grammars, making grammaticality distinctions is simply a requirement of a sound theory, not the purpose of the theory itself. The primary purpose of a theory of grammar is to give an intuitively appealing analysis of language, just as in a theory of music. However, building a theory of grammar that correctly distinguishes grammatical sentences is difficult enough that the primary way of evaluating such a theory is testing whether it "works." Whether the theory is intuitively appealing is a secondary concern. On the other hand, intuitive appeal is the primary concern in a music theory, so that the mode of evaluation for the theories of Rahn, Smoliar, and Lerdahl is quite different from the mode of evaluation of Chomsky's theory.

Comparisons between the MOP and Phrase-Structure Models of Prolongation

It probably appears on the surface that the formalization by MOPs and the phrase-structure formal models differ fundamentally and are incommensurable. In fact, it is possible to show that the two approaches in some sense are comparable apples-to-apples.

First (in “The MOP Model of Prolongation as a Binary Phrase-Structure Model”), I'll show that a MOP analysis can be thought of as a sort of phrase-structure analysis, except than the objects of the phrase-structure are not the events themselves, but intervals between events. This comparison corresponds to the difference between static and dynamic concepts of prolongation as I describe it in part one: static prolongation is a relation between events whereas dynamic prolongation is a relation between motions from one event to another.

For a more direct comparison, I will consider phrase-structure models as a set of assertions about the relative backgroundness of events and the possible melodic reductions of the music (in “Relative Backgroundness in Phrase-Structure Models”). In order to do so, it’s necessary to separate phrase-structure models into different types. Some detailed consideration of different formalisms and their semantics in the sections “Unstratified Phrase-Structure Models” and “Backgroundness Partial Orderings for Phrase-Structure Trees” shows that the most plausible and useful types of phrase-structure models fall into the categories of “stratified” or “strictly binary” models. In the latter section, I present some detailed formal considerations to eliminate another possible type of unstratified model. These considerations are interesting but the reader may skim over them and still get a basic grasp on why I limit unstratified models to the “strictly binary” type.

My eventual goal in this discussion of relative backgroundness is to construct a meaningful mapping from phrase-structure to MOP analyses. A mapping (or function) is a way of associating each object in one set (e.g. the set of labeled MOPs), called the “domain,” to exactly one object in another set (e.g. the set of phrase-structure trees), called the “range.” In other words, it’s a process of “getting from x (in the domain) to y (in the range)”, where I never need to make a decision that affects what y is. We will be especially interested in *bijective* mappings. A bijective mapping is one that can be inverted. That is, if I have a bijective mapping from set A to set B, I can define a mapping from B to A such that if the first mapping maps a to b, the second maps b to a. (Or, to put it differently, if I apply the two mappings in sequence, I always get back to the object I started with). A bijective mapping is interesting because it defines a one-to-one correspondence between two sets with the same number of members.

In constructing such a mapping, it's particularly important to keep in mind the difference between the abstractness of objects in the two analytical models, which is a critical barrier to their comparison. In the analysis by MOP, the objects in the graph are all literal: a particular musical event labels each node. Contrariwise, in the analysis by phrase-structure tree, the graph, properly speaking, shows literal events only in the leaves of the tree. Thus, in phrase-structure trees we have to distinguish between foreground events (the leaves of the tree) and abstract copies of foreground events (the non-terminal vertices of the tree), while in MOPs all events are foreground events. The non-terminal vertices of a phrase-structure tree show a literal event standing in for or governing a passage of music, and are thus abstract in nature.

The first comparison (in "The MOP Model of Prolongation as a Binary Phrase-Structure Model") can be expressed in terms of a bijective mapping from binary phrase-structure trees to MOPs. However, this mapping ignores the nature of the vertex labels so it isn't properly a mapping from phrase-structure *analyses* to MOP analyses. Therefore, this comparison skirts the abstractness-of-objects problem by finding the nature of the objects itself to be the distinction between the two analytical methods.

The following sections look for common ground between the analytical perspectives in relative backgroundness and reduction-lists for the purpose of constructing a mapping that preserves the analytical objects (the events). The method of surmounting the abstractness-of-objects problem here is to view phrase-structure trees in terms of their foreground events—that is, the events that label the leaves of the tree. Therefore much of the discussion in these sections is concerned with deriving relationships between these foreground events from the form of the phrase-structure tree. This allows us ultimately to compare these relationships with those between events as represented in the MOP, and hence arrive at a better understanding of the kinds of relationships proposed in the two models and how they differ.

The comparison of relative-backgroundness assertions through prolongation trees discussed in “Backgroundness Partial Orderings for Phrase-Structure Trees” and “Backgroundness Partial Orderings for MOPs” proves unsatisfactory because of its misrepresentation of the semantics of the MOP model. However, prolongation trees serve as a good starting point for a more successful comparison on the basis of reduction lists, described in the section, “Comparing MOPs and Phrase-Structure Trees through Reduction-Lists.” This section also goes into many technical details to semantically justify the particularities of the mapping. For the interested reader, all of the formal subtleties are worked out here. However, it’s also possible to skim over some of these technicalities and get a general idea of the mechanism of the mapping and its more significant semantic implications. The idea of “consecutivity” worked out here is particularly worthy of attention. This is discussed further in the following section, “Semantics of the Mapping from Phrase-Structure Trees to MOPs.”

The combinatorial discussions in “Combinatorial Comparisons of MOPs and Binary Phrase-Structure Trees” and “Unstratified Phrase-Structure Models” are revealing but not essential to understanding main narrative of this part.

In the end, the distillate of the following sections should be a more comprehensive appreciation of the differences between the dynamic and static concepts of prolongation introduced in part one and their implications. We will see that the concept of “possible reductions” gives the most secure common ground between the two concepts and can therefore be regarded as the essential core of the idea of prolongation. The differences between the two models, however, gets at the crucial issue of what it means for a particular event to appear in a particular possible reduction and the kinds of relationships between events that are asserted by a set of possible reductions.

The MOP Model of Prolongation as a Binary Phrase-Structure Model

The first comparison of the MOP and phrase-structure models is by means of a single bijective mapping, which I’ll call the edge-to-vertex mapping, that transforms a

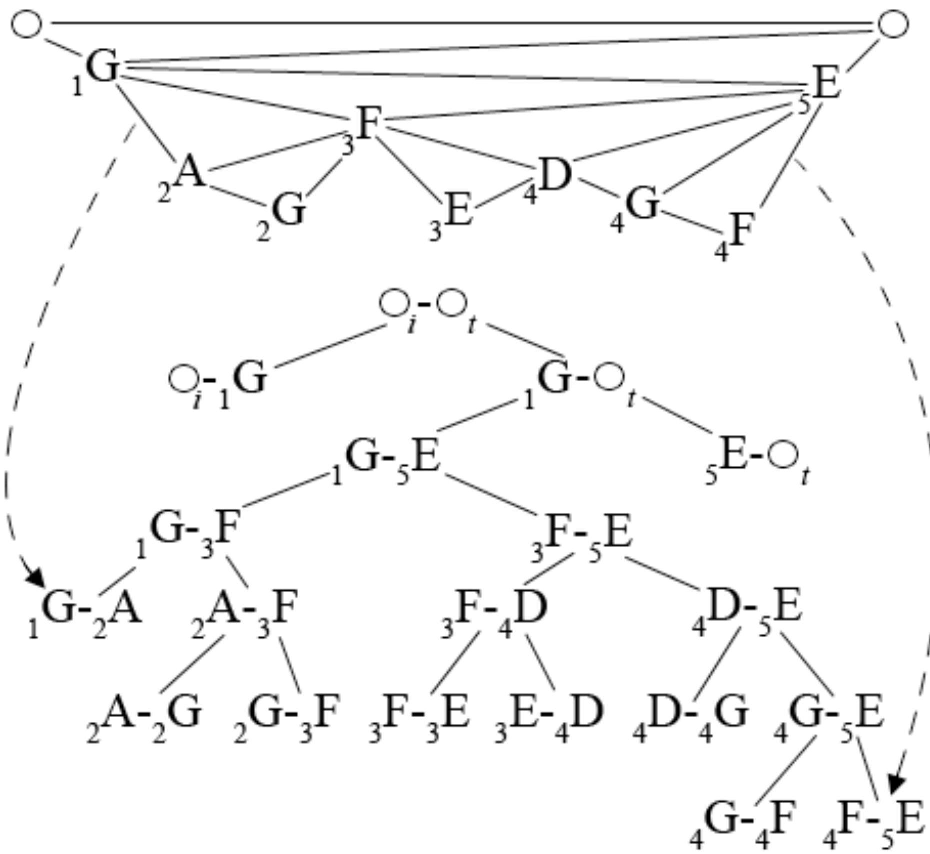

FIGURE 2.11: THE EDGE-TO-VERTEX MAPPING OF A MOP TO A PHRASE-STRUCTURE TREE

MOP into a binary plane tree. The mapping is demonstrated in figure 2.11 for the analysis of the C major fugue subject. For each edge in the MOP, there's a corresponding vertex in the binary planar tree. The vertices of the tree are therefore labeled with the notes that label the endpoints of the corresponding edge in the MOP. There is an edge between every pair of vertices in the tree whose corresponding vertices share a triangle in the MOP. For example, there's an edge from $_1$G-$_5$E to $_1$G-$_3$F, reflecting the fact that {$_1$G, $_3$F, $_5$E} is a triangle of the MOP.

This describes the tree obtained from the edge-to-vertex mapping as a simple graph. However, the tree is not only a simple graph but also a rooted plane tree. This means that we need to define a root of the tree and designate the children of each vertex as left and right children. Recall that the MOP is rooted by the oriented edge from the initiation to the termination event. The root of the tree is the vertex corresponding to this edge in the MOP.

Consider, then, the MOP as a directed graph as shown in figure 2.12. The orientation of the root edge is given, and the orientation of the other edges is defined byapplying a simple recursive rule: if two edges make up a triangle with an oriented edge, give them orientations so that the edge which shares a vertex with the tail of the oriented edge points away from it, and the edge which shares a head with the oriented edge points towards it. It is easy to see from figure 2.12 that this unambiguously defines an orientation to all edges of the MOP, and that these orientations correspond to precedence in the melody. Comparing figure 2.12 to figure 2.11 shows how these orientations define a left-to-right ordering in the binary plane tree. For example, ${}_1$G-${}_5$E has the children ${}_1$G-${}_3$F and ${}_3$F-${}_5$E. Since the arrow in the MOP points from ${}_1$G to ${}_5$E, ${}_1$G-${}_3$F is on the left and ${}_3$F-${}_5$E on the right.

From this perspective the MOP analysis is in fact a phrase structure analysis, one whose elements are not the notes of the melody, but the intervals between successive notes of the melody, which make up the leaves of the tree derived from the edge-to-vertex mapping. This is not exactly correct as I have labeled the tree in figure 2.11, since the leaves of the tree seem to have a mutual dependence which has nothing to do with the phrase structure; for instance, since the ${}_2$G-${}_3$F is the third leaf of the tree, reading from left to right, the fourth leaf must be an interval from ${}_3$F to some other note, even though ${}_2$G-${}_3$F and ${}_3$F-${}_3$E are distantly related in the phrase structure. However, this seeming dependence is easily eliminated as figure 2.13 shows: the first part of this figure presents

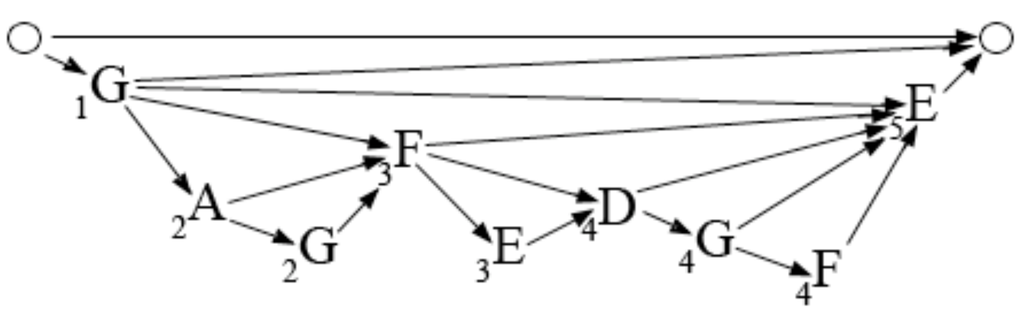

FIGURE 2.12: THE PRECEDENCE ORIENTATION OF THE EDGES OF A MOP

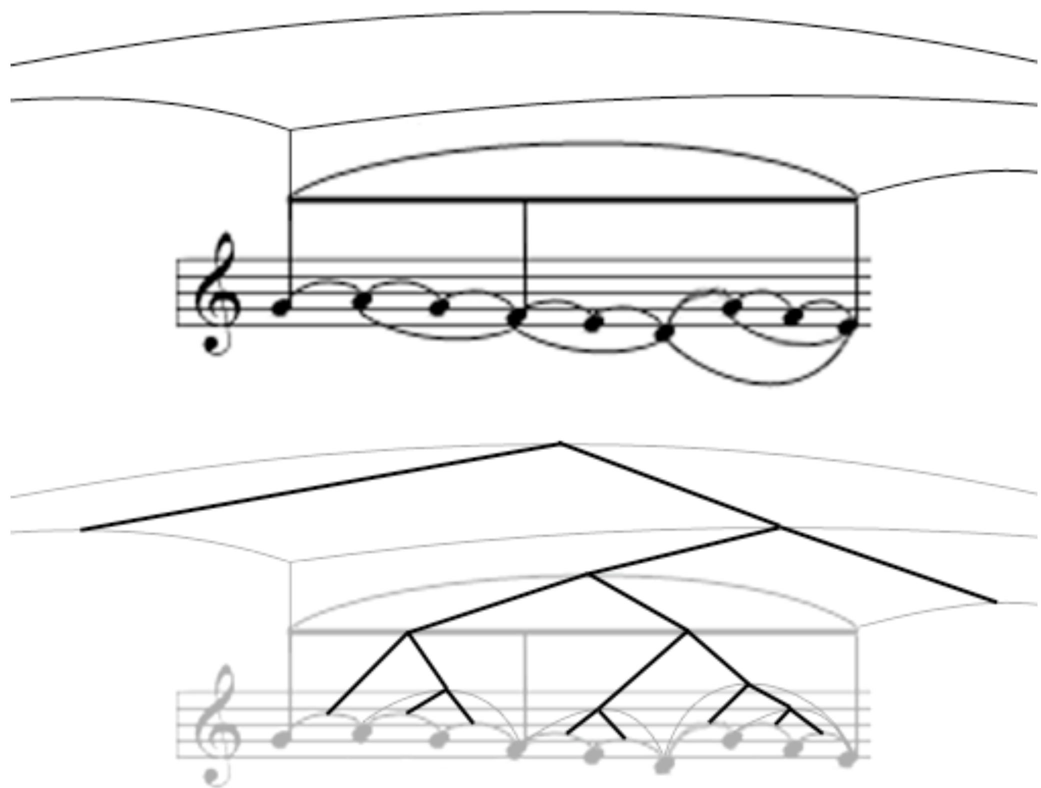

FIGURE 2.13: THE PHRASE-STRUCTURE TREE OF A MOP WITH SLURS AS VERTICES

the reduced fugue subject with slurs between all notes that share an edge in the MOP analysis, as well as slurs for initiation and termination edges. Considering these slurs as objects representing melodic motions, the relationship of direct containment between slurs defines a tree structure on these objects (where "direct containment" means: "slur A contains slur B and no other slur that also contains B"). The second half of the figure shows this tree explicitly with lines connecting each slur.

Combinatorial Comparison of MOPs and Binary Phrase-Structure Trees

As I mentioned above, the edge-to-vertex mapping is in fact a bijective mapping. This means that there are the same number of MOPs on n vertices as binary trees on (2n – 3) vertices (where 2n – 3 is the number of edges in a MOP—for instance, the MOP in figure 2.2 has 11 vertices and 2(11) – 3 = 19 edges, so the corresponding tree has 19 vertices) and the edge-to-vertex mapping gives a one-to-one correspondence between these.

Thus, the number of possible MOP and binary phrase-structure analyses for a given set of events are counted similarly. For a passage with n foreground events, the tree corresponding to a MOP analysis has $2(n + 2) – 3 = 2n + 1$ vertices (—$n + 2$ counts the vertex for each event plus the two formal vertices—), whereas a binary phrase-structure tree has $2n – 1$ vertices. Or, equivalently, the tree corresponding to a MOP analysis has $n + 1$ leaves for the $n + 1$ intervals between successive events, while a binary phrase-structure tree has n leaves. The number of plane binary trees is counted by a sequence called the "Catalan numbers." The number of plane binary trees with $n – 1$ leaves (or the number of MOPs on $n + 2$ vertices) is equal to the nth Catalan number, denoted C_n, which is given by the formula (2n choose n)/(n + 1) or $(2n)!/(n + 1)!n!$. The first ten Catalan numbers are 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796. (See Stanley (1997b), 216ff.)

However, this is not the complete story as far as binary phrase-structure trees are concerned: it only takes into account the way in which the events are grouped, but ignores the labels on the "abstract" vertices. The labeling of the leaves is given by the order of events, but if a vertex has children, it can be labeled with the event of either of its children. Therefore there are two choices of label for each non-leaf vertex, and since there are always $(n – 1)$ such vertices for a binary tree with n leaves, each binary tree with n leaves can be labeled in 2^{n-1} different ways. Therefore the number of conceivable binary phrase-structure tree analyses for a passage with n events is $2^{n-1}C_{n-1}$.

In the case of the phrase-structure tree corresponding to a MOP, the labeling of all vertices is given by the sequence of intervals and the form of the tree. So the number of conceivable MOP analyses on n foreground events is simply C_n.

It's useful to compare these numbers of possibilities to our analytical choices to see that the formal model accurately represents the analytical idea. However, at any reasonable length of musical passage, the number of possibilities is too enormous to come to terms with directly. However, the nature of the combinatorial series shown in table 2.1 makes it easy to characterize in a general way the proportionate change in

TABLE 2.1: A COMBINATORIAL COMPARISON OF MOPS AND BINARY PHRASE-STRUCTURE TREES

Foreground Events	MOPs	Binary Phrase-Structure Trees
1	1	1
2	2	2
3	5	8
4	14	40
5	42	224
6	132	1,344
7	429	8,448
8	1,430	54,912
9	4,862	366,080
10	16,796	2,489,344

number of possibilities as we add events. This proportion increases rapidly at small numbers of events and gradually levels off as it approaches an asymptotic value. In the case of the number of MOPs, this value is four. For binary phrase-structure trees, the value is eight.[57]

Focusing on the proportionate change in the number of analyses allows us to give a sort of recursive account of the choices made in arriving at a particular analysis. Imagine that someone has built the analysis by adding one event at a time and we're trying to reverse the process with only the knowledge of the resulting analysis looks like. Let's call the set of events that could have been added in the last step for a given analysis the "foreground-originating events." In a MOP, a foreground-originating event is one whose vertex is degree 2 (i.e. is incident on only two edges). The number of foreground-originating events in the resulting MOP tells us how many ways there are to get this analysis from one with one fewer event. Furthermore, there are (n – 1) ways to add an event to a MOP with n vertices, corresponding to the (n – 1) foreground intervals in the

---

[57] The convergent of four for MOPs is arrived at by expanding C_n/C_{n-1} via the formula $C_n = (2n)!/(n + 1)!\, n!$, which gives, after some manipulation, $4(n – 1)/(n + 1)$. Letting n go to infinity gives the value 4. For binary phrase-structure trees the value is $2^nC_n/2^{n-1}C_{n-1}$, which is obviously equal to $8(n – 1)/(n + 1)$. The added factor of two reflects the doubling of the number of possible labelings as we add leaves to the tree.

MOP. (That is, the new event can be placed in between any two adjacent events already present in the MOP).

Therefore, if r_n is the ratio comparing the number of MOPs on n vertices to the number of MOPs on (n – 1) vertices and f_n is the average number of foreground events in a MOP on n vertices, then $r_n = (n – 1)/f_n$, or $f_n = (n – 1)/r_n$. This means that the number of foreground-originating events in a large MOP is on average about a quarter of the number of vertices.

We can give a similar account for binary phrase-structure trees. A foreground-originating event in a phrase-structure tree is one that is prolonged by no other event and whose leaf in the tree has a leaf for a sibling. For instance, in figure 2.2, the foreground-originating events are ${}_3$E and ${}_4$F. ${}_2$G is not foreground-originating because its sibling ${}_3$F is not a leaf. ${}_4$G is not foreground-originating because it is prolonged by ${}_4$F. Notice that a foreground-originating event occurs always and only when there are two leaves that are siblings (although the actual foreground event could be either of these). The situation is the same for MOPs when we transform them into phrase-structure trees by the edge-to-vertex mapping. Here, the foreground events also occur exactly where there are two leaves that are siblings. For instance, in figure 2.11, there are three such pairs, ${}_2$A-${}_4$G / ${}_2$G-${}_3$F, ${}_3$F-${}_3$E / ${}_3$E-${}_4$D, and ${}_4$G-${}_4$F / ${}_4$F-${}_5$E. This tells us that ${}_2$G, ${}_3$E, and ${}_4$F are foreground-originating events. The only difference here is that we think of both of these leaves as foreground motions (trivial prolongational spans), whereas in the event-labeled phrase-structure tree, only one of the two leaves represents a foreground-originating event.

Therefore the number of foreground events is the same in MOPs as in binary phrase-structure trees. The combinatorial difference between the two comes from the fact that in the process of constructing a phrase-structure tree, we not only must choose from (n – 1) locations for each new vertex, but we also must choose whether to attach it to the preceding or following leaf. Thus there are twice as many choices for each step in the construction of a binary phrase-structure tree.

Thus the process of construction of a binary phrase-structure analysis, from this perspective, is quite similar to the process of constructing a MOP: at each step, the analyst precedes from background to foreground by choosing the interval of the current analysis into which to fit the next event. In the case of a phrase-structure tree, the new event must prolong an event already in the analysis, so the analyst must make an additional choice between the events making up the interval into which she has placed the new event, whereas in the case of MOPs, the new event is seen as prolonging that interval itself, so no additional choice need be made.

This comparison takes into account only phrase-structure analyses restricted to binary plane trees, as is the case in Lerdahl and Jackendoff's analytical method. Other authors, however, allow for a more general class of plane trees. This of course greatly expands the number of possible analyses. However, general plane trees in phrase-structure analyses should not deviate greatly from strict binary branching, so the comparison is relevant even for this analytical method. A phrase-structure tree can deviate from strict binary in two ways: by having vertices with single children (1-branchings) or by having vertices with more than two children.

The first possibility is demonstrated in figure 2.14, which adds vertices with single children to the analysis of figure 2.2. In this case, the $_5$E vertex just below the root has only one child, and the $_3$F below the second $_1$G also has only one child. These are called 1-branchings. The other 1-branchings in the tree serve to extend each event to the foreground level. 1-branchings are common in the analytical methods outlined by Rahn and Smoliar. Note that such a vertex is always labeled with the same note as its single child, so if we are interested in the way in which the edges of the graph represent particular notes prolonging others, removing a vertex with a single child removes no such information from the graph. The reason for having such vertices in the analysis is that the distance of a vertex from the root may be used to represent the level (of relative backgroundness) of the note labeling the vertex. For instance, the purpose of using the analysis of figure 2.14 may be to show that the note $_4$D arises at a more foreground level than $_3$F, a distinction that is not made in the analysis of figure 2.2. Without this notion of "level" it would be impossible to make such a comparison between these vertices, since they are in entirely different parts of the tree.

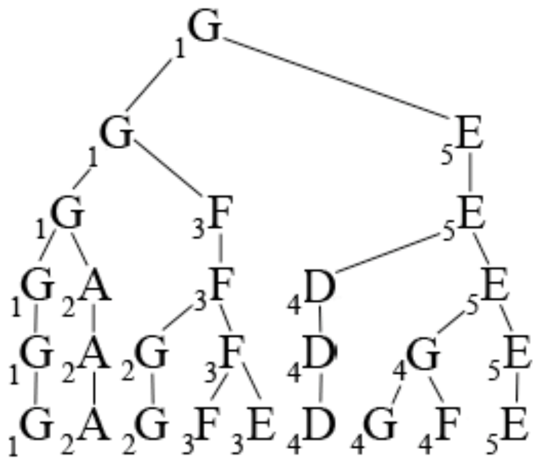

FIGURE 2.14: A PHRASE-STRUCTURE TREE WITH 1-BRANCHINGS

The second possibility is that a vertex has more than two children. For instance, in figure 2.15 the note $_3$F has three children, $_2$G, $_3$F, and $_3$E. The difference between this analysis and a strictly binary one such as that of figure 2.2 is that in figure 2.15, we make no choice between $_2$G and $_3$E as to which is the more background event. Thus, allowing a vertex to have more than two children has the opposite effect of allowing 1-branchings: it allows the analyst to avoid making a relative-backgroundness comparison between two events. Obviously, there is a limit to how much an analyst would want to do this. I will consider some precise limitations below.

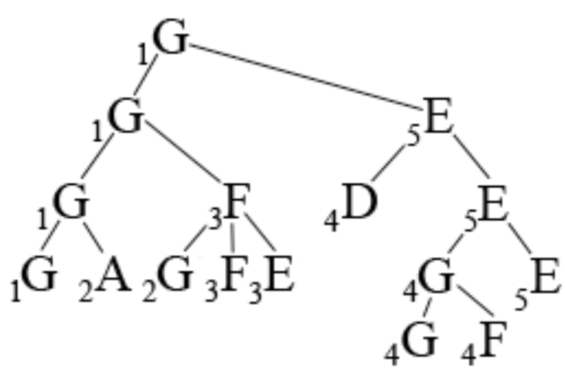

FIGURE 2.15: A PHRASE-STRUCTURE TREE WITH A 3-BRANCHING

Relative Backgroundness in Phrase-Structure Analyses

For a second comparison between the MOP approach to formalized analysis and the phrase-structure approach, let's return to the issue of relative backgroundness or level of an event. As I indicated above, the "events" of a MOP are comparable to the foreground events of a phrase-structure tree, since the phrase-structure concept of events is more general, including abstract events. Therefore, in the case of phrase-structure trees, we aren't interested so much in the relative backgroundness of "events," but the relative backgroundness of the origin of foreground events.

More precisely, let an *origin-event* be the root of the phrase-structure tree or any event whose parent has a different label. Then the *origin-event for some event x* is the earliest origin-event ancestor of x. (In other words, the origin-event that has the same label as x). It is not hard to see that there is a one-to-one correspondence between the foreground events of the tree and their origin-events. I will develop the relative backgroundness relationships of phrase-structure trees, then, in terms of origin-events, it being understood that the foreground events inherit these relationships from their corresponding origin-events.

I have already pointed out that it is possible in the analytical systems of Rahn and Smoliar to represent the analytical level of an event with that distance from the root of the vertex labeled by the event. Here, the "distance" between two vertices is the length of the path between them. (In a general graph, distance would be the length of the *shortest* path between two vertices, but in a tree there is only one such path.) Thus, we can assign an integer to each origin-event representing its level under this interpretation. However, the integer itself has no absolute meaning—for instance, level 3 might contain only background events in one analysis and in another analysis might contain the most foreground events. Rather, the level numbers are a means of comparison between two events, to assess their relative backgroundness. The numbers might be thought of as an indexing of melodic reductions, so that if object A occurs at level n and object B occurs at level m where m > n, then A is more background than B.

For example, the analysis of figure 2.14 above asserts six levels. Every non-terminal event has a similarly-labeled event as a child and all leaves are on the same level. So, for instance, since “$_5$E” labels an event in the level 3 reduction, it also labels an event in every more foreground reduction. One could perhaps relax the restriction that all events extend to the foreground (i.e. that all leaves are on the same level) in order to represent “imaginary notes” or, alternatively, something akin to the conjugation and agreement symbols used in phrase-structure grammars, but for the time being we should retain the restriction in order not to digress. Thus, we can characterize a foreground event by its level of origin—i. e. the earliest level at which a similarly labeled event occurs. For example, the foreground event $_1$G originates at level one while the foreground event $_3$F originates at level three in figure 2.14.

Note that the choice of what exactly constitutes an analytical level is up to the analyst. For instance, the analyses in figure 2.16 are similar except that the first includes fewer analytical levels. That is, both analyses share a basic set of reductions, but the first makes more detailed assertions about the relative backgroundness of different prolongations. Thus, if one says “$_3$E originates three levels below $_5$E” in the second analysis, this is roughly the same claim as “$_3$E originates seven levels below $_5$E” with reference to the first analysis. Therefore, in general, even the numerical difference

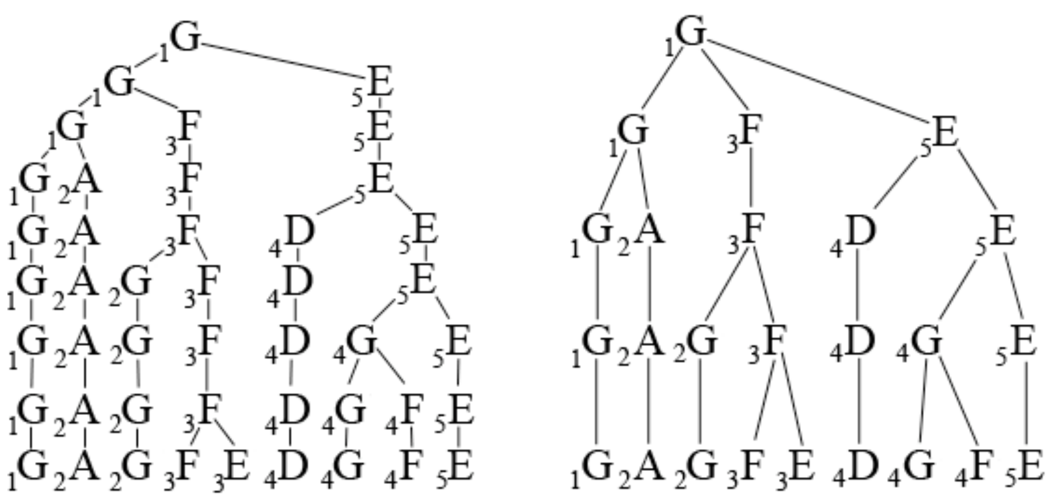

FIGURE 2.16: TWO SIMILAR PHRASE-STRUCTURE ANALYSES WITH DIFFERENT NUMBERS OF LEVELS

between levels has no absolute meaning, and we should regard "analytical level" in this context as an ordinal scale—i. e., one in which assertions such as "m is greater than n" or "m is equal to n" have meaning, but numerical assertions such as "m is five" or "m is two greater than n" have no meaning.

That being said, it would be possible to regard "relative backgroundness level" as an interval scale given a suitable set of restrictions on how one can get from one level to the next. (In an interval scale, assertions such as "m is five" have no absolute meaning, but assertions such as "m is two greater than n" do have some such absolute meaning). Rahn's analytical system is a good example of how such a set of rules might look. Since it's enough for us to focus here on the structure of the analysis itself, I won't discuss such systems of rules, though they're the primary focus of Rahn, Smoliar, and Lerdahl and Jackendoff. The important point here is that level distinctions in phrase-structure analytical systems are primarily ways of comparing events in terms of relative backgroundness. The purpose of adding a vertex with a single child in figure 2.14 is not to say that ${}_4$D arises at level 4 rather than level 3, because "level 3" and "level 4" have no absolute meaning. Rather, figure 2.14 asserts that ${}_4$D arises at a more foreground level than ${}_3$F, and at the same level as A.

If we consider the matter more carefully, however, there are a number of different ways in which two origin-events can be distinguished by relative backgroundness, each of which is semantically quite different. These semantic differences are plowed over by using only differences in distance from the root as a way of comparing events. To recover these differences, consider the following division of relative backgroundness distinctions into three categories:

(1) The strongest relative backgroundness distinction is between an origin-event and the origin-event of its parent. For instance, there is an edge in figure 2.14 between a ${}_1$G event and the origin-event for ${}_5$E. This indicates that ${}_5$E prolongs ${}_1$G. This distinction also extends to any ancestor of an origin-event. For instance, in figure 2.14 ${}_4$D prolongs ${}_5$E while ${}_5$E prolongs ${}_1$G, making ${}_1$G more background than ${}_4$D. Thus, the (1)-relation can be stated succinctly:

Origin-event x is type-(1) more background than origin-event y iff x is an ancestor of y in the phrase-structure tree.

(2) Weaker, but also important, are the relations between two events that are not (1)-related but each prolong a third event at a different level. For instance, in figure 2.14, $_3$F and $_5$E both prolong $_1$G. However, the tree is drawn so that the origin-events of $_3$F and $_5$E are adjacent to different $_1$G-events. In particular, $_3$F prolongs $_1$G at a more foreground level than does $_5$E. Therefore, the phrase-structure analysis asserts that $_5$E is more background than $_3$F. Moreover, if A is more background than B in this sense, the distinction should also carry over to any events prolonging B. Therefore, a good definition of the type-(2) backgroundness distinction is:

Origin-event x is type-(2) more background than origin-event y iff x has a sibling z which is not itself an origin-event and is an ancestor of y.

(3) Finally, the weakest distinctions are those made between remote events by means of measuring a distance from the root of their origin-events, as we did in figure 2.14 for $_4$D and $_3$F. By "remote" here, I mean those for which it is impossible to make distinctions of types (1) or (2). More precisely, let two events, x and y, be *remote* iff they are not (1)-related and the parents of the origin-events of x and y do not themselves share the same origin-event. (In other words, if we move upwards in the tree from x and y one step beyond their origin-events, we arrive at differently labeled events). As we will see below, if two events are not comparable in the tree by (2) but also not remote, it is possible to change the tree to make a type-(2) distinction between these events while preserving all the other type-(1) and type-(2) distinctions. Here is the definition of the type-(3) backgroundness distinction:

Origin-event x is type-(3) more background than origin-event y iff all ancestors of y that aren't ancestors of x are remote from x and the distance from the root to x is smaller than the distance from the root to y.

By restricting the possible types of phrase-structure trees, it is possible to eliminate each of these weaker backgroundness distinctions. This is the sense in which distinctions of type (3) are weaker than those of type (2) and those of type (2) are weaker than those of type (1). An analytical framework that excludes trees with 1-branchings eliminates distinctions of type (3) without affecting distinctions (1) and (2). That is, under this framework, after having drawn the tree in order to make relative backgroundness distinctions of types (1) and (2) between events, the analyst has no control over type-(3) relations between remote events. Consequently, if one imposes a restriction against 1-branchings, there is no reason to read type-(3) distinctions from the tree.

Similarly, allowing a maximum of two vertices per event-label eliminates distinctions of type (2) without affecting distinction (1). That is, if the type-(1) relations are considered as a given, the only way to add type-(2) distinctions between events which are not (1)-related is to have at least three manifestations of some event-label in the tree.

Clarifying these semantic distinctions between types of relative backgroundness makes the job of comparing MOPs and phrase-structure trees much easier. First, consider the weakest type of distinction. This type of distinction is not possible in Lerdahl and Jackendoff's system of analysis, but is quite a salient feature of Rahn's. In analysis by MOP, these "level" types of distinctions are impossible to infer from the structure of the graph. The MOP model of prolongation asserts that distant vertices are incomparable as far as any prolongational relationship is concerned. Or, to put it differently, it is an essential feature of the MOP approach that the order of presentation of distant events in a sequence of melodic reductions is unconstrained. (I will define "distant" more precisely below). In Rahn's system, the analysis prescribes a definite sequence of melodic reductions. A MOP, on the other hand, gives a large set of possible sequences of melodic reductions, differing precisely in the order in which distant vertices appear. (Note that this does not mean, either in Rahn's system or in a sequence of melodic reductions consistent with a MOP analysis, that there must be a total ordering on the appearance of events in the sequence. However, in Rahn's system the analysis must definitely fix which events appear simultaneously in the sequence, whereas a MOP could be interpreted so that distant events appear at the same time or at different times.)

For example, the analysis of figure 2.14 (reproduced in figure 2.17) might be a possible representation of an analysis using Rahn's model. This analysis asserts that there are six reductions of the fugue subject: it can all be thought of as a prolongation of

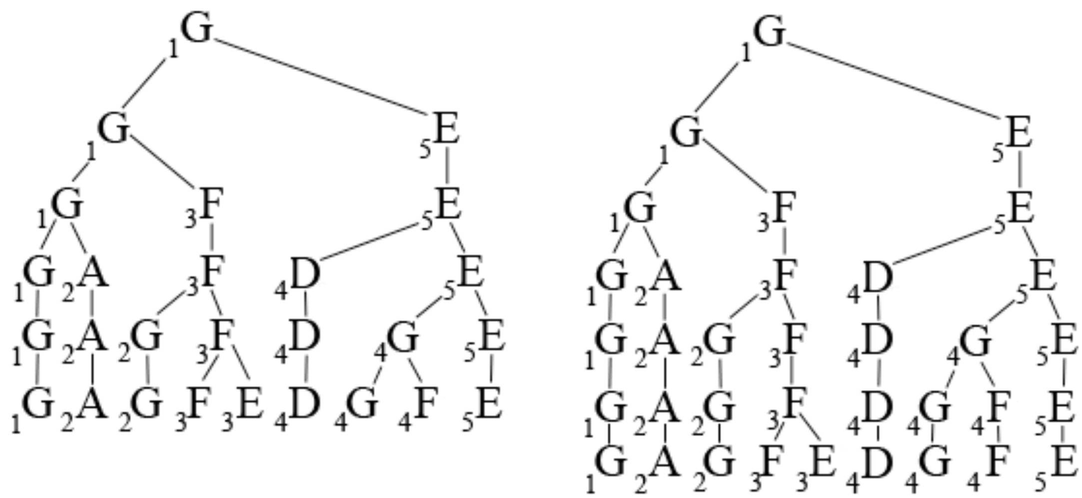

FIGURE 2.17: THE PHRASE-STRUCTURE TREE OF FIGURE 2.14 AND A SIMILAR TREE WITH ONE MORE LEVEL

${}_1$G, it's an arpeggiation from ${}_1$G to ${}_5$E, it's a stepwise descent, ${}_1$G-${}_3$F-${}_5$E, and so on. Of course, there are many other ways of fixing the possible set of reductions. For instance, the second graph of figure 2.17 is like the first, but it asserts that the stepwise descent from ${}_4$G is more background than the one from ${}_3$F.

In the MOP of figure 1.13 (also in figure 2.11), on the other hand, all of these melodic reductions (with at least three notes) occur as cycles: ${}_1$G-${}_3$F-${}_5$E], ${}_1$G-${}_2$A-${}_3$F-${}_4$D-${}_5$E], ${}_1$G-${}_2$A-${}_3$F-${}_4$D-${}_4$G-${}_5$E], ${}_1$G-${}_2$A-${}_2$G-${}_3$F-${}_4$D-${}_4$G-${}_5$E], and ${}_1$G-${}_2$A-${}_2$G-${}_3$F-${}_4$D-${}_4$G-${}_4$F-${}_5$E]. It's even possible to have two reductions that couldn't possibly occur in the same phrase structure tree: for example ${}_1$G-${}_2$A-${}_3$F-${}_5$E] and ${}_1$G-${}_3$F-${}_4$D-${}_5$E]. A phrase-structure tree would force the analyst to choose whether ${}_2$A is more background than ${}_4$D, ${}_4$D more background than ${}_2$A, or if they are both equally background. Therefore, a MOP makes fewer assertions about what reductions of the passage are possible.

Unstratified Phrase-Structure Models

Because an analysis with 1-branchings, as those in figure 2.17, separates events into well-defined layers, or levels, I will call these "stratified models." Both Rahn's and Smoliar's models of prolongation are stratified. It is also possible to assert an unstratified model with phrase-structure trees. For instance, Lerdahl and Jackendoff's model of prolongation is unstratified. The formally distinguishing characteristic of an unstratified phrase-structure model is that its phrase-structure trees do not have 1-branchings, because the only semantic purpose of a 1-branching is to increase the distance to the root for some event below the 1-branching. In unstratified phrase-structure models the distance of a vertex from the root is not a relative measure of backgroundness. Thus, for instance, all of the reductions I mentioned above as ones that could be found in the MOP of figure 1.13 (2.11) are also consistent with the phrase-structure tree of figure 2.2 when it is considered in the framework of an unstratified model.

The advantage of an unstratified model is that the analyst is not forced to make particular decisions as to whether events that may be far apart in the music arise at the same or different levels. Stratified models assert that a specific set of reductions is a necessary condition for the analysis of prolongations, so an allowance for ambiguity in exactly how a set of reductions may be formulated can only be made through a comparison of different possible analyses of the same passage.

I pointed out above that phrase-structure trees without 1-branchings can also deviate from strict binary branching by having vertices with more than two children, but also that such deviations should be minimal in the case of musical analysis. Let's examine in more detail why this is so.

First, let's consider the situation combinatorially, as we did to compare MOPs and binary trees. Table 2.2 compares the number of binary trees to the total number of unstratified phrase-structure trees, first ignoring the possible labelings (the second and third columns) and then including them (the fourth and fifth columns). The series in the third column is called the Schröder numbers. (See Stanley (1997b), 176-8).[58]

Table 2.2 shows that the total number of unstratified phrase-structure trees grows at a significantly faster rate than the number of binary phrase-structure trees. In the

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[58] My method of enumeration in the third and fifth columns is to separate the total set of trees according to how many 2-branchings, 3-branchings, 4-branchings, and so on, that they have. The formula on page 34 of Stanley (1997b) then calculates the number of plane trees in each case. For the third column, each of these is multiplied by the number of possible labelings, given by $2^{(\text{\# of 2-branchings})}3^{(\text{\# of 3-branchings})}4^{(\text{\# of 4-branchings})}$ . . . .

TABLE 2.2: A COMBINATORIAL COMPARISON OF UNSTRATIFIED PHRASE-STRUCTURE TREES AND MOPS

Foreground Events	Unlabeled Binary Plane Trees	Unlabeled Unstratified Plane Trees	Labeled Binary Phrase-Structure Trees	Labeled Unstratified Phrase-Structure trees	MOPs
1	1	1	1	1	1
2	1	1	2	2	2
3	2	3	8	11	5
4	5	11	40	74	14
5	14	45	224	556	42
6	42	197	1,344	4,472	132
7	132	903	8,448	37,667	429
8	429	4,279	54,912	328,010	1,430
9	1,430	20,793	366,080	2,929,230	4,862
10	4,862	103,049	2,489,344	26,679,916	16,796

discussion above, I characterized the number of binary phrase-structure analyses in terms of a recursive process of construction, so that each factor of the number of possible analyses would correspond to a choice made in the process of analysis. Each step of the process consists of choosing an interval in the existing set of events to place a new event. Then one must also choose whether this event prolongs the preceding or following event. Finally, the number of foreground-originating events in the resulting analysis tells us how many ways there are to construct it from an analysis with one fewer event.

General unstratified phrase-structure trees differ from strictly binary in two ways. First, the average number of foreground-originating events in a general phrase-structure tree is greater than in a strictly binary one, which actually reduces the proportional change in the number of analyses in the general case. Second, when we add an event to a general phrase-structure tree, we may have more than two choices of how to attach this event to the tree. As in the case of binary phrase-structure trees, we can assert that it is a prolongation of the preceding or following event, but if either of these events has only leaves as siblings we can also attach it directly to the parent of the preceding or following leaf. Thus, there are between 2 and 4 choices of how to attach the new event.

This accounts for larger rate of growth for general phrase-structure trees. Yet it fails to address the question of what analytical reasons one might have for attaching an event to the parent of an adjacent leaf rather than attaching it to the leaf itself.

Consider the two triple branchings in the analysis of figure 2.18: $_3$F and $_5$E are prolonged at the foreground by $_2$G-$_3$F-$_3$E and $_4$G-$_4$F-$_4$E respectively. In terms of relative backgroundness the purpose of analyzing the music this way is to assert that $_2$G and $_3$E and $_4$G and $_4$F are incomparable. In the former case, then, figure 2.18 avoids choosing between $_2$G and $_3$E with a type-(2) distinction, as shown in figure 2.19. In the latter case, on the other hand, the only type-(2) distinction which we can make between $_4$G and $_4$F (given that they both prolong $_5$E) is one which asserts that $_4$G is more background than $_4$F, as in figure 2.19. This is because the prolongational relationship from $_4$G to $_5$E occurs

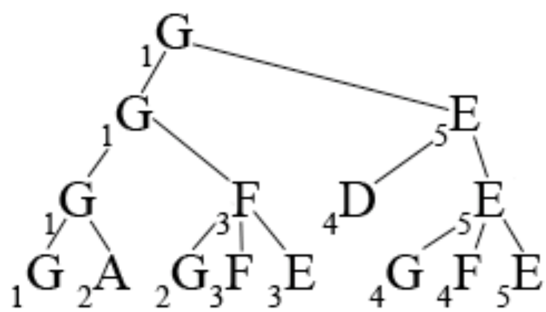

FIGURE 2.18: A PHRASE-STRUCTURE ANALYSIS WITH TWO 3-BRANCHINGS

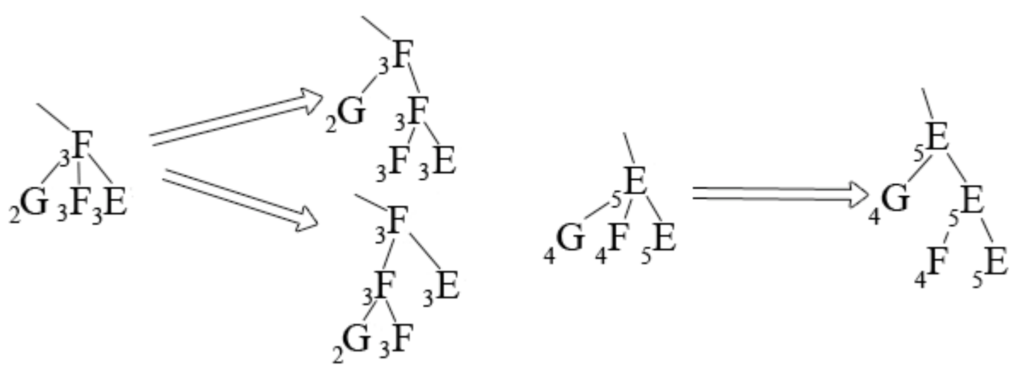

FIGURE 2.19: THE POSSIBLE EXTENSIONS OF A 3-BRANCHING TO TWO 2-BRANCHINGS

over a time-span that includes $_4$F. Since it difficult to imagine that $_4$F is more background than $_4$G and yet $_4$G is a prolongation of $_5$E and not of $_4$F, it is appropriate that the phrase-structure model makes such type-(2) distinctions impossible. However, by the same token it seems unreasonable to make the assertion which figure 2.18 seems to make—that there is no backgroundness distinction between $_4$G and $_4$F —when the span of $_4$G-prolonging-$_5$E includes the event $_4$F.

Therefore, it makes sense to constrain phrase-structure trees in such a way that analyses like figure 2.18 are impossible. To do this, let us recall our definition of "remoteness" above and contrast it with another way of asserting that two events are "far away" from one another, which I will call "distance." Two events, x and y, are *distant* if another event that is more background by (1) than both x and y occurs between them in the sequence of events. Notice that two events can be remote but not distant, and also can be distant but not remote. For example, in figure 2.18, the foreground events $_3$E and $_4$D are remote, because they are not prolongationally related and directly prolong different events ($_3$F and $_5$E). However $_3$E and $_4$D are certainly not distant, since $_4$D directly follows $_3$E in the sequence of foreground events! On the other hand, the foreground events $_2$G and $_3$E are not remote, because they both prolong the same event $_3$F directly. However, they are distant because $_3$F is between them and is more background than either of them.

Now we can make a restriction which requires that all pairs of events which are neither distant nor remote to be comparable by a type-(1) or type-(2) distinction. This means that a vertex can only have two children labeled differently than their parent if they are distant, making all non-binary branchings impossible except for triple branchings where the middle child is labeled the same as the parent. Since this constraint limits the possibility of multiple children, I'll call it the "family-planning" model.

We go further and say that all events that are not remote should be comparable. Thus, in figure 2.18, not only would the analysis of the passage $_4$D-$_4$G-$_4$F be inadmissible, but so would be the analysis of the passage $_2$G-$_3$F-$_3$E, because it prevents a type-(2) distinction between $_2$G and $_3$E, which are not remote (but are distant). In fact, this constraint is equivalent to saying that the tree must be strictly binary.

In terms of our recursive process of analysis construction, the family-planning constraint says that the new event must be a direct prolongation of a time-adjacent event. (Note that this event won't necessarily be time-adjacent at some later stage of analysis-construction, but will always be more background than any intervening events). Yet if the parent of the leaf corresponding to that time-adjacent event has the same label we can attach the new event to the parent rather than the leaf. This allows the analyst to assert that the new event A prolongs some time-adjacent event B, and there is no relative backgroundness distinction to be made between the A and any other event prolonging B.

Comparing the combinatorial situation of binary and family-planning phrase-structure trees to the general unstratified case, the results are quite different, as table 2.3 shows. The family-planning phrase-structure analyses are closer to binary phrase-structure analyses in their rate of growth, showing that the constraint is quite strong.

TABLE 2.3: A COMBINATORIAL COMPARISON OF TYPES OF PHRASE-STRUCTURE TREES[59]

Foreground	Labeled Unstratified Phrase-Structure Trees		
Events	Binary	Family-Planning	Total
1	1	1	1
2	2	2	2
3	8	9	11
4	40	50	74
5	224	311	556
6	1,344	2,072	4,472
7	8,448	14,460	37,667
8	54,912	104,346	328,010
9	366,080	772,255	2,929,230
10	2,489,344	5,829,583	26,679,916

[59] The process of calculating the number of family-planning trees is the same as that described above for finding the total number of unstratified trees, except that we can ignore any possibility of 4-branchings, 5-branchings, and so forth. Also, the number of labelings for a family planning tree is simply $2^{(\# \text{ of 2-branchings})}$, since there is only one choice of label for the parent event of a 3-branching.

Backgroundness Partial Orderings for Phrase-Structure Trees

Backgroundness distinctions define what is called a *partial order* on the foreground events in an analysis. Partial orderings are a common mathematical generalization of the idea of an ordering. In a *well ordering* of a set of objects, for any two distinct members of the set, A and B, either A < B or B < A in the ordering. To properly define an ordering, the relation < must be *antireflexive*, *antisymmetric* and *transitive*. "Antireflexivity" means that it is not the case that A < A for any A. "Antisymmetry" means that if A and B are distinct and A < B, then it cannot also be the case that B < A. "Transitivity" means that if A, B, and C are distinct members of the set such that A < B and B < C, then A < C. The reader can verify that all of these are traits of type-(1), -(2), and -(3) relative backgroundness distinctions.[60]

Obviously, in a well ordering on a set of size n, each member of the set can be indexed with an integer from 1 to n so that A < B if and only if A's index is less than B's index. This means that every well ordering on n objects is of the same form. However, the situation is more complex if we allow some pairs of elements of the set to be *incomparable*, meaning that there is no ordering relation between them. This is what we call a partial ordering.

Partial orderings are typically represented with digraphs called "Hasse diagrams." In a Hasse diagram, a directed edge from A to B means B < A. Because of the transitivity of <, however, it's not necessary to represent all such relations with an edge in the Hasse diagram, since any directed path from A to C such as A → B → C tells us that C < A. (More to the point, an edge in the Hasse diagram from A to B means B < A and there is no X such that B < X < A).

Figure 2.20 illustrates the backgroundness partial ordering derived from the analysis of figure 2.2 (reproduced in figure 2.21) as a Hasse diagram by including all type-(1) and type-(2) backgroundness distinctions. The direction of edges in figure 2.20

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[60]See Stanley (1979a), chapter 3. Stanley defines the basic relation as ≤ instead of <, which is reflexive rather than antireflexive. Although Stanley's approach is standard, I have chosen to define < straightaway, since it is sufficient for our purposes while ≤ is not especially useful.

is indicated by the relative vertical position of vertices, while the horizontal positions represent the chronological order of events. Thus, the edge between $_1$G and $_5$E is from $_1$G to $_5$E, since $_1$G is drawn above $_5$E. Note the following two aspects of figure 2.20:

First, the Hasse diagram has one vertex for each foreground event analyzed in the tree of figure 2.2. Thus, figure 2.20 has the same number of vertices that a MOP analysis of the same passage would have, and like the MOP, has no vertices corresponding to abstract events. Thus, the transformation of the tree of figure 2.2 into the tree of figure 2.20 brings us a step closer to being able to draw a comparison between the two analytical methods. Furthermore, when discussing such partial orderings, we can drop the qualifier "foreground" on events, since all events in the partial ordering, like all events in the MOP, are foreground.

Second, the Hasse diagram in figure 2.20 is a directed rooted tree. This is not true for all partial orderings, but it is true for all relative backgroundness partial orderings derived in this way from phrase structure trees.

Figure 2.20 is missing important information about the analysis of figure 2.2 though because it draws no distinctions between backgroundness relations of type (1) and (2). For instance, the second phrase-structure tree in figure 2.21 would yield the same backgroundness partial ordering, although it shows $_4$G prolonging $_4$D rather than $_5$E. For a clearer picture of the prolongational relationships in the analysis, we need to either label the edges of the Hasse diagram to indicate what kind of relations they represent or present a second partial ordering which includes only the type-(1) relations. Figure 2.22 presents the first solution. The two labeled Hasse diagrams here correspond to the two trees of figure 2.21 respectively, and differ in the label on the edge from $_4$D to $_4$G.

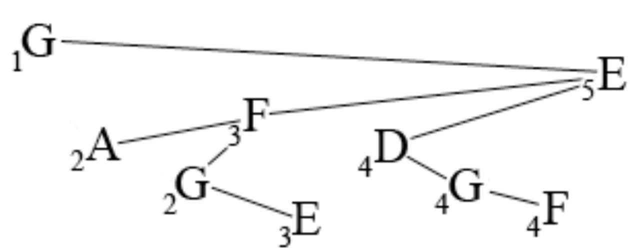

FIGURE 2.20: THE HASSE DIAGRAM OF A BACKGROUNDNESS PARTIAL ORDERING

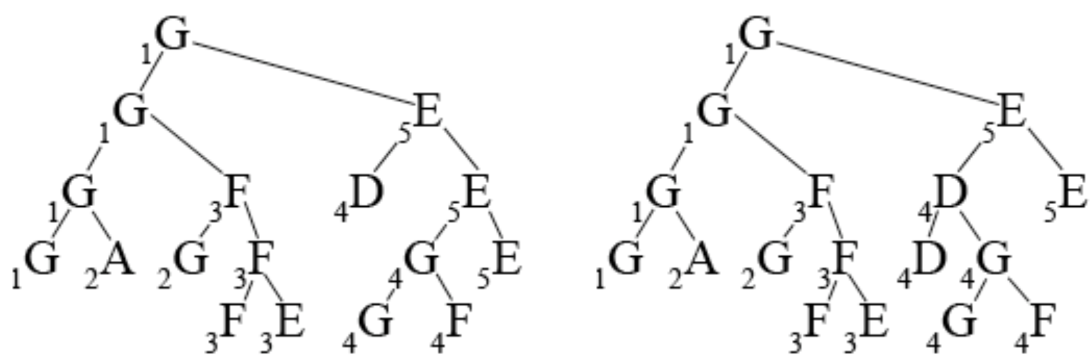

FIGURE 2.21: THE PHRASE-STRUCTURE TREE OF FIGURE 2.2 AND A DIFFERENT TREE WITH THE SAME BACKGROUNDNESS PARTIAL ORDERING

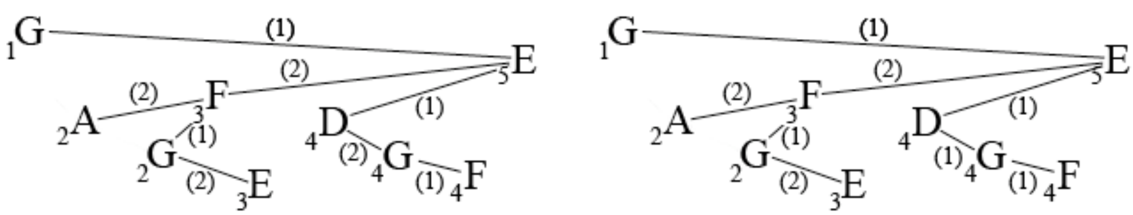

FIGURE 2.22: EDGE-LABELED HASSE DIAGRAMS FOR BACKGROUNDNESS PARTIAL ORDERINGS

To interpret figure 2.22, we need to know what relations exist between vertices in the diagram that don't share a labeled edge but instead are connected by a directed path. For instance, what's the relation between $_1$G and $_3$F? Fortunately, these relations are implied by those already labeled. In particular, if the (downward) directed path from A to B begins with an edge labeled (2), then A is more background than B by (2), whereas if the path begins with an edge labeled (1), then A is more background than B by (1). For example, the relation from $_1$G to $_3$F is of type (1). In general, to find the immediate prolongational parent of an event, take a path upward from that event in the edge labeled Hasse diagram that ends with an edge labeled (1) and has no other edges labeled (1). Observing this rule, we can take diagrams like figure 2.22 to be suitable representations of a partial ordering which distinguishes between the two different kinds of relations.

Here's a reading, then, of the first tree of figure 2.22, starting from the root: the entire passage is a prolongation of $_1$G (all events prolong $_1$G). $_1$G is most directly prolonged by $_5$E. $_3$F prolongs $_1$G at a more foreground level than $_5$E, and $_2$A prolongs $_1$G at a more foreground level than either $_3$F or $_5$E. Here, the parents of $_3$F and $_2$A are (2)-related to them, so they each prolong their nearest ancestor to which they connect with an edge labeled (1), which is $_1$G. And so on.

Figure 2.23 shows the Hasse diagram of a po-set consisting of only the type-(1) relations in the tree of figure 2.2. I'll call such a po-set a prolongational po-set. Together, the partial orders of figure 2.20 and figure 2.23 give the same information as the first labeled Hasse diagram in figure 2.22.

Let's consider this matter as one of mapping phrase-structure trees to partially ordered sets ("po-sets"). We have already demonstrated three possible mappings for the tree of figure 2.2. First, we mapped it to a po-set reflecting type-(1) and -(2) relations, but found that other analyses mapped to the same po-set. In the terminology of set theory, this mapping from binary phrase-structure analyses to po-sets is not *injective*. Furthermore, the mapping is not *surjective*, meaning that there are certain po-sets that correspond to no binary phrase-structure trees. For our purposes injectivity and surjectivity are desirable in a mapping because they're the two components of bijectivity.

By mapping instead to the edge-labeled po-sets as in figure 2.22, we solve the problem of injectivity; that is, there is a unique edge-labeled po-set for each binary phrase-structure tree. However, the mapping is still not surjective, because certain ways of labeling the edges of a tree correspond to no possible phrase-structure analysis. Rather than try to further circumscribe the set of possible edge-labeled po-sets, however, it's more enlightening to instead consider the mapping from phrase-structure trees prolongational po-sets such as that of figure 2.23.

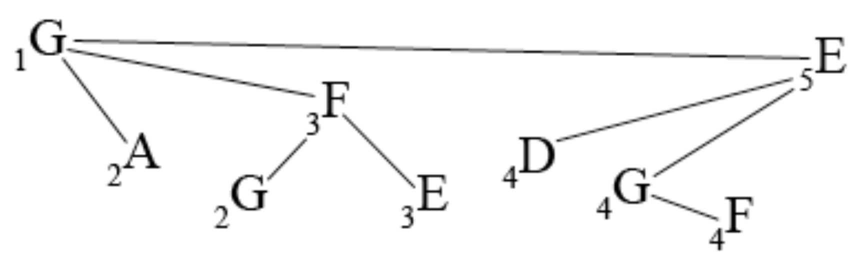

FIGURE 2.23: THE PROLONGATION TREE OF A PHRASE-STRUCTURE ANALYSIS

The constraints on prolongational po-sets are much easier to state than those on labeled backgroundness po-sets. First, prolongation po-sets always have Hasse diagrams in the form of a tree, as in figure 2.23, because an event can only directly prolong one other event. Therefore I'll use the term "prolongation tree" for such a Hasse diagram. Second, if two non-adjacent events, X and Y, are related with X > Y, all events that occur between them must be below X. (That is, if Z is chronologically between X and Y, then X > Z). If a prolongation tree meets this condition, I will call it *crossing-free*. Thus, the prolongation mapping from phrase-structure trees to crossing-free prolongation trees is surjective.[61]

However, this mapping is not injective. Consider the problem of finding a labeled backgroundness po-set to correspond to a given prolongation po-set. The inverse of this it certainly a mapping: just remove the (2) relations from the po-set. But how do we put the (2) relations back into the set? First, we need to know what (2) relations are possible. In order for two events to be (2)-related, either they must prolong the same event—that is, they must be siblings in the prolongation tree—or one must be a descendent of a sibling of the other in the prolongation po-set. In the latter case we say that the two events are remote *zeroeth cousins* (where two vertices are nth cousins in a tree, extending our familial analogy, iff there is a path from one to the other consisting of n upward edges and m downward edges, with $m \geq n$—in other words, a zeroeth cousin is a sibling, "aunt," "great-aunt," et c.). However, if two events X and Y are zeroeth cousins in the prolongation tree such that Z ($\neq$ X) is a sibling of Y and ancestor of X, it is impossible to make a (2) relation between X and Y without also making a (2) relation between Y and Z. Nor is it possible to make $Y >_2 Z$ without making $Y >_2 X$, or to make $Z >_2 Y$ and also relate X and Y. (I use the symbol "$>_2$" here to distinguish type-(2) backgroundness distinctions from type (1) distinctions, which may be indicated by "$>_1$".)

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[61] Formally speaking, a prolongation tree is a set of events that are ordered in two independent ways. There is a well-ordering on them reflecting temporal order and a partial ordering in the form of a tree reflecting prolongations. The crossing-free condition places a mutual dependency between these two orderings.

Therefore the possible (2) relations between remote events are entirely dependent upon those between siblings in the prolongation tree. Yet, as we noted above, if two siblings are on the same side of the parent in the chronological order of events then the more distant cannot be below the closer sibling in a (2) relation. Since we are interested in family-planning and binary phrase-structure trees, we should also add restrictions to ensure that (2) relations between non-distant, non-remote events are always included. (Obviously, if we the labeled po-set were to correspond to any phrase-structure tree, there are numerous possibilities for which events are (2) related, including none at all).

One of the advantages of working with prolongation trees rather than directly with labeled po-sets is that it's easy to define distance and remoteness of events in the prolongation po-set. In particular, two events X and Y are distant iff there is an event Z such that Z > X, Z > Y, and Z is chronologically between X and Y, and two events X and Y are remote iff $X \neq Y$ and X is neither a sibling, ancestor, or descendent of Y in the prolongation tree. Note that all distant events must be unrelated in the prolongation tree by the rule against crossings, and all remote events are unrelated by definition.

Using these definitions, we can say that for a family-planning analysis, if events X and Y are non-distant siblings such that Y is chronologically between X and the parent of X and Y, then $X >_2 Y$. By the transitivity of the partial ordering we also must add relations $X >_2 Z$ wherever there are events X, Y, Z such that $X >_2 Y >_1 Z$. Let's give names to these rules for reference:

> *Expansion Rule 1 (Transitivity of (2)-relations)*: Let X, Y, Z be events in a backgroundness partial ordering such that $X >_2 Y >_1 Z$. Then $X >_2 Y$.

> *Expansion Rule 2 (Prolongation at a distance)*: Let X, Y, Z be events in a backgroundness partial ordering such that Y and Z prolong X, and Y is chronologically between X and Z. Then let $X >_2 Y$ and apply expansion rule 1.

Then there are also optional (2)-relations between any incomparable events which are siblings in the po-set after applying expansion rules 1 and 2. In the case of binary trees, "optional" (2)-relations must be added until only remote events are incomparable in the po-set. In this sense, the addition of these relations is not optional for binary trees, but there is still a choice, wherever X and Y are incomparable non-remote events in the po-set, between adding X $>_2$ Y or Y $>_2$ X.

However, this is not quite the complete story. Consider the possible prolongation tree for the C major fugue subject shown in figure 2.24, which shows ${}_2$A as a prolongation of ${}_3$F (instead of ${}_1$G) and ${}_4$G as a prolongation of ${}_4$D (instead of ${}_5$E). Figure 2.24 also shows the process of expanding this prolongation tree into a backgroundness po-set. In the first step, rule 1 recognizes that ${}_5$E prolongs ${}_1$G over a span including ${}_3$F. (Rule 2 also relates ${}_5$E to ${}_2$A, ${}_2$G, and ${}_3$E here.) In the second step, by rule 1, ${}_2$A is more background than ${}_2$G, while rule 2 relates ${}_3$F to ${}_4$G. However, this last step also adds a relation between distant vertices ${}_3$E and ${}_2$G that should be optional, but is not, as the phrase-structure tree of figure 2.25 shows. Had we chosen the relation ${}_2$A $>_2$ ${}_3$E between distant events, we could have left ${}_2$G and ${}_3$E incomparable. It is also possible to put ${}_2$G $>_2$ ${}_3$E rather than ${}_3$E $>_2$ ${}_2$G. It is only impossible to leave *both* ${}_2$A and ${}_2$G incomparable to ${}_3$E. Therefore we must add the following rule, to be applied after all optional (2)-relations

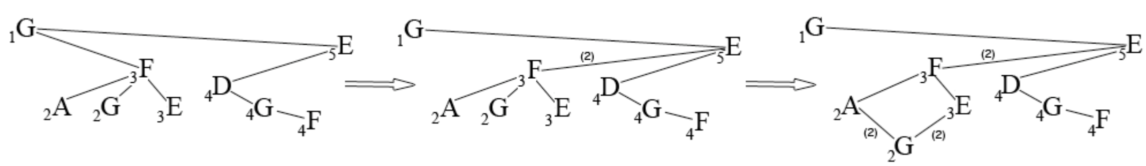

FIGURE 2.24: AN EXPANSION OF A PROLONGATION TREE INTO AN EDGE LABELED HASSE DIAGRAM

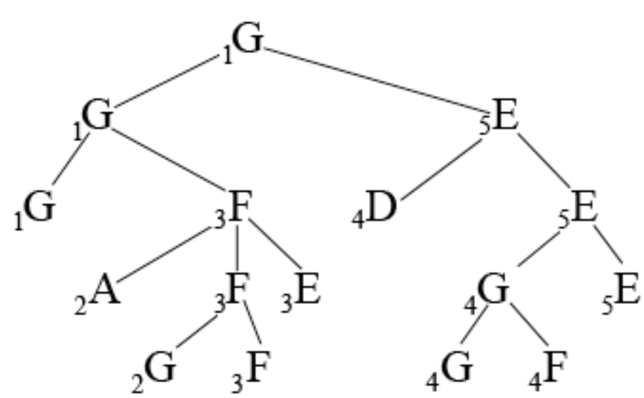

FIGURE 2.25: THE PHRASE-STRUCTURE TREE CORRESPONDING TO THE BACKGROUNDNESS PARTIAL ORDERING OF FIGURE 2.24

have been added: if there are events X, Y, Z such that X and Y are siblings and Y $>_2$ Z, then X $>_2$ Y. Let's call this expansion rule 3:

> *Expansion Rule 3 (Forced level relations)*: Let X, Y, Z be events in a backgroundness partial ordering such that X and Y are siblings and Y $>_2$ Z. Then X $>_2$ Z.

The expansion of the prolongation po-set into a backgroundness po-set then consists of first applying expansion rules 1 and 2, then adding any optional (2)-relations by rule 2, and finally applying rule 3 to the result. Therefore, in general, the transformation of a prolongation tree to a backgroundness po-set is not determinate either for family-planning or binary phrase-structure analyses, and also consequently not a mapping (—or, to put it differently, the mapping from labeled backgroundness po-sets to prolongation trees is not bijective because it is not injective).

This third rule, however, is semantically problematic. Purely in terms of backgroundness orderings, there is no reason, musically or formally, that a relation should be forced between such distant events. It's only the way we've defined (2)-relations in terms of the phrase-structure tree that forces this rule upon us. Furthermore, there is no satisfactory way to redefine (2)-relations in terms of the phrase-structure tree that circumvents this issue. Therefore, we must count this as a flaw in the derivation of a backgroundness partial ordering from the family-planning phrase-structure tree. The problem doesn't exist for binary trees, and is not a problem for trees with 1-branchings where type-(3) relations are relevant. Speaking metaphorically, one could say the problem is a vestige of the "reductional levels" approach to backgroundness that is allowed to leak into type-(2) relations by the weak restrictions of the family-planning approach as opposed to the strictly binary approach.

The semantic problems are serious enough that it would be wise to put the idea of a family-planning model in which the phrase-structure tree is not strictly binary but also doesn't imply a fixed set of reductional levels out of its misery at this point. If the possibility of 3-branchings blurs the distinctions between (2)-relations and (3)-relations, then it makes little sense to discard the (3)-relations in the first place. This is probably why only certain types of phrase-structure models have been proposed to model prolongation: Lerdahl and Jackendoff use a strictly binary system, Smoliar uses a general phrase-structure model in which 1-branchings play a significant semantic role, and Rahn's model (restricted to the neighbor operation, as discussed in "The General Phrase-Structure Model of Prolongation" above) allows only 1-branchings and 2-branchings.

Backgroundness Partial Orderings for MOPs

Fortunately, the derivation of a backgroundness partial ordering from a MOP is much more straightforward than it is for phrase-structure trees. Recall from above (in the section "Combinatorial Comparison of MOPs and Binary Phrase-Structure Trees" that the MOP analysis can be thought of as a directed graph where the orientation of edges reflects melodic precedence. There is another way to represent a MOP as a directed graph where the orientation of edges represents prolongational rather than temporal precedence.

This digraph, shown in figure 2.26 for the MOP analysis of figure 2.11, is defined as follows: first, let the orientation of the root edge go from the initiation to the termination event (by arbitrary convention). Now there is one vertex in the MOP that makes a triangle with ○*i* and ○*t*, which is $_1$G. Orient the edges ○*i*-$_1$G and $_1$G-○*t* towards the new vertex, $_1$G. Now find new triangles including the newly oriented edges ○*i*-$_1$G and $_1$G-○*t* towards, and direct the two other edges of these towards its third vertex. Thus, for $_1$G-○*t* we find the triangle $_1$G-$_5$E-○*t*] and direct the edges $_1$G-$_5$E and $_5$E-○*t* towards $_5$E. Continue this process until all edges have an orientation, and the result is a digraph of the MOP oriented according to the relative backgroundness of events.

As I pointed out in part one, these two ways of orienting a MOP are reflected by the way they're drawn: the horizontal direction of an edge indicates the direction of melodic precedence, while the vertical direction indicates prolongation. We could think of the MOP, then, as a doubly-directed graph where two vertices sharing an edge, u and v, may be related in one of four ways: u may be left of and above v, left of and below v, right of and above v, or right of and below v. It's significant that the prolongational and temporal orientations of all edges of a MOP follow from the definition of a root edge with a prolongational and temporal orientation.

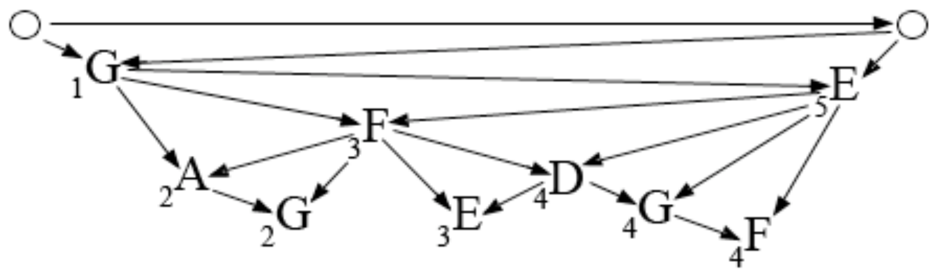

FIGURE 2.26: THE PROLONGATIONAL ORIENTATION OF A MOP

The resulting digraph could be interpreted in terms of static prolongational relationships for the purpose of comparing the MOP structure with phrase structures. That is, we could read an edge A → B in the vertically oriented MOP as denoting a prolongational relationship between events, B prolongs A. Of course, this is really a convenient misinterpretation of the analytical meaning of the MOP, since, as the model is developed in part one of this paper, the MOP is designed to model dynamic rather than static prolongational relationships.

From this process of defining prolongation between events in a MOP, it's obvious that, while each event (excluding the formal events) is defined as prolonging exactly two others, these two are always themselves prolongationally related. Therefore, if we use these relations to define a prolongational partial ordering on events, every event directly prolongs only one other, and the Hasse-diagram of such a po-set is in the form of a tree. I'll call this the prolongation tree for the MOP. The prolongation tree for the analysis in figure 2.11 is shown in figure 2.27.

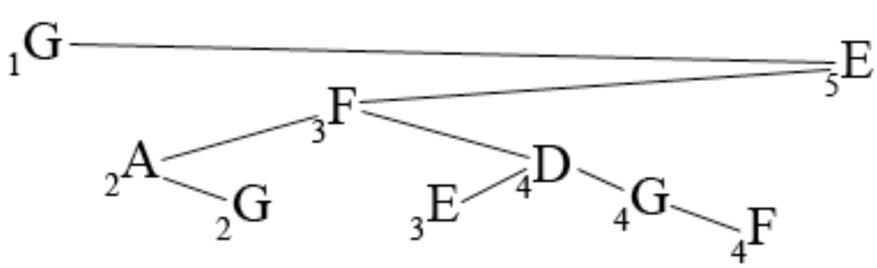

FIGURE 2.27: THE PROLONGATION TREE OF A MOP

The form of this tree is less obviously determined by analytical choices than the structure of the prolongation tree for a phrase-structure analysis. For instance, in constructing this analysis in part one, we chose $_2$A as a prolongation of $_1$G-$_3$F on the grounds that $_2$A was an incomplete upper neighbor to $_1$G. However, it is the "loose-end" relationship of $_2$A to $_3$F that appears in the prolongation tree, simply because $_3$F prolongs $_1$G, and not vice versa.

Observe that every edge in the MOP has at most one event as a "child," and that child must occur chronologically between the events incident on the edge. This is true, by extension, of any descendent of an edge, and if a vertex is not a descendent of some edge, its event cannot be between those on the edge (because it must be between some other two events). This means, for one thing, that the prolongation tree of a MOP is crossing-free. (Recall that "crossing-free" for a prolongation tree means that for any events X, Y, Z such that X > Y in the tree, and Z is chronologically between X and Y, X > Z). It also means that if two events are incomparable in the prolongation tree, there must be some more background event between them. In other words, all events that are not distant are prolongationally related, and all events not prolongationally related are distant. I will call such a prolongation tree *complete crossing-free*, (or, for short, a *complete prolongation tree*).

This definition of prolongation between events in a MOP can be seen as a mapping from MOPs to prolongation trees. In fact, as a mapping from MOPs to complete crossing-free prolongation trees with the initial event as the root and the final event a child of the root it is bijective. In other words, a complete crossing-free prolongation tree is in a mathematical sense a faithful representation of the MOP analysis. (Keep in mind that a prolongation tree by definition includes a well ordering on its vertices that represents the temporal order of events). For a MOP, then, the backgroundness partial ordering is just the prolongation tree; there is no need to add anything like type-(2) or type-(3) relations.

The reader may have noticed that the term "crossing-free," applied here to prolongation trees, is similar to the term "crosschord-free" in part one, which refers to a characteristic of graphs. This isn't accidental. Consider three vertices of a graph $_1$G, x, y, z, that all participate in a cycle, C, in that order. According to the derivation of prolongational orientations to the edges of a MOP, if there is an edge between y and a vertex, v, that precedes x on C, then y is necessarily more background than x—that is, y is above x in the prolongational partial ordering. The same is true of y and z if there's an edge between y and some vertex, v, following z on C. This is precisely the situation in which crossing edges may arise in a graph: an edge between x and z would cross the edge between y and v, and so if $_{1}G$ is a MOP it cannot include the edge xz. This is formally no different from saying that x and z cannot be comparable in the prolongation partial ordering because a more background vertex intervenes between them chronologically.

The MOP model and phrase-structure models agree on this point: in both cases the prolongation tree is crossing-free. However, the MOP model goes one step further: not only does "X prolongs Y" or "Y prolongs X" imply "X and Y are not distant"; in a MOP the converse also holds: "X and Y are not distant" implies "X prolongs Y" or "Y prolongs X."

Comparing MOPs and Phrase-Structure Analyses through Reduction-Lists

The common denominator of all theoretical dispositions concerning prolongation is the idea of reductions. This is the typical association of the term in Schenker's later writings, after his development of the *stimmfuhrungsprolongation* method of analysis (though not in the early history of the term, in the two volumes of *Kontrapunkt*; see "Concept(s) of Prolongation" in part one above). In other words, "prolonging" always indicates something that appears in a later, less "reduced," voice-leading level. It is in the particular nature of the relationship between reductional levels that differences between concepts of prolongation, such as the static-dynamic difference, lie.

Therefore, viewing these analytical models in terms of melodic reductions helps to reveal the semantic implications of the differences between the backgroundness partial orderings of phrase-structure and MOP analyses. I'll represent a way of reducing a passage with a *reduction-list*. A *reduction* of a passage is simply a subset of the set of events in the passage. A *reduction-list* is a set of reductions ordered from highest to lowest such that every reduction properly contains all those above it. In addition, a reduction-list always includes the complete set of events as a reduction. Two events are *consecutive* or *next to one another* in a reduction if no other event in that reduction is between them in the time-ordering. Two events are consecutive in a reduction-list if they are consecutive in any reduction on the list.

One possible way to interpret a prolongation tree in terms of reductions is to consider all reduction-lists consistent with the prolongation tree in the sense that an event is always introduced in a lower reduction than any event it prolongs. One interesting consequence of such a definition of consistency of a set of reductions is that a prolongation tree being crossing-free is equivalent to it being consistent with some reduction-list in which each event is next to its parent in the most background reduction in which it occurs.[62]

For instance, consider the prolongation tree of figure 2.23, reproduced in figure 2.28. Figure 2.28 shows a possible reduction-list for this prolongation tree that introduces each event in a reduction where it's next to its prolongational parent. Such a reduction-list is possible because the prolongation tree is crossing-free. Figure 2.28 also shows a stratified phrase-structure tree that asserts this reduction-list.[63]

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[62] It's simple to show that any crossing-free analysis has such a reduction-list through an algorithm: let the first reduction be the root by itself. Then add to this in turn the most distant left child of the root, followed by the next most distant, and so on, and do the same for the right children of the root. Then do the same for the children of the root in turn from left to right, and so on until all events are included. To show the converse, assume that **T** is some prolongation tree with a crossing. Then there are events X, Y, and Z in **T** with $X > Y$, $X \not> Z$, and Z chronologically between X and Y. Let U be the nearest ancestor of Z that isn't between X and Y, and let V be the next nearest ancestor of Z (i. e. the child of U). If there is no such U then the root is chronologically between X and Y, and the proposition follows immediately. Otherwise, U must be in a higher reduction than V, and X in a higher reduction than Y. However, X and Y must both be in higher reductions than V (because V is between them), and U and V must be in higher reductions than X (if X is between them) or Y (if Y is between them). In either case this is impossible.

[63] A stratified phrase-structure tree such as this is uniquely determined by the combination of the prolongation tree and the reduction-list given the semantically trivial restrictions that from one level to the next in the tree there must be at least one multiple branching and every leaf must be the same distance from the root.

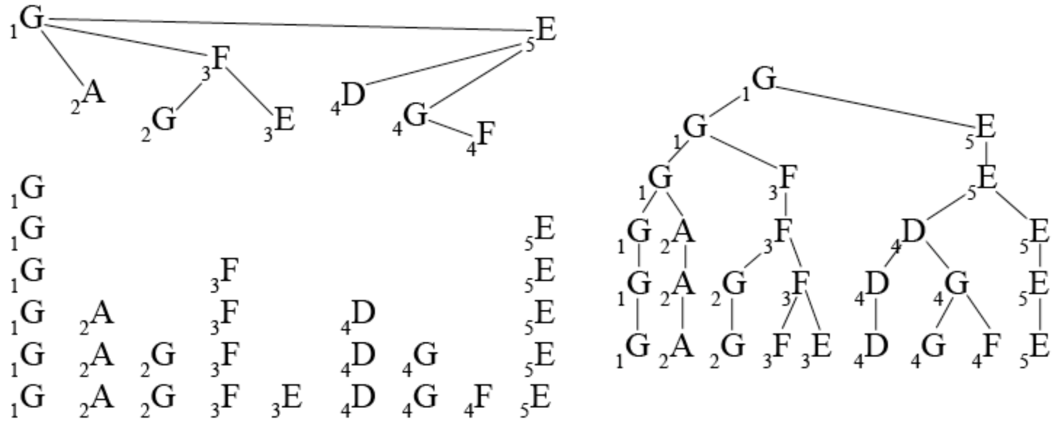

FIGURE 2.28: A PROLONGATION TREE, REDUCTION LIST, AND STRATIFIED PHRASE-STRUCTURE TREE

It's also worth noting the conditions under which any two events can be adjacent in such a reduction-list. Recall that two events are distant iff there is an event between them that prolongs both of them. Obviously two distant events cannot be time-adjacent in any possible reduction. However, many non-distant events also cannot. For instance, it is impossible to put $_3$E next to $_1$G in a reduction-list consistent with the prolongation tree in figure 2.28. Therefore we need a weaker form of distance: let two events be *weakly distant* if an event occurs between them that prolongs either of them. Then two events being weakly distant is equivalent to there being no set of reductions in which they occur next to one another.

However, this sense of "consistent with a prolongation tree" is not strong enough. Figure 2.29 shows a reduction-list in which every event is introduced after every event it prolongs. But this reduction-list doesn't make an especially compelling analysis and doesn't seem like a very faithful realization of the relationships in the prolongation tree. Not only that, it is impossible to convert such a reduction-list into a stratified phrase-structure analysis that also expresses the relationships of the prolongation tree, as the crossing edges in the phrase-structure tree of figure 2.29 show. The problem here is that $_3$F and $_2$A both prolong $_1$G according to the prolongation tree, and $_2$A occurs chronologically between $_1$G and $_3$F, yet $_2$A is introduced before $_3$F in the reductions.

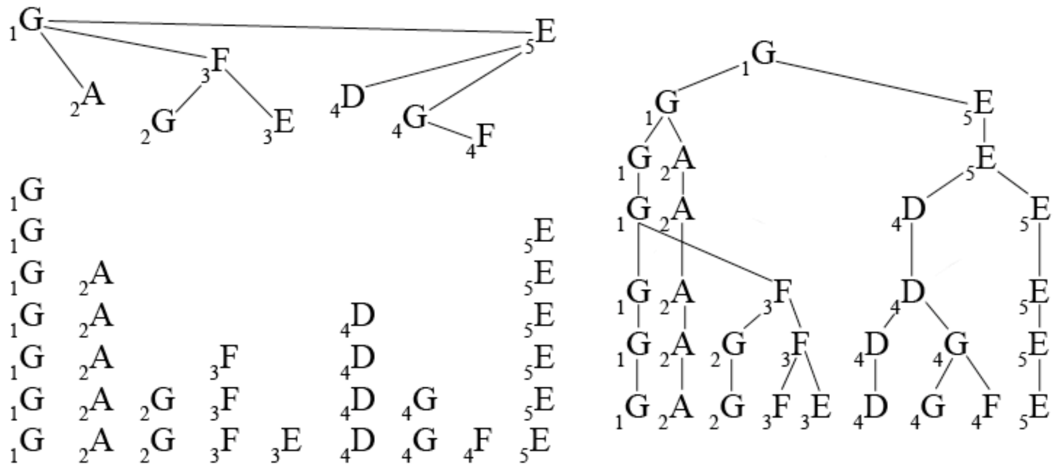

FIGURE 2.29: A PROLONGATION TREE AND REDUCTION LIST WITH NO ASSOCIATED STRATIFIED PHRASE-STRUCTURE ANALYSIS

When we try to hear $_3$F prolonging $_1$G in such a reduction-list, the event $_2$A "gets in the way," as figure 2.29 visually illustrates. Therefore, it makes sense to redefine consistency of a reduction-list with a prolongation tree so that for each event X (other then the root), the reduction just above the origin-reduction of X (i.e. the highest reduction in which X occurs) includes the prolongational parent of X and no other event chronologically between X and itsprolongational parent. A stronger version of this, which I'll call conformity, requires that each event be next to its prolongational parent in the reduction in which it's introduced. Let me restate these definitions for reference:

> A reduction-list, **R**, is *consistent* with a prolongation tree, **T**, iff they analyze the same time-ordered set of events and for any event Y with parent X in **T**, the lowest reduction in which Y does not occur includes X and no event between X and Y in the time-ordering.

> A reduction-list, **R**, *conforms* to a prolongational tree, **T**, iff they analyze the same time-ordered sequence of events and for any event Y with parent X, Y occurs in some reduction that doesn't include X, and X and Y are consecutive in some reduction.

It will also be helpful to define a new, stronger kind of non-distance relation to go with this stronger definition of consistency of a reduction-list with a prolongation tree. I will call this "proximity." Two events, X and Y, *proximate* iff there is no event, Z, chronologically between them such that Z > X, Z > Y, or Z is incomparable with both X and Y. In other words, if Z is between X and Y, either X or Y prolongs Z, and Z is not prolonged by either X or Y. According to this definition of proximity, two events being proximate in a prolongation tree is equivalent to there being some reduction-list consistent with the tree in which they are consecutive.

Another interesting formal aspect of proximity is that it can substitute for distance in the definition of a complete crossing-free prolongation tree. That is, not only can a complete crossing-free prolongation tree be defined as a crossing-free prolongation tree in which all incomparable events are distant, it can also be defined as one in which no incomparable events are proximate.[64]

The reduction-list in figure 2.28 is conformant with its prolongation tree according to our refined definition, while that of figure 2.29 are neither conformant nor consistent. However, figure 2.30 shows another reduction-list that is conformant with the prolongation tree but asserts quite a different analysis. The fundamental difference

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[64] To see this, note that all proximate pairs of events in a complete crossing-free prolongation tree are comparable (because they are not distant). Now assume that **T** is a crossing-free prolongation tree in which no incomparable events are proximate. Let X and Y be any two siblings with parent Z, such that X is to the left of Y and they have no other siblings chronologically between them. Assume Z is left of X. Then we can show that X and Y are proximate by the crossing-free property: Let W be any event between X and Y. W is also between Z and Y, so it must prolong Z by the crossing-free property. But there is no child of Z between X and Y, so it must also prolong either X or Y (otherwise X or Y will cross an edge on the path from Z to W). Obviously then W also cannot be prolonged by either X or Y (because they're siblings). Yet X and Y are by definition incomparable, so they can't be proximate by the definition of **T**. The same is true if Z is to the right of Y. Therefore Z must be between X and Y. Consequently, in general a vertex in **T** can have no more than two children, and if it does have two children, the parent must be chronologically between them.

Now let U and V be any two incomparable vertices in **T**, such that U precedes V chronologically and A is their nearest common ancestor. Let U' be the child of A that's an ancestor of U and V' be the child of A that's an ancestor of V. According to the conclusions above, U' must be left of A and V' to the right of A. By the crossing-free property, U and V must also be to the right and left of A (because no edge on the path from U to U' or V to V' can cross A). Therefore, U and V are distant.
This proves that all incomparable vertices of **T** are distant and **T** is thus complete crossing-free.

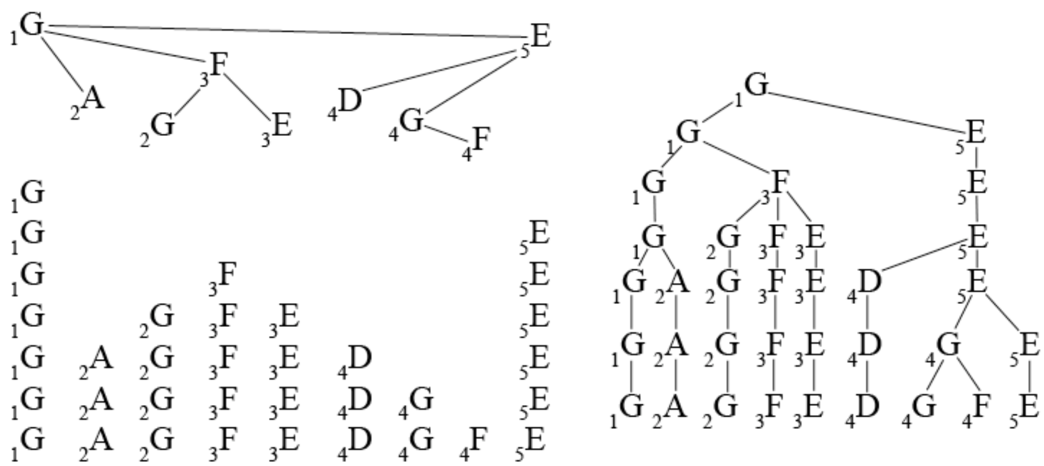


FIGURE 2.30: A DIFFERENT REDUCTION-LIST CONFORMANT WITH THE SAME PROLONGATION TREE AS FIGURE 2.28

between these two reduction-lists is that different incomparable pairs of events in the prolongation tree are consecutive in each reduction-list. We have already established that consecutivity in some reduction is closely tied to prolongation in the sense that an event must always occur next to its prolongational parent in some reduction (when the reduction-list is conformant), and the possibility of such a conformant reduction-list is equivalent to the prolongation tree being crossing-free. However, in realizing a prolongation tree like the one in figure 2.28 in a reduction-list, we are forced to also make some events consecutive which are incomparable in the prolongation tree. A comparison of figures 2.28 and 2.30 shows that such decisions are musically quite consequential.

The important question then is, given a crossing-free prolongation tree, what consecutivity relations between incomparable events are possible? Actually, we already have a general answer to this question: two events can be made consecutive in a reduction if and only if they are proximate. This means that the consecutivity relations in a complete crossing-free tree are fixed, and given an incomplete crossing-free tree, certain choices can usually be made about consecutivity in a conformant reduction-list. Yet it's possible to take this a bit further and show exactly which combinations of choices about consecutivity are possible. The easiest way to do so is to observe the correspondence between making a decision about consecutivity in the set of reductions and transforming a prolongation tree by adding relations between proximate incomparable events. Thus, such a transformation narrows the field of possible reductions to those that include a particular relation. The transformations that can be applied to the resulting tree represent relations compatible with those that have already been added. A series of such transformations can turn any crossing-free prolongation tree into a complete crossing-free tree.

Recall that such complete prolongation trees correspond bijectively with MOPs, so this process also gives a method of transforming any prolongation tree into a MOP. We can interpret this process as showing which MOP analyses are compatible with a particular incomplete prolongation tree. The semantic interest of the transformation of an incomplete tree in to a complete one lies in the fact that the same procedure that relates a prolongation tree to one of its conformant reduction-lists also relates a prolongation tree with a possible MOP analysis of the same set of events.

For any set of reductions, **R**, there is a prolongation tree conformant with **R**, $\mathbf{T_R}$, that is maximal with respect to prolongational relationships—that is, the set of relations in every prolongational po-set conformant with **R** is a subset of the set of relations in $\mathbf{T_R}$. $\mathbf{T_R}$ is just the prolongational tree such that, for events X and Y, X > Y iff X and Y are consecutive in some reduction and X originates above Y or there's some event Z such that X > Z > Y. Let's call this the *maximal tree* for a reduction-list.

For example, consider the prolongation tree of figure 2.28 and the sets of reductions in figures 2.28 and 2.30. Studying the prolongation tree reveals the following complete list of proximate incomparable events: $_2$A-$_2$G, $_2$A-$_3$F, $_3$F-$_5$E, $_3$F-$_4$D, $_3$E-$_4$D, $_3$E-$_5$E, and $_4$D-$_4$G. The reductions in figures 2.28 and 2.30 make many of these events consecutive. For instance, the maximal tree for figure 2.28 must include $_5$E > $_3$F, because $_5$E and $_3$F are consecutive in the third reduction where $_3$F originates and $_3$F is in the second reduction above that. Similarly, the tree must include $_3$F > $_2$A, $_3$F > $_4$D, $_2$A > $_2$G, $_4$D > $_3$E, and $_4$D > $_4$G. The remaining relation, $_5$E > $_3$E, follows from these by transitivity: $_5$E > $_3$F > $_3$E. Therefore the maximal tree for this reduction-list is complete, as figure 2.31 shows. The MOP corresponding to this prolongation tree, also shown in figure 2.31, is our original MOP analysis of the passage from part one.

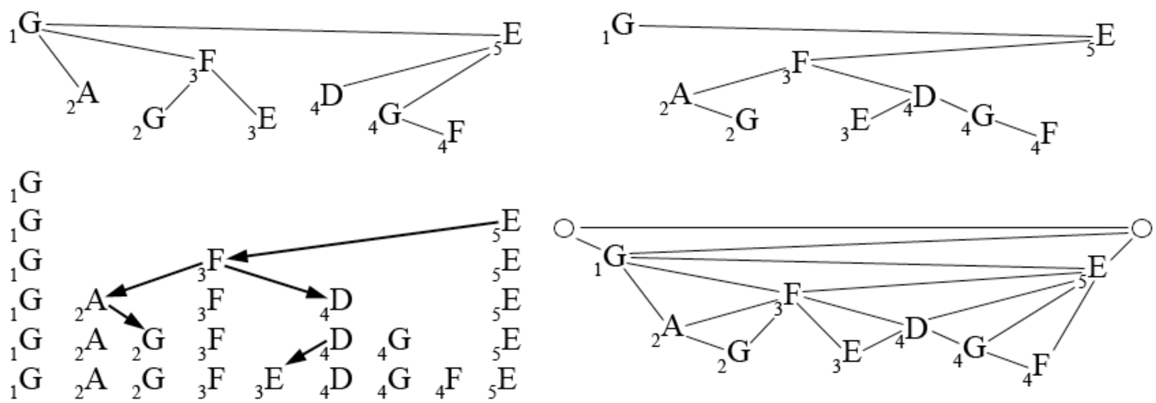


FIGURE 2.31: THE REDUCTION-LIST OF FIGURE 2.29 AND ITS COMPLETE TREE AND MOP

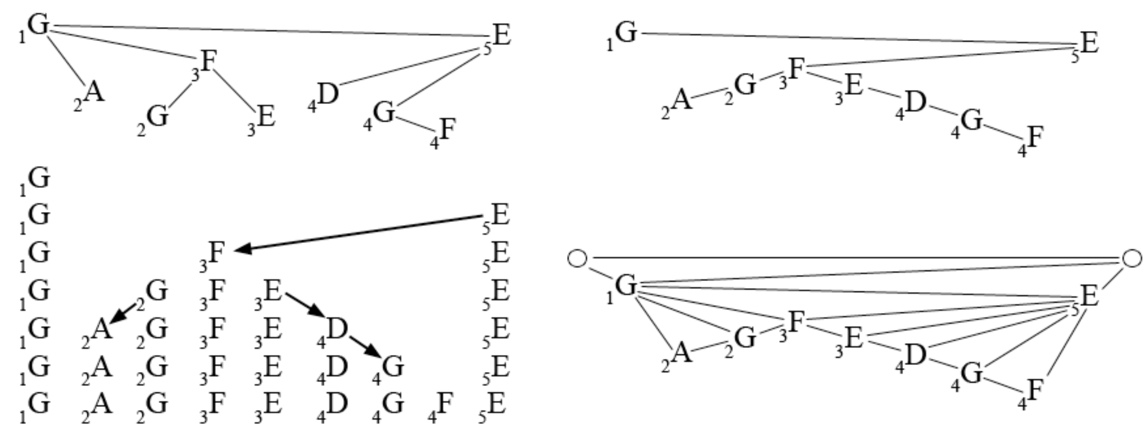


FIGURE 2.32: THE REDUCTION-LIST OF FIGURE 2.30 AND ITS COMPLETE TREE AND MOP

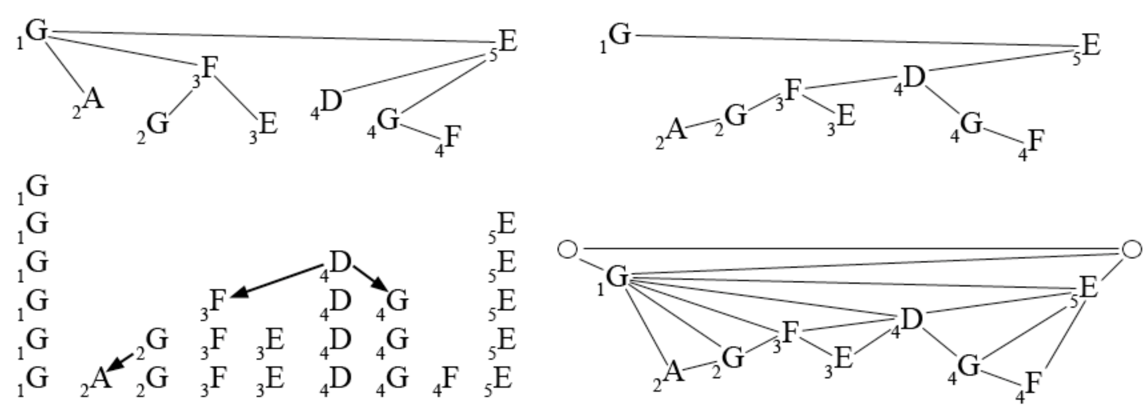


FIGURE 2.33: ANOTHER REDUCTION-LIST, ITS COMPLETE TREE AND MOP

Notice that the MOP in figure 2.31 can be obtained directly by adding edges to the prolongation tree between all of the pairs of events that are consecutive in the reduction-list. This is true for all such reduction-lists, permitting the interpretation of the edges of a MOP as a representation of consecutivity in a reduction-list.

Figure 2.32 shows similarly that the reduction-list presented in figure 2.30 also has a maximal tree that's complete. This tree is, of course, different from that of figure 2.30 and corresponds to a substantially different MOP analysis. This analysis asserts that $_2$G and $_3$E are a repetition and anticipation of $_1$G and $_5$E respectively rather than passing tones from $_2$A to $_3$F and $_3$F to $_4$D. According to this analysis, it makes more sense to think of $_2$G and $_3$E most directly as prolongations of $_1$G and $_5$E rather than of $_3$F, as in the prolongation tree of figure 2.30. Therefore, the reduction-list of figure 2.30 doesn't give a very faithful realization of this prolongation tree even though it doesn't contradict it.

Figure 2.33 shows a third possible reduction-list, also conformant to the same incomplete prolongation tree, that gives another completely different analysis of the passage. In this case, $_1$G-$_3$F-$_4$D appears as a dominant arpeggiation to the lower neighbor of $_5$E. Again, while this is certainly a plausible analysis, it doesn't seem to correspond very well to the prolongation tree of figure 2.23, although it doesn't contradict it. If we hear the passage according to the reduction-list in figure 2.33, it would make more sense to have $_4$D as a direct prolongation of $_1$G rather than of $_5$E.These three examples demonstrate how a complete prolongation tree can represent a reduction-list. I will show momentarily how the conformance of each of these reduction-lists with the prolongation tree of figure 2.23 can be represented as a transformation of this tree into those corresponding to each reduction-list. First, we should note however that our examples show that the idea of conformance of a reduction-list to a prolongation tree is still too admissive, because a conformant reduction-list has the potential to significantly alter the musical analysis suggested by a prolongation tree.

Figures 2.34 and 2.35 each show a reduction-list conformant to the tree in figure 2.23 but having an incomplete maximal tree. The ambiguity of these reduction-lists is caused by the simultaneous introduction at some level of two events consecutive at that level, one of which prolongs the event to its left while the other prolongs the event to its

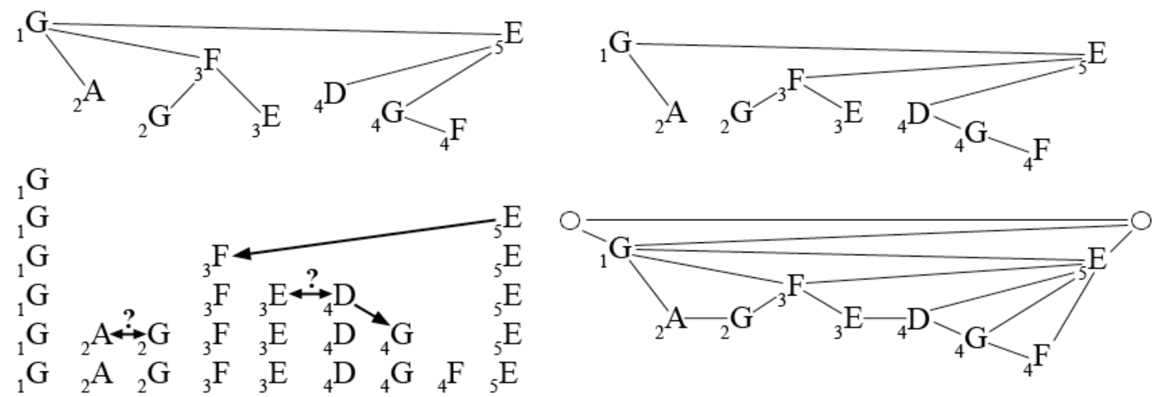


FIGURE 2.34: A REDUCTION-LIST WITH AN INCOMPLETE MAXIMAL TREE

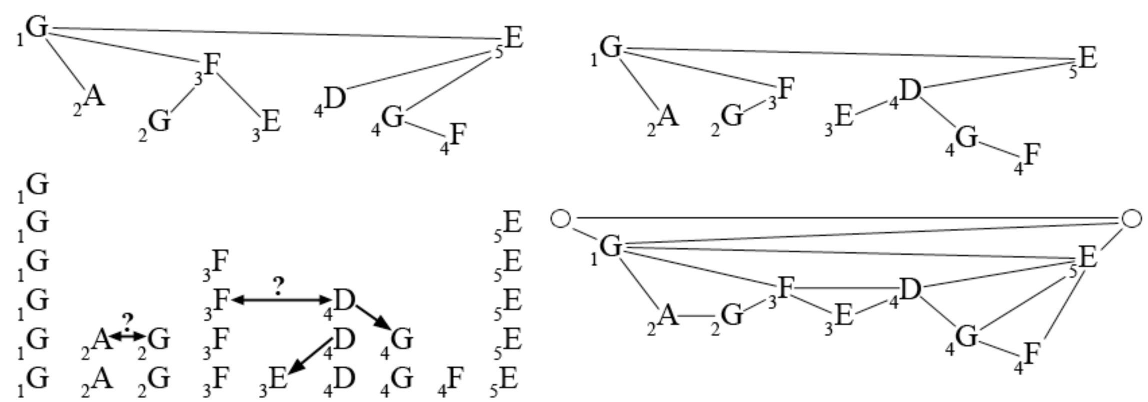


FIGURE 2.35: ANOTHER REDUCTION-LIST WITH AN INCOMPLETE MAXIMAL TREE

right. For instance, in figure 2.34 the reduction-list introduces $_3$E and $_4$D simultaneously in reduction 4, where $_3$E prolongs the previous event $_3$F and $_4$D prolongs the following event $_5$E. The result is that while $_3$E and $_4$D are consecutive, no directed relationship can be established between them from the reduction-list. Furthermore, the proximate events $_4$D and $_3$F are allowed to remain incomparable.

This is the only circumstance in which a reduction-list can have an incomplete maximal tree. It is impossible, for instance, to give a conformant reduction-list that leaves larger holes in its associated outerplanar graph than those in figures 2.34 and 2.35. Also, these reduction-lists are substantially different semantically from those of figures 2.31-33, which have complete maximal trees, because they assert that two events can be consecutive but incomparable in terms of their levels of origin. Therefore it is formally useful to separate out such reduction-lists as ambiguous. More precisely, let an *unambiguous reduction-list* be one in which no two consecutive events have the same origin-reduction (where an origin-reduction of an event is the highest reduction in which that event occurs). The unambiguous reduction-lists, then, are just those that have complete maximal trees.

Figures 2.36-38 correspond to figures 2.31-33, and show how the complete trees of these examples can be constructed from the prolongation tree of figure 2.23. The transformations in these figures always relate two proximate events and add any other relations that follow from this through transitivity.

The analysis of figure 2.36 requires the most steps (six). Note that each step in figure 2.36 relates two events that are siblings in the previous tree. Figure 2.37 shows three transformations, two of which relate non-siblings. Those transformations that relate non-siblings necessarily create more relations through transitivity, explaining why fewer steps are needed overall in this case. The first transformation, ${}_2G > {}_2A$, relates a second descendent of ${}_1G$ to a child of ${}_1G$, and forces ${}_3F > {}_2A$ by transitivity. The fourth transformation, ${}_3E > {}_4D$, relates a second descendent of ${}_5E$ to a child of ${}_5E$, forcing ${}_3F > {}_4D$ by transitivity. In figure 2.38 also, the first and third step relate proximate events that aren't siblings. In both cases, there's no way to arrive at the same complete tree by relating only siblings at each step, as in figure 2.36.

This provides an explanation as to why the reduction-list of figures 2.31 and 2.36 seems to better represent the prolongation tree of figure 2.23 than those of figures 2.32-3 and 2.37-8. In figures 2.37 and 2.38 we can arrive at the correct complete tree only by at some point placing an event below the child of its sibling, whereas the transformations in figure 2.36 always place an event below one of its siblings. The latter transformation is precisely like expansion rule 2 from "Backgroundness Partial Orderings for Phrase-Structure Trees," the prolongation-at-a-distance expansion, except that in the transformed prolongation tree we don't distinguish between the original (type-(1)) relations and the new (type-(2)) relations. We also need a transitivity rule like expansion rule 1. Let's then call these transformation rules 1 and 2:

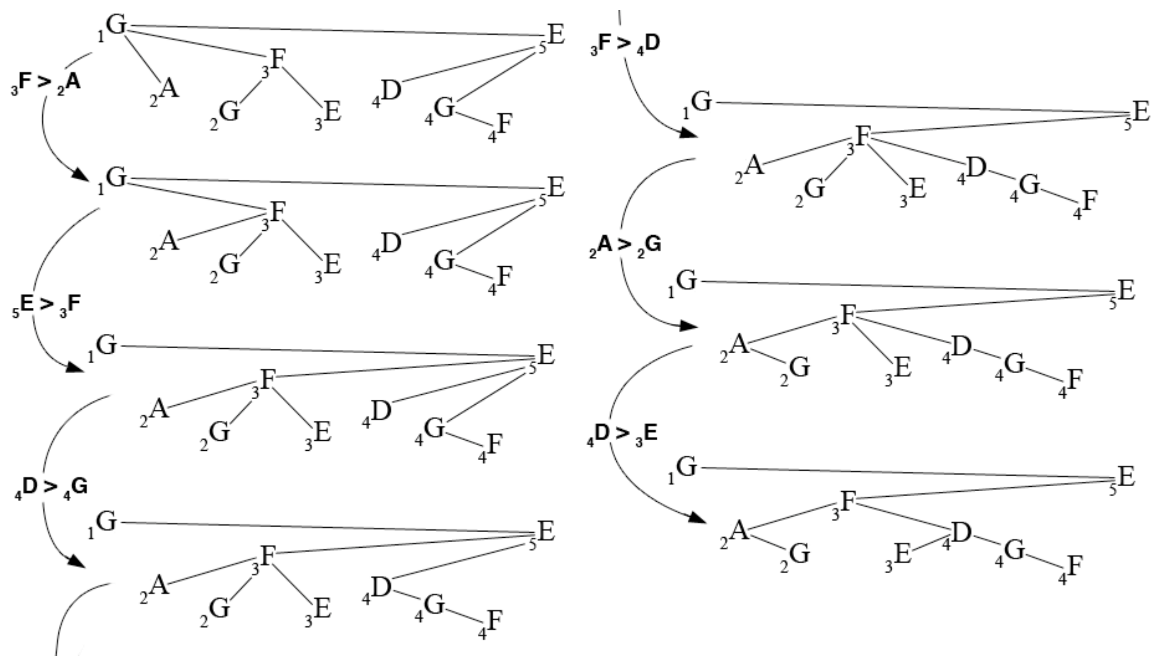
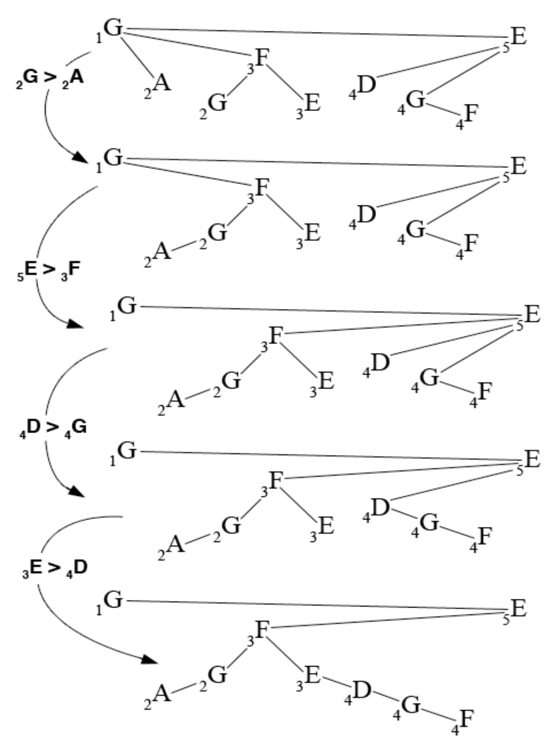
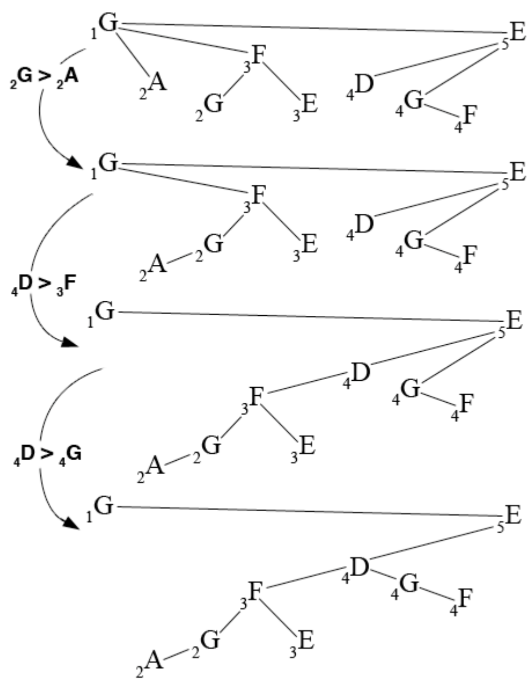


FIGURE 2.36: THE TRANSFORMATION OF THE INCOMPLETE PROLONGATIONAL TREE INTO THE COMPLETE TREE OF FIGURE 2.31





FIGURES 2.37 AND 2.38: SERIES OF TRANSFORMATIONS RESULTING IN THE COMPLETE TREES OF FIGURE 2.32 AND FIGURE 2.33

Transformation Rule 1 (Transitivity of Prolongations): Let X, Y, Z be events in a prolongation tree such that X > Y and Y > Z. Then let X > Z.

Transformation Rule 2 (Prolongation at a distance) [strong version]: Let X and Y be proximate events in the crossing-free prolongation tree, **T**, that share the same parent, Z, such that Y is chronologically between X and Z and no other child of Z is chronologically between Y and Z. Then transform **T** by making X > Y and apply transformation rule 1 to the result.

An incomplete crossing-free prolongation tree always has at least two events that are siblings that both precede or both follow their parent chronologically.[65] Therefore, it is possible to transform any incomplete crossing-free tree into a complete tree by successive applications of transformation rule 2. Furthermore, this procedure always results in the same complete tree given some incomplete tree, no matter how it is carried out.[66] This therefore gives a mapping from phrase-structure trees to MOPs, which I'll discuss in more detail below.

While the application of this strong version of transformation rule 2 gives only one possible transformation of an incomplete tree into a complete tree, there's a weaker

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[65] To prove this, let **T** be a crossing-free prolongation tree such that no event in **T** has two siblings on the same side of their parent. Then a vertex can only have two children in **T**, and if it has two they are on opposite sides of the parent chronologically. Assume, then, that X and Y are two incomparable events in **T**. Let Z be the nearest common ancestor of X and Y and let X' and Y' be the children of Z which are ancestors of X and Y respectively. X' and Y' must be on opposite sides of Z chronologically, so the same must be true of X and Y by the crossing-free property (otherwise an edge on the path from X to X' or Y to Y' would cross Z). Therefore X and Y cannot be proximate, and **T** must be complete.

[66] The strong version of transformation rule 2 is stated so that its application never makes two siblings into non-siblings unless they are chronologically on different sides of the parent or are non-proximate (because they have some other sibling chronologically between them). (Note that the only event that gets a new parent through the rule is Y). Thus, the application of the rule to one pair of siblings never prevents its application to another pair. The rule does, however, make events into siblings that were not before; more precisely, if X has any children, they become siblings with Y. By the crossing-free property, if Y is left of X, it is proximate with X's leftmost child in the resulting tree, and if it is right of X it becomes proximate with X's rightmost child. Therefore, no applications of the rule to any other pairs of vertices can change the vertex to which Y becomes a proximate sibling, because no applications of the rule can change the parent of an event that is further right or left from its parent than any of its siblings.

version that allows for more than one possible complete tree, in general, given some incomplete tree:

> *Transformation Rule 2 (Prolongation at a distance) [weak version]*: Let X and Y be incomparable proximate events in the crossing-free prolongation tree, **T**, such that X prolongs the parent of Y, Z, and Y is chronologically between X and Z. Then you can transform **T** by letting X > Y and applying transformation rule 1 to the result.

For instance, if **T** is the incomplete tree of figures 2.23-38, then ${}_2A$ and ${}_2G$ are incomparable proximate events both prolonging ${}_1G$, the parent of ${}_2A$, and ${}_2A$ is chronologically between ${}_1G$ and ${}_2G$. Therefore it is possible to transform **T** by the weak version of transformation rule 2 by adding the relation ${}_2G > {}_2A$, and ${}_3F > {}_2A$ by transitivity, as in figures 2.37 and 2.38. While the strong version of this rule acts like expansion rule 2, the weak version doesn't necessarily because of cases like this one.

A closer look at the examples reveals that the applications of strong version of the rule (figure 2.36) can be divided into stages. The first stage compares all proximate siblings of the original tree: in figure 2.36, first ${}_3F > {}_2A$, then ${}_5E > {}_3F$, then ${}_4D > {}_4G$. These applications of the rule turn some proximate non-siblings of the original tree into siblings. The second stage then compares these: ${}_3F > {}_4D$ and ${}_2A > {}_2G$. Finally, the addition of ${}_4D > {}_3E$ to the po-set in the third stage completes the tree.

The three additions of the first stage are necessary comparisons in any completed version of the tree (i. e. any completion of the tree with the weak version of rule 2). Note that in figures 2.37 and 2.38, the relation ${}_3F > {}_2A$ is added indirectly by transitivity from ${}_2G > {}_2A$, and in figure 2.38 the relation ${}_5E > {}_3F$ is added by transitivity from ${}_3F > {}_4D$. As a result, these relations are second-generation in the resulting complete tree, and indirect in the associated MOP. In place of them, there are edges of the MOP representing indirect relationships of the original prolongation: in figure 2.32, ${}_1G > {}_2G$ and ${}_5E > {}_3E$, and in figure 2.33, ${}_1G > {}_2G$ and ${}_1G > {}_4D$. The construction of figure 2.38 also relates through transitivity two events that are incomparable in the original tree: ${}_4D$ and ${}_3E$.

I pointed out above that the analyses implied by the reduction-lists of figures 2.37 and 2.38 seem to correspond better to different prolongation trees. In fact, it is possible to construct an incomplete prolongation tree that maps to the same complete tree and

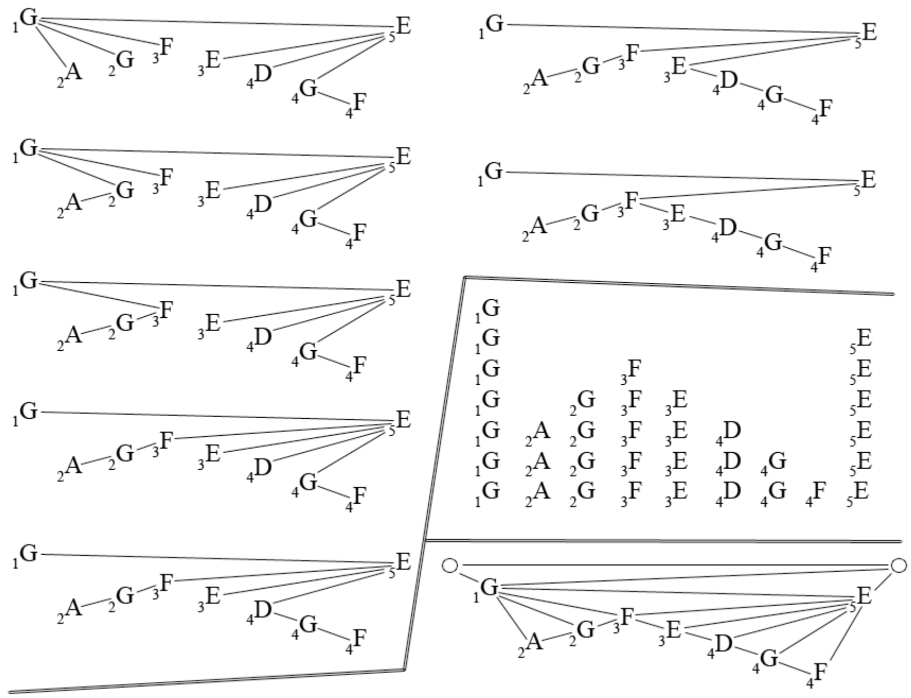

FIGURE 2.39: A PROLONGATION TREE THAT YIELDS THE COMPLETE TREE OF FIGURE 2.32 THROUGH THE APPLICATION OF THE STRONG VERSION OF TRANSFORMATION RULE 2

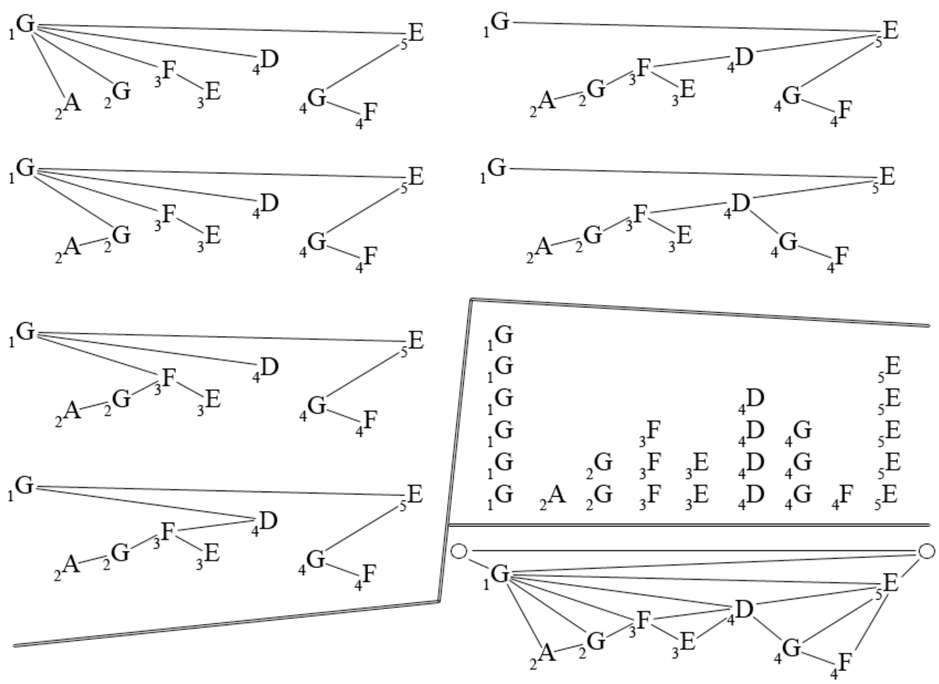

FIGURE 2.40: A PROLONGATION TREE THAT YIELDS THE COMPLETE TREE OF FIGURE 2.33 THROUGH THE APPLICATION OF THE STRONG VERSION OF TRANSFORMATION RULE 2

MOP under the strong version of transformation rule 2. Figures 2.39 and 2.40 show these prolongation trees and the application of the transformation rule to them. The number of steps in the transformation is larger in these cases than in figures 2.35 and 2.36, so that more of the edges in the resulting MOP represent relations added to the prolongation tree directly by rule 2. Also, the number of stages in figures 2.37 and 2.38 is fewer than in figure 2.36, which had three stages. In figure 2.39 there's a second stage represented by a single transformation, ${}_3F > {}_3E$, and in figure 2.40 there's only one stage.

The weak version of transformation rule 2 allows the construction of any MOP that includes all the edges of the original prolongation tree. However, when we add comparisons between non-siblings, the resulting consecutivity relations are not as obvious as with applications of the strong version of the rule. For instance, if we add a comparison X > Y, where X is a second-generation descendant of Z, the parent of Y, we must add a consecutivity relation between Z and X as well as X and Y, and also between X and any events prolonging Y that are proximate to X. For instance, in step 2 of figure 2.38, let $X = {}_4D$ and $Y = {}_3F$, then $Z = {}_1G$ and Y has a child, ${}_3E$, proximate to ${}_4D$. Therefore in addition to ${}_4D$-${}_3F$, this step adds consecutivity relations ${}_1G$-${}_4D$ and ${}_4D$-${}_3E$.

In applications of the strong version of rule 2, the only added consecutivity relations are those explicitly added between siblings in the tree. Therefore, the semantics of the strong version of the rule are relatively clear and have already been discussed above in reference to type-(2) relations of phrase-structure trees. That is, in the first stage strong rule 2 adds a consecutivity where two events directly prolong the same third event and the time-span of one of these prolongations includes the other. Thus, if strong rule 2 adds a relation X > Y, weak rule 2 must either add X > Y or X' > Y where X' is some descendent of X proximate to Y. If weak rule 2 adds X' > Y rather than X > Y, X and Y do not become a consecutivity. Instead, in order to raise the structural status of X', in effect, weak rule 2 adds a consecutivity between X' and Z, the nearest common ancestor of X' and Y. In this case, the time span of X' prolonging Z does include the time span of Y prolonging Z, but X' prolongs Z indirectly.

Therefore, the strong rule 2 always adds the more obvious consecutivity relations (among all the possible ones). This explains why the reduction-lists associated with the strong transformational rule provide an analysis that hews closer to the sense of the original prolongation tree. It also makes the strong rule the obvious choice for a mapping between phrase-structure trees and MOPs, which by definition requires a single target given a single input (whereas the weak rule can transform an incomplete prolongation tree, in general, into many possible MOPs).

Semantics of the Mapping from Phrase-Structure Trees to MOPs

I've spent a great deal of space erecting a formal structure to relate phrase-structure trees and MOPs through backgroundness po-sets and reduction-lists to get to the point where we could construct a meaningful mapping between phrase-structure trees and MOPs. Now it's necessary to unravel this construction to get some kind of grasp on the semantics of the mapping.

Here is an outline of the phrase-structure tree to MOP mapping: let P be a phrase-structure tree. (1) Extract a prolongation tree, **T**, from P. This prolongation tree must be crossing-free, but it may be incomplete. (2) Apply the strong version of transformation rule 2 to the prolongation tree until it is a complete prolongation tree, $\mathbf{T_c}$. (3) Let ${}_1G$ be the MOP corresponding to $\mathbf{T_c}$.

The first step in this mapping removes a certain amount of information from the phrase-structure tree. In particular, it removes all type-(2) and type-(3) backgroundness relationships from the tree. All of the type-(2) relationships are reintroduced into the tree in step 2 except for those between distant events, but, unlike the expansion procedure for prolongation trees in the earlier section, this step doesn't distinguish the new relationships from the old ones. If P is stratified, then all type-(3) relationships (necessarily between distant vertices) are removed.

Due to the combinatorial situation, which we briefly examined above, it is impossible to map phrase-structure trees to MOPs without removing a great deal of information that might distinguish one phrase-structure tree from another. That is, many different phrase-structure trees must map to the same MOP. It makes sense that any differences between phrase-structure trees that serve to draw level comparisons between distant events should be removed by the mapping, since the MOP model doesn't constrain the relative backgroundness of distant events, as we have already discussed.

However, this is a symptom of a more general semantic difference between MOPs and phrase-structure trees. When I derived a prolongation tree for MOPs in the previous section, I imposed the dynamic usage of the word "prolongation" onto MOPs, where prolongation is a relationship between two events, one prolonging the other. It's correct to think of an edge in a phrase-structure tree or a prolongation tree as representing a particular prolongation. But the same isn't true of MOPs; prolongation in a MOP can be described as a relationship between edges (as above, in the section "The MOP Model of Prolongation as a Binary Phrase-Structure Model"), and it can be described as a relationship between a vertex and an edge (where the event represented by the vertex prolongs the interval represented by the edge, as I explain the model in part one). However, it violates the dynamic prolongation conceptual basis of the MOP model to describe prolongation as a relationship purely between two events, or to interpret an edge in a MOP as representing a prolongation, as we do for phrase-structure trees. Thus, although in a formal sense the prolongation tree corresponding to a MOP faithfully reproduces the "information" of the MOP (because of the bijective correspondence), it suggests a false semantic interpretation of that information.

Can the edges of a MOP then be said to represent anything at all by themselves? In fact, the idea of *consecutivity* that the last section developed is a good description of what an edge in a MOP represents. That is, an edge between two events in a MOP means that we can hear them as being consecutive. For events far apart in the music, it is helpful to think of them being "consecutive in some potential reduction consistent with my hearing." Thus, the "prolongation tree of a MOP" might be better called a "consecutivity tree," except that "consecutivity" is not a directed relationship, as it must be to form a po-set. ("Consecutivity" obviously implies the existence of a directed relationship in time, although it deliberately avoids specifying it. But this is beside the point, because the directed relationship we need here is a "vertical" rather than a "horizontal" one). Nor is consecutivity a transitive relationship.

The prolongation po-set for a MOP could perhaps represent the relationship, where X > Y, of “Y prolongs a span bounded by X,” but this is too indirect of a locution to make for a straightforward interpretation of the prolongation tree. More simply, “X is more background than Y” comes to mind. But “more background than” really means “originates at a deeper reductional level than.” That is, it implies a fixed list of reductions, which is not characteristic of a MOP. (Note that consecutivity relationships distinguish between possible and impossible reduction-lists, but don’t fix a particular reduction-list as “the one true”). We could patch up “is more background than” to read “cannot be less background than”; this would be perfectly accurate, but again not especially lucid.

It would be better, then, to sidestep the prolongation tree of a MOP and rephrase part (3) of the phrase-structure-to-MOP mapping outline to read: (3) Let ${}_1G$ be a graph including all edges of **T** (the prolongation tree of the phrase-structure analysis), all edges added to **T** directly by transformation rule 2, and those edges added by transitivity between proximate events. This derives the MOP graph directly without assuming a complete prolongational tree to represent it. We can assume, then, that these edges are meant to represent consecutivity.

The entire mapping process, thus, is a process of discovering consecutivity relationships from a phrase-structure tree. The first step is then unproblematic: distant events cannot be consecutive, so level comparisons between them don’t figure into a MOP analysis. The type-(2) relations eliminated from the phrase-structure analysis by this step are reinstated in step 2 if and only if they are between non-distant events. However, the immediate type-(1) relationships must correspond to consecutivities.[67] Therefore the edges of the prolongation tree of the phrase-structure analysis are retained and will become part of the edge set of the MOP that results from the mapping.

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[67] I refer the reader here back to the idea of *conformity* of a reduction-list to a prolongation tree in the previous section. The weaker property of *consistency* of a reduction-list would allow direct prolongations to not correspond to consecutivities in certain circumstances. However, given the dependency of prolongation on consecutivity in a MOP, for a mapping of a prolongation tree to a MOP the property of conformity is the relevant one.

The second step of the mapping adds relations to the prolongation tree until it becomes a complete tree. We have already discussed the possible ways this can be done in the previous section, and shown how the strong version of rule 2 adds relations in the most direct way. Recall that we're actually not interested in the comparisons added by the transformation rules so much as the consecutivities they add, since these undirected consecutivities are sufficient to characterize the resulting MOP. The strong version of transformation rule 2 adds a consecutivity between any two proximate events that are both consecutive with some third event.

The mapping from a phrase-structure analysis to a MOP illustrates the similarities and differences between the two approaches. There is no way to compare them directly in terms of prolongation because prolongation is conceived as a different kind of relation in the two models. However, they can be compared through the idea of consecutivity. In the MOP model, consecutivity is closely tied to prolongation: if an event prolongs an interval, then the event must be consecutive with each member of the interval (and the interval itself must also represent a consecutivity). Furthermore, every consecutivity represented in the MOP, except the root edge, must result from a prolongation.

In the phrase-structure model, consecutivity is also tied to prolongation, but more weakly. In a binary phrase-structure analysis, immediate prolongational relationships between events must correspond to consecutivities, but in the more general model this may be violated in isolated instances (though in such a way that the selective insertion additional reductions adds the missing consecutivities). Furthermore, the type-(2) relationships, characteristic especially of the binary model, also tend to represent potential consecutivities.

Prolongation Models and Musical Intuition

To conclude this comparison of prolongation models, it would be well to briefly discuss their fidelity to certain common musical intuitions. One of the problems with the phrase-structure model is its representation of certain common melodic paradigms of prolongation: the passing tone and the neighbor note. Phrase-structure models force the analyst to represent a passing tone as a prolongation of either the note it passes from or the note it passes to. In such an analysis, then, a descending passing tone (for instance) becomes, somewhat counterintuitively, an incomplete lower neighbor to the previous note, or an incomplete upper neighbor to the following note. For instance, in the fugue subject we have used as a test case, the overall motion is passing from $_1$G to $_5$E. The MOP model represents the $_3$F as prolonging the motion from $_1$G to $_5$E, which is essentially what we mean when we say "$_3$F is a passing tone between $_1$G and $_5$E." All of the phrase-structure analyses, on the other hand, represent $_3$F as an incomplete lower neighbor to $_1$G, leaving it only indirectly related to $_5$E. The same is true of a neighbor note figure: the analyst must choose whether it's a neighbor to the preceding or the following note.

All of the authors I have mentioned have their own way of dealing with this. Lerdahl and Jackendoff simply ignore it, because they are more concerned with defining a procedure that produces an analysis than the semantics of the analysis it produces. When pressed on the matter, Lerdahl (1997), in a response to a paper by Steve Larson critical of Lerdahl's theory, falls back on "theoretical parsimony." Lerdahl is correct that it's difficult to make a coherent model that asserts prolongation as a relationship between events and also can accurately portray the sense of "being a passing tone." The question then is whether to discard this usage of the term prolongation or to discard the common-sense notion of a passing tone.

I have already discussed Smoliar's model of passing and neighbor motion in "The General Phrase-Structure Model of Prolongation" above. Smoliar makes a valiant effort to represent passing and neighbor motion accurately in a phrase-structure system, but as we saw in the discussion above his model falls short because the circumstances under which a passing tone or neighbor note can occur are too narrow to be of general use.

In Rahn's model, a passing tone is technically represented as an incomplete neighbor to the preceding or following note. However, Rahn's arp operation mitigates this to some extent because it allows the preceding and following notes to become a simultaneity at a next-background level. A passing tone, thus, could be distinguished from an incomplete neighbor by observing whether the "note being passed from" and "the note being passed to" become a simultaneity at some more background level. The same could be said of neighbor notes, except that the “simultaneity” is a unison rather than a third in that case.

All of this could be turned around on the MOP model: if phrase-structure models fail to represent passing tones and neighbor notes, by the same token the MOP model fails to represent such phenomena as escape tones and appoggiaturas—in other words, those types of prolongation that I characterize as “incomplete progressions” in “Some Conceptual Problems in Theories of Prolongation” and “Refinements of the MOP Model” in part one. In those discussions, I gave an account of incomplete progressions that adheres to the dynamic usage of “prolongation” and the mathematical structure of the MOP analysis. This account offers a potentially enlightening musical interpretation of incomplete progressions, an account that follows the Schenkerian model of explaining these prolongations as transformed versions of more basic types, with the passing tone of species counterpoint as the ultimately most fundamental type. The phrase-structure model of passing tones, however, doesn’t offer such an explanation of the passing tone or neighbor note. The passing tone as an incomplete neighbor is more of a marginally acceptable compromise than a musical insight.

PART 3: FORMAL MODELS OF CONTRAPUNTAL ANALYSIS

Criteria for a Contrapuntal Model

No formal model of prolongation would be adequate if it failed to deal with prolongation in a contrapuntal context. So far, I have strained to avoid the topic to maintain a degree of simplicity to the presentation of the basic principles of modeling prolongation. In this part, I will consider a few different approaches to the analysis of counterpoint in formal models of prolongation. First, I will compare the ways in which the theories of Smoliar and Rahn, which I presented in a simplified form in the previous part, deal with contrapuntal considerations. Then I will derive a method for representing a multi-voice analysis from the MOP model of prolongation.

Before getting to the contrapuntal models, let’s consider briefly the models of Keiler and Lerdahl and Jackendoff, which I consider non-contrapuntal models even though their authors apply them to contrapuntal music. With these models, the characterization I gave in the previous section is wholly adequate: all they require formally is a phrase-structure grouping of a sequence of events.

Keiler deals with contrapuntal music by first giving a harmonic (roman-numeral) analysis of it and then adopting these roman numerals as a sequence of events. This approach simply avoids the problems of a contrapuntal analysis by hiding them under the bed in a harmonic-analysis procedure that is not itself formalized. Therefore Keiler’s model is not contrapuntal simply because what it analyzes is not the counterpoint itself, but a string of roman numerals that the analyst asserts as somehow representing the (contrapuntal) music.

Lerdahl and Jackendoff deal with counterpoint by taking simultaneities as events. That is to say, they ignore the problems of counterpoint by treating a piece of contrapuntal music as if it were a sequence of events that may include any number of simultaneous pitches, rather than a collection of distinct event sequences (different voices) whose purposes may agree at some points but not necessarily at all. As it turns out, while their model nominally works this way—and it seems at least a reasonable proposition when a chorale is the object of analysis—in practice, their solution is more like Keiler's. In most of Lerdahl and Jackendoff's analyses, the object of analysis is not the music itself, but a sequence of chords derived from the music. When they analyze the C major prelude of WTC I, the process of deriving a sequence of chords from the musical surface is relatively unproblematic. But for more contrapuntal textures, the process is not at all trivial, as Lerdahl's discussion in *Tonal Pitch Space*, pages 35-40, makes abundantly clear. Like Keiler, Lerdahl and Jackendoff sweep the problems of counterpoint into a corner where the formal model doesn't reach.

In fairness to Lerdahl and Jackendoff, the reason that they try to circumnavigate counterpoint in their formal models is not because they don't recognize the necessity of understanding a piece's counterpoint to fully appreciating the music, nor is it because they underestimate the complexity of the task of including counterpoint in the analytical model. In fact, it is precisely because of the potential complexity of a contrapuntal model that they hesitate to construct one. Lerdahl and Jackendoff set themselves the daunting goal of giving a highly, if not completely, deterministic formal model of prolongational analysis. That is, given a piece of music the model should give exactly one analysis, or at least a small number of analyses. To do this requires formalizing the process of prolongational analysis as well as the analysis itself. Like Schachter, whom I quoted in the introduction, I am dubious about the wisdom of such an endeavor, although not so much, like Schachter, because I believe that the process through which an analyst arrives at "the correct" prolongational analysis is too complex and unpredictable to represent formally, but because according to my own understanding of the concept of prolongation the idea of a "correct" prolongational analysis of a particular piece of music or passage of music is simply not well-defined. A formalization of prolongation along the lines of Lerdahl and Jackendoff could produce a well-defined notion of correctness for prolongational analyses, but unless we find a deterministic model of prolongation that relies upon some especially important or interesting insights about tonal music beyond those of a non-deterministic model, the concept of prolongation should remain non-deterministic.

As we saw in part one (in "The Concept(s) of Prolongation"), for Schenker, prolongation was a way of linking the voice-leading in free composition to the law-abiding world of species counterpoint. This is essentially what we mean when we ask for a contrapuntal representation of music: a description of the music that models it as something like a species counterpoint example. Therefore, as criteria of a contrapuntal representation, I'll look for these two basic characteristics of species counterpoint:

(1) The representation separates musical events into multiple *independent voices*. These are independent in the sense that an event cannot participate in more than one voice (i. e., the voices partition the events), events are well-ordered within a voice, and each voice can be described as having its own prolongational structure (potentially different than that of other voices).

(2) The model relates events between voices in terms of *simultaneity*, *consonance*, and *dissonance*. That is, it can be said for two events in different voices whether or not they're simultaneous, and if they are simultaneous, whether they're consonant or dissonant. The consonance/dissonance distinction should also be non-trivial, so that, e. g., at least some types of event-pairs should be characterizable as dissonant simultaneities.

Keiler's and Lerdahl and Jackendoff's procedures, for instance, don't allow for multiple independent voices. Even if multiple voices could be defined by breaking up the simultaneity-events, each voice will have exactly the same prolongational structure as each other voice (and hence they aren't truly independent). Lerdahl and Jackendoff recognize this problem and assign it's resolution to "future research." (116) They suggest a model in which different voices may receive contrasting structural descriptions (273-7), but don't tackle the problem of how to coordinate these contrasting descriptions into a single analysis.

The road-block that such a "refinement" of Lerdahl and Jackendoff's procedure to a contrapuntal one inevitably runs into is in their too-literal view of simultaneity. Prolongation doesn't work this way in Schenkerian analyses (as Lerdahl and Jackendoff observe about Schenker's own analyses, prompting their comment, "we feel that Schenker sometimes ascribes too great an independence to the outer voices," 276): melodic events that do not literally coincide in the music may coincide at some level of the analysis, and, more problematically, events that do coincide in the music may not actually be related.

The recognition of agreement and disagreement between prolongational structures operating simultaneously in multiple voices, it seems to me, provides the best starting point for advancing a concept of consonance and dissonance for Schenkerian analysis. Consonance is, accordingly, the coordination of the varying structural descriptions of each voice. More precisely, there's some way of reducing the individual prolongational analyses of each voice so that all of them are equivalent and corresponding events in all of the voices can be described as consonant simultaneities. This is analogous to taking a fifth-species counterpoint in multiple voices and simplifying each voice to present an underlying first species counterpoint. This first species counterpoint is the model of consonance, whereas the divergent elaborations that distinguish the prolongational structures of the voices are the model of dissonance.

This is a normative definition of consonance and dissonance: the terms are defined only in the context of an analysis. Conflating this normative sense of the terms consonance and dissonance with other senses of the terms, where they are properties of abstract intervals rather than particular events in a particular piece of music, causes a great deal of confusion. To be certain, the reason why we use the same terms in both cases is that there is some relation between normative consonance and certain definitions of consonance as a property of abstract intervals. But this relationship is complex, and it's not my business here to untangle it.

The Representation of Counterpoint in Smoliar's Model

Smoliar's model introduces counterpoint through the label SIM in the phrase-structure tree. This indicates that all the children of the SIM vertex occur simultaneously, ordered left-to-right from lowest to highest pitched event. Smoliar uses this SIM label to separate the music into distinct voices. This is illustrated in Smoliar's complete middleground analysis of Mozart K. 283, 1, mm. 1-10, shown in Schenkerian notation in figure 3.1 and in Smoliar's tree-form in figure 3.2. The unlabeled vertices in figure 3.2 are SEQs, and there are ten SIM vertices. The SIM on the root vertex shows that the analysis consists of essentially two voices, a lower and an upper voice. The other SIMs break the upper part and the lower part into parallel thirds in certain places.

However, these other SIM vertices actually serve a different function than the one on the root vertex. The SIM on the root vertex has two SEQs as children. This SIM separates the events below it into two subtrees corresponding to a lower and an upper voice. Yet beyond this it is impossible to compare the events in either subtree, either in terms of consonance or in terms of literal coincidence. (Though the comparison in terms of consonance is more important, since one can determine literal simultaneity of events if the labels on the leaves of the tree are made to correspond to particular events in the music).

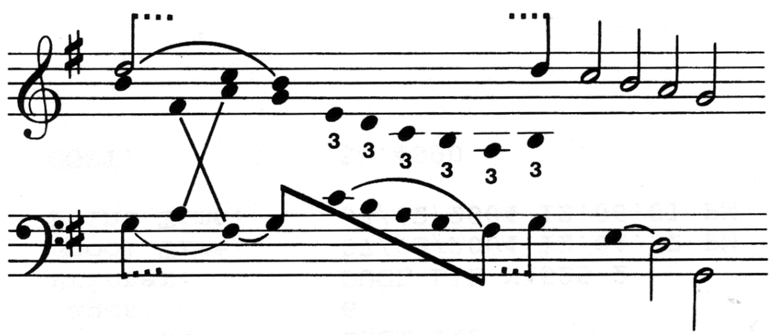

FIGURE 3.1: SMOLIAR'S MIDDLEGROUND ANALYSIS OF MOZART'S K. 283, 1, M. 1-10

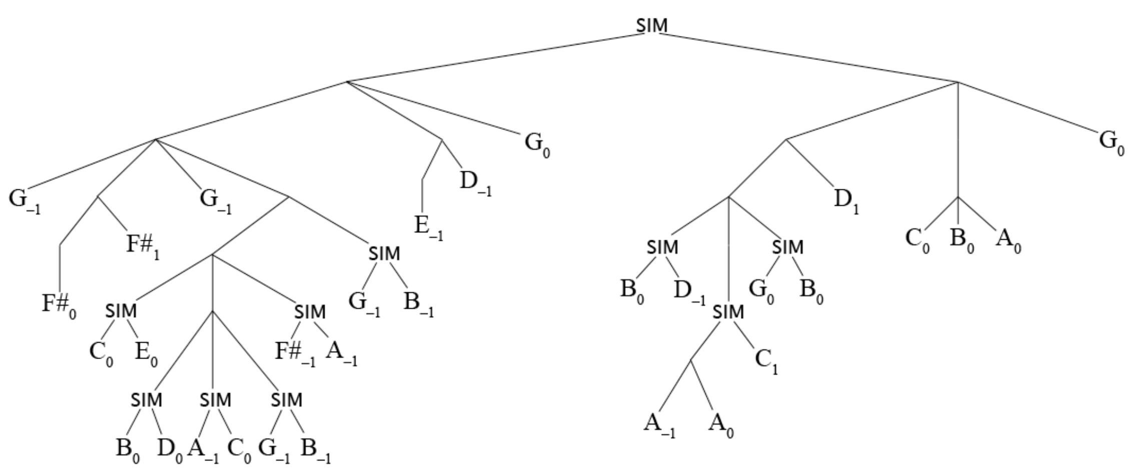

FIGURE 3.2: SMOLIAR'S ANALYSIS OF MOZART K. 283 IN TREE FORM

Consider, on the other hand, the SIMs on the left-hand side of the tree, which have SEQs as parents and pitches as children. These SIMs, unlike the SIM on the root vertex, do in fact show the simultaneity of pitches. In principle SIMs like these could show something like contrapuntal consonance if they were used at deeper levels of the tree. But then they give up their primary function of separating the events into voices.

To see how this works in a simpler case, consider Smoliar's G major *Ursatz* tree shown in figure 3.3. In this tree there is one SIM that separates the events into two voices (and corresponds to the SIM on the root of figure 3.2). This type of analysis does not meet our simultaneity/consonance/dissonance criterion for a contrapuntal analysis from "Criteria for a Contrapuntal Model," because it is impossible to tell from the tree that there is a consonant relationship between the second G_{-1} and G_0, or the first G_{-1} and D_1,

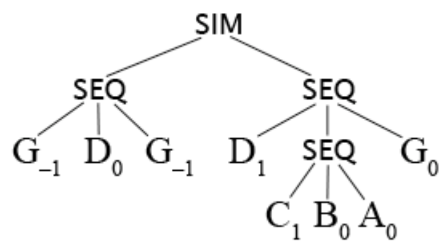

FIGURE 3.3: SMOLIAR'S 5-LINE URSATZ

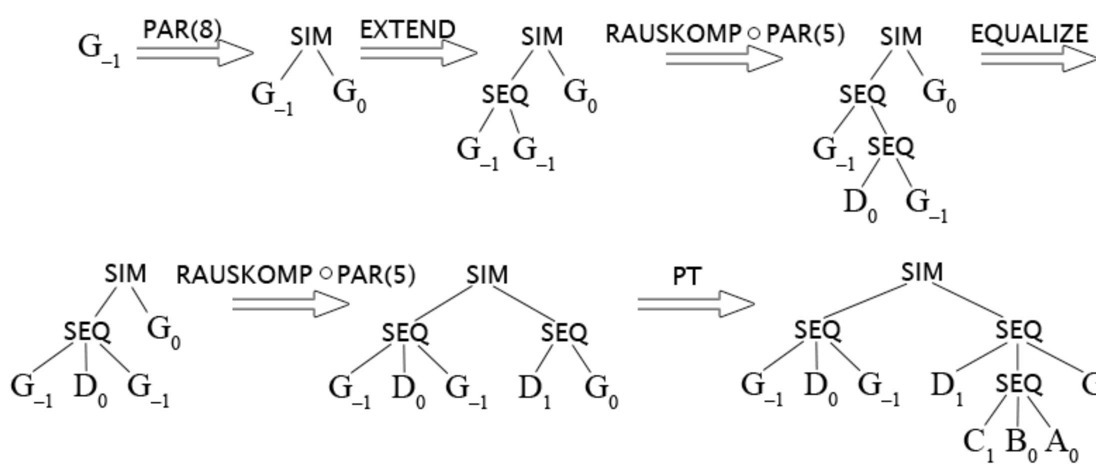

FIGURE 3.4: THE CONSTRUCTION OF SMOLIAR'S 5-LINE URSATZ

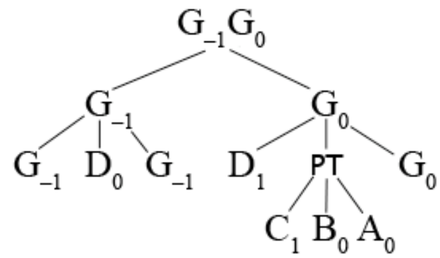

FIGURE 3.5: NON-TERMINAL VERTEX LABELS FOR SMOLIAR'S URSATZ *TREE*

or D_0 and A_0, or G_{-1} and B_0, or a dissonant relationship between G_{-1} and C_1. Figure 3.4 shows the derivation of the *Ursatz* tree (not given by Smoliar, but inferable from the form of the tree), and figure 3.5 shows the resulting labels on non-terminal vertices. (I discuss this process in part two, "The General Phrase-Structure Model of Prolongation").

Now consider the alternative five-line *Ursatz* tree constructed in figure 3.6 that tries to represent the consonant relationships missing in Smoliar's tree. In this tree, we construct the prolongational relationships between the consonantly supported notes of the upper voice before splitting off the lower voice. Then we add the lower voice notes as simultaneities with the upper voice notes they consonantly support. Finally, we add the dissonant note C_1. Figure 3.7 shows the resulting event labeled tree. This analysis has the flaw of fracturing the lower voice, which should be a continuous line. The linear

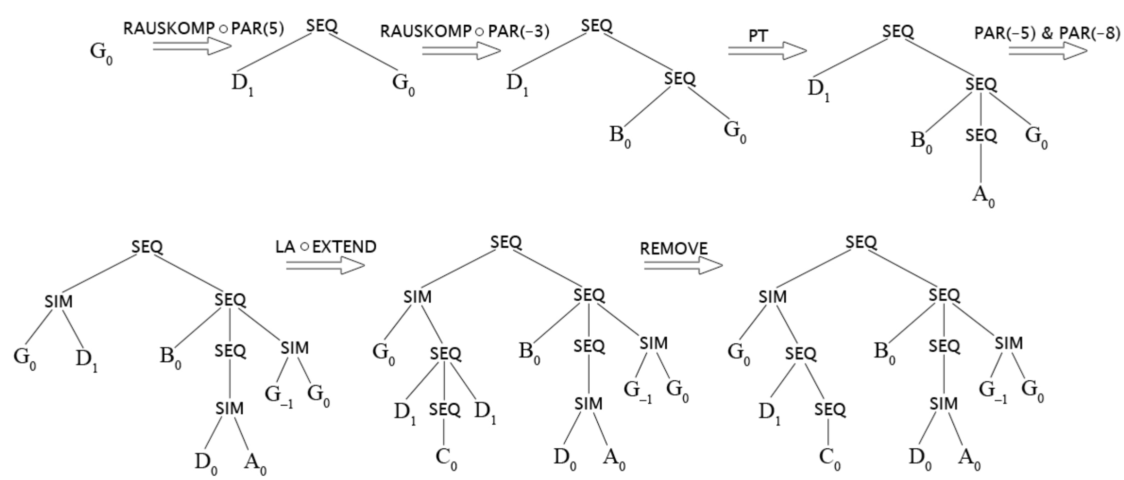

FIGURE 3.6: CONSTRUCTION OF AN ALTERNATE 5-LINE URSATZ

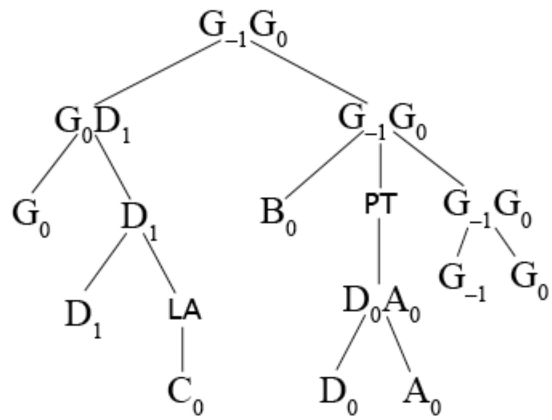

FIGURE 3.7: NON-TERMINAL VERTEX LABELS FOR THE URSATZ *TREE OF FIGURE 3.6*

relationship of G_{-1}-D_0-G_{-1} cannot be shown at the same time as the linear relationships of the upper voice and vertical relationships between the two. Furthermore, the status of B_0 in relation to the lower voice is unclear. Therefore the analysis of figure 6 is also not an entirely satisfactory representation of the prolongational relationships of voices in counterpoint.

Thus, in general, Smoliar's model is not an adequate contrapuntal model of prolongation by the criterion I have proposed, since it is not possible to represent consonant relationships between events without sacrificing the separation of events into voices. (See also Rahn 1989b on this topic).

The Representation of Counterpoint in Rahn's Model

In part two ("The General Phrase-Structure Model of Prolongation") I showed that the analytical system formalized by Rahn in "Logic, Set Theory, Music Theory" could be modified to fit the general phrase-structure model. An important component of this modification was to eliminate the arp operation and replace it with a triadic version of the neighbor operation. In this section I'd like to explore whether Rahn's definitions give us the tools to turn this phrase-structure version of Rahn's theory into a contrapuntal phrase-structure model along the lines proposed above in "Criteria for a Contrapuntal Model."

We'll be interested, then, in finding a way to partition a set of events into separate voices, and in relating events in different voices in terms of consonance. I will propose an interpretation that uses time-points to identify consonant relationships, and consider how the arp operation may fit in with this contrapuntal phrase-structure model. We'll find under this framework that there are three potential solutions to contrapuntal representation of an analysis in Rahn's model, one that bends Rahn's model—employing a more circumscribed version of it—and two that bend the concept of a contrapuntal voice to fit the more general version of the model.

In Rahn's model, each pitch event includes a time-point of initiation and release. Thus the model is potentially not only a model of tonal relationships in the music but also of rhythmic articulation. Although we aren't interested in the potential of Rahn's system to model aspects of theories of rhythm, the time-point information is important both for the derivation of a phrase-structure analysis from and the definition of contrapuntal relationships for an analysis in this system. In part two I used these time-points to associate pitch-events in the analysis with literal events in the score. They also function as a differentiation of events in terms of abstractness. That is, an event whose time-span encompasses the time span of some other event is in general more abstract and occurs at a higher level in the phrase-structure tree for the analysis.

In terms of contrapuntal consonance, time-points provide a concept of contrapuntal consonance that can potentially satisfy the second criterion in "Criteria for a Contrapuntal Model." According to the time-points of two events at the same level, they can be described as simultaneous (i. e. their durations overlap) or non-simultaneous. Simultaneous events (at some level) that remain so at the next level could be defined as consonant, while if one or both of the events is eliminated by the neighbor operation then they are dissonant. (More precisely, if events X and Y are simultaneous at level k and at level k + 1 there is an event Y' with the pitch of Y present throughout the time-span of Y, then X is consonant with Y if there is an event with the pitch of X simultaneous with Y' at level k + 1, and otherwise is dissonant with Y).

For instance, consider Rahn's analysis of the theme from Mozart's K. 333 Piano Sonata, shown in figure 3.8. The analysis takes the form of a series of reductional levels.

FIGURE 3.8: RAHN'S ANALYSIS OF MOZART K. 333, MM. 1-8

From each level to the next, the events are either identical, or there is a way to derive the more background from the more foreground events by the neighbor operation or the arp operation.

To pick out a few examples of dissonance from figure 3.8 somewhat arbitrarily: the sixteenth-note E_1 in measure 4 of reduction 1 is dissonant with the bass D_{-1}. In contrast, the interval $F\#_{-1}$-E_0 in measure 3 is consonant up through level five and becomes dissonant at level six (where $F\#_{-1}$ is defined as the dissonant note). At level 2 in measure 1, the passing tones B_{-1} and D_1 aren't technically dissonant with one another (because neither is present at level three), but they're each dissonant with the inner voice E.

Given these notions of consonance and dissonance the next task is to find a way to partition events into voices to meet the other criterion proposed above. A first condition on the partition of events into voices is that it makes the phrase-structure analysis of each voice possible, as described in part two ("The General Phrase-Structure Model of Prolongation"). One significant characteristic of the phrase-structure tree is that an event occurring at some relatively background level always has a corresponding event with the same pitch at each more foreground level. We can capture this in Rahn's system by assigning events to voices according to the following rule:

> *Voice-Assignment Rule 1:* if events at two consecutive levels have the same pitch, and the duration of the more background event includes the duration of the more foreground event, then assign them to the same voice.

This first rule takes care of events that are "the same" from one level to the next of the phrase-structure tree. We also need a way to assign newly originating events at some level to a voice. In Rahn's system, the number of events increases at more foreground levels primarily through the neighbor operation, the operation that generated phrase-structure trees in the discussion of Rahn's model in part two. Thus, for our partition of events into voices, we need a second rule:

Voice-Assignment Rule 2: if two events are related by the neighbor operation, then put them in the same voice.[68]

The only other way in which the number of events may increase from a more background to a more foreground level in Rahn's system is when the arp operation acts on repeated pitches. This results from some of the details of Rahn's logical construction of the model: his definition I provides no way to distinguish pitch events other than by pitch and time-point of initiation and release. Therefore definition VII for arp-prolongation doesn't require that the prolonging and prolonged sets have the same cardinality, just the same set of pitches. In part two, because it was important for the sake of comparison with the MOP model to use an event-label system for phrase structure analyses, I suggested replacing the arp operation on repeated notes with a special operation that could allow the analyst to distinguish which of the repeated notes was the more background-originating event. However, the event-label system isn't necessary for

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[68] A slight complication here is that it is possible for there to be ambiguity in an analysis in Rahn's system as to whether two notes are in fact related by the neighbor operation. This is apparent from the wording of Rahn's definition VIII, which requires only "*at least one* one-to-one correspondence" between partitions of two next-background levels. For an example of how this could leave the definition of neighbors ambiguous, consider a hypothetical analysis in which at some level a close position BDFG in whole notes moves to a close position CEG in whole notes, and at the next-background level, these are replaced by BDFG in breves. C and E could each be defined as diatonic neighbors to one of two notes of the dominant seventh, and could also be a circle-of-fifths neighbor to at least one note of the dominant seventh. For instance, one might imagine that C is a neighbor to D, E a neighbor to F, and G and D are arp-prolonged, but it's equally possible that C is a circle-or-fifths neighbor to F, E is a neighbor to D, and B and G are arp-prolonged; the analysis doesn't technically need to distinguish between these interpretations, not to mention the other possible ones.

While one might potentially want to avail oneself of such ambiguities to a limited degree, in general they're undesirable (as the hypothetical example shows) because the way you get from one level to the next is important for the meaning of the analysis. Of course, such ambiguities are not especially common given a few assumptions; for instance, there are no ambiguities in the analysis of figure 3.8 given the assumption that the neighbor operation is only used with the A major scale or circle-of-fifths as reference collections, and the circle-of-fifths neighbors are confined to the bass. Where such ambiguities do occur, however, we can assume that the analysis specifies partitions for each pair of next-background levels and a one-to-one correspondence between them that satisfies the definitions, preserving the simple and straightforward form of voice-assignment rule 2.

a phrase-structure analysis, as both Rahn's and Smoliar's work demonstrates: it's possible also to view the pitches themselves as being prolonged, rather than particular events that may be characterized by their pitch. Discarding the event-label system for the moment, voice-assignment rule 1 above takes care of repeated notes (related by the arp operation) by assigning them to the same voice.

These two voice-assignment rules apply only to pitched events, but Rahn also defines silent events, or rests. There is only one rest in the analysis of figure 3.8, which can be seen in reduction 5. This rest also must exist at levels 1-4, but Rahn shows it only at level 5 because it is needed here to extend the E through the fourth measure.

Rahn's inclusion of rests as events is an interesting but problematic aspect of his formal model. They obviously have a certain utility, as the rest in Rahn's Mozart analysis demonstrates. But what does it mean to assert that there is a silent event simultaneous with pitched events? The term "rest" implies that it is intended to show that some particular *voice* is silent for that duration, though Rahn doesn't discuss voices explicitly. For instance, the rest that shows up in reduction 5 of the Mozart analysis seems to indicate that an inner voice that has been repeating the note E_0 is momentarily silent.

This suggests that rests should be assigned to particular voices along with pitched events. Yet silent events don't quite work this way because in general it isn't possible to tell from the event itself which voice is silent for that time span. For instance, it's impossible to have two rests in the same reduction with the same initiation and release, so if multiple voices are silent for the same span of time one silent event must represent the absence of all of these voices. This means that rests cannot always be assigned to a specific voice, and it is better to let the partition of events into voices apply only to pitched events.

The result of our two voice-assignment rules on Rahn's analysis of the Mozart theme is shown in figure 3.9 (showing foreground events only). Note that figure 3.9 always assigns events related by the arp operation in the analysis to different voices, except where they share the same pitch (that is, when a pitch is repeated). The resulting

FIGURE 3.9: THE PARTITION OF EVENTS INTO VOICES GIVEN BY THE APPLICATION OF VOICE-ASSIGNMENT RULES 1-2 TO THE ANALYSIS OF FIGURE 3.8

partition into voices is somewhat implausible: it fractures this simple four-measure passage, written by Mozart as a counterpoint of three voices, into a hocket of seven separate voices.

This is the first possible solution to the problem of a contrapuntal representation of analyses in Rahn's system that I posed at the beginning of this section. This solution stretches the common sense notion of a "voice" by forcing all notes to be stepwise related to adjacent notes in their voice—except in the bass, assuming that circle-of-fifths neighbors are restricted to the bass—, thereby breaking ordinary voices into numerous parts with narrow tessituras. Of course, one could imagine someone with a particular theoretical orientation liking such an analytical system, someone that had a very abstract notion of "voice" and perhaps viewed most music as an imperfect attempt to express an elaborate hocket with a limited number of instruments, but I think it's worth exploring whether there is a way to derive a more common-sense assignment of events to voices from Rahn's analysis.

One solution to the problem is to restore a common-sense notion of voice by tinkering with the analytical system Rahn sets up. In part two I pointed out that the arp operation could be replaced by expanding the definition of neighbor to include neighbors within some chord. In fact, Rahn suggests such an expansion in "Logic, Set Theory, Music Theory," and in "Theories for some *Ars Antiqua* Motets" he employs a system that replaces the arp operation entirely with a neighbor operation defined for thirds. As a result, the two voice-assignment rules above recover the original three voices from Rahn's *ars antiqua* motet analysis. Similarly, figure 3.10 shows how it is possible, by replacing the arp operation with a triadic neighbor operation, to revise Rahn's analysis of the Mozart theme so that voice-assignment rules 1 and 2 produce the three voices shown in the original score. This analysis differs from Rahn's (figure 3.8) in particular in getting from level 3 to 4 and from 6 to 7 (comparable to Rahn's levels 8 and 9), where figure 3.10 uses the neighbor operation on arpeggiations rather than the arp-operation that figure 3.8 uses.

Figure 3.10 also shows how the analysis implies a phrase-structure to the upper and lower voices. (The inner voice is not especially interesting.) The exclusive use of the neighbor operation in an analysis guarantees that the reductions of each voice take the form of a phrase-structure tree in this way. (Given the use of the arp operation for the reduction of repeated notes, violations of phrase-structure are technically possible, but very unlikely in practice.)

Another way that the analysis of figure 3.10 differs from that of figure 3.8 is in how it deals with the suspension evident in reduction 5. There's an operation implied by the series of reductions in figure 3.8 that Rahn doesn't define in the paper: in the process of getting from reduction 5 to reduction 6, the notes on the upper staff tied from measure 3 to measure 4 must be separated into those in measure 3 and those in the first two eighths of measure 4, so that measure 4 can be reduced without swallowing measure 3 along with it. That is, there needs to be a level "5½" where E-A-C# on the upper staff are separate events in measure 3 from the first two eighths of measure 4.

FIGURE 3.10: AN ANALYSIS THAT REPLACES THE ARP RELATIONS OF FIGURE 3.8 WITH TRIADIC NEIGHBOR RELATIONS

This "splitting rule" is problematic for the interpretation of Rahn's analysis as a phrase-structure analysis. A phrase-structure analysis by definition takes the form of a tree so that in moving from foreground to background multiple events combine into single events, but single events do not split into multiple events, as they do in moving from level 5 to level "5½" of Rahn's analysis. This rule is important semantically (in interpreting the phrase-structure tree to represent prolongational relationships) because it prevents the odd situation where one thing prolongs two things.

Fortunately, since we are at present interested in Rahn's system as a model of prolongation and not of rhythmic articulation, we can ignore the splitting rule and the complications in the derivation of a phrase-structure tree that follow from it. Note, however, that the representation of a suspension, as in figure 3.10, is somewhat different without the rule: it is no longer a contrapuntal dissonance, as in figure 3.8, but instead shows up as a metrical displacement of the upper voices against the bass.

This analysis in figure 3.10 demonstrates the second solution to the problem of contrapuntal representation of analyses in Rahn's system I posed at the beginning of the section. This solution preserves the common sense notion of a voice by altering the analytical system itself. The result is nice in that the resulting phrase-structure analyses of each voice give a good overall representation of the analysis. However, I think the analysis in figure 3.10 is unconvincing as an analysis. This is in part because some of the quirks of Rahn's original analysis lose their *raison d'être* when important features of the analytical system—the splitting rule that interprets the C#$_1$ on the first beat of measure 4 as a contrapuntal dissonance, and the fuller simultaneities in the background produced by the arp operation—are pulled out from under them. It would be possible to produce a better analysis by discarding Rahn's analysis and starting from scratch in the retooled system. But let's rather give one more try at a contrapuntal representation that remains true to the original analysis in figure 3.8.

The fact that the analysis of figure 3.10 results in a simpler set of voices under voice-assignment rules 1 and 2 makes the problem with the these rules in interpreting the analysis of figure 3.8 evident: events in the same reduction are assigned to the same voice primarily through the second rule, which recognizes only neighbor relationships between events. Yet two simultaneous events cannot be defined as neighbors. So arp reductions, which generally increase the number of simultaneous events while maintaining the total number of events (except in the case of repeated notes), reduce the number of possible neighbor relationships at more background levels, and hence increase the number of voices under the interpretation provided by voice-assignment rules 1 and 2.

The obvious solution then is to introduce another voice-assignment rule to put arp-related notes in the same voice. This is problematic, however, because the arp operation can relate any number of notes from different registers at once. For instance, in reductions 3-4 of figure 3.8, we might like to say that in measure 1, C# and E in the right hand are related by the arp operation and therefore should be put in the same voice, and the same of A and C# in the left hand. However, there are other ways that the arp relations might be defined in this analysis. All of the notes, A_{-1}, $C\#_0$, $C\#_1$, and E_1, might be arp-related as a group, or A_{-1}-E_1 could be one arpeggiation and $C\#_1$-$C\#_0$ another.

For another example, consider reductions 7-8 of figure 3.8. It is possible that the second chord of reduction 7 should include some rests to extend E_1 and $C\#_0$ into measures 3-4 in reduction 8. However, since Rahn doesn't show these rests, we must assume that E_1 and $C\#_0$ are extended by being included in the arp relation of the repeated E_0's. This shouldn't imply, however, that all of these notes are in the same voice.

Figure 3.11 shows a set of three voices for the analysis of figure 3.8 obtained by selectively assigning arp-related notes to the same voice and the resulting phrase-structure "almost-trees" for the upper and bass voices (—these aren't quite phrase-structure trees because of Rahn's use of the splitting rule). One could formulate many possible voice-assignment rules for arp-related notes of various degrees of complexity, but here is a relatively simple one that will produce the partition of figure 3.11:

> *Voice-Assignment Rule 3*: If two pitched events, a and b, at some level can be arp-related, have pitches that are at most a fourth apart, and the release of a is equal to the initiation of b, then assign them to the same voice.

The phrase-structure (almost-) trees of figure 3.11 demonstrate some of the unusual features of the contrapuntal model produced by applying voice-assignment rules 1-3 to an analysis such as Rahn's analysis of the Mozart theme, especially in comparison to the analysis of figure 3.10. One striking feature is the status of simultaneity in the model. Common sense would lead us to expect some sort of orthogonality between simultaneity and participation in a voice. That is, no two pitches should be both simultaneous and in the same voice, but each pair of different voices should have at least some simultaneous

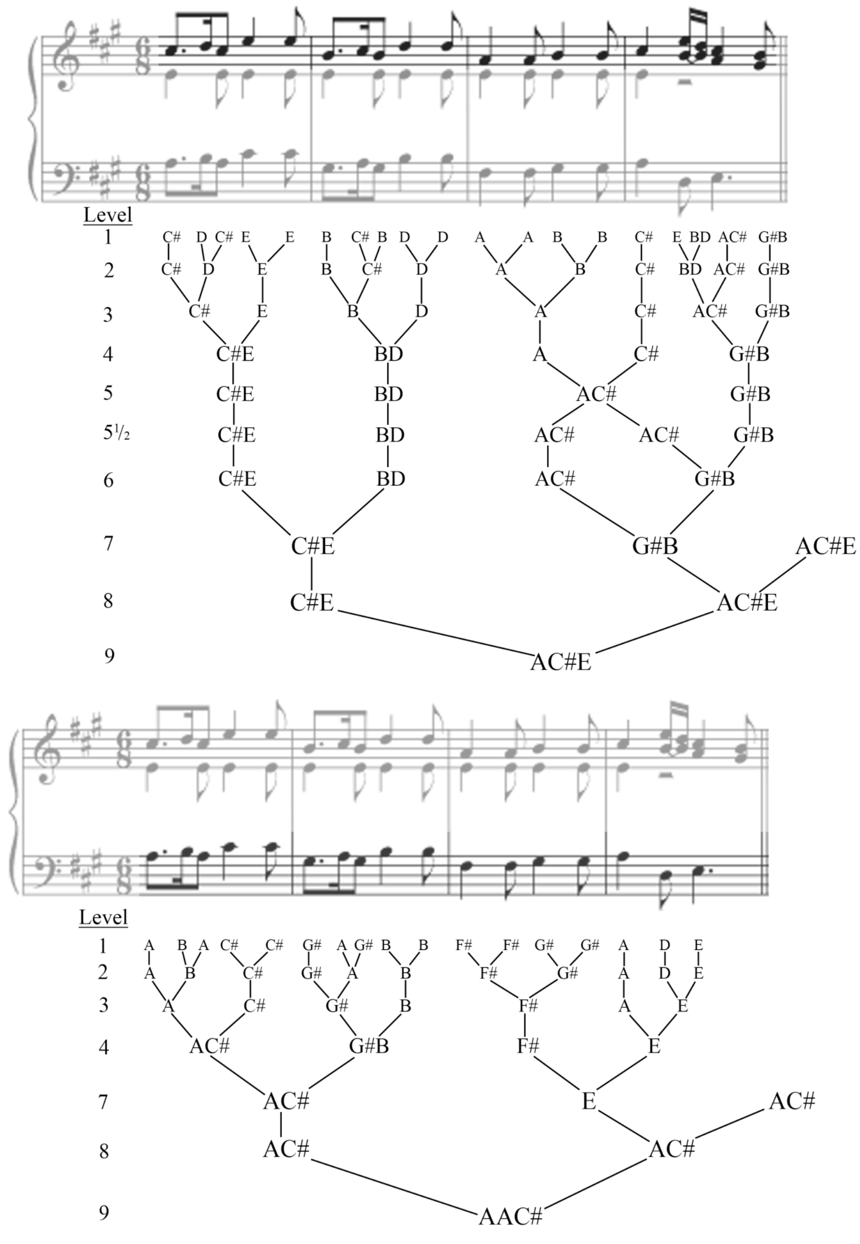

FIGURE 3.11: PHRASE-STRUCTURE TREES FOR THE OUTER VOICES OF THE ANALYSIS OF FIGURE 3.8

pitches between them. The first solution to the contrapuntal representation of Rahn's system as shown by the partition into voices of figure 3.9 violates the second aspect of orthogonality of simultaneity and voice-membership: many pairs of "different" voices share no simultaneous pitches at the foreground (although they do at some sufficiently background level). In contrast, the analysis of figure 3.11 violates the first aspect of orthogonality: simultaneous notes may occur in the same voice, even notes that are simultaneous at the foreground.

This sense of orthogonality with simultaneity, it seems to me, is essential to what we mean when we talk about "voices" in music. After all, the reason given above in "Criteria for a Contrpuntal Model" for using the "independent voices" criterion as a measure of whether an analytical system is contrapuntal is that voices are essential to species counterpoint. In species counterpoint the orthogonality of voice and simultaneity is a critical aspect of separation into voices and is strictly maintained. Consequently the "voices" of figure 3.9 and 3.11 should perhaps be called "registral groupings" or something of the like rather than voices. For this reason none of the three possible contrapuntal interpretations of Rahn's system I have offered are wholly satisfactory. While the third interpretation yields a good representation of Rahn's analysis in terms of (almost-) phrase structures, the lack of orthogonality between the registral groupings and simultaneities makes it inadequate as a contrapuntal representation, at least the understanding of "contrapuntal" I've advanced here.

This doesn't mean, however, that in adopting Rahn's analytical system (complete with arp relationships) one must disavow any discussion of voices altogether. It's possible, for instance, to assert that the set of voices is different at each level. According to this perspective we could derive phrase structures such as those of figure 3.11 for the analysis, yet these phrase structures aren't analyses of a particular voice but of registrally-associated (or however-associated) collections of different voices. Or, in fact, one could assert that the membership of notes in voices is completely independent of their relationships in the analysis. Under such interpretations, the contrapuntal representation is to some extent separate from the analysis of prolongation, so that prolongation and counterpoint can be seen to interact in various ways, but a contrapuntal representation is ultimately inessential to the understanding of prolongational relationships.

In this sense Rahn's system is not a contrapuntal system of prolongational analysis, although it's non-contrapuntal in a radically different way than Lerdahl and Jackendoff's. Lerdahl and Jackendoff's model is non-contrapuntal because it "chunks" events vertically before relating them horizontally, whereas Rahn's model, which always maintains the independence of events vertically, may be considered non-contrapuntal because of its resistance to the horizontal "chunking" of events into voices.

The Extension of the MOP Model of Prolongation to Contrapuntal Analysis

My extension of the MOP model of part one to contrapuntal analysis will follow the suggestions made in "Criteria for a Contrapuntal Model." I'll give a general outline of the system before proceeding to a more detailed treatment of some of its parts.

First, the events of the music are separated into voices ordered from lowest to highest (although there are no constraints concerning ranges or voice crossing).

Second, the event sequence of each voice has a MOP prolongational analysis as described in part two.

Third, there is a harmonic prolongational analysis of outer voice consonances, also in the form of a MOP. This is the sort of analysis presented in "Refinements of the MOP Model" in part one to show unfolding transformations, and represents the "underlying first-species counterpoint" described in the analogy of "Criteria for a Contrapuntal Model" above. This MOP doesn't necessarily include all events in either voice, but includes all those events that participate in a consonance with an event in the other voice. Actually, the latter part of this statement is tautological, since "consonance" here is normatively defined. That is, a consonant pair of events is by definition one that occurs in the harmonic prolongational analysis. The choice of which event pairs make up the consonances, then, is primarily an analytic decision that isn't prescribed by the formal model. In particular, a consonant pair of events need not necessarily sound simultaneously on the musical surface.

Fourth, there is a harmonic prolongational analysis that extends the MOP of consonances by adding the dissonant events of each voice. This includes all events not represented in the MOP of consonances, producing a complete prolongational analysis of the outer voice counterpoint. However, as a graph this prolongational analysis is not necessarily a MOP, but a member of a more general class of graphs called "2-trees."

Finally, each additional voice can be combined with the complete harmonic prolongational analysis in a similar fashion, by first identifying consonant relationships between this voice and the elements of the outer-voice contrapuntal analysis, and then adding the remaining dissonances. In principle, the process can begin with any two voices, not just the outer two. In fact, it is not actually necessary to break the process down voice-by-voice in this way; I have chosen to do so only to facilitate the presentation of the system.

There are three important constraints on the definition of a set of consonant event-pairs between two voices. First, the consonant event-pairs must make a well-ordered sequence that agrees with the ordering of the events within their individual voices. More precisely, if (w, y) and (x, z) are consonant event-pairs such that (w, y) precedes (x, z) (in the ordering of consonant event-pairs), then either w = x or w precedes x in the lower voice and y = z or y precedes z in the upper voice. (Note that the set of consonant pairs is defined as a set of two-element lists where the first element is an event in the lower voice and the second is an event in the upper voice. As a result, it is impossible for both x = w and y = z, since that would imply (w, y) = (x, z).)

The rationale for this constraint is fairly obvious: the condition for interpretation of a MOP prolongational analysis is that the events make up a time-ordered sequence in the music, reflected by the Hamiltonian cycle of the MOP. Yet the ordering of the sequence of consonances would not make musical sense if it contradicted the ordering of any of the individual voices.

Second, the initiation events in each voice make a consonant pair, as do the termination events. This indicates that the event sequences of the two voices occur within the same general time-span.

Third, the prolongational analyses of the events making up the consonant pairs must also agree. That is, for any two consonant pairs, (w, y) and (x, z), (w, y)(x, z) is an edge of the harmonic MOP if and only if either w = x or wx is an edge of the lower-voice MOP and either y = z or yz is an edge of the upper-voice MOP. If every event of some voice participates in no more than one consonant pair, then this constraint says that the mapping of an event in this voice to the event-pair it participates in gives an isomorphism from a subgraph of the individual voice's MOP to the consonant harmonic MOP. Also, note that the edges of the harmonic MOP are completely determined by the set of consonant event pairs and the analysis of each voice.

Perhaps it is clear why this last condition is necessary, but it may be worthwhile at this point to clarify the semantics of a harmonic prolongational analysis. The construction of a prolongational analysis of consonances makes the implicit claim that at some sufficiently background level, the events in each individual voice combine "vertically" into harmonic events, and that these harmonic events can be said to relate to one another prolongationally. In order to construct such an analysis, then, at least the most background events in each voice (the initiation and termination events) must participate in harmonic events. And, in addition, the voices making up these harmonic events necessarily inherit their prolongational relationships, leading us to posit the third condition.

Let me further clarify these points in the context of an example. Figure 3.12 shows an episode of the fugue whose subject I analyzed in part 1, and William Renwick's reduction and analysis of the episode in *Analyzing Fugue*. (157) Figure 3.13 shows the consonant harmonic MOP for the outer voices (with event names that include both measure numbers and register numbers). At the bottom of Renwick's analysis he provides the basic harmonic progression of the passage: I-IV-V-VI. This progression is represented in the background of the consonant MOP: $({}_{13}E_{-1}, {}_{13}G_{1})$-$({}_{14}F_{-1}, {}_{14}A_{1})$-$({}_{16}G_{-1}, {}_{16}B_{1})$-$({}_{18}A_{-1}, {}_{18}C_{2})$. At this level, the two melodies move exclusively in parallel tenths. However, the inner voice motion, ${}_{13}C_{1}$-${}_{16}D_{1}$-${}_{18}E_{1}$, not yet included in the analysis, reveals the V as a dividing dominant, making the basic motion $({}_{13}E_{-1}, {}_{13}G_{1})$-$({}_{16}G_{-1}, {}_{16}B_{1})$-$({}_{18}A_{-1}, {}_{18}C_{2})$ with passing motion through $({}_{14}F_{-1}, {}_{14}A_{1})$.

In measures 15 and 17 there are subordinate applied dominant harmonic events. These are represented in the MOP by the vertices $({}_{15}D_{-1}, {}_{15}A_{1})$ and $({}_{17}E_{-1}, {}_{17}B_{1})$. Notice that these pairs of events do not occur simultaneously in the music, but comprise the "real" (most background) outer voices for the harmonic activity of each of these two measures.

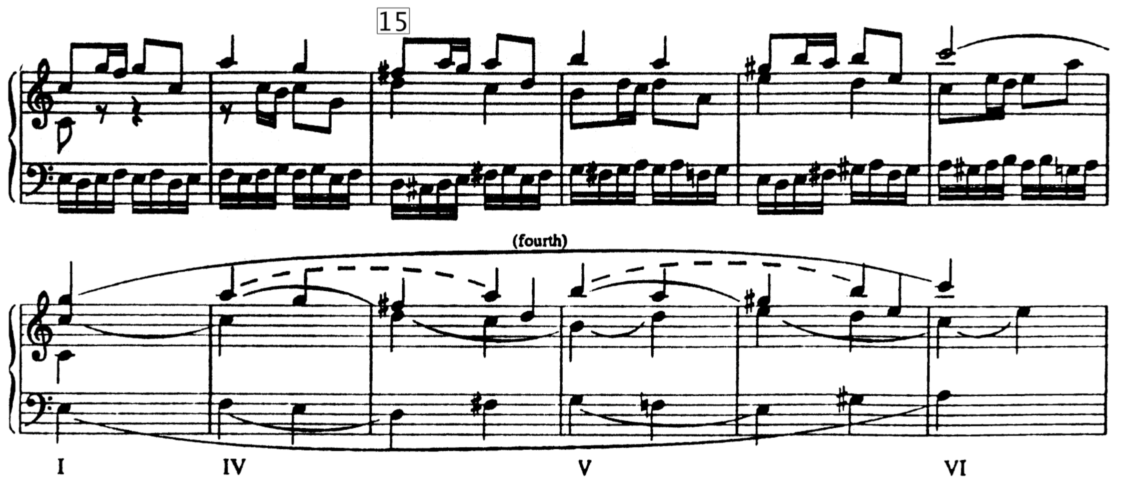

FIGURE 3.12: A FUGUE EPISODE AND RENWICK'S ANALYSIS

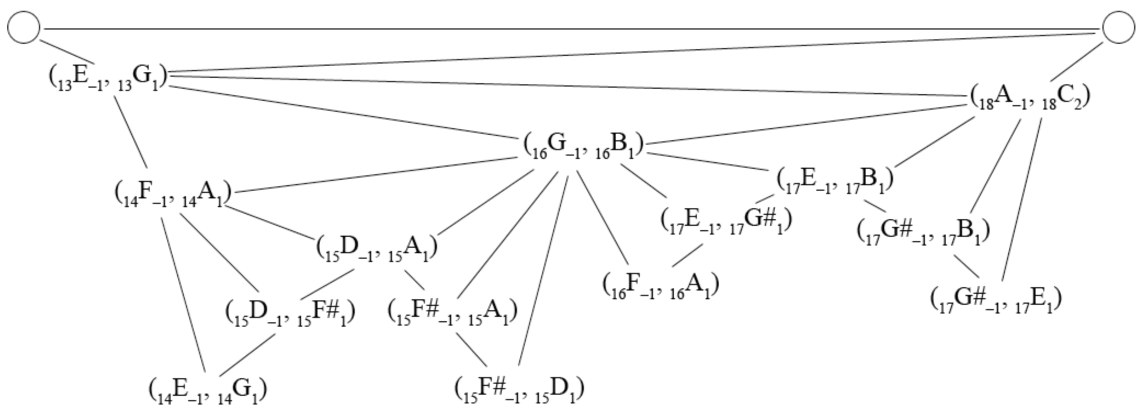

FIGURE 3.13: THE CONSONANT HARMONIC MOP FOR THE OUTER VOICES OF THE FUGUE EPISODE

The F#'s in both voices in measure 15 also participate in this applied dominant harmony of this measure, although at a slightly more foreground level. Therefore, they can also be included as consonant events, ${}_{15}F\#_1$ in consonance with the lower voice ${}_{15}D_{-1}$ and ${}_{15}F\#_{-1}$ in consonance with the upper voice ${}_{15}A_1$. In addition, the leap to ${}_{15}D_1$ in the upper voice is consonantly supported by ${}_{15}F\#_{-1}$ (not ${}_{15}D_{-1}$, because it is a relatively foreground event, as the isolation of ${}_{15}D_1$ from ${}_{15}D_{-1}$ in the MOP of figure 3.13 shows). Measure 17 is analyzed similarly to measure 15.

Finally, the pairs of passing tones (${}_{14}E_{-1}$, ${}_{14}G_1$) and (${}_{16}F_{-1}$, ${}_{16}A_1$) are shown in the MOP as consonances, since they move together in parallel tenths in Renwick's analysis (although they do not ultimately participate in distinct harmonies, properly speaking, as we will see below).

Thus the outer voices in Renwick's reduction are entirely consonant with one another. That is, every event participates in the consonant prolongational analysis, making it equivalent to the complete prolongational analysis. (Of course, this is due to the way Renwick reduces the passage: there are many dissonances in the outer voices of the actual music). The only complication in the consonant MOP is the duplication of the notes ${}_{15}D_{-1}$ and ${}_{15}A_1$ (and the corresponding duplication in measure 17). This is necessary to show that not only do ${}_{15}D_{-1}$ and ${}_{15}A_1$ bear consonant relationships to ${}_{15}F\#_1$ and ${}_{15}F\#_{-1}$ respectively, but also, at a more background level, to one another.

Figure 3.14 shows the MOPs analyzing each of the outer voices individually for comparison with the consonant harmonic MOP of figure 3.13. This example demonstrates the need for the conditions on the consonant harmonic MOP described above. The voices must inherit the prolongational relationships of the harmonic events

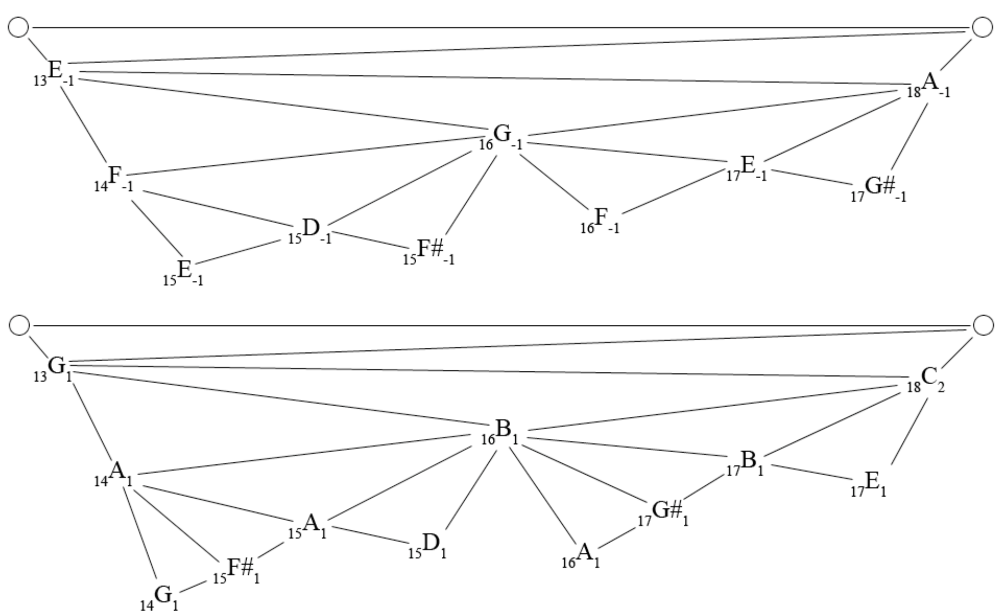

FIGURE 3.14: MOP ANALYSES OF THE OUTER VOICES OF THE FUGUE EPISODE

they constitute. For instance, if the consonant MOP shows a motion from a subdominant event, $({}_{14}F_{-1}, {}_{14}A_1)$, to a dominant event, $({}_{16}G_{-1}, {}_{16}B_1)$, being prolonged by an applied dominant event $({}_{15}D_{-1}, {}_{15}A_1)$, then the bass motion from the subdominant to the dominant, ${}_{14}F_{-1}$ to ${}_{16}G_{-1}$, must also be prolonged by the bass of the applied dominant, ${}_{15}D_{-1}$. However, in certain cases an event in one voice may relate consonantly to more than one event in another voice. For instance, ${}_{15}F\#_{-1}$ is consonant both with ${}_{15}A_1$ and ${}_{15}D_1$ in the upper voice. In this case the fact that some motion from $({}_{15}F\#_{-1}, {}_{15}A_1)$ is prolonged by $({}_{15}F\#_{-1}, {}_{15}D_1)$ doesn't itself say anything about relationships between events of the lower voice (since it involves only one such event), just the upper voice.

The next step in the analysis is to add the inner voice. For the sake of comparison with the relatively consonant outer voice counterpoint, I will first combine the inner voice with the upper voice alone as a slightly more complex example of two-voice counterpoint. This will illustrate the difference, in general, between a consonant harmonic prolongational analysis and a complete harmonic prolongational analysis.

Figure 3.15 gives the consonant MOP for the upper voices of the fugue episode. This includes the background tonic, subdominant, dominant, and submediant events $(({}_{13}C_1, {}_{13}G_1), ({}_{14}C_1, {}_{14}A_1), ({}_{16}D_1, {}_{16}B_1)$, and $({}_{18}E_1, {}_{18}C_1))$, as well as the applied dominant events $(({}_{15}D_1, {}_{15}A_1), ({}_{17}E_1, {}_{17}B_1))$. Furthermore, ${}_{15}F\#_1$ and ${}_{17}G\#_1$ in the soprano also participate consonantly in the applied dominants. However, there are many events in the soprano and alto of Renwick's reduction not included in the consonant MOP of figure 3.15. These can be seen in the MOPs for each of these individual voices in figure 3.16: in the alto, ${}_{15}C_1$, ${}_{16}B_0$, ${}_{17}D_1$, and ${}_{18}C_1$, and in the soprano, ${}_{14}G_1$, ${}_{15}D_1$, ${}_{16}A_1$, and ${}_{17}E_1$. The harmonic prolongational analysis must include these events as dissonances.

These dissonant events are incorporated into the harmonic analysis as prolongations of consonant events. The complete harmonic prolongational analysis of figure 3.17 illustrates this process. Most such dissonant events prolong the motion between two consonant events directly. In this case, the edge between the prolonged events in the individual voice's MOP reflects an edge of the consonant harmonic MOP, so the dissonant event is joined to this edge. For example, the passing tone ${}_{14}G_1$ in the

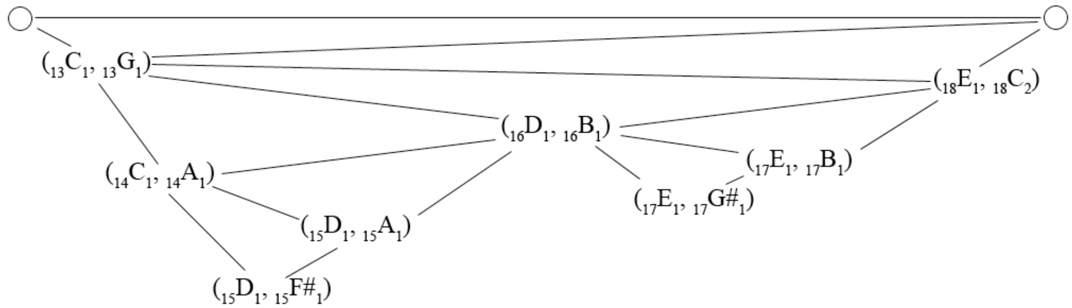

FIGURE 3.15: THE CONSONANT HARMONIC MOP FOR THE UPPER VOICES OF THE FUGUE EPISODE

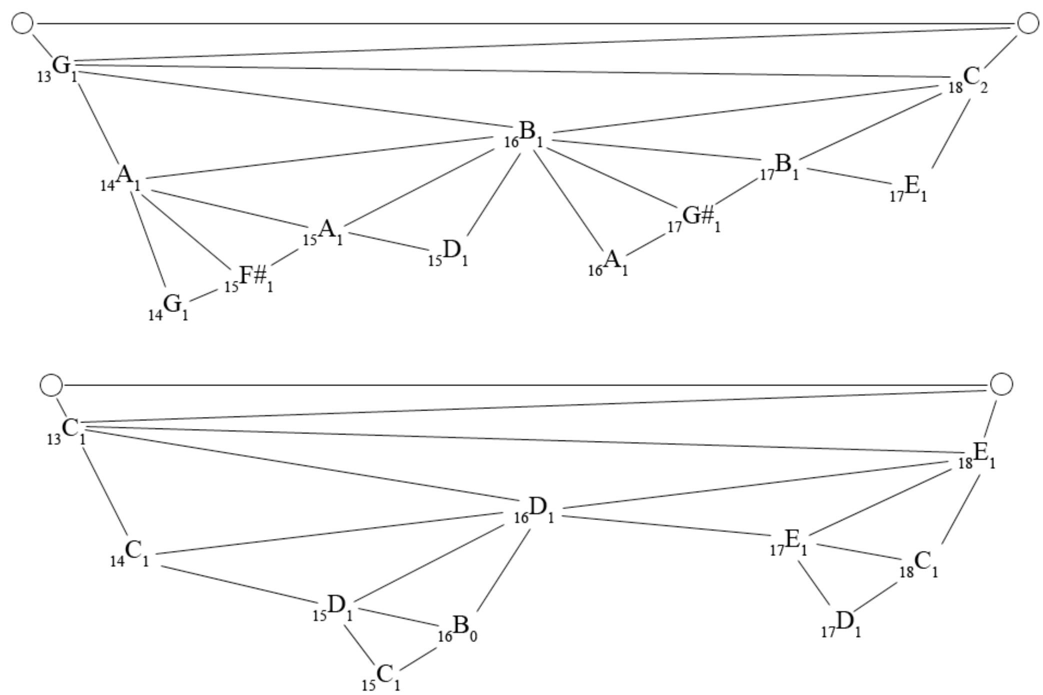

FIGURE 3.16: MOP ANALYSES OF THE ALTO AND SOPRANO OF THE FUGUE EPISODE

soprano prolongs the interval from ${}_{14}A_1$ to ${}_{15}F\#_1$. The consonant harmonic MOP represents the motion from ${}_{14}A_1$ to ${}_{15}F\#_1$ with the edge $({}_{14}C_1, {}_{14}A_1)({}_{15}D_1, {}_{15}F\#_1)$. Therefore, there is a vertex in the complete harmonic analysis labeled $(–, {}_{14}G_1)$, where the dash asserts that ${}_{14}G_1$ is dissonant with the alto voice, and there are edges from $({}_{14}C_1, {}_{14}A_1)$ and $({}_{15}D_1, {}_{15}F\#_1)$ to this vertex.

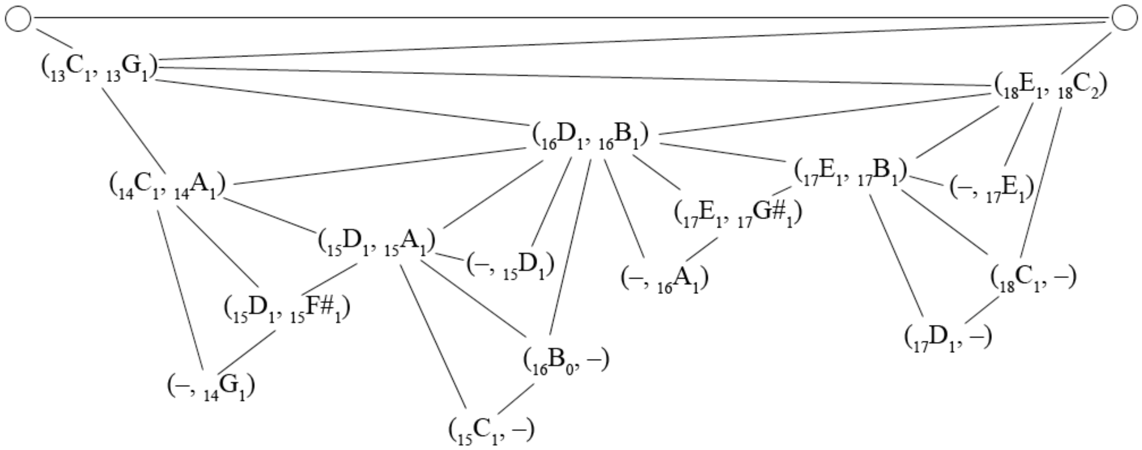

FIGURE 3.17: THE COMPLETE HARMONIC ANALYSIS OF THE UPPER VOICES OF THE FUGUE EPISODE

Occasionally a dissonant event in some voice prolongs the motion to or from another event that is also dissonant in that voice. An example in figure 3.17 is ${}_{15}C_1$ in the alto, which prolongs the motion to ${}_{16}B_0$. The event ${}_{16}B_0$ is dissonant with the soprano because the proper voicing of the dominant chord to support the soprano ${}_{16}B_1$ is with the D_1 that follows ${}_{16}B_0$ in the alto. This is not a problem for the construction of the complete harmonic analysis, since ${}_{16}B_0$ itself directly prolongs the motion between two consonant events, so that $({}_{15}C_1, –)$ can be added with an edge to $({}_{16}B_0, –)$ after $({}_{16}B_0, –)$ itself is added. Since the initial and final events must be in the consonant MOP, any chain of dissonances prolonging dissonances must eventually terminate in a dissonance that prolongs the motion between two consonant events.

Note that the graph in figure 3.17 is not itself a MOP, because the dissonances $(–, {}_{15}D_1)$ and $(–, {}_{17}E_1)$ are not on the outer perimeter in the drawing. (It would be possible to draw the graph differently to put them on the outer perimeter, but this would take the alto dissonances off the perimeter). It is, rather, a member of the more general class of 2-trees, which will be defined formally in part 4. In general, a 2-tree violates MOP-hood wherever multiple events directly prolong the same interval, as $(–, {}_{15}D_1)$ and $({}_{16}B_0, –)$ both prolong $({}_{15}D_1, {}_{15}A_1)({}_{16}D_1, {}_{16}B_1)$ in figure 3.17. This is allowed to happen in harmonic analyses when the prolonging events are dissonances in distinct voices. Recall from part one that MOPs can be defined as Hamiltonian crossing-free chordal graphs. 2-trees are crossing-free chordal graphs, like MOPs, but unlike MOPs are not necessarily Hamiltonian (rather they are "2-connected," a weaker property; see "H2-Intrasymmetry" in part four). This reflects the fact that there may not be a completely well ordered sequence of events when those events can occur in different voices.

Note also how the complete harmonic prolongational analysis of figure 3.17, because it includes at least one vertex corresponding to each event of each individual voice, completely determines the MOP analyses of each voice included in it. You can see this by comparing the analyses of the alto and soprano voices in figure 3.16 with the complete two-voice harmonic analysis of figure 3.17. I will formalize this process below as a direct way of defining the complete harmonic prolongational analysis for any number of voices.

But before I do so, let's consider how these analyses combine into a complete three-voice analysis. Figure 3.18 reproduces the analysis of the bass voice from figure 3.14 and the harmonic analysis of the upper voices from figure 3.17. The complete three-voice harmonic analysis the passage can be thought of as a combination of these two analyses, treating the two-voice harmonic analysis as if its events were those of a single voice. Figure 3.19 shows a "consonant analysis" for the combination, including all three-voice consonances and all places where the bass gives consonant support to an event in an upper voice that is dissonant with the other upper voice. The only events missing are the passing tones of the alto, ${}_{15}C_1$ and ${}_{17}D_1$, which are dissonant against the bass. These are added as dissonances in the complete harmonic analysis of figure 3.20.

The complete analysis of figure 3.20 includes a foundational background structure of harmony in which all three voices participate, consisting of the overall motion from I to VI being prolonged by the progression from IV to V and the applied dominants prolonging the motion from IV to V and V to VI. There are also arpeggiations in the outer voices within these applied dominants that make up the most foreground part of the consonant structure of the analysis.

The remaining events appearing in the analysis are somehow dissonant. However, some are purely dissonant, while others are dissonant only with one of the two

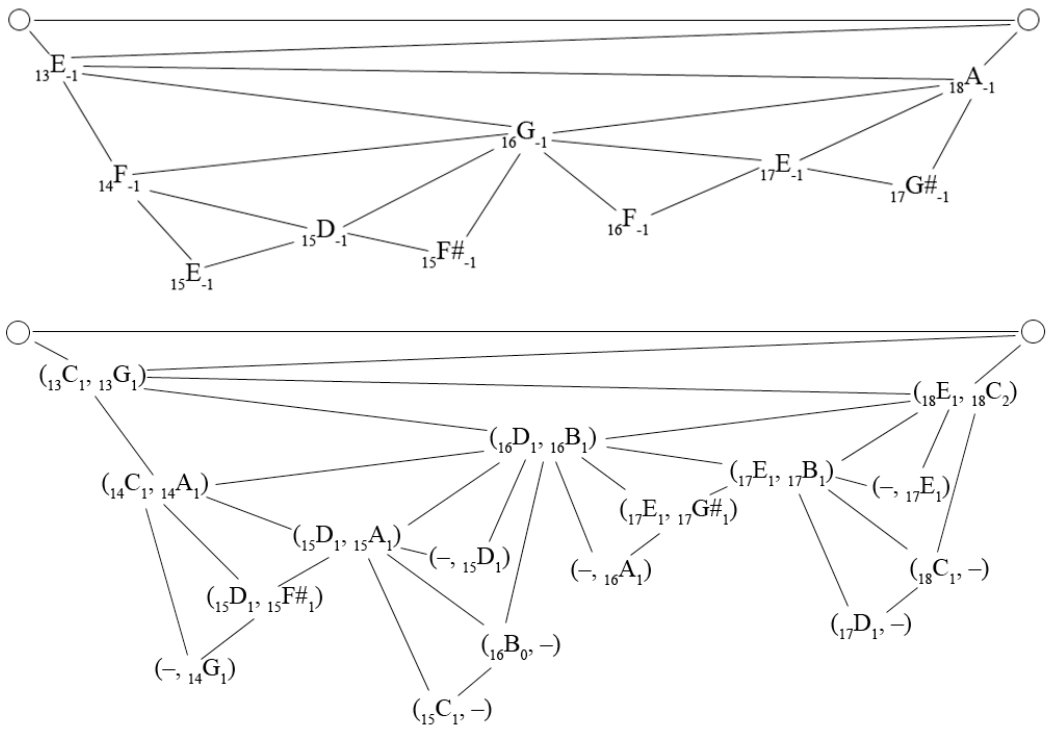

FIGURE 3.18: THE MOP ANALYSIS FOR THE BASS VOICE AND COMPLETE HARMONIC ANALYSIS FOR THE UPPER VOICES OF THE FUGUE EPISODE

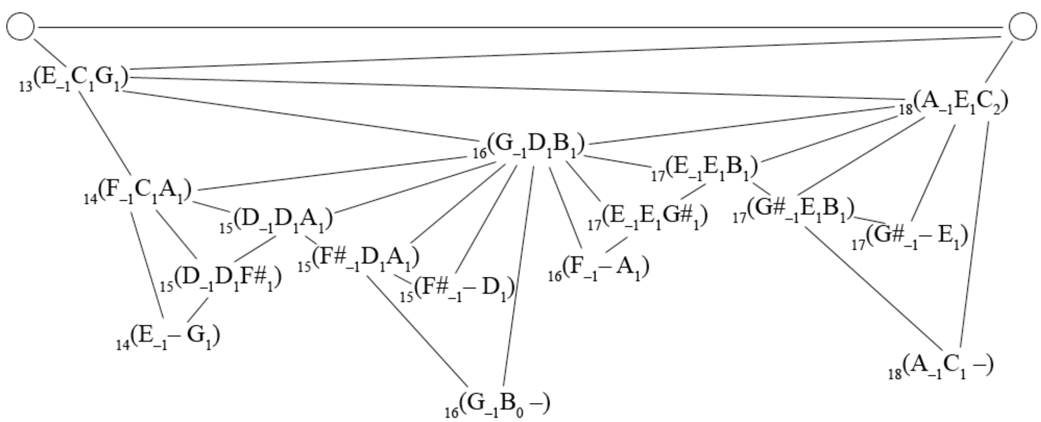

FIGURE 3.19: THE "CONSONANT ANALYSIS" OF BASS AND UPPER VOICES FOR THE FUGUE EPISODE

other voices. Consider just the events in the prolongation of IV-V (the event prolonging V-VI are analyzed equivalently): the passing tones $_{14}E_{-1}$ and $_{14}G_1$ in the outer voices move together in consonant parallel tenths, but they don't participate in the main

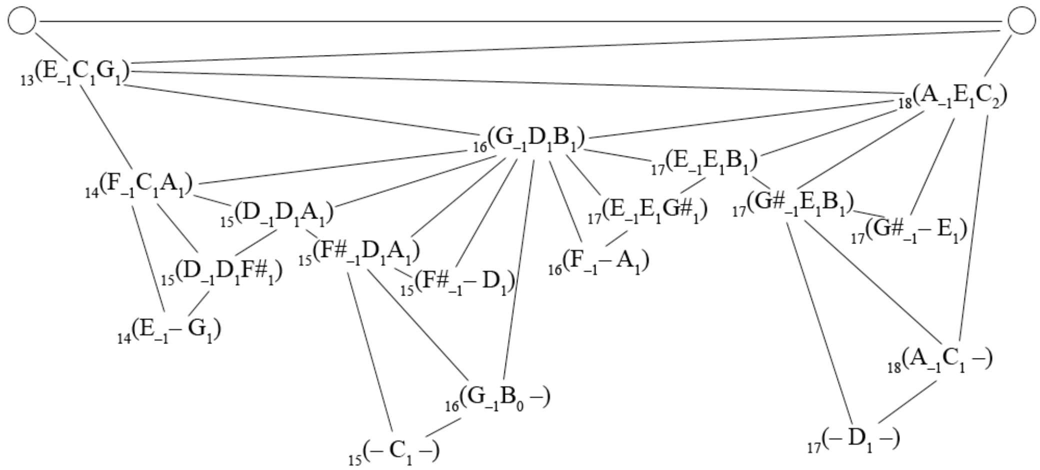

FIGURE 3.20: THE COMPLETE HARMONIC ANALYSIS OF THE FUGUE EPISODE

harmonic event of the measure (the IV), and thus are dissonant with the inner voice. The note ${}_{16}B_0$ in the alto voice, as I noted above, isn't the correct inner voice support of ${}_{16}B_1$ in the soprano, so, although both of these note are consonantly supported by the V in the bass, the alto note ${}_{16}B_0$ is dissonant with the soprano. Finally, the note ${}_{15}D_1$ in the soprano is a consonant arpeggiation within the applied dominant, but is too local of an event to be supported by the ${}_{15}D_1$ of the alto. The only pure dissonances are the passing tones ${}_{15}C_1$ and ${}_{17}D_1$ in the alto.

We can also read a multivoice harmonic analysis like that of figure 3.20 holistically. For instance, the graph indicates a motion from an applied dominant, (${}_{15}D_{-1}$, ${}_{15}D_1$, ${}_{15}A_1$), to a V, (${}_{16}G_{-1}$, ${}_{16}D_1$, ${}_{16}B_1$). This is prolonged first by a consonant arpeggiation of the bass to F#. Then, the motion from (${}_{15}F\#_{-1}$, ${}_{15}D_1$, ${}_{15}A_1$) to (${}_{16}G_{-1}$, ${}_{16}D_1$, ${}_{16}B_1$) is prolonged in two (independent) ways: by a consonant leap in the soprano to ${}_{15}D_1$ and a resolution of the alto voice to ${}_{16}B_0$ of the V filled in by a passing tone ${}_{15}C_1$.

Notice that something unusual occurs here that is impossible in a two-voice analysis: the upper voices have two dissonant prolongations (${}_{15}D_1$ in the soprano and ${}_{16}B_0$ in the alto) of the motion between two consonant groups, one of which is supported by the preceding bass note, ${}_{15}F\#_{-1}$, and one supported by the following bass note, ${}_{16}G_{-1}$.

A Formal Definition of the Complete Harmonic Prolongational Analysis

I have suggested up to this point defining harmonic analyses for more than two voices by a process of construction that involves adding each voice successively, although I have not rigorously formalized the process. Although this is possible, it's actually a rather clumsy way of going about it. It's easier—and more enlightening—to give a more direct characterization of a harmonic analysis for any number of voices.

The formal definition of a harmonic analysis I will give below is based on the fact that the analysis of each voice can be derived directly from the harmonic analysis. For instance, let's derive a MOP analysis of the inner voice of the fugue episode from the three-voice analysis of figure 3.20. First, we *delete* any vertices dissonant with the inner voice (that have a "–" in the second place). These vertices are $({}_{14}E_{-1}, –, {}_{14}G_1)$, $({}_{15}F\#_{-1}, –, {}_{15}D_1)$, $({}_{16}F_{-1}, –, {}_{16}A_1)$, and $({}_{17}G\#_{-1}, –, {}_{17}B_1)$. (Note that each of these is incident on only two edges of the graph). To *delete* a vertex means to remove it and any edges it participates in from the graph; figure 3.21 shows the result.

Next, we need to combine all the vertices that involve the same alto event. These are all the vertices with ${}_{15}D_1$ and ${}_{17}E_1$ in the second place. We combine them with a process called *contraction*. To *contract* two adjacent vertices means to replace them with a single vertex and add an edge between the new vertex and every vertex adjacent to one of the old vertices. (Recall that two vertices being *adjacent* means they share an edge in the graph). Figure 3.22 shows the result of this process applied to the graph of figure 3.21, with the new vertices labeled by the alto voice event they represent. Now we have a graph in the form of a MOP and has one vertex for each alto voice event. Therefore we only need to switch the labels of the vertices and we have the MOP analysis for the alto voice, as shown in figure 3.23.

This process serves as a basis for a relatively straightforward definition of a complete harmonic analysis. Besides deletion and contraction, which I just explained, we also need the idea of a *neighborhood* in this definition. The neighborhood of a vertex v is the set of all vertices adjacent to v (but not equal to v). The idea of an overlap of

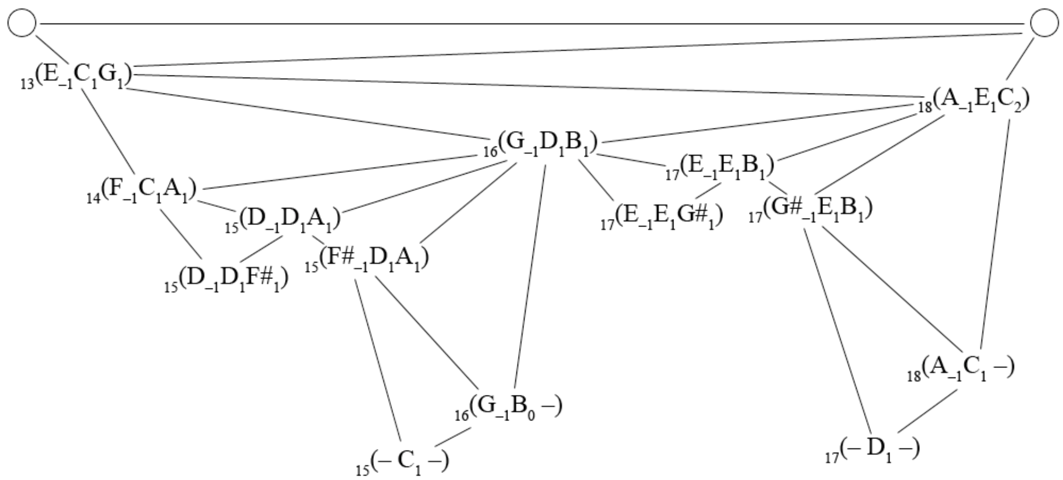

FIGURE 3.21: THE COMPLETE HARMONIC ANALYSIS WITH EVENTS DISSONANT WITH THE ALTO REMOVED

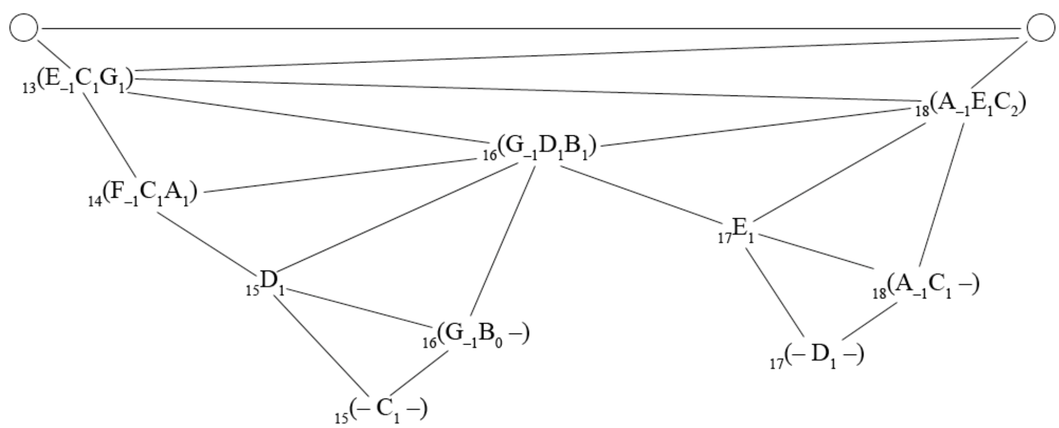

FIGURE 3.22: THE GRAPH OF FIGURE 3.21 WITH EVENTS DUPLICATING THE ALTO CONTRACTED

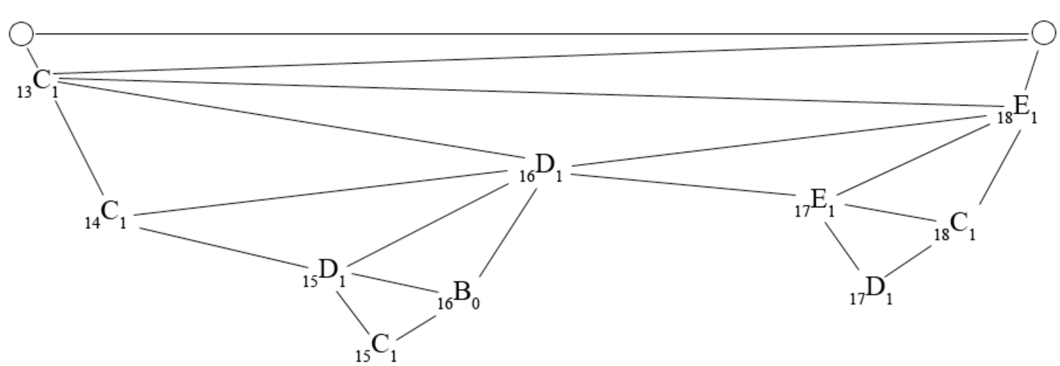

FIGURE 3.23: THE MOP ANALYSIS OF THE ALTO VOICE OF THE FUGUE EPISODE

neighborhoods for two adjacent vertices will be useful for the definition below (where "overlap" just means the intersection of sets).

To see the usefulness of a concept of overlap of neighborhoods, examine any MOP or harmonic analysis I have discussed thus far and consider the following: the overlap of the neighborhoods of any two adjacent vertices (that is, the set of vertices that the two neighborhoods share in common) is just the set of events that bear some direct prolongational relationship to the interval defined by the two vertices. Let x and y be the two vertices and S the set of vertices in the overlap of their neighborhoods. If xy is theroot edge, then S includes exactly one vertex, which is the event that directly prolongs the span of the passage. If xy is not the root edge, then S includes at least one vertex z, where either x prolongs yz or y prolongs xz. Any additional vertices in S are events that directly prolong xy. (In a MOP there can be only one such event, but in a harmonic analysis there can be up to one such event for each voice).

A complete harmonic analysis on a set of voices with complete MOP analyses V^1, V^2, V^3, . . . , V^n can then be defined as a graph G with the following characteristics:

(1) Each vertex of G is an ordered list of n events, where the i^{th} event is either an event in V^i, or is the null event (indicated by a dash). Also, G has an oriented root edge.

(2) For each voice V^i, there's a subgraph G^i of G such that:

(a) G^i is isomorphic to V^i such that each vertex, v, in V^i corresponds to a vertex in G^i with v in its i^{th} place, and G^i has an oriented root edge inherited from G that corresponds to the oriented root edge of V^i via the isomorphism.

(b) G^i can be obtained from G by successively eliminating vertices as follows:

First, delete any vertex that is null in the i^{th} position, is adjacent to exactly two other vertices, and is not on the root edge.

Second, contract every pair of adjacent vertices that have the same event in the i^{th} place, have neighborhoods overlapping in exactly one vertex, and aren't on the root edge. Label the contracted vertex with the event of either contracting vertex.

Furthermore, every vertex in G is in some G^i.

A complete MOP analysis of a voice is one that includes all the events of that voice and, of course, has no duplications of any events. Therefore the process of (2)(b) must delete all vertices that are dissonant with V^i and contract all vertices that duplicate some event in V^i. This definition restricts the form of the harmonic analysis, then, in that in order to qualify as a harmonic analysis, the process of (2)(b) must eliminate these dissonances and duplicates for each voice. For instance, motion to or from a dissonance cannot be prolonged by a consonance; otherwise the dissonance will never be adjacent to as few as two vertices in the deletion process. Also, when two harmonic events include a duplicate in some voice, the interval between these two harmonic events can only be prolonged by an event dissonant with the voice in which the duplicate occurs. Otherwise, the two events will have neighborhoods overlapping in more than one vertex, and the process in (2)(b) won't contract them.

The identification of the root edges of the G^i's with that of G in (2)(a) forces the initial and final events in each voice to correspond to initial and final events in the harmonic analysis. (In particular, if two voices are not completely dissonant with one another, then they're consonant in some initial and final event). However, by rewording this and allowing for the deletion of background vertices in (2)(b) one can allow for voices that do not participate in the entire prolongational span:

(2) [background-deletion version] For each voice V^i, there's a subgraph G^i of G such that:

(a) G^i is isomorphic to V^i such that each vertex, v, in V^i corresponds to a vertex in G^i with v in its i^{th} place, and G^i has an oriented root edge that corresponds to the oriented root edge of V^i via the isomorphism.

(b) G^i can be obtained from G by successively eliminating vertices as follows:

First, delete any vertex that is null in the i^{th} position and is adjacent to exactly two other vertices. If this vertex, x, is on the root edge, xy, then let z be the vertex in the overlap of the neighborhoods of a and y, and let zy be the new root edge, oriented in the same way with respect to y as the previous root edge.

Second, contract every pair of adjacent vertices that have the same event in the i^{th} place, have neighborhoods overlapping in exactly one vertex, and do not

> make up the root edge. Label the contracted vertex with the event of either contracting vertex.

Furthermore, every vertex in G is in some G^i.

This redefines the root edge so that it retains its orientation, and eliminates the possibility of reordering the events of some voice by rotation or retrogression (which could produce some very queer analyses indeed!).

Broadening the definition of the G^i's in this way makes the definition of harmonic analyses somewhat more general by allowing voices to enter and leave off in the middle of the phrase rather than requiring all initial and final events to coincide. In other words, some voices may participate only in certain more local prolongational spans and not throughout the entire phrase. Note, however, that such an analysis requires two divergent semantic interpretations of the null event in a voice: it can indicate either some foreground activity dissonant with the voice or some background level at which the voice is absent.

This may be particularly useful in the case of unfolding transformations, where one might often want an inner voice to enter only fleetingly. For example, in "Maximality and Chordality" in part one, I pointed out some possible unfoldings in the upper voice of the slow movement of Haydn's Symphony #99 (according to the analysis of Schachter given in "A Comparison of Analyses Using the MOP Model"). These might be best introduced in an inner voice that enters only briefly. Figure 3.24 shows a possible set of voices for the "folded" analysis of the upper voice and they're combination in a harmonic analysis. (I omit the dashes that indicate null events here, since there's no danger of confusion). In deriving the inner voice from the harmonic analysis according to the process described above, it's necessary to delete the initiation and termination vertices of the harmonic analysis in turn. The vertices ${}_{7}B$ and ${}_{12}B$ then take on the function of initiation and termination events respectively for the inner voice. In other words, this inner voice, a simple third progression, is heard in the immediate context of the principal voice's retention of B from measure 7 to 12 of the piece. Finally, the unfolding transformation is shown in the derivation of the last MOP of figure 3.24 from the "folded" analysis (again, not a process that I've explicitly formalized in this paper).

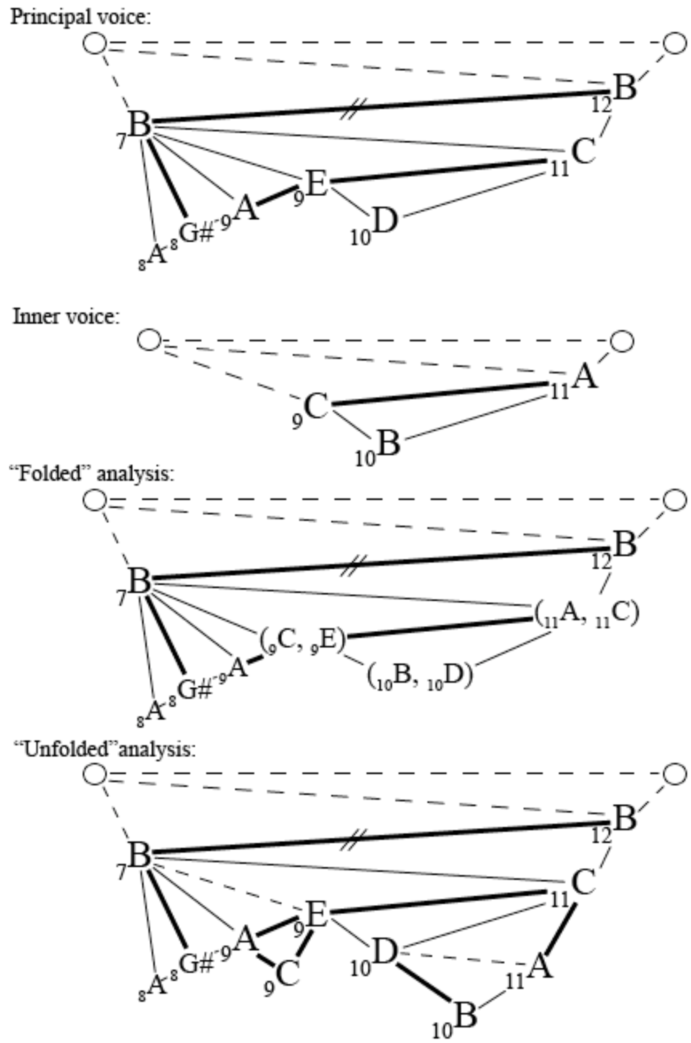

FIGURE 3.24: UNFOLDING TRANSFORMATIONS IN SCHACHTER'S HAYDN ANALYSIS

This is just one simple way in which a mathematical model, once it is formulated, suggests new possibilities and asks us to provide a semantic interpretation of them. Some of these will yield a useful broadening of the model while others will not.

Analytical Decision-Making in the Contrapuntal Model

In part one I illustrated how the MOP model of prolongation could formalize aspects of the decision making process of analysis using the discussion of a passage in Haydn's Symphony 99 from Carl Schachter's article “Either/or.” Now we can see how giving a more complete contrapuntal representation of the analyses Schachter discusses influences the picture.

First, consider the analysis I presented in the first section as the one Schachter rejects. Schachter doesn't give a full realization of this analysis, so when I formulated this analysis in part one, I tried to retain as many details of the analysis that Schachter does present. However, rather than using ${}_{10}D_1$ as the principal melodic tone of the tonic that is supposed to complete the tonic prolongation, I used ${}_{10}B_1$, even though Schachter puts more importance on ${}_{10}D_1$ to establish the linear connection from ${}_{9}E_1$ to ${}_{11}C_1$. The primary reasons for choosing ${}_{10}B_1$ invoke harmonic and contrapuntal aspects of the passage. In order for the II chord of measure 9 to prolong the tonic convincingly, there must be a dominant preceding the end of the tonic prolongation. If ${}_{10}D_1$ is chosen to represent the tonic of measure 10, then there is no melodic note between E and D to serve as the upper tone of a dominant.

This is resolved by identifying ${}_{10}B_0$ as the actual melodic tone of the tonic in measure 10, and identifying ${}_{10}D_1$ rather as the delayed resolution of the suspension ${}_{10}E_1$, enabling ${}_{10}D_1$ to serve as the upper note of the dominant. Figure 3.25 shows this as a four-voice harmonic analysis. (For the sake of convenience, I will call these voices soprano, alto, tenor, and bass). Each voice is shown in musical notation on a separate staff and roughly aligned with the graph below to show which events the pitch names on the graph refer to while avoiding a profusion of superscripts and subscripts. The musical notation uses Schenkerian symbols somewhat loosely, solely for the purpose of representing the prolongational information in the MOP analyses as accurately as possible without being too difficult to read, and not to introduce any more refined Schenkerian concepts into the analysis.

I also extend the notational distinctions of edges to contrapuntal analysis here, as suggested in "Refinements of the MOP Model" of part one. The (intentionally somewhat loose) rules for this extension are as follows: an ordinary line can be used to represent a stepwise progression in any voice, following the Schenker's principle of "ensnaring the leap," according to which a leaping motion in a voice that occurs in conjunction with a passing motion in another voice can be heard as a "leaping passing tone" due to the greater strength of the passing motion. (See “Prolongations as Passing Events” in part one). Thick lines can then generally be read as the motion of voices within a retained harmony, although they can sometimes be used for motion that is primarily arpeggiating but also involves a change to a closely related harmony, as in the progression from $_{7}$(GDGB) to $_{8}$(B–DG#) in figure 3.25. If a harmony is retained and includes only relatively insignificant arpeggiations, then the double-slashed line should be used, as in the large retention from $_{7}$(GDGB) to $_{12}$(GGGB) in the Haydn analysis. Figure 3.25 includes no broken lines other than the formal ones. However, these do occur in Schachter’s preferred analysis of the passage below.

Schachter’s rejected analysis is relatively chord-heavy in its representation of measures 7-12: its basic structure is a progression in four-voice harmony, I-II-V^6-I-II^6-V^7-I. Most of the dissonance in the passage occurs in getting from the II to the V^6, which of all these chords are the most fragmented at the musical surface. The prolongation of the motion from II to V^6 occurs independently in the alto and in the lower two voices. The alto has a motion from A_0 to C_1 and back, filled in by passing tones. What’s remarkable about this is that at the surface the alto’s second B_0 here appears to be an accompaniment in parallel thirds to the soprano, but according to this analysis the B_0 is a dissonant passing tone in the dominant harmony while the D_1 is consonant in this harmony. Yet, like the D_1 in the soprano, the resolution of this passing tone is delayed, so that, although the A_0 to which it resolves is consonant in the dominant harmony, by the time it actually sounds the bass has moved on the tonic.

For comparison, figure 3.26 displays a four-voice version of Schachter’s preferred analysis. This analysis obviously is less chord-heavy—or, more precisely, it locates functional harmonic activity at a more global level. Its basic structure for measures 7-12 is a I-II^6-V^7-I progression. Most of the other consonant activity consists of arpeggiations within the II^6 harmony, which take up most of the passage. Schachter’s “apparent tonic” appears as a direct prolongation of the two main voicings of the II^6 harmony. The analysis shows this as passing motion in the upper voices harmonized by a prolongation of the bass C by the “leaping passing tone,” $_{10}G_{-1}$, mimicking a dividing-dominant type of progression. This harmony isn’t a “real” tonic in this analysis is because it isn’t directly

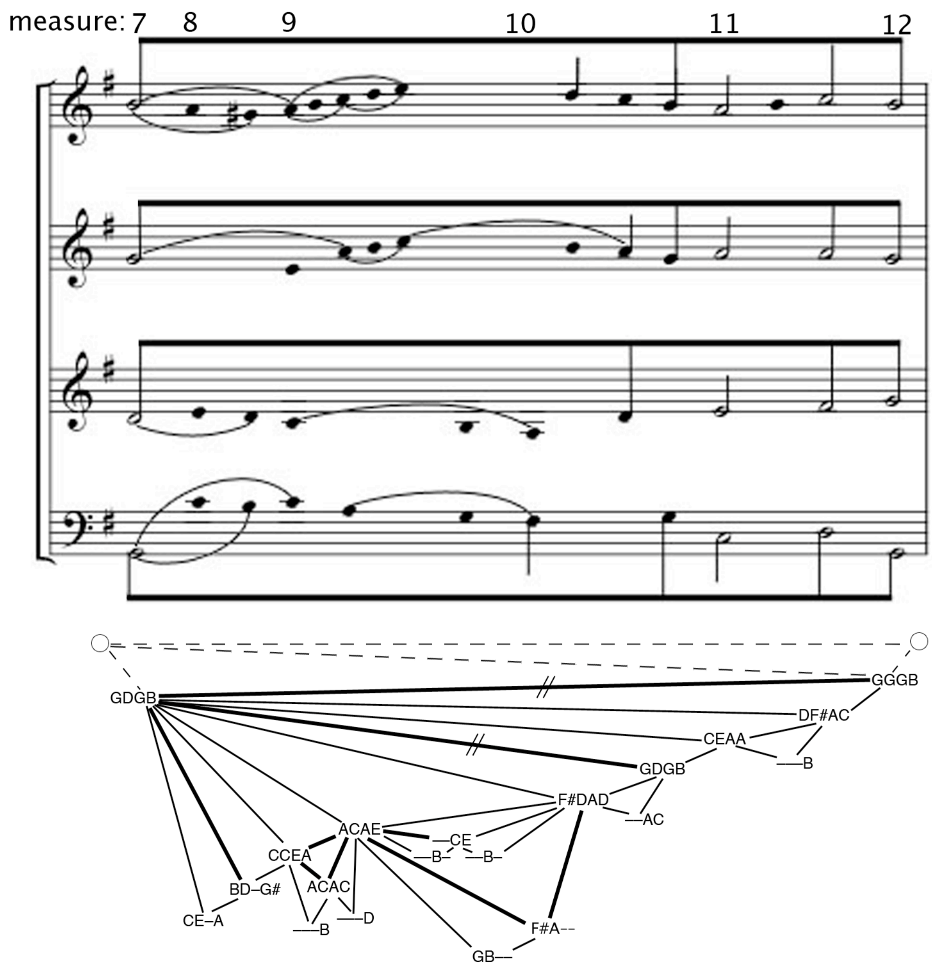

FIGURE 3.25: A VERSION OF SCHACHTER'S REJECTED ANALYSIS OF THE HAYDN ADAGIO

related as a retention or arpeggiation to the tonic harmony that structurally frames the passage, as the graph shows. Rather, it acts as a kind of dominant to the II^6 harmony. In the analysis of figure 3.25, on the other hand, the corresponding chord appears as a retention of the initial tonic, making it a genuine tonic chord.

Schachter's preferred analysis also identifies $_{11}C_1$ as the primary melodic tone over both the II^6 and V^7 chords of measure 11. To preserve the continuity of stepwise motion from $_9E_1$ to $_{10}B_0$ in the soprano, the voicing of II^6 in measure 11 with $_{11}A_0$ in the soprano is demoted to a prolongation of the motion from the apparent tonic to the II^6 of measure 11. The analysis of figure 3.25, on the other hand, retains the $_{11}A_0$ at a relatively background level as part of a double neighbor motion around B_0 in the soprano.

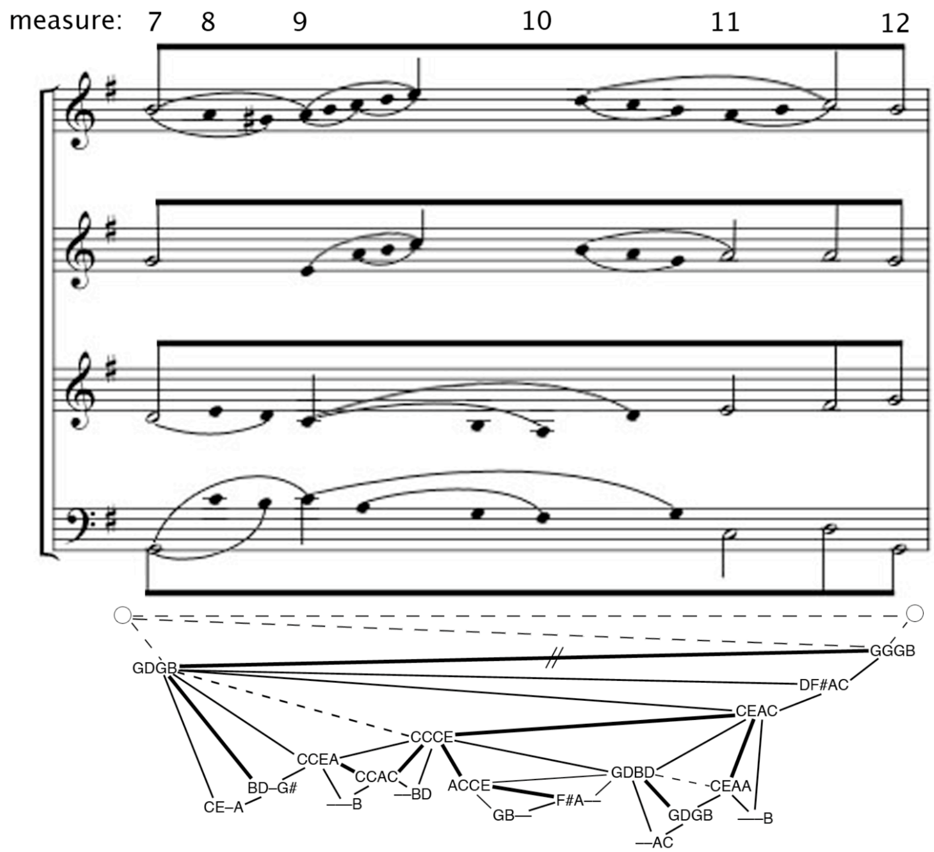

FIGURE 3.26: SCHACHTER'S PREFERRED ANALYSIS OF THE HAYDN ADAGIO

Unlike figure 3.25, the analysis of figure 3.26 asserts two incomplete progressions. These both correspond to incomplete progressions in the upper voice as discussed in "A Comparison of Analyses Using the MOP Model" in part one.

These are some of the more interesting differences between the analyses of figures 3.25 and 3.26. I could discuss them in more detail, but instead I invite the reader to inspect the graph-theoretic presentation of them and find the prolongations they assert in the music. I think they both ultimately represent plausible and interesting hearings although Schachter's gives a more global analysis.[69]

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[69] An indicator of a more global analysis in the graph is the *depth* at which events occur in the analysis, where the depth of a vertex v is the length of a minimum path from either of the root vertices to v. For instance, both of the analyses find a pair of dissonant passing tones in parallel thirds in the upper voices of measure 10 indicated by the vertex

Ambiguity and Fomalization: A Summary

As the MOP versions of Schachter's Haydn analyses in the previous section illustrate, with the development of the contrapuntal version of prolongational analysis by MOP we've arrived at a model sophisticated enough to express interesting musical insights. The long road of careful considerations that brought us to this point demonstrates the high degree of ambiguity in the concept of prolongation that I pointed out in the introduction.

The brief historical survey of part 1 showed the instability of the concept of prolongation and showed two broad and incompatible senses in which the word may be used: the static and the dynamic. The MOP model developed in part 1 to formalize the dynamic sense allowed us to break down the concept into its constituent parts, represented by the graph theoretic properties of Hamiltonicity, outerplanarity, and chordality. The next part will describe other equivalent ways to circumscribe the concept of dynamic prolongation and thus provide a deeper and fuller understanding of the formal properties of the MOP model of prolongation.

Part 2 further illustrated the ambiguity of the concept of prolongation by showing myriad ways in which one may interpret the idea of static prolongation. A comparison of these with the MOP model revealed the stable core of the concept of prolongation in the ideas of relative backgroundness and levels of reduction. Factoring out this common starting point provided one perspective on the fundamental difference between static and dynamic prolongation: the focus on structural dominance relations between events in different reductions versus consecutivity relations within reductions.

– –AC. Considering just the passage from measures 7-12, indicated by the prolongational span from $_7$GDGB to $_{12}$GGGB, the event – –AC in figure 23 directly prolongs two events adjacent to $_7$GDGB, making it a depth-two event. In Schachter's analysis, on the other hand, this passing dissonance relates directly to events that are at least one step removed from $_7$GDGB and $_{12}$GGGB, making it a depth-three event. Schachter's analysis has four depth-three events, seven depth-two events, and six depth-one events in this passage while the alternate analysis has no depth-three events, ten depth-two events and eight depth-one events.

The consideration of contrapuntal models at the beginning of this part shows that at the level where prolongational analysis becomes truly musical, the number of ways of pinning down the concept of contrapuntal prolongations becomes virtually limitless. However, a consideration of the essential components of the idea of counterpoint makes it possible to outline some basic principles of representing the contrapuntal nature of music in analysis. I've pursued one application of these principles using the dynamic sense of prolongation to construct a formal system that can satisfactorily represent the basic prolongational structures of a Schenkerian analysis.

As I pointed out in "A Formal Definition of the Complete Harmonic Analysis," once a model is built it immediately suggests extensions. One such extension, which I've only begun to develop here, is a formalization of the concept of unfolding through transformations on MOP analyses. Another is the classification of types of elemental prolongation, suggested in "Refinements of the MOP Model" in part one.

Although I've portrayed ambiguity in places as a flaw of a theory, it's not the ambiguities themselves but the refusal to address them that constitutes a flaw. In fact ambiguities are important and essential component of theory. These are what suggest new developments, generating new perspectives and concepts. These new concepts in their turn will require their own developments, further deepening our engagement with the process of hearing and understanding music.

PART 4: MATHEMATICAL CHARACTERIZATIONS OF MOPS

Properties of MOPs

In this part I'll describe twelve mathematical ways of defining the graph class of MOPs. I've touched on parts of the graph properties that make up these characterizations in the previous three parts of the paper, and in this and the following part, I'll provide more precise definitions of these and show the mathematical relationships between them and between other properties of the class of MOPs that have semantically interesting consequences.

In particular, I described MOPs in two ways in part one: viewing them as maximal outerplanar graphs suggests a way of drawing them and makes it easy to recognize them, but doesn't tell us why they make a good model of prolongation. Viewing them as crosschord-free Hamiltonian chordal graphs, on the other hand, isolates three distinct properties of the graphs, each of which corresponds to some property that the MOP model ascribes to prolongation. This mathematical transformation was invaluable in the exposition of part one for a number of reasons: it revealed precisely the claims that the MOP model makes about prolongation in restricting the analyses to this specific graph class. It showed us ways of interpreting the graphs, such as reading the cycles of the graph as independent phrases and chords as analyses of those phrases. It also allowed us to isolate the property of chordality for scrutiny in the section "Maximality and Chordality."

Other properties explicated in this part of the paper have figured in less explicit ways in discussions in the earlier parts. For instance, in the first constructions of the analysis in "Maximal Outerplanar Graphs" in part one, I roughly followed the recursive procedure that defines 2-trees (See "(1) Unary 2-trees" below). I also described complete harmonic analyses as 2-connected 2-trees in part three ("The Extension of the MOP Model of Prolongation to Contrapuntal Analysis"). (See "(10) H2-Intrasymmetry") Also, in "A Comparison of Analyses Using the MOP Model" and "Maximality and Chordality" in part one, I linked outerplanarity to the temporal nature of prolongation. More specifically, the outerplanarity of the graph makes it possible to consistently assign temporal orientations to each prolongational span described by the graph. This fact can be properly understood by reference to the property of *confluence* defined below ("(8) Confluence"). Finally, in part two ("The MOP Model of Prolongation as a Binary Phrase-Structure Model") I showed that a MOP can be transformed into a binary plane tree, giving another perspective on it as a description of musical structure (; see "(2) Maximal Cliques, 2-Overlap Clique Graphs, and Clique Trees").

All of these different characterizations of the graph class of MOPs thus deepen our understanding of them as a model of prolongation. In this part I'll precisely define all the properties described thus far and add a few other ways of defining the graph class of MOPs that give somewhat different perspectives on the semantics of MOPs as representations of prolongational structure.

Basic Terms and Definitions

The definitions of preexisting mathematical terms here follow the model of Brandstädt, Le, and Spinrad (1999) although most of the terms and definitions I use are quite standard.

I have already discussed in part one the semantics of graphs in the MOP model, where the vertices correspond to musical events and the edges show relationships between those events. In this section, we will need only the concept of a simple undirected graph,

Definitions A *graph*, G, is a set of vertices, V, and a set of edges, E, where V is any set of objects and E is a set of two-element subsets of V. I'll indicate this by saying "G is a graph on vertex set V (with edge set E)."

Sometimes I will say, e. g., "G' is a graph on the vertex set V(G) + v," meaning that the vertices of G' are those of G plus one more, v.

A *subgraph* of a graph, G, is a graph, G', whose vertex set is a subset of V(G) and whose edge set is a subset of E(G).

Notation An edge is indicated by concatenating the names of its incident vertices. For instance, if u and v are vertices, then uv is an edge connecting u and v.

Let G be a graph. Then V(G) is the set of vertices of G, E(G) is the set of edges of G. The sizes of these set are $|V(G)|$ and $|E(G)|$, respectively.

If uv $\in$ E(G), then I will say that the vertices u and v are *adjacent* in G, and the vertices u and v are *incident upon* the edge uv.

One semantically important aspects of a graph is its *paths*. Paths are ways of getting from one vertex to another, and thus represent possible indirect relationships between events.

Definition Let G be a graph, let n be an integer, $n \geq 2$, and let v_1 and v_n be vertices of G. A *path from v_1 to v_n*, v_1Pv_n is a sequence of distinct vertices $v_1v_2v_3 \ldots v_n$ such that each $v_{i-1}v_i$ is an edge of G. The edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$ are the edges of P. The vertices v_1 and v_n are the *endpoints* of P. The *inverse* of P is a path $v_nP^{-1}v_1 = v_nv_{n-1} \ldots v_2v_1$.

Notation I will represent the path defined by the sequence of vertices $(v_1, v_2, v_3, \ldots, v_n)$ by simply concatenating the sequence of vertex names. For instance, if a, b, c, and d are edges of a graph G and ab, bc, and cd are edges, then abcd is a path of G. For indeterminate paths between two vertices I will concatenate the endpoints of the path with a capital roman letter as a variable representing the intermediary vertices. For instance, aPd and aP'd might represent two different paths from a to d.

The notation for edges and paths, as well as the following notation for cycles, is my own invention. I have found it useful because it avoids the need to use variable names for edges, and allows for a relatively brief indication of a path or cycle that fully determines it and also immediately shows both the sequence of vertices and sequence of edges involved in the path or cycle.

One aspect of paths that is worthy of note is that the inverse of a path is a distinct path even though it includes precisely the same vertices and edges (and, in fact, is the only such path). The term "path" is also sometimes used for a kind of graph. To avoid ambiguity I will use the term "path-graph."

Definitions A *path-graph*, P, is a graph whose entire edge set and vertex set are included in a single path.

A *path-subgraph* of a graph G is a subgraph of G that is a path-graph.

The idea of a path-subgraph will be useful because it doesn't distinguish between a path and its inverse.

The cycles of a graph are especially important in the music-analytic interpretations of MOPs. As discussed in part one, a cycle of a MOP represents a sequence of events that make up a complete prolongational span.

Definitions A *cycle* of a graph G is a sequence of at least 3 vertices, $(v_1, v_2, \ldots, v_n)$ such that v_1v_2, v_2v_3, . . . , $v_{n-1}v_n$, and v_nv_1 are edges of G. An *n-cycle* is a cycle of n vertices. A *trivial cycle* is a 3-cycle.

Two vertices are *adjacent on a cycle* iff they make up one of the edges of the cycle.

Notation I will write a cycle $C = (v_1, v_2, \ldots, v_n)$ by concatenating the vertices in order and appending a right bracket: $v_1v_2v_3 \ldots v_n]$. The bracket serves to distinguish the cycle from the path $v_1v_2v_3 \ldots v_n$ by indicating the edge from the last vertex to the first.

According to this definition of a cycle, the sequence of vertices is a characteristic of the cycle, so that if a, b, c, d are vertices of a graph and abcd] is a cycle, then bcda] is a distinct cycle, as is dcba]. Thus, as with paths, we will separately define *cycle-graph* and *cycle-subgraph*:

Definitions A *cycle-graph* is a graph whose edges and vertices are all included in a single cycle.

A *cycle-subgraph* of a graph, G, is a subgraph of G that is a cycle-graph.

As with path-subgraphs, the cycle-subgraph is a useful concept because it collects together the many possible rotations and inversions of a cycle.

One special type of graph that is of special interest is a tree. In part two, we discussed trees extensively, but these trees were always rooted directed trees and often planar trees. In this part, we will deal only with simple undirected graphs, so we will define trees as undirected graphs without a root vertex.

Definitions A graph is *acyclic* if it has no cycles.

A graph G is *connected* iff for any two vertices u, v $\in$ V(G) there is a path with u and v as endpoints.

A *tree* is a connected acyclic graph.

Statement of Theorem 1 (Characterizations of MOPs)

As I indicated above, a MOPs can be seen to represent an analysis in a number of different ways depending on exactly how the graph is interpreted. For instance, we can look a graph as showing a set of prolongational relationships, as showing the possible melodic reductions consistent with the analysis, or representing the melodic hierarchy implied by the analysis. The following theorem, which gives twelve equivalent characterizations of MOPs, will organize the discussion of these different perspectives:

Theorem 1 (Characterizations of MOPs) Let G be a graph on 4 or more vertices. The following are equivalent:

(1) G is a unary 2-tree.

(2) All of the maximal cliques of G are triangles and the 2-overlap clique graph of G is a tree.

(3) G is maximal outerplanar.

(4) G is MOP(2)

(5) G is chordal and HOP(2).

(6) G is minimal Hamiltonian-chordal.

(7) G is cycle-connected.

(8) G is maximal Hamiltonian-confluent

(9) G is Hamiltonian, chordal, and confluent

(10) G is H2-intrasymmetric.

(11) G is HOP-intrasymmetric.

(12) G is HC-intrasymmetric.

I will motivate each of these characterizations in turn, showing how each leads one to a different perspective on how the graph represents the analysis, then give the precise mathematical definition for each. Finally, I will prove the theorem in part five. Note that an appendix includes many propositions, all of which play some role in the proof of part five and some of which will be useful in the sections below. Each of these sections is numbered according to the part of theorem 1 to which it is relevant.

(1) Unary 2-Trees

The first characterization of MOP invokes the top-down construction of a MOP used in part one, in the section "Maximal Outerplanar Graphs."

More specifically, in part one we described the MOP model as a formalization of the idea of dynamic prolongation. The dynamic usage of the term prolongation describes an event occurring as prolonging a motion from a preceding event to a following event. If an event, Y, prolongs the motion from X to Z, then there must be a motion from X to Y and Y to Z also. Therefore, dynamic prolongation suggests a recursive construction: beginning with one most background motion, we can add prolonging events one after another, where each new prolonging event creates two new motions that themselves can become the object of a subsequent prolongation.

In part one I recommended that this most background prolongation always be given to a pair of formal vertices called the "initiation" and "termination" events, ○*i* and ○*t* (denoted by circles in the drawing of the graph). This is useful because it makes it possible to extend the analysis in the background as well as the foreground direction, by viewing the formal vertices as substitutes for a pair real events that constitute a motion in a wider musical context. Thus the formal vertices allow us to assert that a series of prolongations constitutes a unified motion without requiring us to specify exactly what that motion prolongs.

For demonstration, consider the construction of the MOP for the fugue subject of part one. Let G_2 be a graph whose vertex set consists of the initiation and termination events and an edge between them. The first step is to the event that most directly prolongs the "motion" indicated G_2—i. e. the most background event of the passage (the one that can represent the entire passage in a wider context by itself). Let G_3 be the graph that adds this event to the vertex set of G_2 and includes edges between the new vertex and each vertex of G_2. In our example G_3 adds the vertex ${}_1$G and the edges ○*i*-${}_1$G and ${}_1$G-○*t*. This is shown in figure 4.1.

The graph on four vertices, G_4, is then defined by adding a vertex for an event that prolongs one of the motions indicated by an edge in G_3. Since, in our example, G_3 includes the "formal" vertices, the new vertex must be one that makes up the most background motion either from ${}_1$G or to ${}_1$G. Since there are no event preceding ${}_1$G, the new vertex must be ${}_5$E prolonging ${}_1$G-○*t*. Therefore G_4 has edges ${}_1$G-${}_5$E and ${}_5$E-○*t*.

This process continues, defining G_5, G_6, and so on. At each step a single event is added to the analysis along with edges between that vertex and the two vertices making up the edge it prolongs. There are generally multiple ways of defining the process to arrive at the same graph. For instance graph G_5 of figure 4.1 has edges ${}_1$G-${}_3$F and ${}_3$F-${}_5$E, which in the G_7 are prolonged by ${}_2$A and ${}_4$D respectively. The order in which we add these events makes no difference to the resulting form of G_7 or any graph following it in the sequence.

The entire process of construction is shown in figure 4.1, concluding with a graph on 11 vertices, G_{11}, the same graph shown in figure 1.13. This demonstrates a recursive

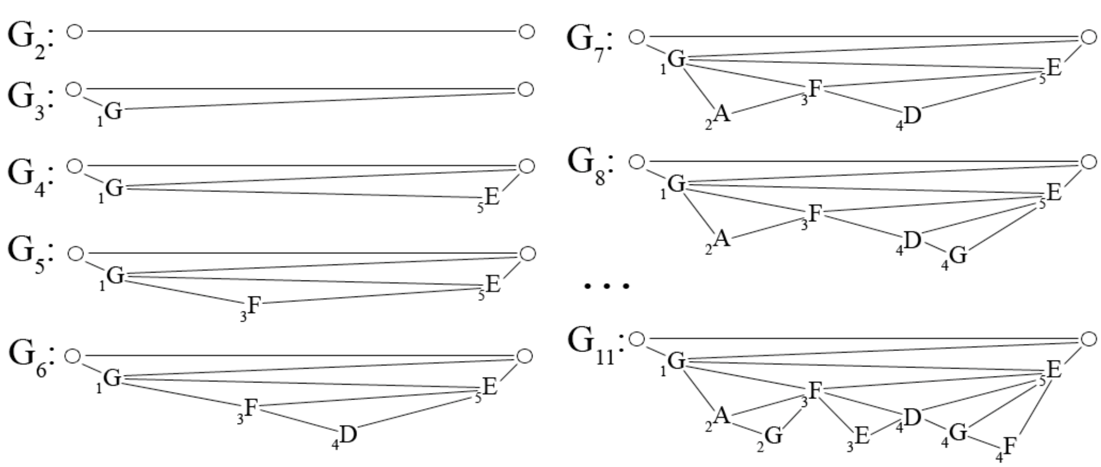

FIGURE 4.1: THE CONSTRUCTION OF A MOP AS A 2-TREE

process that could continue for any number of steps, depending on the size of the analysis. Graph-theoretically, this recursive process defines the type of graph known as a 2-tree. To understand why this is called a 2-tree, first we need to define the terms clique and complete graph:

Definitions Let G be a graph. A *clique* of G is a set of vertices of G such that every two distinct vertices are adjacent in G. A clique with only one vertex is a 1-clique (which is trivial; any single vertex makes a 1-clique), a clique of two vertices (i.e. a pair of adjacent vertices) is a 2-clique, a clique of three vertices (also called a *triangle*) is a 3-clique, and so on.

A *maximal clique* of a graph G is a clique that is not a proper subset of any clique of G.

A graph, G, on n vertices is a *complete graph* (denoted K_n) iff every pair of vertices in G are adjacent. K_1 is a graph with one vertex and no edges, K_2 is graph with two vertices and one edge between them, K_3 is a graph of three vertices and three edges, K_4 is a graph on four vertices with six edges, and so on.[70]

A complete graph could also be defined as a graph with one maximal clique. Using the idea of cliques, let us redefine a tree recursively as a 1-tree.

Definition A *1-tree* is either a graph isomorphic to K_1, or is a graph G_i constructed from a 1-tree G_{i-1} as follows: let Q_i be any 1-clique of G_{i-1} and v_i any vertex not in G_i. Let G_i be a graph with the vertex set $V(G_{i-1}) + v_i$ and the edge set consisting of all the edges of G_{i-1} plus an edge from v_i to every vertex of Q_i.

Of course, the clique in this definition is trivial, since it consists of only a single vertex. However, the phrasing of the definition allows us to generalize it to an n-tree by simply replacing "1-clique" with "n-clique." So a 2-tree is defined in the following way:

Definition A *2-tree* is either a graph isomorphic to K_2, or is a graph G_i constructed from a 2-tree G_{i-1} as follows: let Q_i be any 2-clique of G_{i-1} and v_i any vertex not in G_i. Let G_i be a graph with the vertex set $V(G_{i-1}) + v_i$ and the edge set consisting of all the edges of G_{i-1} plus an edge from v_i to every vertex of Q_i.

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[70] This way of speaking, though standard in graph theory literature, is actually somewhat confusing. K_n is not actually a graph but an isomorphism class of graphs, because its vertex set isn't defined as a particular set of objects.

Furthermore, the notation I have used in the definitions allows us to characterize a 2-tree in terms of its history: Let G_n be any 2-tree on n vertices, $n \geq 4$. Note that the only 2-trees on 3 vertices are isomorphic to K_3. By the definition, there is some way to construct G_n beginning with a graph $G_3 \approx K_3$. Furthermore for each G_i, $4 \leq i \leq n$, there is a Q_i, which I will call the ith *supporting clique*, and a v_i, the ith vertex (arbitrarily assigning 1 and 2 to the vertices of G_2). Then the 2-tree G_n can be characterized by two sequences, $\mathbf{V}_G = v_1, v_2, v_3, \ldots, v_n$ and $\mathbf{Q}_G = Q_4, Q_5, \ldots, Q_n$.

The process of construction of a 2-tree represents the analytical process described above of adding notes to a melody in turn as they prolong the interval of the notes before and after them. However, for MOPs we need to add one more constraint: an interval cannot be prolonged independently by two different notes of the melody, so a MOP is a special kind of 2-tree which we will call a *unary 2-tree*, where only one vertex is joined to any clique in the process of construction:

Definition A *unary 2-tree* is either a graph isomorphic to K_2, or is a graph G_i constructed from a 2-tree G_{i-1} with supporting clique set $\mathbf{Q}_G = Q_4, Q_5, \ldots, Q_{i-1}$ as follows: let Q_i be a 2-clique of G_{i-1} distinct from $Q_4, Q_5, \ldots, Q_{i-1}$, and v_i any vertex not in G_i. Let G_i be a graph with the vertex set $V(G_{i-1}) + v_i$ and the edge set consisting of all the edges of G_{i-1} plus an edge from v_i to every vertex of Q_i.

The reader will recall from part three that while a simple melodic analysis always takes the form of a unary 2-tree (a MOP), it is possible to have “multiple branchings” in a contrapuntal analysis; see “The Extension of the MOP Model of Prolongation to Contrapuntal Analysis.” It’s possible to define “binary 2-trees,” “tertiary 2-trees,” and so on by allowing a clique to appear in the supporting clique set twice, three times, et c. Then one could say that a two-voice analysis must take the form of a binary 2-tree, while a three-voice analysis takes the form of a tertiary 2-tree, and so forth.

(2) Maximal Cliques, 2-Overlap Clique Graphs, and Clique Trees.

As Figure 4.1 makes clear, a unary 2-tree is made up of triangles that share common edges. Each of these triangles represents an “elemental” prolongation of three events, as described in “Maximality and Chordality” in part one. This observation suggests a way to represent a unary 2-tree as a kind of tree, a tree of elemental prolongations. More specifically: for any unary 2-tree there is a tree with a vertex for each triangle (representing an elemental prolongational), and an edge between each triangle sharing two vertices (showing where two prolongations share an interval, which is a prolonging interval in one case and a prolonged interval in the other). This is called a 2-overlap clique graph. For example, figure 4.2 shows the MOP for the fugue subject and the corresponding tree of prolongations.

If we instead were to take each vertex of this tree to represent the prolonged interval, and add new leaves to represent the trivial prolongational spans (foreground intervals), then we would have the phrase-structure tree of intervals corresponding to the MOP as discussed in "The MOP Model of Prolongation as a Binary Phrase Structure Model" in part two. In the tree of elemental prolongations of figure 4.2, for instance, there's an edge $\{_1G, _3F, _5E\}\{_1G, _2A, _3F\}$, which means "the prolonged motion in $\{_1G, _2A, _3F\}$ is a prolonging motion in $\{_1G, _3F, _5E\}$." In the phrase-structure tree of intervals, there is a corresponding edge $(_1G\text{-}_5E)(_1G\text{-}_3F)$ that says essentially the same thing: the motion $_1G\text{-}_3F$ is in the elemental prolongation of $_1G\text{-}_5E$. The only difference is that trivial prolongations are included in the phrase-structure tree (as leaves) but not as prolonged motions in the tree of elemental prolongations (since they aren't prolonged).

Before defining a 2-overlap clique graph, we will need a few basic terms:

Definition A *maximal clique*, Q, of G is a clique of G such that no other clique of G contains Q.

Notation For any graph G, **K**(G) is the set of maximal cliques of G.

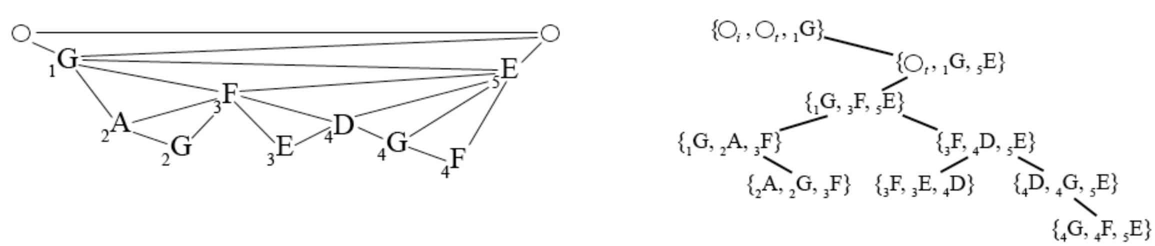

FIGURE 4.2: THE TREE OF ELEMENTAL PROLONGATIONS FOR A MOP

Definitions A *clique graph* of a graph G is a graph with a vertex corresponding to each maximal clique of G such that if two vertices are adjacent then their corresponding maximal cliques share vertices in common.

The *2-overlap clique graph* of a graph G is a graph with a vertex for each member of **K**(G) and an edge between the vertices corresponding to maximal cliques that share at least 2 vertices.

To make clear distinctions between the vertices of a clique graph and the cliques they represent, I will use the following notation:

Notation Let T be a clique graph of a graph G. Then for any $K \in \mathbf{K}(G)$, k_T is the vertex of T corresponding to K.

This notation will be important in the proof of parts 2 and 3 of the theorem below. Another kind of clique graph is a clique tree, following Blair and Peyton (1993):

Definition A *clique tree*, T, of a graph G is a tree that is a clique graph for G such that for any two cliques K, K' $\in \mathbf{K}(G)$, every clique along the path connecting k_T and k'_T in T contains $K \cap K'$.

Thus, the second characterization of MOPs says that G should have a clique tree (equivalent to its 2-overlap clique graph) representing the linking of elemental prolongations in the analysis. The fact that the 2-overlap clique graph is a tree means that this linking can be defined as a strict hierarchy on prolongations. By replacing the 3-cliques of this clique graph with chordless cycles, we could give a similar definition for the "holey" HOP analyses described in "Maximality and Chordality" in part one.

(3) Maximal Outerplanar Graphs, First Definition

The definition of maximal outerplanar is not itself particularly useful in the present application of MOPs to analysis, except that it gives a way of representing the graphs in a two-dimensional plane. I include the definition primarily because it is the way in which this specific class of graphs is discussed in the literature of graph theory. Characterization (4) is an alternate definition of maximal outerplanarity that will be more useful. I discuss both of these definitions in part one, "Maximal Outerplanar Graphs," but will define them more precisely here. A drawing of a graph is defined as an *embedding* of the graph in a plane.

Definitions An *embedding*, E, of a graph G, is a drawing on a two dimensional plane with the following properties:

(1) E has a unique point corresponding to each vertex of G.

(2) For each edge, uv, of G, E has a continuous curve (not necessarily a line) with endpoints at u and v.

(3) No point of E corresponding to a vertex falls on a curve of E except as an endpoint.

An embedding is *planar* if it satisfies one further property:

(4) No two curves of E cross in the plane.

A planar embedding, E, has an *outer face*, which can be defined as follows: let R be the union of regions of the plane completely encircled by any number of curves of E. The outer face of E consists of the edges and vertices of the graph that are outside of R or on its outer perimeter. A planar embedding is *outerplanar* iff

(5) All of the points of E corresponding to vertices of G are on the outer face.

Finally, a graph G is *planar* or *outerplanar* iff it has, respectively, a planar or outerplanar embedding.

A number of the characterizations of theorem 1 use the terms “maximal” and “minimal.” Note that there is an important distinction between these terms and the terms “maximum” and “minimum.” When saying a graph G is maximal with some property, I will always mean that it is impossible to add an edge to G and retain the property. If I were to use the term “maximum” instead I would mean that no graph on $|V(G)|$ vertices with the property has more edges than G.

To deal with maximality and minimality we will need the following notation for adding or removing edges from a graph:

Notation Let G be a graph. If uv ∈ E(G) then G – uv is a graph on the same vertex set as G that includes all edges of G except uv. That is, V(G – uv) = V(G) and

$E(G - uv) = (E(G) - uv)$. Similarly, if $u, v \in V(G)$ and $uv \notin E(G)$, then $G + uv$ is a graph on the same vertex set as G which adds uv to the edge set of G.

We can use a similar notation to indicate vertex deletion. Vertex deletion is defined in part three ("A Formal Definition of the Complete Harmonic Prolongational Analysis"), but I will define it again here for reference.

Definition Let G be a graph, and v be a vertex of G. The graph obtained by *deleting* v from G, denoted $G - v$, is a graph on the vertex set $(V(G) - v)$ and includes all edges of G not incident on v. That is, $E(G - v) = \{\forall xy \in E(G): x, y \neq v\}$

Characterization (3) thus says that G is outerplanar and for any non-adjacent vertices of G, u and v, $G + uv$ is not outerplanar.

As I pointed out above, characterization 4 gives a definition of maximal outerplanarity that avoids reference to embeddings. Another such characterization of maximal outerplanarity is given by proposition 7 which says that a graph is MOP(1) if it contains no subgraph which is a subdivision of K_4 or $K_{2,3}$. This type of characterization is important in the proof to theorem 1, so it is worth defining its terms right away. Recall that K_4 is the complete graph on 4 vertices. $K_{2,3}$ is the *complete (2, 3)-bipartite graph*:

Definition For integers m and n, the *complete (m, n)-bipartite graph*, $K_{m,n}$, is a graph on the vertex set $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, such that no two vertices in the set $U = \{u_1, u_2, \ldots u_m\}$ are adjacent, no two vertices in the set $V = \{v_1, v_2, \ldots, v_n\}$ are adjacent, and all vertices in U are adjacent to vertices in V.

The graphs K_4 and $K_{2,3}$ are shown in figure 4.3.[71]

Definition Let G be a graph with edge uv. A graph, G', obtained from G by *subdividing* uv is a graph with the vertex set $V(G) + x$ (where x is any vertex not in V(G)) and the edge set $E(G) - uv + ux + xv$. In other words, subdividing the edge uv is replacing it with a path uxv (with a vertex x not in G).

A *subdivision* of a graph G is a graph in a sequence $G_0, G_1, G_2, \ldots, G_n$ where $G_0 = G$ and for all i, $1 \leq i \leq n$, G_i is obtained from G_{i-1} by subdividing some edge of G_{i-1}.

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[71] The language here is actually somewhat imprecise, because K_4 and $K_{2,3}$ are actually isomorphism classes of graphs. That is, giving the vertices of K_4 different names makes it a different graph, even though it's still a complete graph on four vertices. I will avoid this imprecise way of speaking where it might potentially cause confusion.

Figure 4.3 illustrates a series of possible subdivisions for each of K_4 and $K_{2,3}$.

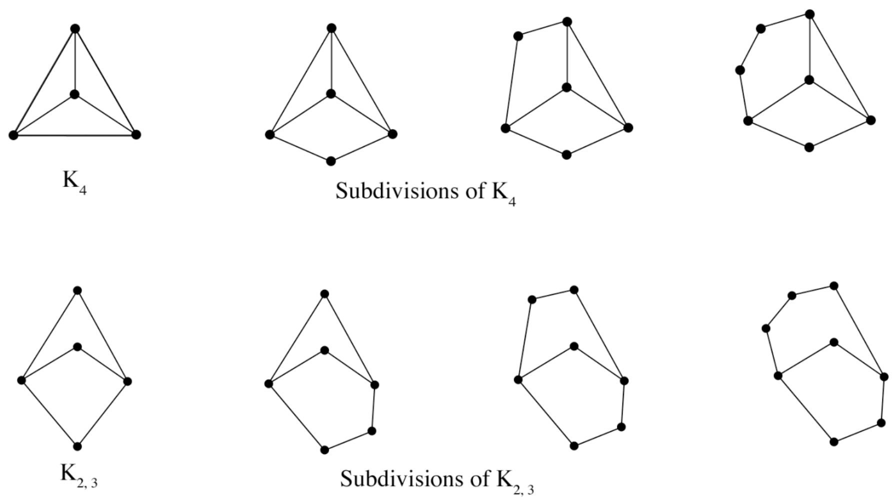

FIGURE 4.3: K_4, $K_{2,3}$, AND SUBDIVISIONS OF THEM.

(4) Maximal Outerplanar Graphs, Second Definition

A fact about maximal outerplanar graphs that is important in many of the interpretations below is the fact that they have a unique Hamiltonian cycle (proved as proposition 9 in the appendix).

Definitions A *Hamiltonian cycle* of a graph G is a cycle that includes all of the vertices of G.

A graph is Hamiltonian iff it has a Hamiltonian cycle.

When a graph has a unique Hamiltonian cycle-subgraph, this specifies an ordering on the vertices (up to rotation and inversion) that we can use to reflect the actual order of events in the melody being analyzed. The Hamiltonian cycle includes an edge for each pair of adjacent events, and one edge for the entire span of the sequence of events.

Assume, then that C is any such cycle. An edge in C indicates that the two events involved can be heard as consecutive, invoking Schenker's notion of retention of the initial tone of a prolongation span (; see the discussion in "Maximal Outerplanar Graphs" in part one). The cycle itself then claims that there is a sequence of three or more events that can be heard as a sequence of *consecutive* events, and also constitutes a single unified motion from the first event to the last (which are themselves consecutive).

If any two events on the cycle are not adjacent on the cycle itself, but are nonetheless heard as consecutive (i. e. they are adjacent in a graph containing the cycle), then these two events constitute a *chord*:

Definition Let C be a cycle of a graph G. A *chord* of G is an edge uv where u and v are non-adjacent on C.

Such a chord, call it uv, splits the cycle C into two smaller cycles, an outer and an inner part. The inner part consists of u and v and all of the events occurring between them on C. The outer part consists of u, v, and all of the events occurring before u and after v on C. Each of these makes up its own cycle, meaning it constitutes a single unified motion (prolonging either uv itself or the same motion that C prolongs) and is a sequence of consecutive events. In the case of the outer part, this sequence includes uv, meaning that the events between u and v have been reduced-out by virtue of the fact that u and v can be heard as consecutive. The inner part of uv is a prolongation of the motion from u to v.

Now assume that xy is another such chord on the same cycle, C. Furthermore, assume that one of these vertices, x, is on uv's inner cycle, while the other, y, is after v on the cycle C. This would mean that there's an event, x, that's part of a prolongation of a motion from u to v, but itself initiates a motion that isn't contained in that prolongation—that is, the motion is completed by an event y that occurs after the motion from u to v is itself completed. Or, to put it differently, there's an event v that completes a motion initiated by an event u in the sequence of events described by C, but there is also a reduction of the sequence of events on C that includes the initiation of the motion, u, but not its resolution v. This situation is defined as *crossing chords* or *crosschords* on the cycle C:

Definition Let C be a cycle of a graph G. Let uv and xy be chords of C such that u precedes v, x precedes y, and u precedes x on C. Then uv and xy are *crosschords* of C iff u, v, x, and y are ordered u . . . x . . . v . . . y on C. Two chords of a cycle *cross* if they are crosschords. (Note that I use the word "cross" in a different but related way when talking about curves in an embedding).

This way of describing prolongation in terms of sequences of events defining complete motions makes it nonsensical to allow the motions to cross in this way. If u and v define a motion, and u is itself part of another motion from x to y, then the progression from x to y ought to also contain v, because the x-to-y motion can't itself be complete if it includes an initiation of a progression, u, without completing it (by including v). Thus, one way of defining such an idea of prolongation is to say that its graph is Hamiltonian (all of the events together make up a single complete motion) and *crosschord-free*, that is,

Definition A graph G is *crosschord-free* iff for any cycle C of G, no two chords of C cross.

This non-geometric definition of crossing, gives an alternate characterization of a Hamiltonian outerplanar graph, or *HOP*,

Definition A graph G is a HOP(2) iff it is Hamiltonian and crosschord-free.

Using this definition we can give a non-geometric definition of MOPs:

Definition A graph G is a MOP(2) iff it is a maximal HOP(2).

Thus, characterization (4) thus says that a prolongational analysis is one with a Hamiltonian cycle ordering all events and in which all non-consecutive pairs of events cross some some prolongational span (that is, if an edge were added to the graph it would cross some other edge).

When necessary to distinguish the two definitions of Hamiltonian outerplanar or maximal outerplanar, I will call them HOP(1) and HOP(2) or MOP(1) and MOP(2). Otherwise I will simply call them HOP or MOP. The equivalence of HOP(1) and HOP(2) is proved as proposition 10, and the equivalence of MOP(1) and MOP(2) follows easily.

(5) Chordality

It is also possible to replace the maximality condition in (4) with chordality:

Definition A graph G is *chordal* iff every non-trivial cycle of G has at least one chord.

A cycle of four or more vertices with no chords is called a *chordless cycle* or *hole*.

In other words, instead of saying "as many events as possible are heard as consecutive," we can say "every complete progression of four or more events can be broken up into two smaller complete progressions (by hearing another pair of events in that progression as consecutive)," according to the semantic interpretation of chords and cycles proposed in the previous section.

(6) Minimality

Furthermore, we can replace the outerplanar (or crosschord-free) condition of characterizations (4)-(5) with minimality. Characterization (6) then says: (again, according to the interpretation of chords and cycles described in the previous two sections) a prolongational analysis is one in which all events constitute a single complete motion (Hamiltonicity) every complete progression of four or more events can be broken up into two more basic motions, and this is done in such a way *every consecutivity relationship included is necessary* (minimality)—that is, any consecutivity proposed by the analysis is either part of the overall (Hamiltonian) progression, or leaves a hole (a chordless cycle) when it's removed.

Looking at it from a different perspective, the equivalence of this condition means that if an analysis is Hamiltonian and chordal and has crossing chords, then its possible to remove a "consecutivity" from the analysis (remove an edge from the graph) without destroying the "completeness" (the Hamiltonicity and chordality) of the analysis. Thus given such an analysis one can remove consecutivities from it one at a time and eventually arrive at a MOP.

This seemingly simple variation on the other MOP characterizations is in fact the most difficult to demonstrate as equivalent (as the second and third parts of the proof in part five should make clear).

(7) Cycle-Connectedness

I pointed out in the "Basic Terms and Definitions" section above that paths are, semantically, ways of indirectly relating two vertices. Thus, if edges are seen as showing "directly related" events, then an obvious condition for a musical analysis might be that it is possible to relate any two events at least indirectly by finding a path between them. This would be equivalent to saying that all analytical graphs must be *connected*. However, if finding paths were a primary way of relating events, we might also want any pair of vertices to be related in a specific way. That is, there should be exactly one path connecting any two vertices. This is equivalent to saying that there's a bijection between paths of the graph and pairs of vertices. We could call such graphs *path-connected*, but in fact the path-connected graphs are just the trees.

Yet, a MOP is intended as a description of dynamic prolongation. Such a prolongational analysis should therefore describe the structural status of an event not so much by showing which other events it's "directly related" to, but rather by showing the structural motions that initiates and completes and otherwise participates in. In other words, following the semantic interpretation of graphs in terms of dynamic prolongation that I've pursued thus far in this part of the paper, it's not so much the edges of the graph themselves that are meaningful but the cycles that they make up. While I've described an edge of a MOP as indicating that two events are "heard as consecutive" or "constitute a motion," this doesn't tell us much about the prolongational analysis unless we can answer the questions "consecutive in what progression?" or "a motion prolonged by what?"

Therefore, one should be able to more meaningfully relate sets of vertices in a MOP by finding cycles that include them together—that is, by finding complete progressions in which they all participate. In the case of cycles, however, we cannot have as clean of a bijection as in the case of paths, but we can define something similar that I'll call cycle-connectedness.

First, there must be a cycle corresponding to each pair of non-adjacent vertices. It is obviously too restrictive to say that every pair of non-adjacent vertices must participate in only one cycle-subgraph, so instead we will say that for every pair of non-adjacent vertices, u and v, every cycle-subgraph containing u and v will contain all the vertices of some minimal cycle-subgraph. That is (for sets of vertices of any size),

Definition Let Ω be a set of vertices of a graph G. A *minimal cycle-subgraph* for Ω, C, is a cycle-subgraph of G such that $\Omega \subset V(C)$ and for any other cycle-subgraph C' such that $\Omega \subset V(C')$, $V(C) \subset V(C')$.

If a graph has a minimal cycle-subgraph for each pair of non-adjacent vertices, then we can interpret the graph as relating each pair of non-consecutive melodic notes by the smallest melodic reduction that contains them both. Then we might define a graph class in which there is a bijection between pairs of non-adjacent vertices and their minimal cycle-subgraphs. This will actually produce a subclass of MOPs, so we will call it strong cycle-connectedness.

Consider the situation in the MOP analysis of figure 4.2: this MOP has a cycle ${}_1$G-${}_3$F-${}_3$E-${}_4$D-${}_4$G-${}_5$E], shown in figure 4.4. Each of the non-adjacent pairs {${}_1$G, ${}_3$E}, {${}_1$G, ${}_4$G}, and {${}_3$E, ${}_4$G} on this cycle has a smaller minimal cycle: ${}_1$G-${}_3$F-${}_3$E-${}_4$D-${}_5$E], ${}_1$G-${}_3$F-${}_4$D-${}_4$G-${}_5$E], and ${}_3$F-${}_3$E-${}_4$D-${}_4$G-${}_5$E]. Therefore there is a cycle-subgraph of this graph that isn't minimal for any pair of vertices, so it isn't strongly cycle-connected. However, this cycle-subgraph is minimal for the larger set {${}_1$G, ${}_3$E, ${}_4$G}, and can be interpreted as representing a progression corresponding to that set. A set such as this, in which no pair of vertices is adjacent, is called an independent set:

Definition An *independent set*, Ω, is a set of vertices of a graph G such that no two vertices of Ω are adjacent.

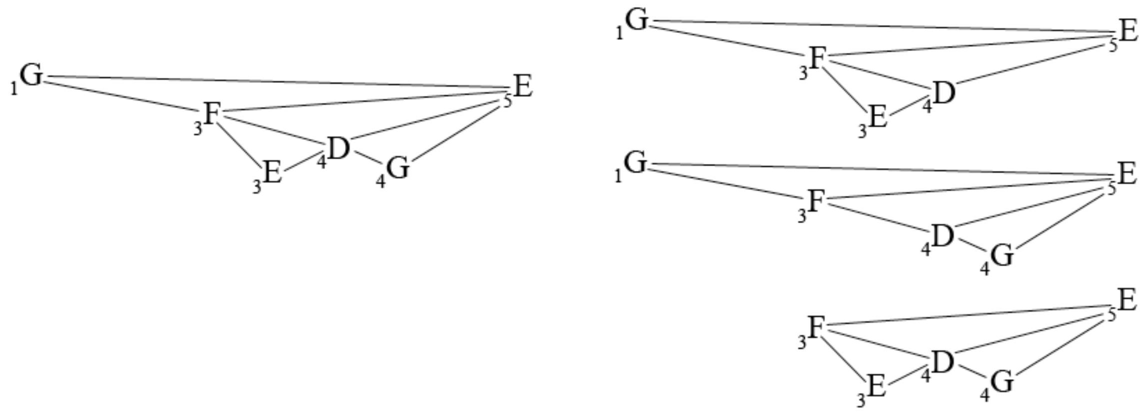

FIGURE 4.4: MINIMAL CYCLES FOR {$_1G$, $_3E$, $_2G$}, {$_1G$, $_3E$}, {$_1G$, $_4G$}, AND {$_3E$, $_4G$}

In order to allow cycles to be minimal for sets of any size, let us weaken the bijection of strong cycle-connectedness in the following way: the mapping from independent *pairs* to their minimal cycle-subgraphs should still be injective, but only the mapping from independent *sets* to their minimal cycles should be surjective. This is stated in parts (2) and (3) of the following definition:

Definitions A graph G is *cycle-connected* iff there is a mapping, σ, from sets of independent vertices of G to cycle-subgraphs of G which satisfies the following three properties:

(1) For each independent set of vertices of G, Ω, σ(Ω) is a minimal cycle-subgraph for Ω.

(2) For any two distinct 2-member independent sets of G, Ω and Ω', σ(Ω) ≠ σ(Ω').

(3) For any cycle-subgraph of G on at least 4 vertices, C, there is some non-trivial independent set of G, Ω, such that C = σ(Ω).

Therefore, translating from cycles to prolongations, cycle-connectedness says that a MOP analysis in one in which (1) every set of events can be described by the minimal prolongation that all the events in the set participate in (that is, every prolongation they participate in shares some essential sequence of events that itself constitutes a complete prolongation), (2) no two non-consecutive events are thus described by the same prolongation, and (3) every complete prolongation identified by the analysis is minimal for some set of at least two events (containing no consecutive events).

(8) Confluence.

Another way to characterize MOPs in terms of their potential to model prolongational spans through their cycle content is through the confluence condition. It is obviously crucial that once the Hamiltonian cycle establishes the order of pitches in the melody, the ordering of vertices in smaller cycles representing complete progressions of events follow the ordering of the Hamiltonian cycle. Otherwise we would have the absurd consequence of melodic reductions with notes in a different order than they actually occur in the melody.

The property of confluence (from Duffin (1965)) ensures that the order of pitches is consistent between any two cycles of the graph. Confluence is defined first for pairs of edges, then for graphs:

Definitions Let uv and xy be edges of the graph G. The edges uv and xy are *confluent* iff every cycle-subgraph including uv and xy orients them in the same way with respect to one another. In other words, uv and xy are confluent iff for every cycle including both uv and xy and beginning with uv, x either always precedes y or always follows y in the cycle.

A graph G is *confluent* iff every pair of edges of G is confluent.

Consider the edges $_3$F-$_3$E and $_4$D-$_4$G in the MOP of figure 4.2. Figure 4.5 shows the Hamiltonian cycle of this graph, and assigns orientations to $_3$F-$_3$E and $_4$D-$_4$G that point in the same direction around the cycle. These orientation represents melodic precedence. Next to the Hamiltonian cycle in figure 4.5 are two other cycles that also

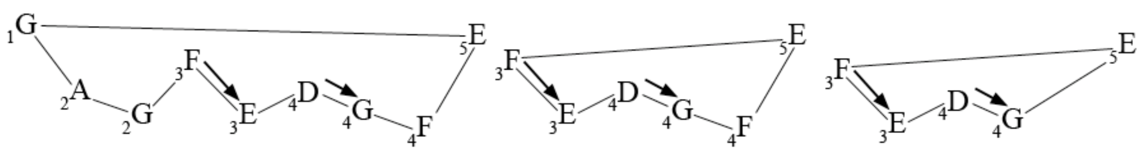

FIGURE 4.5: AN ILLUSTRATION OF CONFLUENCE FOR $_3F$-$_3E$ AND $_4D$-$_4G$

contain the edges $_3$F-$_3$E and $_4$D-$_4$G. The orientations also point in the same direction around these cycles, reflecting the fact that $_3$F-$_3$E and $_4$D-$_4$G are confluent edges. This is true of all intervals in the Hamiltonian cycle other than the root edge.

The reader will no doubt have noticed, however, that there is one interval in every cycle that doesn't represent a melodic interval but the interval from the first note of the melody to the last. When an orientation is assigned to *this* edge to match the orientation of any other edge in the cycle, it points in the direction opposite the temporal precedence in the melody being analyzed. This may come into play with confluence when comparing an edge representing a prolongational span to an edge representing an interval prolonging that span. For instance, in the MOP of figure 4.2 the interval $_3$F-$_3$E prolongs the interval $_1$G-$_5$E (though indirectly). These two intervals occur together in a number of cycles: $_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_5$E], $_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_4$G-$_5$E], $_1$G-$_2$A-$_2$G-$_3$F-$_3$E-$_4$D-$_5$E], $_1$G-$_2$A-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_5$E], $_1$G-$_2$A-$_3$F-$_3$E-$_4$D-$_4$G-$_5$E], $_1$G-$_2$A-$_3$F-$_3$E-$_4$D-$_5$E], $_1$G-$_3$F-$_3$E-$_4$D-$_4$G-$_4$F-$_5$E], $_1$G-$_3$F-$_3$E-$_4$D-$_4$F-$_5$E], and $_1$G-$_3$F-$_3$E-$_4$D-$_5$E]. In all of these cycles, $_3$F-$_3$E is oriented towards $_3$E (following melodic precedence) and $_1$G-$_5$E is a spanning interval so is oriented towards $_1$G (against melodic precedence).[72] Therefore such edges are always confluent.

The reader might worry that this is a problem for confluence, since an edge might in one cycle represent a melodic interval (in which case it is oriented temporally forwards) and in a different cycle represent a prolongational span (in which case it is oriented temporally backwards). For instance, figure 4.6 demonstrates how the interval

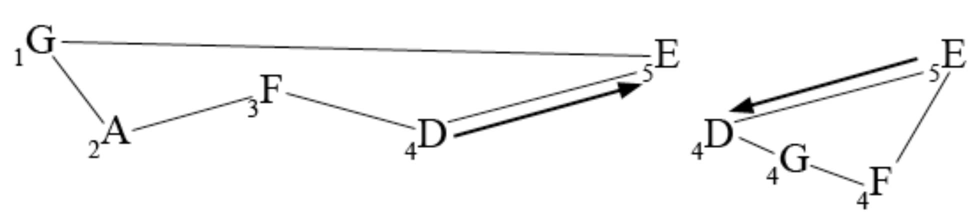

FIGURE 4.6: DIFFERENT ORIENTATIONS FOR THE EDGE $_4$D-$_5$E

[72] It would be possible, of course, to orient all edges in the opposite way, but this is immaterial as far as confluence is concerned.

${}_4$D-${}_5$E is melodic in the cycle ${}_1$G-${}_2$A-${}_3$F-${}_4$D-${}_5$E] (which represents a relatively background melodic reduction) and is a prolongational span in the cycle ${}_4$D-${}_4$G-${}_4$F-${}_5$E] (which represents a relatively foreground phrase). But since, in the confluence definition, edges are evaluated pairwise, such contradictions of confluence are prevented by the fact that no one interval can be more foreground than ${}_4$D-${}_5$E in one context (in one cycle) and more background in another context (in a different cycle).

In other words, a Hamiltonian confluent analysis is one in which a single large prolongation defines a well-ordering on events, with the initial and final events making up the prolonged motion, the order of events in all other prolongations agree with the general ordering, and furthermore any two consecutivities identified by the analysis can be described as either one following the other or one containing (i. e. being prolonged by) the other (where motion xy follows motion uv iff uv always comes before xy in any prolongation including them both, and motion xy contains uv iff every prolongation including them both is a prolongation of xy).

Characterization (8) thus describes a MOP analysis as one that is maximal with respect to this property. That is, it's a Hamiltonian confluent analysis in which any two non-consecutive events cannot be described as following (or being followed by) or containing (or being contained by) some pair of consecutive events.

Note that by proposition 6, taken from Duffin (1965), confluence is equivalent to having no subgraph that is a subdivision of K_4. I will use this fact extensively in the proof to circumnavigate any use of the definition of confluence itself, since the exclusion of K_4 subdivisions is a more direct property that is easier to work with and is closely related to outerplanarity. See the section "(3) Maximal Outerplanar Graphs, First Definition" above for definitions of K_4 and subdivisions.

(9) Chordality and Confluence

As with the outerplanar characterizations, the maximality condition on Hamiltonian-confluence can be replaced with chordality. According to characterization (9) a MOP analysis is one where all prolongations can be broken down into elemental prolongations of 3 members and agree with one another in the temporal orientation of their edges, and if two prolongational spans overlap in time, one is a prolongation of the other and not vice versa.

(10) H2-Intrasymmetry

The last three characterizations, (10), (11), and (12), define MOPs with a different approach in an attempt to capture the hierarchic nature of prolongational analysis.

There are two possible ways to construe the "hierarchical nature" of a Schenkerian analysis. The simpler construal is that each direct relationship between events is one of subordination of one event to another. In other words, the hierarchy is a hierarchy of events. This is an obvious characteristic of the phrase-structure trees and prolongation trees discussed throughout part two. There's another way to think of hierarchy in prolongational analysis, however, which is in terms of a hierarchy of possible analyses. That is, an analysis contains within itself a number of smaller analyses of various sizes. These analyses are related hierarchically, some being more background and others more foreground. But regardless of how foreground or background two analyses are, they share something in common which makes them qualify as well-formed analyses.

In other words, a prolongational analysis is saturated with smaller possible analyses of every size, and regardless of the way in which one reduces the overall analysis to get one containing, say, five notes, the general principles of how these five notes are analyzed remains the same.

One can envision this in terms of the voice-leading graphs that make up Schenker's *stimmfuhrungsprolongation* analyses. A single large voice-leading graph can be broken down into several smaller ones. In fact, every note in the analysis could be said to participate in many possible voice-leading graphs of every size implied by the analysis. These voice-leading graphs, regardless of their size, share essential properties.

More generally, the principles of analysis are the same between the entire analysis and parts of the analysis of various sizes. This obviously cannot be a strict identity, since a part of an analysis necessarily includes fewer events than the whole. However, this is still a kind of self-resemblance or symmetry, which I will call *intrasymmetry*.

Let's translate this into graphical terms:

(1) We first need to formalize what I have called the "principles of analysis," which will give us some set of basic conditions on what constitutes an analysis. This depends of course on how we interpret elements of a graph musically. These conditions will partly determine a class of graphs, **C**, which are analytically useful according to our way of interpreting graphs.

(2) The graph should be "saturated" with graphs of every size that can be interpreted as analyses. We formalize this by saying every vertex should take part in a series of subgraphs of every size that also qualify as analyses. These subgraphs should be determined by the vertex sets alone, so we specify that they must be *induced* subgraphs:

Definition An *induced subgraph*, H, of a graph G is a subgraph of G that contains every edge of G between vertices in V(H).

(3) Finally, we need some way of distinguishing which induced subgraphs should be interpretable as analyses. In other words, any vertex set satisfying certain conditions should have an induced subgraph in **C**.

One way we might flesh out these conditions is to use connectedness. I defined connectedness in the "Basic Terms and Definitions" section above but didn't generalize the notion to k-connectedness.

Definition A graph G is *k-connected* (for some integer k) if it has at least k vertices and there is no set **S** of k – 1 or fewer vertices such that G – **S** is an unconnected graph.

In other words, if a graph is k-connected then one must remove at least k vertices (and their incident edges, of course) from the graph to eliminate every path between any two vertices of the graph. A 1-connected graph is just a connected graph.

We will be mainly interested in 2-connectedness when discussing MOPs, which can also be defined in terms of *cut-vertices*.

Definition A *cut-vertex* of a connected graph G is a vertex, v, whose deletion disconnects the graph. That is, v is a cut-vertex iff G – v is unconnected.

(Vertex deletion is defined in "(3) Maximal Outerplanar Graphs, First Definition" above). A 2-connected graph can be defined as a connected graph with no cut-vertices.

Consider the following interpretation of a graph: if the vertices corresponding to two different events are adjacent, then they are "directly related." If two vertices, x and y, are not adjacent, then we relate them by finding a path from one to the other, so that they are related by a series of intermediary notes which are directly related one to the next. Then we can interpret a graph by finding vertices that are on all x-y paths, telling us which events mediate the relationship between a particular two events. Then for a graph to represent the relationship between two events represented by vertices x and y, all events in the graph should be connected to x and y, and there should be some event which is on all paths from x to y. In other words, a graph $G \in \mathbf{C}$ must be connected and have a cut-vertex, or equivalently, G must be 1-connected but not 2-connected. Let's call any class of graphs **C** which satisfy the following properties *1-intrasymmetric*:

Definition Let **C** be a class of graphs. **C** is 1-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is connected but not 2-connected.

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $2 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is connected, $H \in \mathbf{C}$.

The *1-intrasymmetric graphs*, then, are the graphs in any such class:

Definition A graph G is 1-intrasymmetric iff there exists some 1-intrasymmetric class of graphs **C** such that $G \in \mathbf{C}$.

Obviously the class of 1-intrasymmetric graphs itself is 1-intrasymmetric. Combining (1) and (3), we can see that 1-intrasymmetric graphs must be connected and have no cycles; that is, 1-intrasymmetric graphs are trees. Furthermore, the class of trees satisfies (2), so the 1-intrasymmetric graphs are just the trees. Note that the premises of this derivation are similar to the premises in the derivation of path-connectedness as a property in the section "(7) Cycle-Connectedness" above, and that we also arrived at a graph class equivalent to trees there. We make roughly the same analogy here from 1-intrasymmetric to *2-intrasymmetric* graphs as we did from path-connectedness to cycle-connectedness in the earlier discussion.

Generalizing the idea of 1-intrasymmetry we can define the notion of a 2-intrasymmetric class of graphs:

Definition A class of graphs **C** is 2-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is 2-connected but not 3-connected.

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is 2-connected, $H \in \mathbf{C}$.

By analogy, the *2-intrasymmetric graphs* are the graphs in any 2-intrasymmetric class and themselves form a 2-intrasymmetric class.

Definition A graph G is 2-intrasymmetric iff there exists some 2-intrasymmetric class of graphs **C** such that $G \in \mathbf{C}$.

The semantics of 1-intrasymmetry say that in order for a graph to function as an analysis, every pair of events must be relatable by finding a path between them (connectedness), and at least one pair of vertices, x, y, must have a vertex, v, that occurs on all paths between them (non-2-connectedness) so that one could interpret the graph as stating that the relationship between x and y is mediated by v. Analogously, the semantics of 2-intrasymmetry is that every pair of vertices must participate in at least one cycle (2-connectedness), and for at least one pair of vertices, x and y, there must be some other vertex, v, that occurs on every cycle including x and y (non-3-connectedness).

The class of graphs in MOP characterization (10) is class of the *Hamiltonian* 2-intrasymmetric graphs. This definition is stronger, saying that in order for a graph to qualify as an analysis, all events must participate in a Hamiltonian cycle:

Definition A class of graphs **C** is H2-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is Hamiltonian but not 3-connected.

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is 2-connected, $H \in \mathbf{C}$.

(11) HOP-Intrasymmetry

The first condition of the definition of H2-intrasymmetric classes is intended to give a possible basic condition on what kind of graph is interpretable as an analysis given a particular set of semantic assumptions. Of course, there are many other conditions one could put here, and some of them will delineate the same class of graphs.

One other approach is to replace "not 3-connected" in part (1) of the definition with "crosschord-free." The definition of HOP-intrasymmetric thus says a graph is interpretable as an analysis if it has a Hamiltonian cycle and no crossing relationships.

Definitions A class of graphs **C** is HOP-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is HOP(2).

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is 2-connected, $H \in \mathbf{C}$.

A graph G is HOP-intrasymmetric iff there exists some HOP-intrasymmetric class of graphs **C** such that $G \in \mathbf{C}$.

(12) HC-Intrasymmetry.

Finally, we can replace "crosschord-free" or "not 3-connected" in the intrasymmetry definition to a requirement that G be confluent.

Definitions A class of graphs **C** is HC-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is Hamiltonian and confluent.

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is 2-connected, $H \in \mathbf{C}$.

A graph G is Hamiltonian-confluent-intrasymmetric iff there exists some Hamiltonian-confluent-intrasymmetric class of graphs **C** such that $G \in \mathbf{C}$.

An Overview of the Characterizations of Theorem 1

The reader has no doubt noticed certain similarities between many of the characterizations of theorem 1. Many properties of MOPs recur in two or three or more characterizations. To help put some order to this observation and to improve our understanding of theorem 1 before proving it, I propose that most of the characterizations can be divided roughly into three properties: Hamiltonicity, maximality, and minimality. This division into three properties is also discussed in "Maximality and Chordality" in part one of the paper. The Hamiltonicity property is readily separable from every characterization with the exception of the first definition of maximal outerplanarity. The maximality and minimality conditions, on the other hand, are not always equivalent from one characterization to the next, but serve similar roles. Characterizations (1) and (2) both fix the number of edges of the graph given the number of vertices (at twice the number of vertices minus three), and so cannot be well described in terms of separate maximality and minimality conditions.

(1) Hamiltonicity. This is explicit in characterizations (4) – (6) and (8) – (12). In (1) the qualifier "unary" ensures Hamiltonicity (see proposition 12). In (2), Hamiltonicity is implied by requiring that the 2-overlap clique graph be a tree, rather than having a subgraph of the 2-overlap clique graph on the same vertex set be a clique tree. Finally, the definition of cycle-connected requires Hamiltonicity by asking for a minimal cycle for all independent sets rather than simply non-adjacent pairs of vertices.

The complete harmonic prolongational analyses of part three relax the Hamiltonicity of MOPs. In that part I described them as 2-trees. They could also be thought of as chordal crossing-free graphs, maximal confluent graphs, and so on.

(2) Maximality. Most of the characterizations have some condition akin to maximality; that is, a condition that is only breached by removing edges from the graph. In characterizations (3), (4), and (8) this is explicit. In (5), (6), and (9) the property of chordality prevents us from removing the chords of a cycle (but doesn't prevent adding new chords). In characterizations (10) – (12) the maximality condition is in part (2) of the intrasymmetry definition that requires a saturation of Hamiltonian induced subgraphs. Removing an edge from an intrasymmetric graph destroys the 2-connectedness of some subgraph necessary to satisfy (2). In the case of cycle connectedness, part (2) of the definition (injectivity) is a maximality-type condition because it requires that the graph has at least as many unique cycles as independent pairs, and it cannot be violated by adding edges (which reduces the number of independent pairs and can only increase the number of cycles).

As I pointed out in "Maximality and Chordality" the relaxation of this condition can allow for holes (chordless cycles) as prolongational building blocks.

(3) Minimality. All the characterizations with some sort of maximality condition also have a condition limiting the number of edges of the graph; one that is violated only by adding and never by removing edges. This is explicit in characterization (6). Other minimality conditions are outerplanarity or being crosschord-free for (3), (4), (5) and (11), and confluence for (8), (9), and (12). These conditions are equivalent for Hamiltonian graphs. In the case of H2-intrasymmetry, having a large set of subgraphs that cannot be 3-connected ensures minimality. Finally, the surjective part (part (3)) of the cycle connectedness definition is a minimality condition, because removing edges reduces the number of cycles and increases the number of independent sets and so cannot violate this part of the definition.

PART 5: PROOF OF THEOREM 1, CHARACTERIZATIONS OF MOPS

Outline of the Proof

Theorem 1 (Characterizations of MOPs) Let G be a graph on 4 or more vertices. The following are equivalent:

(1) G is a unary 2-tree.

(2) All of the maximal cliques of G are triangles and the 2-overlap clique graph of G is a clique tree.

(3) G is maximal outerplanar.

(4) G is MOP(2)

(5) G is chordal and HOP(2).

(6) G is minimal Hamiltonian-chordal.

(7) G is cycle-connected.

(8) G is maximal Hamiltonian-confluent

(9) G is Hamiltonian, chordal, and confluent

(10) G is H2-intrasymmetric.

(11) G is HOP-intrasymmetric.

(12) G is HC-intrasymmetric.

The proof is organized in twelve parts as follows:

Part:	1	2	3	4	5	6	7	8	9	10	11	12

(1) ⇒ (6) ⇒ (2) ⇒ (7) ⇒ (8) ⇒ (9) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (11) ⇒ (12) ⇒ (10) ⇒ (1)

I present the proof of each part below in order. Some of the proofs require propositions or lemmas; if so the statements of the propositions and lemmas precede the proof. The proofs of lemmas follow the larger proofs that incorporate them, and the proofs of propositions are in the appendix.

Part 1

(1) ⇒ (6) G is a unary 2-tree ⇒ G is minimal Hamiltonian-chordal.

Proposition 12 Let G be a 2-tree. Then G is a unary 2-tree iff G is Hamiltonian. Furthermore there's a unique Hamiltonian cycle-subgraph of G that consists of all edges that aren't in the clique sequence defining G as a unary 2-tree.

Proof of Theorem 1, Part 1

Let $G = G_n$ be a unary 2-tree on n vertices, with ordered vertex set $\mathbf{V}_G = v_1, v_2, \ldots, v_n$ and ordered clique set $\mathbf{Q}_G = Q_4, Q_5, \ldots, Q_n$. If n = 3 then G is a triangle and is minimal Hamiltonian-chordal. Assume inductively that every unary 2-tree on n – 1 vertices is minimal Hamiltonian-chordal. According to this assumption, the unary 2-tree $G_{n-1} = G - v_n$ is minimal Hamiltonian-chordal. Let $Q_n = \{x, y\}$.

By proposition 12, G is Hamiltonian with a unique Hamiltonian cycle-subgraph, C_{ham}.

Let C be any non-trivial cycle of G. If C is also a cycle of G_{n-1}, then it has a chord by the assumption that G_{n-1} is chordal. Otherwise C must include xv_n and v_ny. In this case it has the chord xy. Thus G is chordal.

Let ab be any edge of G. I will prove that G – ab is either not chordal or not Hamiltonian.

Because v_n is degree 2, xv_n and v_ny must be part of C_{ham}. So if ab is xv_n or v_ny then G – ab is not Hamiltonian. Otherwise ab is an edge which G shares with G_{n-1}. Since G_{n-1} is minimal Hamiltonian-chordal, G_{n-1} – ab is either not chordal or lacks a Hamiltonian cycle. If G_{n-1} – ab is not chordal then G – ab is also not chordal (because any cycle of G_{n-1} – ab is also a cycle of G – ab and has no chords in G – ab which it does not have in G_{n-1} – ab).

On the other hand, assume that G_{n-1} – ab lacks a Hamiltonian cycle and G – ab has a Hamiltonian cycle (= C_{ham}). Since xv_n and v_ny must be in C_{ham}, if G_{n-1} – ab includes xy then it has a Hamiltonian cycle the same as C_{ham} but substituting xv_n and v_ny with xy. Therefore ab = xy.

We need only show, then, that G – xy is not chordal. Assume without loss of generality that x precedes y in $\mathbf{V}_G$ and let y = v_i for some integer i. Additionally, let Q_i = {x, z}. Then G has a triangle {x, y, z}, and a 4-cycle $xv_nyz]$ with chord xy. The vertex v_n is not adjacent to z, so $xv_nyz]$ has no chord in G – xy. Therefore G – xy is not chordal.

This proves that G – uv non-Hamiltonian-chordal for any edge uv, and G is minimal Hamiltonian-chordal. ♦

Part 2

(6) ⇒ (2) G is minimal Hamiltonian-chordal ⇒ All of the maximal cliques of G are triangles and the 2-overlap clique graph of G is a clique tree.

Definitions A *simplicial* edge of a graph G is an edge whose endpoints share only one maximal clique of G in common.

A *leaf* of a tree is a degree-one vertex. (Note that any tree must have at least two leaves).

Definition A clique graph of a graph G is a graph with a vertex corresponding to each maximal clique of G such that if two vertices are adjacent then their corresponding maximal cliques share vertices in common.

Notation Let T be a clique graph of a chordal graph G. Then for any K ∈ **K**(G), k_T is the vertex of T corresponding to K.

Definition The 2-overlap clique graph of a graph G is a graph with a vertex for each member of **K**(G) and an edge between the vertices corresponding to maximal cliques which share at least 2 vertices.

Definition A clique tree, T, of a graph G is a tree that is a clique graph for G such that for any two cliques K, K' ∈ **K**(G), every clique along the path connecting k_T and k'_T in T contains K ∩ K'.

Lemma 2.1 Let G be a minimal Hamiltonian chordal graph. Every vertex of G is incident upon at least 2 simplicial edges of G.

Lemma 2.2 Let G be a chordal graph. Then (G – uv) is also chordal if uv is simplicial.

Proposition 3 (Blair and Peyton (1993)) A graph is chordal iff it has at least one clique tree.

Proposition 5 Let T be a clique tree of a chordal graph G, and let k_T be a leaf of any subtree S of T. Then K contains a vertex not contained in any other clique corresponding to a vertex of S.

Proof of Theorem 1 Part 2

Let G be a minimal Hamiltonian-chordal graph. By lemma 2.2 and the minimality of G, every simplicial edge of G must be part of all Hamiltonian cycles of G. Hence no vertex of G may be incident on more than two simplicial edges. However, by lemma 2.1, all the vertices of G are incident upon at least two simplicial edges. Therefore all vertices of G are incident upon exactly two simplicial edges.

Let n = $|V(G)|$. We will define a series of subgraphs of G, G = $G_0, G_1, \ldots,$ G_{n-3}, and show that they have clique trees T = $T_0, T_1, \ldots, T_{n-3}$.

By proposition 3, G_0 has a clique tree T_0. Let $\ell^0{}_{T0}$ be a leaf of T_0. By proposition 5 L^0 has vertex x_0 not contained in any other clique of G_0. Since x_0 must be incident upon exactly two simplicial edges, L^0 must be a 3-clique, $\{w_0, x_0, y_0\}$. Because G_0 is Hamiltonian, w_0x_0 and x_0y_0 must be part of a Hamiltonian cycle. By replacing $w_0x_0y_0$ with w_0y_0 in this cycle, we get a Hamiltonian cycle for $G_1 = (G_0 - x_0)$.

Clearly w_0 and y_0 are in at least one clique of G_0 other than L^0, since they connect to other vertices on the Hamiltonian cycle. Hence $\mathbf{K}(G_1) = \mathbf{K}(G_0) - L^0$, and the vertex set of $T_1 = (T_0 - \ell^0{}_{T0})$ corresponds to the set of maximal cliques of G_1. Furthermore, if $k^0{}_{T0}$ is the vertex adjacent to $\ell^0{}_{T0}$ in T_0, then K_0 must include w_0 and y_0 by the definition of a clique tree. This means that the edge between $k^0{}_{T0}$ and $\ell^0{}_{T0}$ represents an overlap of two vertices in G_0.

Finally, since $\ell^0{}_{T0}$ is a leaf of T_0, T_1 is a tree and inherits the clique-tree property from T_0. Therefore T_1 is a clique tree for G_1, and G_1 must be chordal by proposition 3.

Assume inductively that G_i is a Hamiltonian-chordal subgraph of G with clique tree T_i. By the same argument as above (assuming $|V(G_i)| \geq 4$), any clique L^i for which $\ell^i{}_{Ti}$ is a leaf of T_i must be a 3-clique with a single vertex, x_i, shared by no other maximal clique of G_i, and an overlap of 2 vertices of G_i with the adjacent vertex of T_i. Then we can define yet another Hamiltonian chordal graph $G_{i+1} = (G_i - x_i)$ with clique tree $T_{i+1} = (T_i - \ell^i_{Ti})$.

By induction, then, (note that G_{n-3} must be a complete graph on 3 vertices), every clique in G must be a 3-clique. Furthermore, each edge of T represents an overlap of 2 vertices between the 3-cliques corresponding to its incident edges. Thus T is the same as the 2-overlap clique graph of G. ♦

Lemma 2.1 Let G be a minimal Hamiltonian chordal graph. Every vertex of G is incident upon at least 2 simplicial edges of G.

Proposition 2 Let G be a Hamiltonian chordal graph. Then G has no maximal 2-cliques.

Proposition 3 (Blair and Peyton (1993)) A graph is chordal iff it has at least one clique tree.

Proposition 4 (Blair and Peyton (1993)) Let v be any vertex of a graph G with clique tree T, and let **K**(v) be the set of maximal cliques of G containing v. Then the vertices of T corresponding to **K**(v) induce a subtree of T, T^v.

Proposition 5 Let T be a clique tree of a chordal graph G, and let k_T be a leaf of any subtree S of T. Then K contains a vertex not contained in any other clique corresponding to a vertex of S.

Proof of Lemma 2.1

Let G be a minimal Hamiltonian chordal graph and let v be any vertex of G. Let **K**(v) be the set of maximal cliques containing v.

Case 1: |**K**(v)| = 1. Then all the edges incident on v are simplicial. By proposition 2 the clique containing v must be at least a 3-clique, so v has degree at least two.

Case 2: |**K**(v)| ≥ 2. By proposition 3 G has a clique tree T. By proposition 4, the set of vertices of T corresponding to **K**(v) induces a subtree of T, T^v. T^v is non-trivial by assumption. Let k_T and k'_T be leaves of T^v. By proposition 5, K and K’ contain vertices, x and x’, respectively, such that there are no cliques corresponding to vertices of

T^v that include x or x' other than K and K'. Therefore there are no cliques corresponding to vertices of T other than K and K' that include both v and x or both v and x'. Therefore vx and vx' are simplicial edges in T. ♦

Lemma 2.2 Let G be a chordal graph. Then (G – uv) is also chordal if uv is simplicial.

Proposition 3 (Blair and Peyton (1993)) A graph is chordal iff it has at least one clique tree.

Proof of Lemma 2.2

Let G be a chordal graph with simplicial edge uv. Let Q be the clique containing u and v. By proposition 3 G has a clique tree; call it T_0. I will prove G' = G – uv is chordal by defining a clique tree T' for G' for each case below.

Case 1: Assume that no maximal clique of G contains (Q – u) or (Q – v) other than Q. Then the set of maximal cliques of (G – uv) is the same as G, but excluding Q and including Q^v = (Q – u) and Q^u = (Q – v). Let T = T_0, and let T' be a tree on the vertex set of T excluding q_T and including $q^v{}_{T'}$ and $q^u{}_{T'}$ (and changing all subscripts to T'). This situation is illustrated in figure 5.1. Define T' such that all edges are the same as those of T, except that for any vertex x_T connected to q_T, if X contains u then $x_{T'}$ is connected to $q^u{}_{T'}$, and if X contains v then $x_{T'}$ is connected to $q^v{}_{T'}$. (Note that because uv is simplicial in G, X cannot include both u and v).

We show that G' is chordal by proposition 3 by showing that T' is a clique tree. Clearly the vertex set of T' is the set of maximal cliques of (G – uv). Furthermore, T' is connected with $|E(T')| = |V(T')| - 1$, since $|E(T')| = |E(T)| + 1$ and $|V(T')| = |V(T')| + 1$, so it is a tree. It remains then to show that for any two vertices $j_{T'}$ and $k_{T'}$, each clique corresponding to a vertex on the path, P', between them contains J ∩ K.

If J and K are also cliques of G, then there's a unique path, j_TPk_T in G. If one of J or K is Q^u or Q^v, then there's a path q_TPk_T or j_TPq_T. If J = Q^u and K = Q^v or vice versa, then P' is trivial.

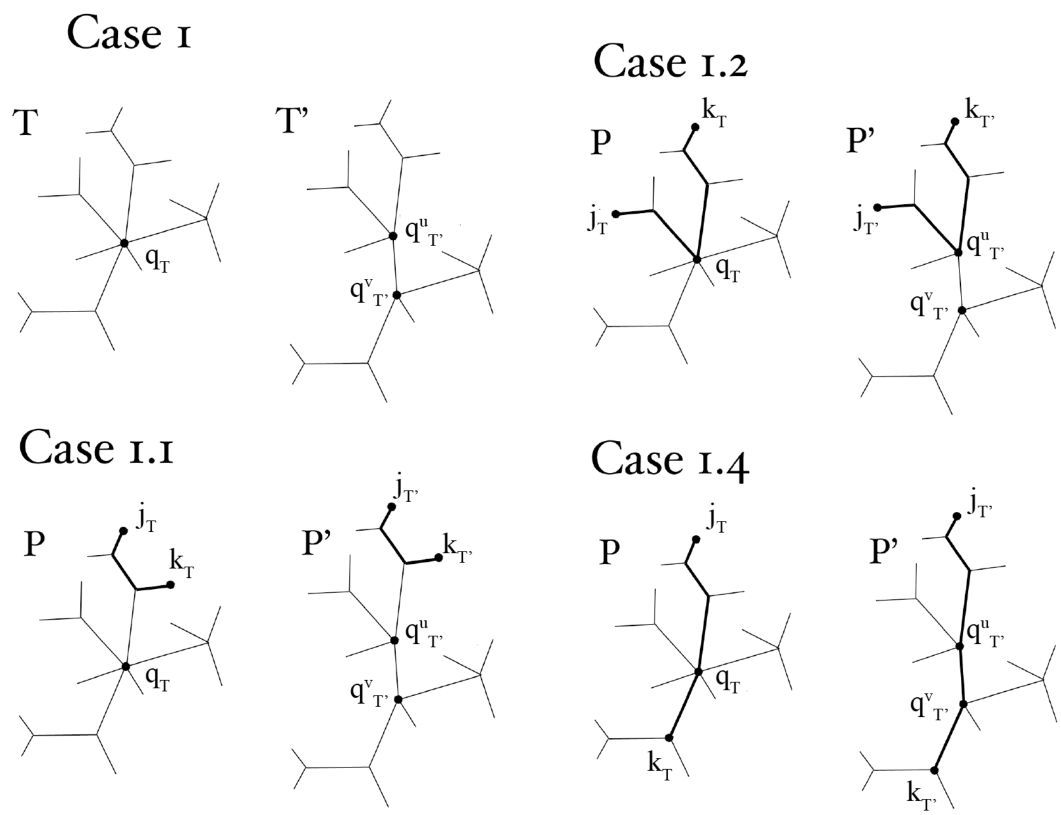

FIGURE 5.1: ILLUSTRATIONS FOR THE PROOF OF LEMMA 2.2

There are four possible cases here, which are illustrated in figure 5.1. In each case we will characterize P' in terms of a corresponding path of G, P, in order to derive the adherence of P' to the clique tree property from that of P. Let **K**(P) and **K**(P') be the sets of maximal cliques corresponding to vertices on P and P' respectively:

Case 1.1: If P does not include q_T, then **K**(P') = **K**(P).

Case 1.2: If P includes q_T and the cliques of adjacent members of P both contain u, then **K**(P') = **K**(P) – Q + Q^u.

Case 1.3: If P includes Q and the cliques of adjacent members of P both contain v, then **K**(P') = **K**(P) – Q + Q^v.

Case 1.4: If P includes Q and the clique of one adjacent member of P contains v while the other contains u, then **K**(P') = **K**(P) – Q + Q^u + Q^v.

By the definition of a clique tree, all cliques corresponding to vertices along P contain $J \cap K$. All the cliques corresponding to vertices along P' are the same as those along P, except that $q_{T'}$ may be replaced with $q^u_{T'}$ or $q^v_{T'}$. By the definition of a clique tree applied to the portions of P preceding and following q_T, and the definitions of P' above, P' includes $q^v_{T'}$ only if $J \cap K$ excludes u, and includes $q^u_{T'}$ only if $J \cap K$ excludes v. Therefore, T' satisfies the definition of a clique tree, and G' is chordal.

Case 2: Assume some maximal clique, Q^v, of G includes $(Q - u)$ but none includes $(Q - v)$. Then $\mathbf{K}(G - uv)$ is the same as $\mathbf{K}(G)$ but excluding Q and including $Q^u = (Q - v)$. If T_0 has a vertex x_{T0} adjacent to q_{T0} and $v \in X$, then $u \notin X$ (because uv is simplicial in G). Therefore there's another clique tree of G, T_1, with x_{T1} adjacent to q^v_{T1} instead of q_{T1}. Extending this reasoning to all such vertices, let T be a clique tree of G such that no vertices adjacent to q_T except q^v_T correspond to a clique containing v.

Let T' be a tree isomorphic to T, but replace q_T with $q^u_{T'}$ (and change the vertex subscripts to T'). This situation is illustrated in figure 5.2.

Consider any two cliques in $\mathbf{K}(G - uv)$, J and K. There's a unique path, P', from $j_{T'}$ to $k_{T'}$. There's a corresponding path, P, by the isomorphism. By the definition of a clique tree, all cliques corresponding to vertices along the path P contain $J \cap K$, and the cliques corresponding to vertices along P' are the same as those on P with the possible substitution of Q^u for Q. Therefore to show T' is a clique tree we only need to show that if P' includes $Q^u_{T'}$, $v \notin J \cap K$.

If P includes q_T and any adjacent vertex other than q^v_T, then by construction of T and the definition of a clique tree, $v \notin J \cap K$. If P includes q_T but no adjacent vertex other than q^v_T, then q_T must be an endpoint of the path. Then either K or K' or both is Q^u, so again $v \notin K \cap K'$. Thus T' is a clique tree for $(G - uv)$ and $(G - uv)$ is chordal.

Case 3: Assume some maximal clique, Q^u, of G includes $(Q - v)$ but none includes $(Q - u)$. By an argument analogous to that of case 2, $(G - uv)$ is chordal.

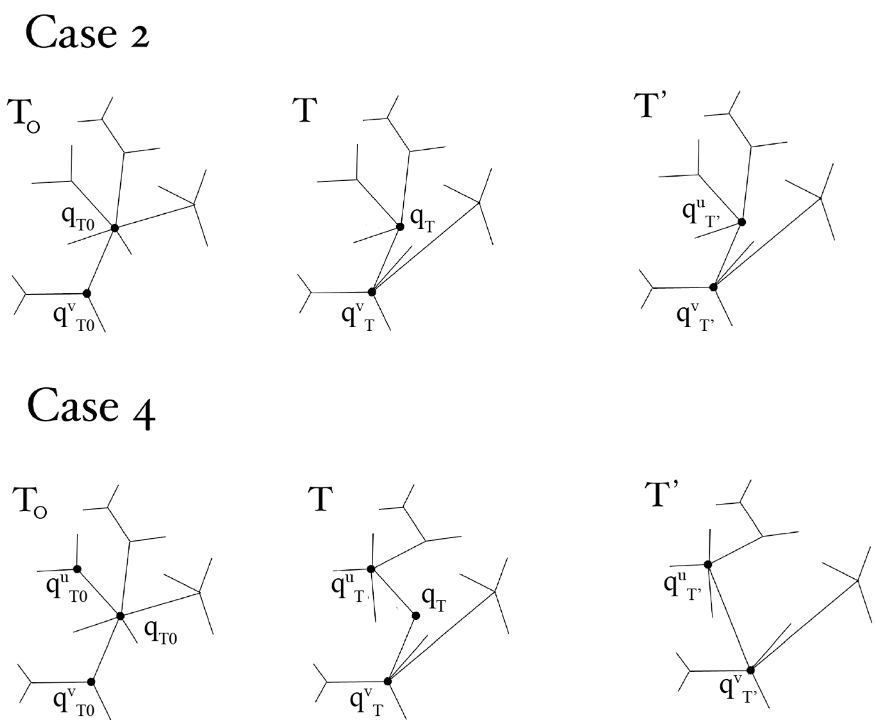

FIGURE 5.2: MORE ILLUSTRATIONS FOR THE PROOF OF LEMMA 2.2

Case 4: Finally, let G include maximal cliques Q^v and Q^u, distinct from Q, which contain (Q – u) and (Q – v) respectively. Then **K**(G – uv) = **K**(G) – Q. Since no maximal clique connected to Q contains both u and v by assumption, it must contain some subset of Q^v or Q^u. Thus, by an argument similar to that of case 2, there's a clique tree of G, T, with q_T connected to only $q^v{}_T$ and $q^u{}_T$.

Let T' be the tree obtained by deleting q_T from T, re-subscripting with T', and adding the edge $q^v{}_{T'}q^u{}_{T'}$. This situation is illustrated in figure 5.2.

Since q_T is adjacent to no other vertices in T, T' is also a tree. Let K and K' be cliques in **K**(G – uv), and let P' be the path connecting them. Let P be the corresponding path in T, replacing $q^v{}_{T'}q^u{}_{T'}$ with $q^v{}_Tq_Tq^u{}_T$ or $q^u{}_{T'}q^v{}_{T'}$ with $q^u{}_Tq_Tq^v{}_T$. Evidently, the intersection of cliques corresponding to vertices along P is the same that of those along P'. Therefore the fact that T is a clique tree implies that T' is also a clique tree, and (G – uv) is chordal. ♦

Part 3

(2) ⇒ (7) All of the maximal cliques of G are triangles and the 2-overlap clique graph of G is a clique tree of G ⇒ G is cycle-connected.

Definition Let Ω be a set of vertices of a graph G. A *minimal cycle-subgraph* for Ω, C, is a cycle-subgraph of G such that $\Omega \in V(C)$ and for any other cycle-subgraph C' such that $\Omega \in V(C')$, $V(C) \in V(C')$.

Definition An *independent set*, Ω, is a set of vertices of a graph G such that no two vertices of Ω are adjacent.

Definition A graph G is *cycle-connected* iff there is a mapping, σ, from sets of independent vertices of G to cycle-subgraphs of G which satisfies the following three properties:

(1) For each independent set of vertices of G, Ω, σ(Ω) is a minimal cycle-subgraph for Ω.

(2) For any two distinct 2-member independent sets of G, Ω and Ω', $\sigma(\Omega) \neq \sigma(\Omega')$.

(3) For any cycle-subgraph of G on at least 4 vertices, C, there is some non-trivial independent set of G, Ω, such that $C = \sigma(\Omega)$.

Lemma 3.1 Let G be a graph such that all maximal cliques of G are triangles and the 2-overlap clique graph of G, T, is a clique tree of G. Then

(1) There's a mapping, ψ, from subtrees of T to cycle-subgraphs of G obtained by taking the ring sum of cycles defined by nodes of the subtree.

(2) ψ is bijective.

(3) Let q_T be a vertex in a subtree S of T and v a vertex in the clique Q. Then v is in the cycle-subgraph ψ(S).

Proposition 4 (Blair and Peyton (1993)) Let v be any vertex of a graph G with clique tree T, and let **K**(v) be the set of maximal cliques of G containing v. Then the vertices of T corresponding to **K**(v) induce a subtree of T, T^v.

Proposition 5 Let T be a clique tree of a chordal graph G, and let k_T be a leaf of any subtree S of T. Then K contains a vertex not contained in any other clique corresponding to a vertex of S.

Definition The *ring sum of cycles* C and C', is the subgraph on the edge set $C \oplus C' = C + C' - (C \cap C')$. [73]

Proof of Theorem 1 Part 3

Let G be a graph such that all maximal cliques of G are triangles and the 2-overlap clique graph of G, T, is a clique tree of G, and let Ω be an independent set of V(G). For any two vertices in Ω, u and v, let T^u and T^v be subtrees of T induced by u and v according to the subtree property described in proposition 4. Because u and v are independent, T^u and T^v share no vertices.

Consider a path connecting any vertex of T^u and any vertex of T^v. Some subpath of this must start on a vertex of T^u, end with a vertex of T^v, and include no other vertex of T^u or T^v inbetween. Let us call this path-subgraph of T a *minimal path* for u and v. I will prove that u and v hae a unique minimal path S^{uv}, as illustrated in figure 5.3.

Assume by way of contradiction that u and v have two minimal paths, say from $\ell^{u1}{}_T$ to $\ell^{v1}{}_T$ and $\ell^{u2}{}_T$ to $\ell^{v2}{}_T$ such that $\ell^{u1}{}_T, \ell^{u2}{}_T \in V(T^u)$ and $\ell^{v1}{}_T, \ell^{v2}{}_T \in V(T^v)$, and

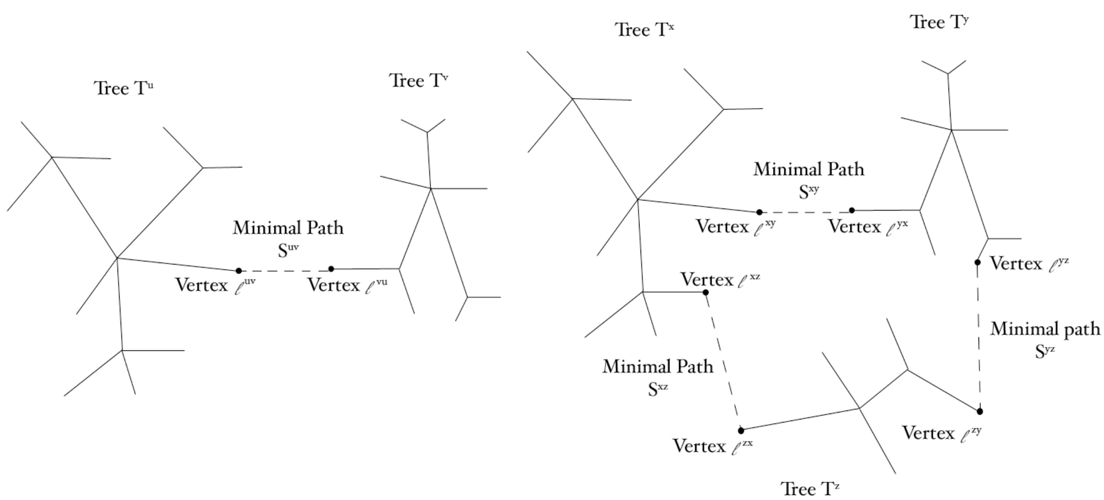

FIGURE 5.3: ILLUSTRATIONS FOR THE PROOF OF THEOREM 1 PART 3

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[73] For a proof that the ring sum of cycles is a well defined ring sum on the set of cycle-subgraphs and unions of edge-disjoint cycle-subgraphs of a graph, see Thulasiraman and Swarmy (1992). This fact, however, is not essential to the current proof.

$\ell^{u1}_T \neq \ell^{u2}_T$. Then there would be another distinct path-subgraph from ℓ^{u1}_T to ℓ^{v1}_T as follows: connect ℓ^{u1}_T to ℓ^{u2}_T within T^u, take the path from ℓ^{u2}_T to ℓ^{v2}_T including no members of T^u or T^v, then connect ℓ^{v1}_T to ℓ^{v2}_T within T^v if $\ell^{v1}_T \neq \ell^{v2}_T$. Yet this is impossible because there cannot be two distinct paths from ℓ^{u1}_T to ℓ^{v1}_T. Therefore all paths any vertex of T^u to any vertex of T^v contain some unique minimal path S^{uv} ($= S^{vu}$) which overlaps T^u and T^v only in its endpoints, ℓ^{uv}_T and ℓ^{vu}_T. (From here on I will use ℓ^{uv}_T and ℓ^{vu}_T to denote the endpoints of S^{uv} on $V(T^u)$ and $V(T^v)$ respectively.)

Let S^Ω be the union of each such minimal path for every pair in Ω. Obviously every vertex of Ω occurs in some clique corresponding to a vertex of S^Ω. We must prove, however, that S_Ω is a connected subtree of T.

If Ω contains only two vertices, then S^Ω is certainly a subtree of T.

Otherwise, consider any triple of vertices of Ω: x, y, and z. Assume one of the minimal paths corresponding to a pair of these vertices, S^{xy}, shares no vertices with the other two, S^{xz} and S^{yz}. Then there are two distinct path-subgraphs from ℓ^{zy}_T to ℓ^{xy}_T: one connecting ℓ^{yz}_T and ℓ^{xy}_T within T^y, and one connecting ℓ^{zy}_T and ℓ^{xy}_T to the endpoints of S^{xz} and S^{yz} within T^x and T^y. (See figure 5.3). This is impossible, since T is a tree.

Thus, for any triple of vertices in Ω, each minimal path shares a vertex with at least one other. Consider any w, x, y, $z \in \Omega$ such that S^{wx} and S^{yz} share no vertices. From the argument above, S^{wx} either shares a vertex with S^{xy} or is connected to it through S^{wy}. Furthermore, S^{yz} either shares a vertex with S^{xy} or is connected to it through S^{xz}. Therefore S^{wx} and S^{yz} are connected through S^{xy}. This holds for any two such minimal paths, so S^Ω must be a connected subtree of T.

Let ψ be the mapping described in lemma 3.1. Let σ be the mapping from independent sets of G to cycles of G, such that if Ω is an independent set of G, $\sigma(\Omega) = \psi(S^\Omega)$. I will demonstrate that G and σ satisfy each property of cycle-connectedness in turn.

(1) For each independent set of vertices of G, Ω, $\sigma(\Omega)$ is a minimal cycle-subgraph.

By lemma 3.1 (3), all the vertices of Ω are on $\sigma(\Omega)$.

Now consider any two vertices of Ω, u and v. Let C be a cycle-subgraph of G and S the subtree of G corresponding to C under the bijection of lemma 3.1. Clearly, u and v can only be in the cycle-subgraph corresponding to a subtree of T if that subtree contains some vertex corresponding to a clique containing u and some vertex corresponding to a clique containing v. That is, S must contain at least one member of T^u and one of T^v (the subtrees of T induced by cliques containing u and v respectively). Therefore S must contain the minimal path S^{uv}. Thus, in general, if C is a cycle-subgraph containing all the members of Ω, then $\psi^{-1}(C)$ must be a subtree of T which includes the minimal path for each pair of vertices in Ω. In other words, S^{Ω} is a subtree of $\psi^{-1}(C)$.

By lemma 3.1 (3), again, if $\psi^{-1}(C)$ contains all the vertices of S^{Ω} then C must contain all the vertices of $\sigma(\Omega)$. Therefore $\sigma(\Omega)$ is a minimal cycle-subgraph for Ω.

(2) For any two distinct 2-member independent sets of G, Ω and Ω', $\sigma(\Omega) \neq \sigma(\Omega')$.

Let $\Omega = (u, v)$ and $\Omega' = (u', v')$ be two-member independent sets of G. Assume $\sigma(\Omega) = \sigma(\Omega')$. By the bijectivity of ψ (lemma 3.1 (2)) $S^{\Omega} = S^{\Omega'}$. By construction, S^{Ω} and $S^{\Omega'}$ are paths. Furthermore, the unique vertices of the cliques corresponding to the endpoints of S^{Ω} are u and v, while the unique vertices of the cliques corresponding to the endpoints of $S^{\Omega'}$ are u' and v'. Since T is a 2-overlap tree and all maximal cliques of G are triangles, each endpoint can only have one unique vertex in its clique. Therefore $\{u, v\} = \{u', v'\}$.

This proves $\sigma(\Omega) = \sigma(\Omega') \Rightarrow \Omega = \Omega'$ and hence also the contrapositive $\Omega \neq \Omega' \Rightarrow \sigma(\Omega) \neq \sigma(\Omega')$

(3) For any cycle-subgraph of G on at least 4 vertices, C, there is some non-trivial independent set of G, Ω, such that $C = \sigma(\Omega)$.

Let C be any cycle-subgraph of G. By lemma 3.1 there is a subtree, S, of T, such that $C = \psi(S)$. Let $\ell^1{}_T, \ell^2{}_T, \ldots, \ell^n{}_T$ be the leaves of S. By proposition 5, there is a series of vertices, $v_1, v_2, \ldots, v_n$, such that v_i is in L^i and in no other clique corresponding to a vertex of S. For each ℓ^i_T and ℓ^k_T, the path from ℓ^i_T to ℓ^k_T is clearly a minimal path of T for v_i and v_k. Therefore S is the subtree S^{Ω}, and $C = \sigma(\Omega)$. ♦

Lemma 3.1 Let G be a graph such that all maximal cliques of G are triangles and the 2-overlap clique graph of G, T, is a clique tree of G. Then

(1) There's a mapping, ψ, from subtrees of T to cycle-subgraphs of G obtained by taking the ring sum of cycles defined by nodes of the subtree.

(2) ψ is bijective.

(3) Let q_T be a vertex in a subtree S of T and v a vertex in the clique Q. Then v is in the cycle-subgraph $\psi(S)$.

Definition The *ring sum of cycles* C and C', is the subgraph on the edge set $C \oplus C' = C + C' - (C \cap C')$.

Proposition 1 Let G be a chordal graph with cycle C. Any edge uv on C is part of a triangle {u, v, w} where w is some vertex on C.

Proposition 3 (Blair and Peyton (1993)) A graph is chordal iff it has at least one clique tree.

Proposition 5 Let T be a clique tree of a chordal graph G, and let k_T be a leaf of any subtree S of T. Then K contains a vertex not contained in any other clique corresponding to a vertex of S.

Proof of Lemma 3.1

Let G be a graph such that all maximal cliques of G are triangles and the 2-overlap clique graph of G, T, is a clique tree of G. Let ψ be a map from subtrees of T to cycles of G defined as follows: For some subtree, S, let $\psi(S) = \sum_{\forall k_T \in V(S)} K$, where the sum refers to the ring sum of cycles.

(1) ψ is a mapping

Let S be a subtree of T. If S is a single vertex, k_T, then $\psi(S) = K$ which is a 3-cycle-subgraph. Assume inductively that all subtrees of T on up to $(n - 1)$ vertices map to cycle-subgraphs of G.

Let ℓ_T be a leaf of S and k_T the vertex adjacent to ℓ_T in S. Let $S' = S - \ell_T$. By the inductive premise $\psi(S')$ is a cycle-subgraph. Because ℓ_T and k_T are adjacent in T, L and K share two vertices; call them x and y. Furthermore, no other maximal clique of G contains both x and y, otherwise ℓ_T and k_T would both be connected to the vertex corresponding to this clique in T, contradicting the fact that T is a tree. Since the edge xy cannot be in the intersection of any two 3-cliques of vertices in S', it must be on the cycle $\psi(S')$. Thus if $L = \{x, y, v\}$, then $\psi(S) = \psi(S') + xv + vy - xy$ which must be a cycle-subgraph since $\psi(S')$ is a cycle-subgraph containing xy. By induction, then, ψ maps every subtree of T to a particular cycle-subgraph in G.

(2) ψ is bijective

Let C be any cycle-subgraph of G. If C is a 3-cycle-subgraph then $C = \psi(k_T)$ for some vertex, K_T, of T. Assume that C is a larger cycle-subgraph. By proposition 3 G is chordal, so by proposition 1 for any edge of C, uv, there's a triangle, {u, v, w}, where w is a vertex in C.

Because C is not a 3-cycle-subgraph, either uw or vw must be a chord of C. Without loss of generality assume that uw is a chord. Then there's a cycle, C', made up of uw and the part of C from u to w not including v. By proposition 1 again, uw is part of a triangle {u, w, z}, where z is on C'. See figure 5.4.

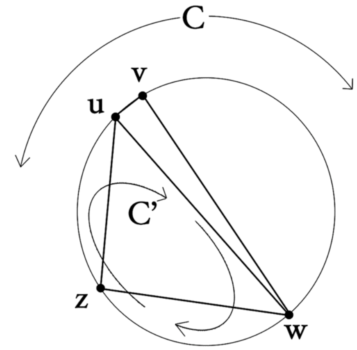

FIGURE 5.4: AN ILLUSTRATION FOR THE PROOF OF LEMMA 3.1

Let C^4 = uvwz], the ring sum of the 2-overlapping 3-cycles uvw] and uwz]. If C is a 4-cycle then C = C^4. Otherwise, either vw, wz , or zu is a chord of C. In that case there's a cycle comprised of this chord and the part of C including no vertices of C^4. By proposition 1 again we can find a 3-cycle (triangle) whose vertices are all in C and 2-overlaps C^4. Let C^5 be the ring sum of C^4 and this 3-cycle.

We can repeat this reasoning until we reach C^n = C. Because each triangle 2-overlaps the previous cycle at each step, the series of triangles induces a subtree of T, S, such that $\psi(S) = C$.

This shows that every cycle-subgraph of G has a corresponding subtree of T under the mapping. To see that the subtree is unique, assume that for some two distinct subtrees of T, S and S', $\psi(S) = \psi(S')$. Define S and S' so that S is not a subtree of S'. Since S and S' are distinct by assumption, there's some leaf of S which is not in S'; call it ℓ_T. By proposition 5, L has a vertex v which occurs in no other cliques corresponding to vertices in S. Let L = {v, x, y}. Because vx and vy cannot be duplicated in any cliques corresponding to vertices of S other than L they only occur once in the ring sum, so v is in the cycle-subgraph $\psi(S)$. According to the assumption $\psi(S) = \psi(S')$, then, vx and vy must also occur in some cliques of S'. Since $\ell_T \notin V(S')$, there must be two distinct cliques, K and K', such that vx $\in$ E(K) and vy $\in$ E(K'). This implies that k_T and k'_T are adjacent to L_T. Since T is a tree, the path from k_T to k'_T must go through ℓ_T, and in order for S' to be a tree, it must contain ℓ_T, contradicting the assumptions.

(3) Let S be a subtree of T. For any $q_T \in V(S)$, $v \in Q$, v is on the cycle $\psi(S)$.

Let $\mathbf{K}_S(v)$ be the set of vertices of S corresponding to cliques containing v. The subgraph of S induced by $\mathbf{K}_S(v)$ cannot have any cycles, since S is a tree. Therefore at least one vertex, ℓ_T, of $\mathbf{K}_S(v)$ is adjacent to no more than one other vertex of $\mathbf{K}_S(v)$. L must have at least one edge incident on v shared by no other clique with a vertex in S. This edge occurs only once in the ring sum and so must be in the cycle-subgraph $\psi(S)$. Therefore v must be in $\psi(S)$. ♦

Part 4

(7) ⇒ (8) G is cycle-connected ⇒ G is Hamiltonian, chordal, and confluent.

Proposition 6 A graph G is confluent iff it has no subgraph that is a subdivision of K_4.

Proposition 1 Let G be a chordal graph with cycle C. Any edge uv on C is part of a triangle {u, v, w} where w is some vertex on C.

Lemma 4.1 If G is a cycle-connected graph with two non-adjacent vertices u and v, there can be no more than two vertex-disjoint u-v paths.

Lemma 4.2 If G is a cycle-connected graph, it has no subgraph which is a subdivision of K_4.

Definition Let G be a graph with vertices u and v. A set of u-v paths is *vertex-disjoint* iff no two of the paths share a vertex other than u or v.

Proof of Theorem 1 Part 4

Let G be a cycle-connected graph.

G is chordal

Assume that G has a chordless cycle, C, of at least four vertices. Let a, b, c, d be vertices on C, in that order. Then {a, c} and {b, d} are independent sets of G. Also, for each of these sets, C is either minimal for that set, or there is a cycle C’ such that V(C’) is a proper subset of V(C). However, for such a cycle to exist, two vertices non-adjacent on

C would need to be adjacent in G. This contradicts the assumption that C is chordless. Therefore C must be minimal for both {a, c} and {b, d}. This is impossible by part two of the definition of cycle-connected.

G is Hamiltonian

Let Ω_0 be any independent set of G. Let C_0 be the cycle-subgraph corresponding to Ω_0. If C_0 is not Hamiltonian, then let v be any vertex not on C_0, but adjacent to some vertex x on C_0 (such a vertex must exist since G is connected). We will define a cycle-subgraph, C_1, which includes all the vertices of C_0 and v.

Since G is certainly 2-connected, v must be connected to C_0 independently of x. That is, there is some vertex $y \neq x$ in C_0, such that there is a path vPy which includes no vertices of C_0 other than y. Define paths xP_0y and xP_1y such that $C_0 = xP_0y + xP_1y$ and $|V(P_0)| \leq |V(P_1)|$, and let $xP_2y = xvPy$. (See figure 5.5). I will show by contradiction that P_0 is trivial.

By definition, P_1 and P_2 are certainly non-trivial, so assume P_0, P_1, and P_2 are all non-trivial. Note that P_0, P_1, and P_2 are vertex-disjoint, so by lemma 4.1, x and y must be adjacent in G. Assume that there is a path, aP'b, from some vertex a on P_0 to some vertex b on P_1 such that P' includes no vertices on xP_0y or xP_1y. Then the subgraph of G made up of xy, P_0, P_1, and P' is a subdivision of K_4 with x, y, a, and b corresponding to the vertices of K_4. This is illustrated in figure 5.5: a and b connect to x and y on P_0 and P_1 respectively, x and y are directly adjacent, and a and b are connected by P'. By lemma

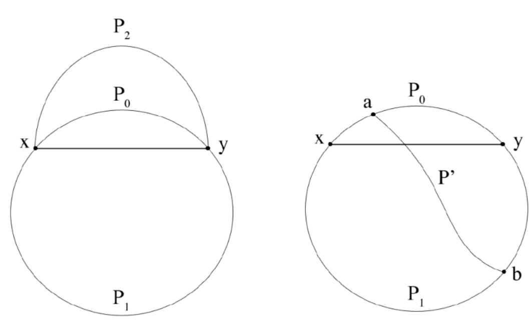

FIGURE 5.5: AN ILLUSTRATION FOR THE PROOF OF THEOREM 1 PART 4

4.2 G cannot have a subgraph which is a subdivision of K_4, so any path from a vertex on P_0 to a vertex on P_1 must go through x or y. This is true also of P_0 and P_2 or P_1 and P_2 by the same reasoning.

Now consider any vertices a, b, and c on P_0, P_1, and P_2 respectively. The set $\Omega = \{a, b, c\}$ is certainly independent, so by part 1 of the definition of cycle reducibility, there must be a cycle-subgraph, C, including these three vertices. Define paths aP_ab, bP_bc, and cP_ca such that $C = aP_ab + bP_bc + cP_ca$. From the reasoning above, either x or y must fall on each of P_a, P_b, and P_c. However, this is impossible, since a vertex cannot repeat on the cycle C. Therefore, by contradiction the path P_0 must be trivial.

Let the cycle $C_1 = xP_1yP_2^{-1}]$. Since P_0 is trivial, xP_1y includes all the vertices on C_0 while the vertex v is on P_2. Let Ω_1 be the independent set corresponding to the subgraph of C_1 by part (3) of the definition of cycle-connected. If C_1 is not a Hamiltonian cycle, then, repeating the same reasoning we can find a larger cycle, C_2. Since G is a finite graph, this process must end eventually with a Hamiltonian cycle.

G is confluent

By lemma 4.2, G has no subgraph which is a subdivision of K_4, so by proposition 6 G is confluent.

Lemma 4.1 If G is a cycle-connected graph with two non-adjacent vertices u and v, there can be no more than two vertex-disjoint u-v paths.

Proof of Lemma 4.1

Let uP_1v, uP_2v, and uP_3v be three vertex-disjoint paths from u to v. Then there are three cycles including u and v: $uP_1vP_2^{-1}]$, $uP_1vP_3^{-1}]$, and $uP_2vP_3^{-1}]$. Since u and v are non-adjacent they must have a minimal cycle-subgraph, C. By the definition of a minimal cycle-subgraph, $V(C) \subset V(uP_1vP_2^{-1}]) \cap V(uP_1vP_3^{-1}]) \cap V(uP_2vP_3^{-1}]) = \{u, v\}$. This is impossible, proving the lemma by contradiction. ♦

Lemma 4.2 If G is a cycle-connected graph, it has no subgraph which is a subdivision of K_4.

Proof of Lemma 4.2

Assume that G has a subgraph, H, which is a subdivision of K_4. Then there is a cycle passing through the four vertices of the subdivided K_4 in H, and by parts 1 and 3 of the definition of cycle-connected, at least one pair of these vertices must be independent in G. Let u, v be these non-adjacent vertices and let x, y be the other two vertices of H corresponding to the vertices of K_4, and let uPv be the u-v path corresponding to the u-v edge of K_4. Then there are 3 disjoint paths from u to v: uPv, the path through x, uP_xv, and the path through y, uP_yv. (See figure 5.6). This is impossible by lemma 4.1. ♦

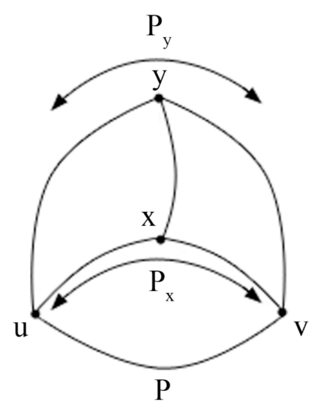

FIGURE 5.6: AN ILLUSTRATION FOR THE PROOF OF LEMMA 4.2

Part 5

(8) ⇒ (9) G is Hamiltonian, chordal, and confluent ⇒ G is maximal Hamiltonian-confluent.

Proof of Theorem 1 Part 5

Let u and v be non-adjacent vertices of G. Let C_0 be a Hamiltonian cycle of G.

Take any chord of C_0, x_0y_0. If x_0y_0 crosses u and v on C_0, then let $x = x_0$, $y = y_0$, and $C = C_0$. Otherwise, let C_1 be the cycle made up of the part of C_0 from x_0 to y_0

including u and v and the edge x_0y_0. C_1 cannot be a triangle, since uv is not an edge of G, so it has a chord, x_1y_1.

We can continue this reasoning until we arrive at an edge x_ny_n which crosses u and v on a cycle C_n. Then let $x = x_n$, $y = y_n$, and $C = C_n$. Figure 5.7 gives an example of this as an illustration.

Consider the graph G + uv. There are paths from u to x, x to v, v to y, and y to u along the cycle C which are vertex-disjoint. Furthermore, uv and xy are edges of G. Therefore C + uv + xy is a subdivision of K_4. By proposition 6, then, G + uv is not confluent. Since the choice of u and v was arbitrary, G is maximal Hamiltonian-confluent. ♦

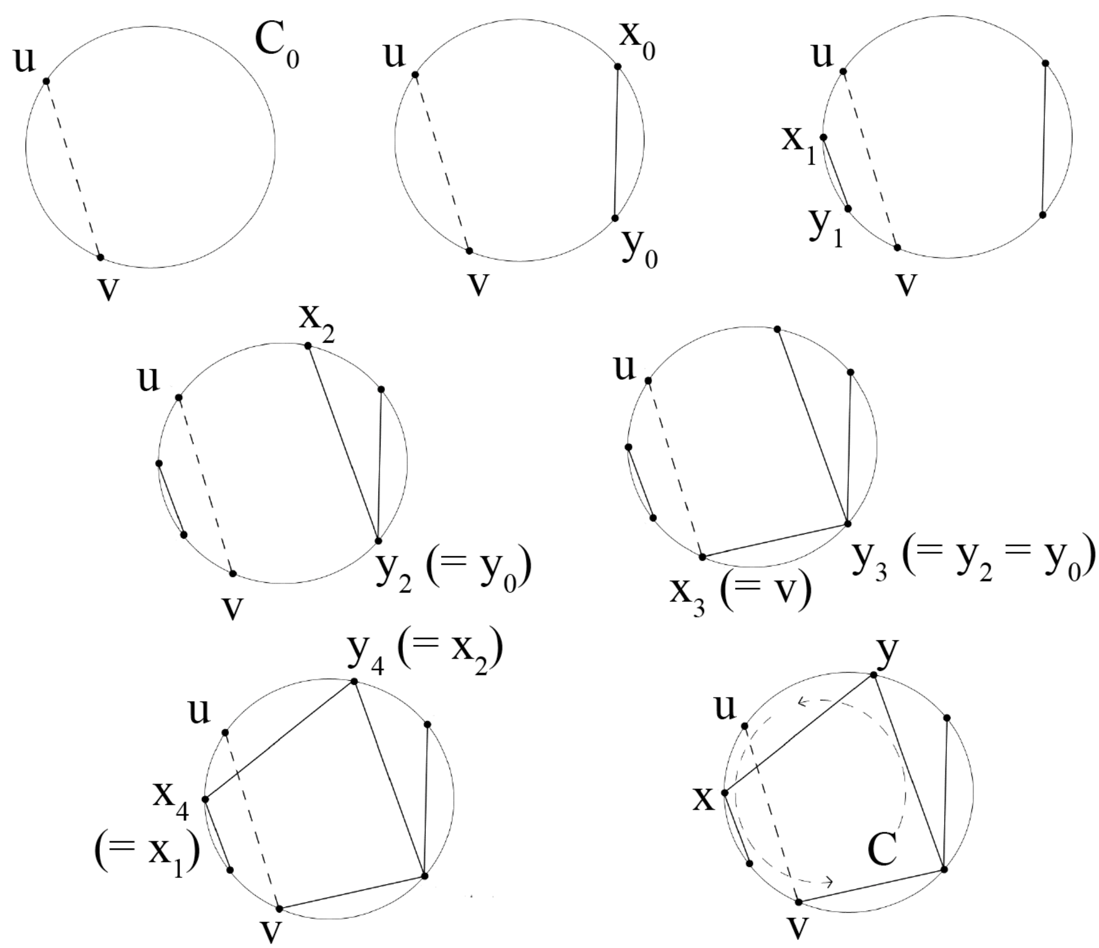

FIGURE 5.7: AN ILLUSTRATION FOR THE PROOF OF THEOREM 1 PART 5

Part 6

(9) ⇒ (3) G is maximal Hamiltonian-confluent ⇒ G is MOP(1).

Proof of Theorem 1 Part 6

Let G be a maximal Hamiltonian-confluent graph and assume G has a subgraph, H, which is a subdivion of $K_{2,3}$. Let u and v be the vertices in H corresponding to the independent pair of vertices of $K_{2,3}$ and x, y, z be the vertices of H corresponding to the independent triple of vertices of $K_{2,3}$. H can be thought of as three vertex-disjoint paths of G connecting u and v: uP_xv, uP_yv, and uP_zv.

If all paths in G from x to y, y to z, and z to x include either u or v, then there can be no Hamiltonian cycle in G. Therefore, without loss of generality, let x and y be connected by a path, xP'y, which includes neither u nor v nor any vertex on P_z. Let a be the vertex on P_x closest to y on P' (if there is no other vertex on both paths then a = x). Similarly let b be the vertex on P_y closest to a on the part of P' from a to y. Then there is a subpath of P', P, from a to b that includes no other members of P_x or P_y. Figure 5.8 illustrates this.

G then has a subgraph which is a subdivision of K_4 with the vertices a, b, u, and v corresponding to the vertices of K_4 as follows: u and v are connected by P_z, a and b are

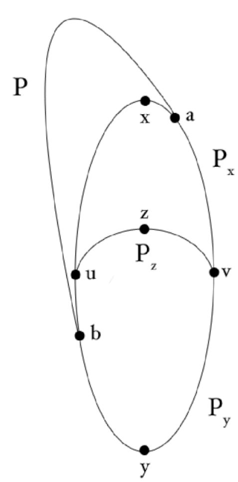

FIGURE 5.8: AN ILLUSTRATION FOR THE PROOF OF THEOREM 1 PART 6

connected by P, a is connected to u and v by P_x, and b is connected to u and v by P_y. By definition all of these paths are vertex-disjoint, so the graph H' = (H + aPb) is a subdivision of K_4, contradicting the assumption that G is confluent, by proposition 6.

This shows that G has no subgraph which is a subdivision of $K_{2,3}$. Since G also has no subgraph that is a subdivision of K_4 by proposition 6, G must be outerplanar by proposition 7.

Let u and v be any non-adjacent vertices in G. By maximality, (G + uv) is either non-Hamiltonian or non-confluent. But (G + uv) must be Hamiltonian since it has the same Hamiltonian cycle as G. Thus (G + uv) is non-confluent, and by proposition 6 it has a subgraph which is a subdivision of K_4. Therefore (G + uv) is not outerplanar by proposition 7. Since u and v are arbitrary vertices, G must be maximal outerplanar.♦

Part 7

(3) ⇒ (4) G is MOP(1) ⇒ G is MOP(2).

Proof of Theorem 1 Part 7

Let G be a MOP(1). G is Hamiltonian by proposition 9. Therefore G is HOP(1), and by proposition 10 G is HOP(2).

Let u and v be non-adjacent vertices of G. By maximality G + uv is not HOP(1), so by proposition 10 it is not HOP(2). Thus G is maximal HOP(2). ♦

Part 8

(4) ⇒ (5) G is MOP(2) ⇒ G is HOP(2) and chordal.

Proof of Theorem 1 Part 8

Let G be MOP(2). Let C be any non-trivial cycle of G and let v be the first vertex of C. The edges of C are either edges or chords of any Hamiltonian cycle of G. Therefore the vertices of C must be ordered in the same way as they are on some inversion or rotation of a Hamiltonian cycle, otherwise two edges of C will cross as chords of the Hamiltonian cycle. So we can choose a rotation and inversion of the Hamiltonian cycle, C_{ham}, that begins on v and has its vertices in the same order as on C.

Assume C is chordless and let u be a vertex of C not adjacent to v. Consider the graph G + vu. Assume by way of contradiction that C_{ham} has a chord xy which crosses vu in G + vu (with x preceding y on C_{ham}). Then, the order of vertices on C_{ham} is v…x…u…y…]. Assume x is not on C. Then x falls between some two adjacent vertices of C, a and b, on C_{ham}, where a = v or follows v on C (and C_{ham}) and b = u or precedes u on C and C_{ham}. So the order of vertices on C_{ham} is (v)…a…x…b…(u)…y…]. This is impossible because ab and xy would then be crosschords of C_{ham}. The same is true if y is not on C. However, x and y cannot both be vertices of C since C is chordless by assumption. Therefore no chord of C_{ham} crosses vu in G + vu.

Since all cycles of G must be ordered in the same way on any cycle as they are on the Hamiltonian cycle of G it is evident that crosschords of any cycle must be crosschords of C_{ham}. So G + vu is crosschord-free, contradicting the maximality of G, and proving by contradiction that no cycle of G is chordless, and G is chordal. ♦

Part 9

(5) ⇒ (11) G is HOP(2) and chordal ⇒ G is HOP-intrasymmetric.

Definition A class of graphs **C** is HOP-intrasymmetric iff

(1) Any graph G ∈ **C** is HOP(2).

(2) For any vertex, v, of a graph G ∈ **C**, and any integer n, $3 \le n \le |V(G)|$, there is an induced subgraph of G, H, on n vertices such that v ∈ V(H) and H ∈ **C**.

(3) For any induced subgraph H of any G ∈ **C**, if H is 2-connected, H ∈ **C**.

A graph G is *HOP-intrasymmetric* iff there exists some HOP-intrasymmetric class of graphs **C** such that G ∈ **C**.

Proposition 11 Let G be a chordal graph, let C be a non-trivial cycle of G, and let v be a vertex on C. Then C has a chord, ab, such that for some vertex u ≠ v, au and ub are in C.

Lemma 9.1 Any 2-connected induced subgraph of a HOP is Hamiltonian.

Proof of Theorem 1 Part 9

It suffices to show that the class of chordal HOPs is a HOP-intrasymmetric class. Let **C** be the class of chordal HOPs and let G $\in$ **C**. G obviously satisfies condition (1) of the definition.

If $|V(G)| = 3$, then (2) is trivially true. Let $|V(G)| = n$ and assume inductively that (2) is true for all graphs on fewer than n vertices.

Let C_{ham} be a Hamiltonian cycle of G and let v be any vertex of G. By proposition 11 there's a chord of C_{ham}, ab, such that for some vertex $u \neq v$, aub is part of C_{ham}. Let G' = G – u. G' is clearly chordal and crosschord-free, because any cycle of G' is also a cycle of G and has all the same chords. Furthermore, G' has a Hamiltonian cycle equivalent to C_{ham} but replacing aub with ab.

By the inductive premise, then, G' has an induced subgraph on n vertices containing v which is in **C**. This is also an induced subgraph of G, so (2) holds.

Let H be a 2-connected induced subgraph of G. By lemma 9.1 H is Hamiltonian. Furthermore, any subgraph of a crosschord-free graph is clearly crosschord-free. Finally, any cycle, C, of H is also a cycle of G, and because H is an induced subgraph, C must have the same chords in H as in G. So H is chordal. Therefore, by the inductive premise, H $\in$ **C**. ♦

Lemma 9.1 Any 2-connected induced subgraph of a HOP is Hamiltonian.

Proof of Lemma 9.1

Let G be a HOP, let C be a Hamiltonian cycle of G, and let H be a 2-connected subgraph of G. Let $V(H) = v_1, v_2, v_3, \ldots, v_n$ such that the order of vertices on C is $v_1 \ldots v_2 \ldots v_3 \ldots \ldots v_n \ldots]$. Assume that two successive vertices of H, v_i and v_{i+1}, are non-adjacent.

Because H is 2-connected, there must be a cycle of H, C', containing v_i and v_{i+1}. C' is also a cycle of G, so its vertices must be ordered in the same way on C' as they are on C (up to rotation or inversion), otherwise the edges of C' which are chords of C will cross. Since there are no vertices of H between v_i and v_{i+1} on C, v_i and v_{i+1} must be adjacent on C'. This contradicts the premise that they are non-adjacent.

Thus by contradiction, every pair of vertices between successive vertices of H are adjacent in H. Therefore $v_1v_2v_3 \ldots v_nv_1$ is a Hamiltonian cycle of H.

Part 10

(11) ⇒ (12) G is HOP-intrasymmetric ⇒ G is HC-intrasymmetric.

Definition A class of graphs **C** is Hamiltonian-confluent intrasymmetric iff

(1) Any graph $G \in$ **C** is Hamiltonian and confluent.

(2) For any vertex, v, of a graph $G \in$ **C**, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in$ **C**.

(3) For any induced subgraph H of any $G \in$ **C**, if H is 2-connected, $H \in$ **C**.

A graph G is *Hamiltonian-confluent intrasymmetric* iff there exists some Hamiltonian-confluent intrasymmetric class of graphs **C** such that $G \in$ **C**.

Proposition 6 A graph G is confluent iff it has no subgraph that is a subdivision of K_4.

Proposition 7 A graph G is outerplanar iff it has no subgraph that is a subdivision of K_4 or $K_{2,3}$.

Proposition 10 A graph is HOP(2) iff it is HOP(1).

Proof of Theorem 1 Part 10

Again, it is sufficient to show that the HOP-intrasymmetric graphs are a HC-intrasymmetric class. Let **C** be the class of HOP-intrasymmetric graphs. Let G be any graph in **C**.

By propositions 6 and 7 all outerplanar graphs are confluent (and G is outerplanar by proposition 10), so **C** certainly satisfisfies condition (1).

Let v be any vertex of G and n any integer between 3 and $|V(G)|$. By part (2) of the definition of a HOP-intrasymmetric class, there's an induced subgraph of G, H, including v which is HOP-intrasymmetric. Therefore $H \in \mathbf{C}$.

Let H be any 2-connected induced subgraph of G. By part (3) of the definition of a HOP-intrasymmetric class, H is HOP-intrasymmetric, so $H \in \mathbf{C}$. ♦

Part 11

(12) ⇒ (10) G is HC-intrasymmetric ⇒ G is H2-intrasymmetric.

Definition A class of graphs **C** is H2-intrasymmetric iff

(1) Any graph $G \in \mathbf{C}$ is Hamiltonian but not 3-connected.

(2) For any vertex, v, of a graph $G \in \mathbf{C}$, and any integer n, $3 \leq n \leq |V(G)|$, there is an induced subgraph of G, H, on n vertices such that $v \in V(H)$ and $H \in \mathbf{C}$.

(3) For any induced subgraph H of any $G \in \mathbf{C}$, if H is 2-connected, $H \in \mathbf{C}$.

A graph G is H*2-intrasymmetric* iff there exists some H2-intrasymmetric class of graphs **C** such that $G \in \mathbf{C}$.

Proposition 6 A graph G is confluent iff it has no subgraph that is a subdivision of K_4.

Proposition 8 If a graph is 3-connected then it has a subgraph that is a subdivision of K_4.

Proof of Theorem 1 Part 11

Here it suffices to show that the HC-intrasymmetric graphs are an H2-intrasymmetric class. Let **C** be the class of HC-intrasymmetric graphs. Let G be any graph in **C**.

G is Hamiltonian by part (1) of the definition of HC-intrasymmetric.

Assume G is 3-connected. Then by proposition 8 it has a subgraph which is a subdivision of K_4, and by proposition 6 it is not confluent, contradicting part (1) of the definition of an HC-intrasymmetric class. Therefore G cannot be 3-connected.

Let v be a vertex of G and n any integer. By part (2) of the definition of an HC-intrasymmetric class, G has an induced subgraph on n vertices containing v in **C**.

Let H be any 2-connected induced subgraph of G. By part (3) of the definition of an HC-intrasymmetric class, H is in **C**. ♦

Part 12

(10) ⇒ (1) G is H2-intrasymmetric ⇒ G is a unary 2-tree.

Proposition 1 Let G be a chordal graph with cycle C. Any edge uv on C is part of a triangle {u, v, w} where w is some vertex on C.

Proposition 6 A graph G is confluent iff it has no subgraph that is a subdivision of K_4.

Proposition 9 If G is maximal outerplanar then it has exactly one Hamiltonian cycle-subgraph, which consists of the edges on the outer face of any outerplanar emedding of G.

Proposition 11 Let G be a chordal graph, let C be a non-trivial cycle of G, and let uv be an edge of C. Then C has a chord, ab, such that for some vertex x ≠ u, v, ax and xb are in C.

Proposition 12 Let G be a 2-tree. Then G is a unary 2-tree iff G is Hamiltonian. Furthermore there's a unique Hamiltonian cycle-subgraph of G that consists of all edges whose endpoints are not a clique in the clique sequence defining G as a unary 2-tree.

Proof of Theorem 1 Part 12

Let G be a H2-intrasymmetric graph on n vertices. If n = 3 then G is a triangle and is a unary 2-tree. Assume inductively that all H2-intrasymmetric graphs on 3 to n – 1 vertices are unary 2-trees.

Let v be a vertex of G such that G – v is an H2-intrasymmetric subgraph of G (invoking part (2) of the definition of an H2-intrasymmetric class). Then G – v is a unary 2-tree by the inductive premise, and by parts one through four of the theorem G is chordal and confluent. By (2) again, v must be part of some triangle in G, {v, x, y}. I will prove that v is adjacent to only x and y in G for the following two possible cases.

Case 1: Let C' be a Hamiltonian cycle of G – v and assume x and y are adjacent in C'. By proposition 11, there's a triangle {a, u, b} where au and ub are edges of C' and u ≠ x, y. If u is connected to some other vertex, c, of G – v, then the subgraph consisting of the vertices a, b, c, u, the edges au, ub, ab, uc, and the cycle C', is a subdivision of K_4. (See figure 5.9). This is impossible, so u is a degree 2 vertex in G – v. Since G – v is a unary 2-tree, G – {u, v} is also a unary 2-tree (u cannot be part of a supporting clique in G – v since it is degree 2, so the vertex sequence of G – v with u removed is still well-formed).

Let H be a subgraph including all vertices and edges of G – u except any edges that may be incident on v other than xv and vy. Since G – {u, v} is a 2-tree, H is clearly a 2-tree whose vertex sequence is that of G – {u, v} followed by v. Let H' be the induced subgraph of G on the vertex set of H. Since a 2-tree always has exactly $2|V| - 3$ edges by construction and H is a subgraph of H', H' can only be a 2-tree if H' = H. But H' is certainly 2-connected, so by part (3) of the definition of an H2-intrasymmetric class and the inductive premise, H' must be a unary 2-tree and equal to H. Therefore v cannot be adjacent to any vertex other than x and y.

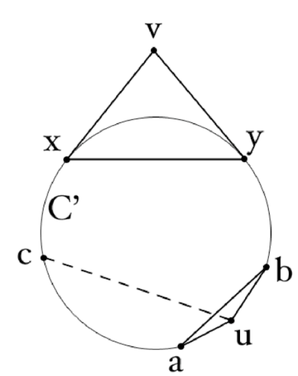

FIGURE 5.9: AN ILLUSTRATION FOR THE PROOF OF THEOREM 1 PART 12

Case 2: Assume x and y are not adjacent on the Hamiltonian cycle of G – v. Assume also by way of contradiction that v is adjacent to some vertex $z \neq x, y$. Let C' be the cycle consisting of xy and the part of the Hamiltonian cycle of G – v including z. Let H be the induced subgraph of G on the vertex set of C' plus v. H is certainly 2-connected, so by part (3) of the definition of an H2-intrasymmetric class and the inductive premise, H must be a unary 2-tree. Then by parts one through four of theorem 1, H is chordal and confluent. However, H has a subgraph which is a subdivision of K_4 consisting of the vertices v, x, y, z, the edges vx, xy, xy, vz, and the cycle C'. This contradicts the confluence of G by proposition 6, proving that v is adjacent to no vertex other than x and y.

This proves that v is a degree 2 vertex and part of the triangle {x, v, y}. By assumption, G is Hamiltonian, so the Hamiltonian cycle of G, C_{ham}, must include edges xv and vy. Therefore G – v has a Hamiltonian cycle that is the same as C_{ham} but replaces xvy with xy. Since G – v is a unary 2-tree, the subgraph of this Hamiltonian cycle is unique by proposition 12. (This means that case 2 above is actually impossible.) Let C' be a Hamiltonian cycle of G – v.

Let $Q_4, Q_5, \ldots, Q_{n-1}$ be a clique sequence and let $v_1, v_2, \ldots, v_{n-1}$ be a vertex sequence for G – v defined as a unary 2-tree. By proposition 12 again, $Q_i \neq \{x, y\}$ for any integer i, since x and y are adjacent on C'. Therefore if $Q_n = \{x, y\}$ and $v_n = v$ then $Q_4, Q_5, \ldots, Q_{n-1}, Q_n$ and $v_1, v_2, \ldots, v_{n-1}, v_n$ are a vertex and a clique sequence defining G as a unary 2-tree. ♦

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APPENDIX: PROOFS OF PROPOSITIONS

Proposition 1 Let G be a chordal graph with cycle C. Any edge uv on C is part of a triangle {u, v, w} where w is some vertex on C.

Proof If $|V(C)| = 3$, then let {u, v, w} = V(C). Otherwise let x_0y_0 be a chord of C. Then let C_0 be a rotation of C beginning with x_0 and define paths such that $C_0 = x_0Py_0P']$. Then there are two cycles $x_0Py_0]$ and $y_0P'x_0]$. Let C_1 be the cycle of these which includes uv. Repeat this process defining cycles $C_1, C_2, \ldots, C_n$ such that $|V(C_n)| = 3$. Note that any C_i must be strictly smaller than C_{i+1}, and must contain the vertices u,v, so the triangle C_n certainly exists and includes the vertices u and v. Let $\{u, v, w\} = C_n$. ♦

Proposition 2 Let G be a Hamiltonian chordal graph. Then G has no maximal 2-cliques.

Proof Let uv be any edge of G. If uv is part of a Hamiltonian cycle then by proposition 1, uv is part of a triangle. If uv is not part of a Hamiltonian cycle, then it's part of a cycle by putting uv together with part of the Hamiltonian cycle from v to u. By proposition 1, again, uv is part of a triangle. Therefore, {u, v} cannot be a maximal 2-clique. ♦

Proposition 3 (Blair and Peyton (1993)) A graph is chordal iff it has at least one clique tree.

Proposition 4 (Blair and Peyton (1993)) Let v be any vertex of a graph G with clique tree T, and let **K**(v) be the set of maximal cliques of G containing v. Then the vertices of T corresponding to **K**(v) induce a subtree of T, T^v.

Proposition 5 Let T be a clique tree of a chordal graph G, and let k_T be a leaf of any subtree S of T. Then K contains a vertex not contained in any other clique corresponding to a vertex of S.

Proof Let k'_T be the vertex of S adjacent to k_T. By maximality of cliques, K has at least one vertex, x, not in K'. For any clique Q in G such that $q_T \in S$, let $k_T P q_T$ be a path in T. By the definition of a clique tree and the fact that $k_T P q_T$ must include k'_T, Q cannot contain x. ♦

Proposition 6 A graph G is confluent iff it has no subgraph that is a subdivision of K_4. (Duffin 1965)

Proposition 7 A graph G is outerplanar iff it has no subgraph that is a subdivision of K_4 or $K_{2,3}$.

This proposition is essentially a version of Kurtowski's theorem. See Harary (1969) or Chartrand and Lesniak (2005).

Proposition 8 If a graph is 3-connected then it has a subgraph that is a subdivision of K_4.

Definition An *n-wheel* is a graph constructed of a cycle (the *outer cycle*) of n vertices, plus a vertex (the *center*) adjacent to each vertex on the cycle. (See figure A.1)

Proof We will need the following lemma from Tutte (1961):

Lemma (Tutte) Let G be a 3-connected graph. Then either G is a wheel or there is a sequence of graphs $G_0, G_1, \ldots, G_n$ where G_0 is a wheel, $G = G_n$, and for each i ($1 \le i \le n$), G_i can be obtained from G_{i-1} by one of the following operations:

(1) Add an edge between two non-adjacent vertices of G_{i-1}.

(2) "Split" a vertex v: let v be a vertex of G_{i-1} of degree at least 4. Partition the vertices adjacent to v into sets V_1 and V_2, each with at least 2 members. Define new vertices v_1 and v_2 and let $V(G_i) = V(G_{i-1}) - v + v_1 + v_2$. Let $E(G_i)$ be $E(G_{i-1} - v)$ plus edges between v_1 and each member of V_1 and between v_2 and each member of V_2.

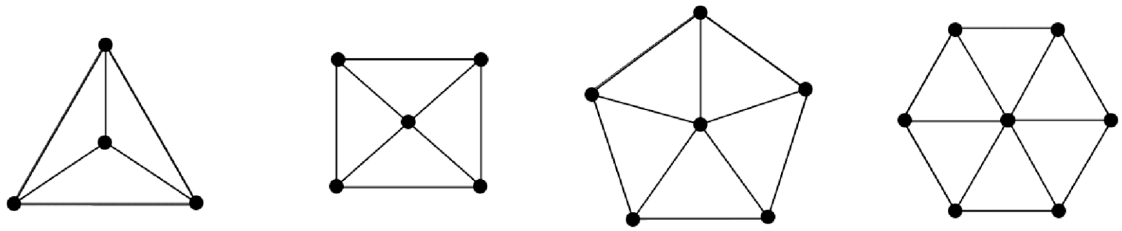

FIGURE A.1: A 3-, 4-, 5-, AND 6-WHEEL

Let W be any wheel. Let c be the center and C the outer cycle(-subgraph) of W. Let x, y, and z be vertices adjacent in C. Let H be the subgraph on V(G) including the edges of C, cx, cy, and cz and no others. Then H is a subdivision of K_4 with the vertices c, x, y, z corresponding to the vertices of K_4.

Assume inductively that all 3-connected graphs obtained by up to n – 1 operations on W have a subgraph that's a subdivision of K_4. Let G be any 3-connected graph constructed from W and let G' be the graph preceding G in the construction. By assumption G' has a subgraph H that is a subdivision of K_4. If G' and G are related by operation (1), then H is also a subgraph of G. So let G' and G be related by operation (2) with $v \in V(G')$ split into v_1 and v_2 of V(G). There are 3 possible cases:

Case 1: If v is not in V(H), then H is also a subgraph of G.

Case 2: Let v be a vertex of H that is not a vertex of the subdivided K_4. Then v must be degree 2 in H; let x and y be the vertices adjacent to v in H. If $xv_1, yv_1 \in G$ or $xv_2, yv_2 \in G$ then G has a subgraph isomorphic to H, replacing v with v_1 or v_2. Otherwise $xv_1, yv_2 \in G$ or $xv_2, yv_1 \in G$. Assume the former without loss of generality. Let H' be a subgraph of G that is equivalent to H except that v is replaced by v_1 and v_2, and the edges xv and vy are replaced by xv_1, v_1v_2, and v_2y. Then H' is a subdivision of H, and hence is isomorphic to a subdivision of K_4.

Case 3: Let v be a vertex of H that is a vertex of the subdivided K_4. Then v is degree 3 in H. Let x, y, and z be adjacent to v in H. If x, y, and z are all adjacent to v_1 or all adjacent to v_2 in G, then G has a subgraph isomorphic to H, replacing v with v_1 or v_2. Otherwise G has two of x, y, and z adjacent to v_1 (or v_2) and the remaining one of x, y, and z adjacent to v_2 (or v_1). The argument is the same in all six cases, so assume x and y are adjacent to v_1 and z is adjacent to v_2. Then let H' be a graph equivalent to H but replacing v with v_1 and v_2, and replacing the edges vx, vy, and vz with v_1x, v_1y, v_1v_2, and v_2z. Thus H' is isomorphic to a subdivision of H (with v_1 corresponding to v and v_2 being the added vertex), and is a subdivision of K_4.

Thus in all cases G has a subgraph which is a subdivision of K_4. ♦

Proposition 9 If G is maximal outerplanar then it has exactly one Hamiltonian cycle-subgraph, which consists of the edges on the outer face of any outerplanar embedding of G.

Proof Let G be a maximal outerplanar graph.

Definition A *circuit* of a graph G is a sequence of vertices, $v_1v_2v_3 \ldots v_n]$, such that each $v_{i-1}v_i$ is an edge of G and v_nv_1 is an edge of G. In other words, a circuit is a cycle which is allowed to have repeating vertices.

G has a Hamiltonian cycle.

Let G be a MOP and E be an outerplanar embedding of G. Let W be a circuit that traces the perimeter of the outer face of G. W must include all vertices of G by outerplanarity. If W includes each vertex of G exactly once (i. e. exactly two edges of G are incident on each vertex of G) then W must be a Hamiltonian cycle.

Assume by way of contradiction then that W includes some vertex, v, twice. Let $W = vx_1x_2 \ldots x_mvy_1y_2 \ldots y_n]$. Then $X = vx_1x_2 \ldots x_m]$ and $Y = vy_1y_2 \ldots y_n]$ are separate circuits making up the outer perimeter of G such that there is no edge x_iy_j in G for any i, j (otherwise, x_iy_j would be on the outer perimeter).

Now let $G' = G + x_my_1$. Then G' has an embedding ${}_3E$ equivalent to E except for the edge x_my_1 which is drawn so that it crosses no other edges in ${}_3E$ (it may be drawn arbitrarily close to the edges x_mv and vy_1, which are not crossed in E). ${}_3E$ is clearly outerplanar, since it has a circuit $W' = vx_1x_2 \ldots x_my_1y_2 \ldots y_n]$ which traces the outer perimeter of ${}_3E$ and includes all vertices of G'. Therefore G' is outerplanar, and G is not maximal outerplanar. By contradiction, W must be a Hamiltonian cycle for G.

W is a unique Hamiltonian cycle for G.

Assume that G has some distinct Hamiltonian cycle-subgraph, C. Then C must include some edge, uv, that is a chord of W. Let P_{w1} and P_{w2} be non-trivial paths such that $W = uP_{W1}vP_{W2}]$. Let P_C be a path such that $C = uvP_C]$. Since W and C are Hamiltonian, $V(vP_Cu) = V(uP_{W1}v) + V(vP_{W2}u) = V(G)$. Therefore there must be at least one edge in P_C with one endpoint in $V(P_{W1})$ and the other in $V(P_{W2})$. This edge will cross uv in any outerplanar embedding with W as the outer perimeter, contradicting the assumptions. ♦

Proposition 10 A graph is HOP(2) iff it is HOP(1).

HOP(2) ⇒ HOP(1): Let G be HOP(2) with Hamiltonian cycle C. Define an embedding, E, of G as follows: draw an n-gon and label its vertices with the vertices of G in the order of C. Draw each remaining edge of G as a straight line between its incident vertices. If any of these edges cross in E, then they must be crosschords of C, which is impossible by assumption. Therefore E is an outerplanar embedding and G is Hamiltonian outerplanar.

HOP(1) ⇒ HOP(2): Let G be Hamiltonian outerplanar with outerplanar embedding E. Let C be any cycle of G. Since the edges of C cannot cross in E, C must define a region R of the plane in E. Let uv be a chord of C. Since E is outerplanar, the curve corresponding to uv in E must be within R (otherwise vertices between u and v on C wouldn't be on the outer face of E). Since the curve corresponding to uv must be continuous, it divides R into two regions, R_1 and R_2. If there's a chord of C that is a crosschord with uv, then its curve also cannot be drawn outside of R and must go from R_1 to R_2, which it cannot do without crossing the curve for uv. Therefore G cannot have crosschords, and it HOP(2) ♦

Proposition 11 Let G be a chordal graph, let C be a non-trivial cycle of G, and let uv be an edge of C. Then C has a chord, ab, such that for some vertex $x \neq u, v$, ax and xb are in C.

Proof Let G be a chordal graph, let C be a cycle of G of four or more vertices, and let v be a vertex on C. Then C has a chord, a_0b_0. Define paths such that $a_0P_0b_0P'_0]$ is a rotation of C. Let C_1 be the smaller cycle, $a_0P_0b_0]$ or $b_0P'_0a_0]$, which does not include the edge uv.

If C_1 is not a 3-cycle, then let a_1b_1 be a chord of C_1. Define paths such that $a_1P_1b_1P'_1]$ is a rotation of C_1. Let C_2 be the cycle, $a_1P_1b_1]$ or $b_1P'_1a_1]$, which does not contain the edge a_0b_0.

This process can be continued, defining C_3, C_4, and so on. At each stage, the vertices of C_{i+1} are a proper subset of those of C_i, so there must be some C_n such that $|C_n| = 3$. Let $a = a_n$, $b = b_n$, and let x be the remaining vertex of C_n.

Note that the only edge of C_1 that is not also an edge of C is b_0a_0. Furthermore, C_2 is defined so that it cannot contain a_0b_0, so the only edge of C_2 that is not also an edge of C is b_1a_1. This is true of each cycle defined, so edges ax and xb on C_n are also edges of C. Furthermore, only one of u or v can be on C_1 (and hence on C_n), and if u or v is on C_1 it can only be on C_i if it is equal to a_i or b_i. Thus if u or v is on C_n it is equal to a_n or b_n and cannot be equal to x. ♦

Proposition 12 Let G be a 2-tree. Then G is a unary 2-tree iff G is Hamiltonian. Furthermore there's a unique Hamiltonian cycle-subgraph of G that consists of all edges that aren't in the clique sequence defining G as a unary 2-tree.

Let G be a 2-tree on n vertices. If n = 3 then the proposition certainly holds. Assume inductively that the proposition holds for all 2-trees on n – 1 vertices. In particular, by this assumption the proposition holds for $G_{n-1} = G - v_n$.

Unary ⇒ Hamiltonian: Let G be defined as a unary 2-tree with vertex sequence $\mathbf{V}_G = v_1, v_2, v_3, \ldots, v_n$ and clique sequence $\mathbf{Q}_G = Q_4, Q_5, \ldots, Q_n$. Then the graph G_{n-1} is a unary 2-tree with vertex sequence $\mathbf{V}_{G-vn} = v_1, v_2, v_3, \ldots, v_{n-1}$ and clique sequence $\mathbf{Q}_{G(n-1)} = Q_4, Q_5, \ldots, Q_{n-1}$, and by the inductive assumption G_{n-1} has a Hamiltonian cycle consisting of all the edges whose endpoints are not one of the cliques in $\mathbf{Q}_{G(n-1)}$. Since G is a unary 2-tree, Q_n is distinct from $Q_4, Q_5, \ldots, Q_{n-1}$ and the vertices of Q_n are adjacent in the Hamiltonian cycle of G_{n-1}. Let C' be this Hamiltonian cycle and let $\{x, y\} = Q_n$. Thus G has a Hamiltonian cycle, C, the same as C' but replacing the edge xy with xv_n and v_ny.

Hamiltonian ⇒ unary: Let G be a 2-tree with vertex sequence $\mathbf{V}_G = v_1, v_2, v_3, \ldots, v_n$ and clique sequence $\mathbf{Q}_G = Q_4, Q_5, \ldots, Q_n$ and a Hamiltonian cycle C. Then G_{n-1} is a 2-tree with clique sequence $\mathbf{Q}_{G(n-1)} = Q_4, Q_5, \ldots, Q_{n-1}$.

Let $\{x, y\} = Q_n$. Then the vertex v_n is degree 2 in G and adjacent to vertices x and y. Therefore C must include xv_n and v_ny, and G_{n-1} has a Hamiltonian cycle, C', the same as C but replacing xv_n and v_ny with xy. By the inductive premise C' includes all edges of $G - v_n$ whose endpoints are not one of the cliques in $\mathbf{Q}_{G(n-1)}$, so Q_n, which is on C', must be distinct from $Q_4, Q_5, \ldots, Q_{n-1}$. Furthermore all of $Q_4, Q_5, \ldots, Q_{n-1}$, are distinct by the inductive assumption that G_{n-1} is a unary 2-tree. Therefore G is a unary 2-tree.

C includes all edges of G that aren't in the clique sequence $\boldsymbol{Q}_G$: By the inductive premise, C' includes all edges of G_{n-1} other than those in $\mathbf{Q}_{G(n-1)}$. Only the clique $Q_n = \{x, y\}$ is an edge of C'. Because v_n is degree 2, xv_n and v_ny are the only edges in G not also in G_{n-1}. Thus the only edges of G whose endpoints are not one of the cliques of $\mathbf{Q}_G$ are all edges of C' excluding xy plus the edges xv_n and v_ny. This is precisely the edge set of C.

The subgraph of C is unique: Assume G has a Hamiltonian cycle-subgraph D distinct from the subgraph of C. D must also include xv_n and v_ny, so we can define a Hamiltonian cycle-subgraph D' for G_{n-1} replacing xv_n and v_ny with xy in D. However, by the inductive premise the subgraph of C' is unique, so C' = D'. By the definitions of C' and D' then C must be equal to D contradicting the assumption. Therefore the subgraph of C is a unique Hamiltonian cycle-subgraph for G. ♦

VITA

Jason Yust was born in St. Louis, Missouri. In 2001 he earned a BA in Music from Brown University. In 2006 he earned a PhD in Music Theory at the University of Washington and married his lovely wife, Kelly Cannon, whose enduring patience with his late-night music theorizing is inspiring. He has been known to heartily play the Irish tenor banjo and wooden flute and Balinese *suling* of various sizes and shapes.