Wreaths for Rahn, and Valuable Exchanges

The years when I studied with John Rahn, 2001–6, were, in retrospect, an inflection point in mathematical music theory. There was a sort of Cambrian explosion with the breaking down of a geographic barrier (the Atlantic Ocean), leading in short order to the founding of an international society and journal, the Society for Mathematics and Computation in Music and the Journal of Mathematics and Music, whose flagstone was the mission of intercontinental dialogue. At around the same time we lost, far too soon, two of the most important American pioneers in mathematical music theory both John Clough and David Lewin died in 2003. The daunting task of honoring their legacy was, in its own way, an important impetus that accelerated and guided subsequent developments in the field.

I would be putting myself in good company to say that discovering Lewin's work was a defining moment in my own intellectual development, and it was Rahn's guidance into this universe of ideas. Early in my studies at University of Washington, he introduced me to the whole idea of groups in music theory; not only Lewin, but other authors, including many gems of mathematical music theory published in the pages of *Perspectives*. When I studied serialism with him, he pointed me towards Mead's (1988) excellent exegesis of the twelve-tone system and an intriguing paper by Stanfield (1984) about the exchange operation (which exchanges the pitch-class numbers and order numbers of a row). Around the same time he brought Michael Leyton to the UW to give a lecture, and was investigating the application of Leyton's mathematical theories of shape to music theory. Leyton (2001) showed how symmetry might be an essential part of the description of the form of an object, even though the object itself might not be literally symmetrical. His memorable analogy is a smashed soda can on the floor of the subway station: one conceives this shape by imagining some ideal symmetrical shape, a cylinder, and applying some deformations to it. The asymmetry of the smashed can encodes a process by which its shape came into being. Leyton's basic mathematical tool was a group-theoretic construction called the wreath product. For Rahn's take on Leyton, see Rahn 2003, 18–25.

Rahn introduced me to Leyton as I was learning about another wreath-product group, Julian Hook's UTT (Uniform Triadic Transformation) group. I first encountered Hook's work at a special session of the American Mathematical Society Spring Sectional in Baton Rouge that John generously brought me to in 2003, but most will know it from Hook 2002. The application of this twelve-tone music, a natural extension, is explored in an excellent paper by Hook and Douthett (2008). Since then, some excellent work extending these that has been supported by the Society of Mathematics and Computation in Music (Fiore, Noll, Satyendra 2013a) and the *Journal of Mathematics and Music* (Fiore, Noll, Satyendra 2013) both of which Rahn was instrumental in helping to get off the ground. (See also Fiore and Noll 2016.)

One of my fond memories of graduate school was the "aha!" moment I had in this serialism seminar. While inventing symmetrical tone rows at the little upright piano in the dungeonous theory TA office in the School of Music (according to the lettering on the door it was actually the "Sprinkler Supply Valve Room") I realized that rotationally symmetrical rows, those that map onto themselves by some combination of rotation of order positions and transposition, could actually be described by a wreath product group! (N.B.: the exclamation point is not for you, dear reader, but for my 25-year-old self.)

Here's how it works: Assume your row has some symmetry such as T_4r_4, where "r_4" means to rotate the order positions ahead by 4 places. This is satisfied precisely if the four augmented triads, the *orbits* of T_4, are assigned to the orbits of r_4, order positions {0, 4, 8}, {1, 5, 9}, et c. The augmented triads can be assigned in any permutation, and they can start from any of the three members, as long as they go in ascending order.

This observation is sufficient to count the T_4r_4-symmetrical rows, but the transformational idea goes a step further and puts a structure around these musical entities by defining a group that acts upon them. The musical objects do not dictate precisely how they may be acted upon—witness the difference between the PLR group and T_nI group, both of which act in a simply transitive fashion over the 24 major and minor triads. Actions that preserve T_4r_4 symmetry include transpositions of each of the r_4 orbits by 4 or 8, and permutations of the four augmented triads between the four r_4 orbits. In fact, we can relate any two T_4r_4-symmetrical rows with some combination of these two kinds of operation, which means that they generate a group that acts *transitively* over this set of rows. The group is a nice example of a wreath product. It contains four copies of a cyclic group of order 3 ($\mathbf{Z}_3$), one for transposing order positions {0, 4, 8}, one for transposing order positions {1, 5, 9}, and so on. Notice that each of these operate independently of one another, so altogether they are a *direct product* of cyclic groups ($\mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3 \times \mathbf{Z}_3$). The group that permutes the augmented triads is called a "symmetric group" or S_4: it contains all possible permutations of four things, and has 24 (= 4!) elements. It does *not* operate independently of the direct product of cyclic groups, because it changes which augmented triad they operate upon. In other words, the two operations do not *commute*: the order in which they occur makes a difference. Say we begin from the trivial row:

0 1 2 3 4 5 6 7 8 9 t e

and transpose orbits 2 and 3:

0 1 6 7 4 5 t e 8 9 2 3

then swap orbits 1 and 2:

0 6 1 7 4 t 5 e 8 2 9 3

If we instead swap orbits 1 and 2 first:

0 2 1 3 4 6 5 7 8 t 9 e

then transpose orbits 2 and 3:

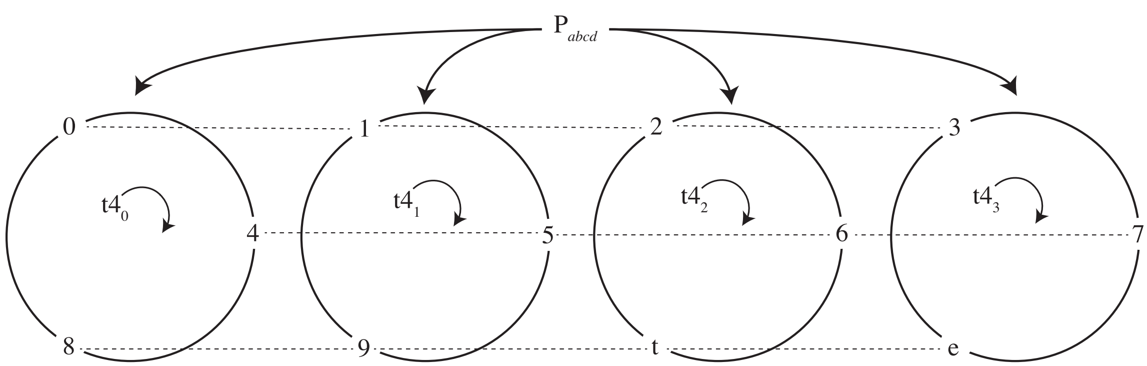
0 2 5 7 4 6 9 e 8 t 1 3

the result is different. Let us represent the order positions of the row with a pair of numbers (x, y), where x gives the tetrachordal segment as 0, 4, or 8, and y gives the position in the tetrachordal segment as 0, 1, 2, or 3. The order position can then be recovered as $x + y$. Now define operations $r4_0$, $r4_1$, $r4_2$, and $r4_3$ where $r4_n$ transposes the pcs in (x, n) by 4 for all x:

$$\mathrm{r4}_n(x,y) = \begin{cases} ((x+4)_{mod12}, y) \text{ if } y = n \\ (x,y) \text{ otherwise} \end{cases}$$

Example 1 illustrates how these work: each circle on the diagram can be rotated independently by one of the $\mathrm{r4}_n$s, and each of these generates a cyclic group of order 3. A direct product of these four cyclic groups, indexed by the set of integers {0, 1, 2, 3}, then operates on the $\mathrm{T}_4\mathrm{r}_4$-symmetric rows, but only relates those with a fixed order to the augmented triads. We then define a group of permutations, S_4, that acts upon the set {0, 1, 2, 3}. For each order position in the row $\{x, y\}$, the elements of S_4 permute the ys. Let us write P_{abcd} for the permutation $\{a \to 0, b \to 1, c \to 2, d \to 3\}$ (where $0 \leq a, b, c, d \leq 3$ and none of a, b, c, or d are equal). This is the inverse of the usual notation, but it will produce more intuitive results below. We define an action on the ordered pairs (x, y):

$$\mathrm{P}_{abcd}(x,y) = (x, \mathrm{P}_{abcd}(y))$$



Example 1: A visualization of the wreath product on $\mathrm{T}_4\mathrm{r}_4$-symmetric rows. The resulting row can be read left-to-right from top to bottom.

In Example 1, P_{abcd} corresponds to a shuffling of the circles. The permutations also operate on the indices of the $\mathrm{r4}_n$s, which is essential for describing how these operations combine:

$$\mathrm{P}_{abcd} \circ \mathrm{r4}_n(x,y) = \begin{cases} ((x+4)_{\mathrm{mod12}}, \mathrm{P}_{abcd}(y)) \text{ if } y = n \\ (x, \mathrm{P}_{abcd}(y)) \text{ otherwise} \end{cases}$$

But:

$$\mathrm{r4}_n \circ \mathrm{P}_{abcd}(x,y) = \begin{cases} ((x+4)_{\mathrm{mod12}}, \mathrm{P}_{abcd}(y)) \text{ if } \mathrm{P}_{abcd}(y) = n \\ (x, \mathrm{P}_{abcd}(y)) \text{ otherwise} \end{cases}$$

Which explains why the two kinds of operation do not commute (the order of application matters). This can be succinctly explained by saying that S_4 acts upon the $\mathrm{r4}_n$s by *conjugation*:

$$\mathrm{P}_{abcd}^{-1} \circ \mathrm{r4}_n \circ \mathrm{P}_{abcd}(x,y) = \begin{cases} ((x+4)_{\mathrm{mod12}}, y) \text{ if } \mathrm{P}_{abcd}(y) = n \\ (x,y) \text{ otherwise} \end{cases}$$

Note how multiplying by inverse permutations on either side cancels out the effect on *y*, but rearranges which orbit each $r4_n$ affects.

These are the general components of a wreath product: one group (the cyclic group of order three) operates some set X (= {0,4,8}) and the other (S_4) operates on a set Y (= {0,1,2,3}). The first group is copied $|Y|$ times and indexed with the elements of Y (= $\langle r4_0\rangle \times \langle r4_1\rangle \times \langle r4_2\rangle \times \langle r4_3\rangle$). This direct product is called the *base group* while the one acting on Y is called the *control group*. The control group acts upon the base group by permuting its factors via its action on the indices.[1]

This is a nice little machine; the only problem is that rotationally symmetrical twelve-tone rows turn out to be of virtually no interest to composers. "Among the virtues of any tool should be listed the virtues of the nails it hits" (Rahn 2007, 8). A survey of serial pieces turns up next to nothing based on rotationally symmetrical series.[2] The reason for this is perhaps self-evident: they are simply too symmetrical. A row that repeats the same intervallic pattern over and over may be tiresome and uninspiring, maybe lacking some essential position-finding features that make it cohere as an object and prevent it from disintegrating into its constituent trichords, tetrachords, or hexachords. Although I thought I had found some pretty pleasing tone rows in the Sprinkler Supply Valve Room, I did not manage to find any composers that shared my enthusiasm for them. The wreath product is an enticing mathematical construction, but it is also a mathematical solution in search of a musical problem.

But wait! (Here is where I_{2017} come to my_{2004} rescue.) Surely derived series, those that partition into a single trichord or tetrachord type (or even dyad-type), are quite important to many serialist composers, not least Webern and Babbitt, and the rotationally symmetrical rows are a kind of derived series. In fact, a derived series relying on a single T_n-type (rather than a T_nI-type, as with some familiar trichordally derived and hexachordally combinatorial series) can be made into a rotationally symmetrical one by permuting its individual trichords or tetrachords. Perhaps instead of applying S_4 uniformly across the r_4 orbits, we could make a direct product of three S_4s, one operating on each of the tetrachords independently. The only problem, then, is that the $r4_n$s, as defined above, will not necessarily preserve the tetrachordal-combinatoriality property, since they are defined to operate on fixed order positions that are no longer required to line up with the T_4-orbits. So instead, we need an operation that acts on the T_4-orbits. Define a two-place notation for pitch-classes, (x, y), like the one for order positions above, where $x \in \{0, 4, 8\}$ and $y \in \{0, 1, 2, 3\}$, and:

[1] For a fuller treatment, see Robinson 1993, which starts from group theory fundamentals and gets to wreath products fairly quickly (32–33). The definition here closely parallels Robinson's. For more advanced material on wreath products, see Meldrum 1995. Note that, like Robinson and Meldrum, I define a general wreath product by specifying a set and an action of the control group. Some sources define a "standard" or "regular" wreath product where the control group is assumed to act upon itself, but that will not work for all of the applications here. Note also that the distinction between "restricted" and "unrestricted" wreath products only comes into play when infinite groups are involved, so is irrelevant here.
[2] The few exceptions are pieces that are only marginally related to the serial tradition, such as the first movement of Lutoslawski's *Musique Funèbre*.

$$t4_n(x,y) = \begin{cases} ((x+4)_{mod12}, y) \text{ if } y = n \\ (x,y) \text{ otherwise} \end{cases}$$

These two groups of operations do commute, because one acts on order positions and one on pitch-classes. Therefore, they generate a direct product, $(\mathbf{Z}_3)^4 \times (S_4)^3$. It has a simply transitive action on the tetrachordally derived rows up to retrograde, since the tetrachords are related successively by T_4, and retrograde will produce a row that relates successive tetrachords by T_8, the other possibility. The "simply transitive" action means that there is exactly one element of the group to take any row of the given type to any other. This implies that we could bijectively map the group onto the rows by having them operate on the trivial row. The group has $3^4 \times 24^3 = 81 \times 13{,}824 = 1{,}119{,}744$ elements, so the total number of tetrachordally derived rows is twice that, 2,239,488. The total number of orderings of the aggregate is 12! = 479,001,600, so the tetrachordally derived rows make up 0.468%—not an especially common property, despite the large number of possibilities. Both numbers are perhaps misleadingly high, however, because they do not take into account the commonly presumed transpositional and inversional equivalences.

Simply transitive actions appear frequently in music theory, and they are nice in a way, but also misleading, a kind of group-theoretic lotus fruit that lulls the unwary into forgetting the difference between the operations and the set being acted upon. For instance, the T_nI group is simply transitive on non-symmetrical set classes, and the PLR group is simply transitive on major and minor triads. With the PLR group, any pair of triads implies a unique transformation because of the simply transitive property. If PLR(C major) = F minor and RLP(C major) = F minor, we can confidently assert that PLR = RLP. However, we might also want to say that T_0I(C major) = F minor, and it is not true that PLR = T_0I. The T_0I operation has no equivalent in the PLR group, and including it generates a group with many more than 24 elements. A similar point might be made about applying the standard T_nI operation to row forms, and applying a "Stravinsky inversion," an inversion that stabilizes the first note of the row. Recognizing either one individually gives a simply transitive group of operations on row forms, but recognizing both possibilities generates a larger group with multiple non-equivalent ways to get from one row form to another. The surjective, non-injective, mapping from group to set in this kind of situation leads to the question of multivalent transformational networks, broached by Rahn (2004, 142–3) and more fully developed as "polysemic networks" in Rahn 2007, and to commutativity in Lewin's (1983, 2007) transformational networks, also investigated by Rahn (2007), and by Hook (2007) as the "path consistency" condition.

In our example of the order-1,119,744 element group with its simply transitive action, the lack of T_n operations in the group might soon start to chafe a bit, since this is a fundamental operation of serialism. For instance, within the simply transitive group we can now derive the row of Webern's String Quartet (Op. 28) from the trivial row, writing the permutations on tetrachord *n* as $P_{abcd}(n)$ ($n \in \{0, 4, 8\}$).

$$0123\ 4567\ 89te \xrightarrow{t4_3^2} 012e\ 4563\ 89t7$$

$$\xrightarrow{P_{0321}(0)\ P_{1230}(4)\ P_{0321}(8)} 0e21\ 5634\ 87t9$$

I use G = 0 here, in accordance with the convention that the initial row form be labeled as P_0, following Rahn 1980 (and mindful of his spirited philosophical defense of this convention in his serialism seminar). Taking this a step further, any tetrachordally derived row could be denoted by a vector that lists the exponents of $t4_0$, $t4_1$, $t4_2$, and $t4_3$ times 4, followed by the permutations of tetrachords 0, 4, and 8. Webern's row, according to this notation, would be (0, 0, 0, 8, 0321, 1230, 0321). The permutations show the retrograde relationship of the tetrachords nicely (hence the inverse notation for permutations). The notation also allows us to evaluate the pc content of the tetrachords, which is determined by the first four numbers, separately from their orderings.

So far the simple transitivity property serves us well. But what if we want to show the row form P_{11} that initiates the first variation in measure 16? In the notation just described this row form is (0, 0, 8, 8, 3210, 0123, 3210). We can use the same notation for the operation that takes us from P_0 to P_{11}, which would be (0, 0, 8, 0, 1230, 3012, 1230). But one would like to think that the relevant operation here is T_{11}, and $t4_2^{\,2} \circ P_{1230}(0) \circ P_{3012}(4) \circ P_{1230}(8)$ is *not* the same operation as T_{11}. The group contains no operation equivalent to T_1 or T_{11} (though it does have an equivalent to T_4 and T_8), so including transpositions requires an abandonment of the simply transitive property and enlarges the group by a factor of four. It also challenges us to understand how transpositions relate to the other operations.

Adding T_1 directly as a new generator is not the clearest way to view the structure of the resulting group, because we have the relation $T_4 = t4_0 \circ t4_1 \circ t4_2 \circ t4_3$. Instead, define an operation τ on pitch classes represented in our $X \times Y$ notation:

$$\tau\,(x, y) = (x, (y + 1)\mathrm{mod}4)$$

The new operation τ is different than T_1 because it fixes x, which means that pitch classes 3, 7, e (= (0, 3), (4, 3), and (8, 3)) go to 0, 4, and 8 (= (0, 0), (4, 0), and (8, 0)) respectively, rather than 4, 8, and 0. It is evident, then, that

$$T_1 = t4_0 \circ \tau.$$

The $t4_n$s do not commute with τ:

$$\tau^{-1} \circ t4_n \circ \tau = t4_{(n-1)\mathrm{mod}4} \quad \text{and} \quad \tau \circ t4_n \circ \tau^{-1} = t4_{(n+1)\mathrm{mod}4}$$

So, for example:

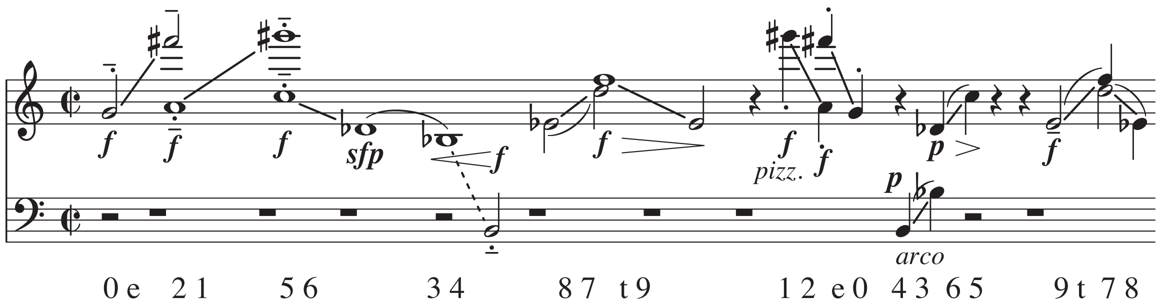
$$T_2 = t4_0 \circ \tau \circ t4_0 \circ \tau = t4_0 \circ \tau \circ t4_0 \circ \tau^{-1} \circ \tau \circ \tau = t4_0 \circ t4_1 \circ \tau^2$$

And similarly, $T_3 = t4_0 \circ t4_1 \circ t4_2 \circ \tau^3$, and $T_4 = t4_0 \circ t4_1 \circ t4_2 \circ t4_3 \circ \tau^4 = t4_0 \circ t4_1 \circ t4_2 \circ t4_3$ (because $\tau^4 = 1$).

Thus, we have a direct product, $\langle t4_0 \rangle \times \langle t4_1 \rangle \times \langle t4_2 \rangle \times \langle t4_3 \rangle$, a base group, and a control group of order 4, $\langle \tau \rangle$, that permutes its indices: a wreath product. The group of permutations, $S_4^{\,3}$, operates on order numbers (rather than pitch class numbers), so it can be added as a direct product to get a full group operating on the tetrachordally combinatorial rows that includes transpositions.

This group, like the previous one, is not transitive on *all* tetrachordally combinatorial rows because it preserves the transposition from one tetrachord to the next, so it cannot relate those where the successive transpositions are T_4 to those where they are T_8. To get a fully transitive group, we might wish to add another one of the standard serial operations, retrograde. This is an order-number operation, so it will commute with the $\langle t4 \rangle$ wr $\langle \tau \rangle$ group, but not with the S_4^3 group. Again, we might best understand this group by generating it from a different operation than the standard retrograde: let ρ be the operation that swaps tetrachords 1 and 3. That is, it reorders the three tetrachords without changing their internal order. Then $R = {}^0P_{3210} \circ {}^4P_{3210} \circ {}^8P_{3210} \circ \rho$. Also, $\langle \rho \rangle$ is order-2 and acts upon the indices of the direct product of S_4s, so it can be understood as the control group of (you guessed it) a wreath product. The T_nI operations may be incorporated in a similar way.

One use of this sort of group is that it helps to loosen up a problem that Rahn complains about in *Basic Atonal Theory:* the "categorization" of a piece of music "may encourage shallow understanding," as manifest in "the still widespread fallacy that a 'serial' piece is nearly completely understood by '12-counting' it" (11). Example 2 shows the first 24 pitch events of Webern's string quartet. Notice the evident break between the two aggregates, expressed by a change of pace (half note to quarter note), and a change of playing technique (*arco* to *pizz.*). Webern partitions both aggregates into a series of wide ic1 dyads (11-, 13-, and 23-semitone melodic intervals), and the registral positions of pitch-classes are fixed, making it particularly evident that the tetrachords of the second aggregate are the same, in the same order, as the first. This fact is also expressed in the resulting inverted contour of the first two tetrachords, and further emphasized by the similar rhythm and pattern of simultaneity. A wealth of factors points to a kind of statement–response formal design here. However, if we take Webern's lead on this, we get two row forms not related by a standard twelve-tone operation. The usual 12-count method tends to categorically reject this possibility, presuming that a piece must be based on a single row type according to the standard canonical transformations. The typical solution (e.g., Bailey 1991, 390–1, Moseley 2013, 192–205) is to regard this passage as constructed by overlapping tetrachords of T_8-related rows, so that these 24 pitch events constitute two overlapping rows plus the first eight notes of a third. While the idea of overlapping row forms may bear some relationship to Webern's compositional process (it is a common method in his serial works), it could hardly be more apparent that this kind of segmentation leads us away from, not towards, the music that Webern actually wrote. The categorical rejection of any operation that falls outside the box of the well-worn $T_n/T_nI + R$ group and its orbits hardly seems worth that, especially in this case where the second aggregate is so clearly a variation on the first. It so happens that the necessary operation, ${}^0P_{3210} \circ {}^4P_{3210} \circ {}^8P_{3210}$, which retrogrades each tetrachord individually without reordering them, is one of the elements of the group above. It commutes with T_n, T_nI, and R, and so augments the usual set of 48 row forms to a set of 96. But for this movement, T_nI operations are of little use given the RI symmetry of the row, so a better approach would be an alternate class of 48 rows given by the orbit of a group generated by T_n, R, and ρ.[3]



Example 2: Webern, *String Quartet*, op. 28, mm. 1–10, with barlines omitted and ic1 dyads connected. These are always played by an individual instrument, except where shown with a dashed line.

The same basic transformational approach might be effective for all-combinatorial hexachordal rows, but for trichordally derived rows, the usual relationships relating the components of the aggregate partition are T_6 and some contextual inversion. We could define new group structures to act on these, but let us (momentarily adopting the habits of mathematicians) leave this for an exercise, so that we can press ahead and inquire into what larger lessons we might take from the way that wreath products seem to pop out of the ground like a fairy ring of mushrooms after a spring rain. The seeming magic of such coincidences often catches our attention and perhaps leaves us vulnerable to the charge of peddling hocus-pocus theory. But usually a concerted investigation of such "magic" is rewarded when the mysticism dissipates and leaves behind a deeper understanding, when we discover that the mushroom is really one large underground organism. The wreath products always seem to come about in similar circumstances, when layered processes operate with respect to one another, and one controls the reference point for the other. Both of these appear to be common features of musical systems: layering, and level-dependent relativity of reference points. Relativity, in particular, relates to the non-commutative aspect of wreath products, the use of a semidirect product to relate the base and control groups.[4] David Lewin recognized this link between relativity and non-commutativity: when he introduces the idea of a non-commutative generalized interval system in *Generalized Musical Intervals and Transformations* with the rhythmic GIS in Chapter 4 (2007, 60–87), his central concern,

[3] Straus (2016, 357) suggests a similar approach, and Hook and Douthett (2008) propose another effective method that deals directly with the tetrachords. An advantage of the row-based approach is that aggregate-groupings play an important role in the piece, although these could also be theorized through Hook and Douthett's method (by recognizing aggregate-generating transformations on the tetrachords).

[4] On semidirect products, see any introductory treatment of group theory, such as Robinson 1993, 27–8.

dominating the discussion of the GIS and its analytical application, is the difference between absolute and contextual reference points for durations.

At this point, having identified a possible abstract source for a recurring mathematical structure, a new test case is useful. For example, a pitch-class set may be represented by a characteristic function, a string of twelve 1s and 0s indicating the presence or absence of a pitch class, such as (100010010000) for a C major triad. The reason for using characteristic functions is that it makes it possible to define an algebra on pitch-class sets. If we adopt the convention that pitch classes cancel one another out, a kind of pitch-class on-off switch, then we can treat the pitch-class sets as a group acting on themselves by addition. The sum of a C major triad and G major triad, for example, would be

$$(100010010000) + (001000010001) = (101010000001)$$

Or {BCDE} (the Gs cancel out). The pitch-class sets then have the structure of a group, $\mathbf{Z_2}^{12}$, the direct product of twelve copies of $\mathbf{Z_2}$. Leaving aside the possible applications of this group for the moment (one might rather define the group on multisets as $\mathbf{Z}^{12}$, getting rid of the cancelation property[5]), one potential problem with it is that it does not include transpositions. Or, to put it differently, the pitch-class positions are overly fixed; the basic equivalence of transposed versions of the same sums is not reflected in the algebra. This can be solved by extending to a wreath product with the group of transpositions: $\mathbf{Z_2}^{12} \ltimes \mathbf{Z_{12}}$. The transpositions act as a control group, cycling the positions of the pitch classes. Again, the wreath product emerges as a way to loosen up the referential framework in the simpler base group.

Concepts of referential framework in music tend to be closely tied to ideas about tonality, and pitch class has been a central feature of all of the examples described so far. But the more abstract considerations about layering and reference points that have emerged should not necessarily have to involve pitch-based relationships. Abstracting fundamental musical principles from the standard pitch-class system of the Western tradition in not just a theoretical concern; it is also a long-standing preoccupation of composers, one that led to, among other things, percussion-based concert music, starting in the 1930s. A very early, and fascinating, example of this trend is the piece *Ostinato Pianissimo*, by influential and free-thinking west-coast composer Henry Cowell. At the time when Cowell wrote *Ostinato Pianissimo*, there was little precedent for the idea of a genre of Western concert music made up entirely of percussion and in which pitch and harmony did not function as structuring elements. The only comparable work that predates it is Varese's *Ionization*, frequently cited as the earliest all-percussion piece in the Western canon. Cowell's piece, like Varese's, had to essentially create its own tradition, its own rules, and therefore its own basis for aesthetic judgment. Although the piece, in its sonic qualities and approachability, could hardly be further from the contemporaneous serial music of Schoenberg and his fellow travelers, it shared this core feature, for Schoenberg and Webern also, with epochal hubris, declared the composers' right to write their own laws of music anew for each piece. They gave life to not only a musical style but a philosophical tradition that populated music theory through Babbitt and then his students, among them David Lewin, Benjamin Boretz, and John Rahn. This important thread of music's structural autonomy is manifest, for instance, in the

[5] See Amiot and Sethares 2011 and Yust 2015 for possible uses of this kind of algebra.

painstaking way that Rahn (2001) shows how the initial measures of the first song of Babbitt's *Du* generate the logic by which the song may be interpreted, seemingly (but not really) *ex nihilo*.

It is fascinating then that Cowell's solution to the problems of creating a musical style from the ground up also shares an essential feature with Schoenberg's solution to writing atonal music: it is based on permutation. I will consider just the foundational pattern played by the woodblocks, tambourine, and guiro—for a fuller explanation of all the ostinati of the piece and how they interact, see Hitchcock 1984. Example 3 shows the nine-measure pattern, which consists of four sounds, each occurring exactly once in each measure, but never in the same order. We might then conceive of the pattern as a sequence of permutations relating one measure to the next—this is shown in Example 3b. The permutations are given in the most compact form, in *cycle notation*, with order numbers labeled 1–4. A cycle such as (xyz) in cycle notation is read as $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$. If an order number does not appear in the cycle notation it is fixed by the permutation.

Example 3: One of the foundational ostinati from Cowell's *Ostinato Pianissimo*, which repeats eight times spanning the entire piece. The measure-to-measure permutations are given below the staff in cycle notation.

Some interesting features are already evident from the successive permutations. The first three patterns are related by successive application of the same permutation, (234). This generates a small cyclic group of order 3 that fixes the first element of the pattern. Since the group is order 3, another (234) permutation would return us to the arrangement of m. 1; to maintain variety, an extension of this group is needed. The next permutation is a different order-2 type, (12)(34), which exchanges the positions of two pairs of sounds. These two operations do not commute:

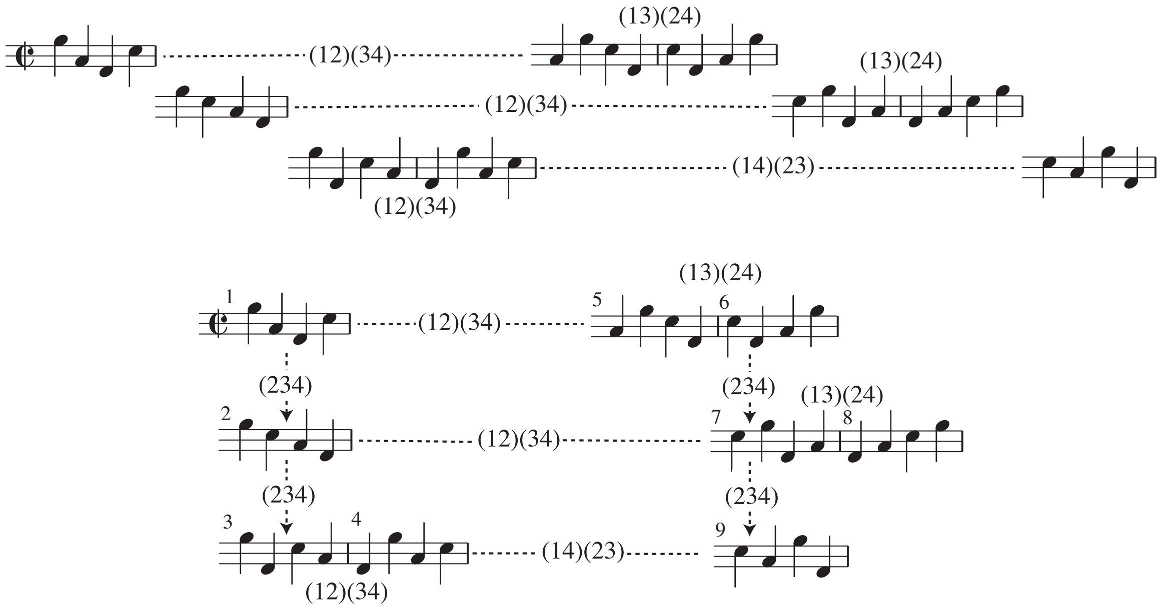
$$(12)(34) \circ (234) \circ (12)(34) = (143)$$

This new order-3 operation, which fixes the second sound, it just so happens, is the next one that Cowell uses, and the fixed sound is once again the high woodblock, which is now in a new position. The same operation occurs at the end of the sequence, fixing the tambourine in position 2, which is where it is in m. 1 (and m. 11 when the pattern repeats). All of this is suggestive of an underlying method.

The limited number of operations (just four) that Cowell uses between successive measures can be partly explained by the idea that he was avoiding any permutation involving 4→1, since it would create a repetition over the barline, and aiming for those that include 3→1 and/or 4→2, which create a kind of "neighbor-note" pattern of alternating sounds over the barline. This kind of pattern is also prevalent in the ostinato of string piano no. 1. All of the permutations except one include 3→1 or 4→2 or both (the exception being (12)(34) between mm. 3–4). However, this does not explain Cowell's avoidance of permutations like (13), (24), or (1243), which also have this property.

One feature that all of Cowell's permutations share, and which the permutations (13), (24), and (1243) do not, is that they are *even* permutations, which means that they reverse the order of an even number of pairs—in (234) it is 2-3 and 2-4, and in (12)(34) it is 1-2 and 3-4. The significance of this is primarily an algebraic one: even permutations always compose to give even permutations, which means that a group generated by even permutations will include no odd permutations. The permutations Cowell uses generate A_4, the *alternating group* on four elements, a special subgroup of the full permutation group, S_4, that is exactly half its size (order 12). It includes the four 3-cycles: (234), (134), (124), and (123), and their inverses; three order-2 operations with no fixed element: (12)(34), (13)(24), and (14)(23); and the identity. The orbit of this group, the number of possible arrangements of the four sounds accessible via these permutations, also has twelve members, of which Cowell selects nine for his pattern.

A good way to understand the structure of a group is to look at its subgroups. For A_4, these include the order-3 subgroups that fix the position of one sound and cycle the other three. Altogether there are four of these, but two appear to be important here: {1, (234), (243)} and {1, (134), (143)}. Example 4(a) splits Cowell's pattern up into ⟨(234)⟩ orbits and 4(b) into ⟨(134)⟩ orbits. Both include two full cycles, one of which is between three adjacent measures (if we count mm. 9 and 1 as adjacent) and one split by a single intervening measure. In fact, the patterns are almost identical, with the pattern of ⟨(134)⟩ orbits shifted two measures behind that of the ⟨(234)⟩ orbits.



Example 4: The measures divided into ⟨(234)⟩ orbits, (a), into ⟨(134)⟩ orbits, (b), into ⟨(12)(34), (13)(24)⟩ orbits, (c), and into ⟨(234)⟩ orbits vertically and ⟨(12)(34), (13)(24)⟩ orbits horizontally, (d)

The other important subgroup of A_4 is the one made up of order-2 elements of the group, which commute with one another, and therefore constitute a normal subgroup isomorphic to $\mathbf{Z_2} \times \mathbf{Z_2}$, a Klein 4-group.[6] Example 4(c) shows the orbits of these, which also make a kind of regular pattern, with two adjacent measures and one isolated one in each orbit. The first two orderings are related by (12)(34) in all orbits. This arrangement is largely oblique to the arrangement of ⟨(234)⟩ orbits in a sense, so that they can be arranged in a grid in a way that mostly preserves temporal order in one dimension or the other, as in Example 4(d)

It may seem that I have derived this group directly from the music, but actually, implicit assumptions about what is important in the music have played an essential role. The discussion of the passage exclusively in terms of permutations implies that what is important about the sounds is how they are ordered, not what kind of sounds they are, or how they relate as sounds. The high woodblock could be replaced by a car horn and it would have no effect on the analysis. It would also be possible to define operations that relate to the sounds themselves. Instead of (234), which fixes the first element and cycles the other three, we could define an operation that fixes the high woodblock and cycles the other three. In measures 1–3, this would function exactly like (234), but in measures 4, 5, and 7, it would function like (143).

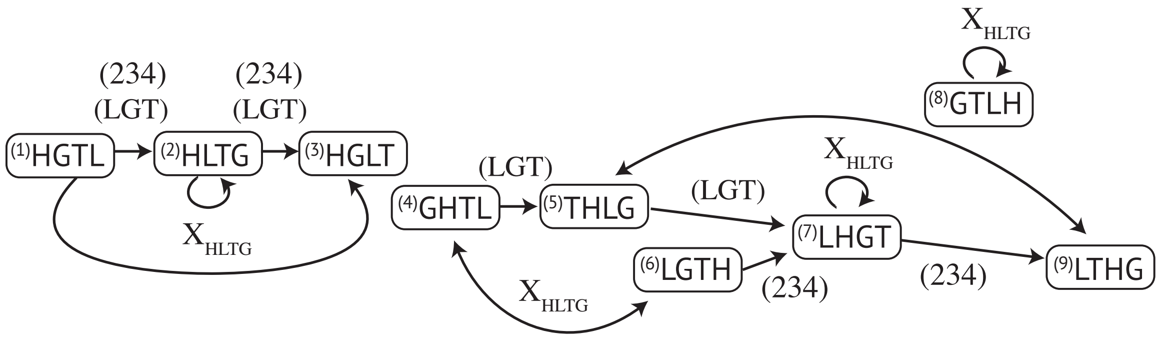
While the sounds-oriented analysis thus seems to reveal something about the pattern, the position-oriented one clearly does also, as we have already seen. Do we need to choose between them? If one demands a simply transitive system, where two measures can be

[6] A normal subgroup is one stabilized under conjugation by the entire group.

related by one and only one operation, then yes. But, assuming that multivalence and polysemy can be a virtue rather than a flaw, in the spirit of Rahn 2007, let us free ourselves of such strictures. It is easy enough to combine both approaches: they are isomorphic, both forming an A_4 group, and commute, so that when combined they generate a direct product, A_4^2. The potential multivalence here is already high, since this group is order $12^2 = 144$, meaning there are potentially 12 ways to relate any two measures of the pattern. But since the two types of operation do not interact, it does not quite satisfy the intuition that the mm. 1–3 cycle relates to the mm. 4–5/7 cycle in one way, by the stabilizing of a common sound, and the mm. 6–7/9 cycle in another, by the stabilizing of a common position. That requires seeing the mm. 1–3 cycle in two different senses simultaneously (polysemy), but we also want to show how the two kinds of cycle are in a sense equivalent. That sense of equivalence has to do with the isomorphism between the sound-permutation and the order-permutation groups. What we need then, is to identify the appropriate isomorphism. And while we're at it, why not add that operation to our group?

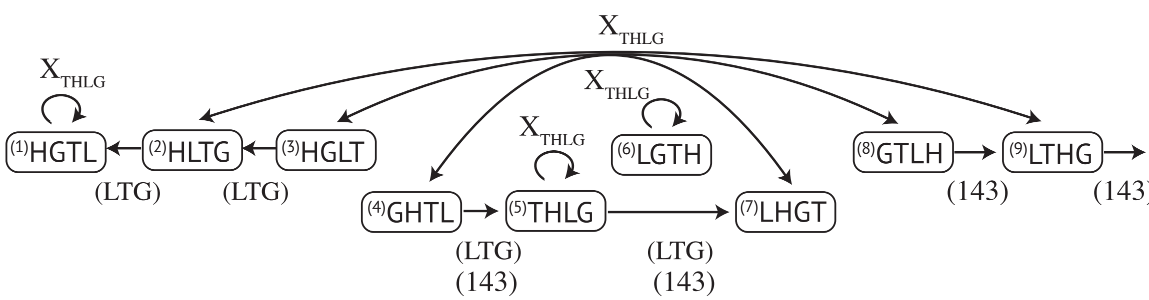
The isomorphism we are zeroing in on is one that will stabilize the initial cycle in mm. 1–3 but map the generating (234) permutation to an (LTG) permutation, where the letters refer to sounds: H = high woodblock, L = low woodblock, T = tambourine, and G = guiro. The needed operation is therefore an *exchange* operation—that is, exactly the kind proposed by Stanfield (1984), except with two differences. The first difference is that Stanfield's E operates on twelve things rather than four. The second is that, for Stanfield, pitch classes and order numbers were already labeled with the same integers, so a single operation is easy to define. Simply write out the row as a mapping from order numbers to pitch classes and reverse the arrows. The "naturalness" of this definition is misleading, however: it presumes a fixed 0 (such as C = 0), and also assumes stepwise-ascending to be a standard ordering of the pitch classes, which may or may not be a legitimate assumption. This is not a problem when one constructs a group generated by E and, say, T_n and T_nI, because ultimately that group will include a number of different exchange operations, which can be obtained by conjugating E with all of the T_ns and T_nIs (or whatever). One is free to change which exchange operation generates the group—the group itself will be the same. This is also true for the exchange operations on the four sounds: there are twelve options, corresponding to all of the ways to order H, L, T, and G within the A_4 orbit, none of which is *a priori* better than any other. We can label these X_{abcd}, where *abcd* is the ordering.

Which exchange, then, will map the (234) operation onto (LGT)? As one might guess, the one that maps L→2, G→3, T→4, and H→1: X_{HLGT}. This matches the ordering of the sounds in measure 2. As Example 5 shows, X_{HLGT} maps the mm. 1–3 cycle to itself, reversing the orientation (which, it turns out, is a necessary condition for stabilizing this kind of cycle with an exchange operation). It then maps the mm. 4–5/7 cycle onto the 6–7/9 cycle, revealing the underlying equivalence.



Example 5: A network using X_{HLTG}

This seems pretty satisfying, except that the analysis orphans measure 8 to some extent. The X_{HLGT} operation happens to stabilize it, but this account says little else about how it relates to the other measures or why it might occur in the position it does. One of the observations made about measure 8 in Example 4(b) is that it makes a (143) cycle from mm. 8–9 back to measure 1, which fixes the tambourine in position 2. This is not reflected in the analysis of Example 5, but it is similar to what happens between mm. 4–5/7 and 6–7/9, two 3-cycles overlapping in a shared measure (m. 7 or m. 1). There should be an exchange operation that swaps these overlapping cycles too, one that equates (234) to (HLG). Indeed, the operation X_{THLG}, based on the ordering of m. 5 at the center of the ostinato, does just this. It also stabilizes the set of nine patterns that Cowell uses, and, as Example 6 shows, makes an appealingly symmetrical pattern of relationships within Cowell's ordering. It also stabilizes the 3-cycle in mm. 4–5/7 (in the same way that X_{HLTG} stabilizes mm. 1–3).



Example 6: A network using X_{THLG}

At this point the reader will perhaps be unsurprised by the punchline: Voilà, a wreath product. The base group, A_4^2, is acted upon by a control group of order 2, the exchange group, which swaps the two A_4s. Again, the group describes leveled processes, different systems of permutation and a global switch the allows us to move back and forth between them, and allows reference points—the privileging of a particular sound, order position, or ordering of sounds—to adapt dynamically.

Perhaps the most important thing that I learned studying with John Rahn is something he taught not explicitly but by example: that mathematical music theory is a kind of dialogue. When mathematical problems arise in music theory, that is music theory listening to mathematics. For instance, a musically motivated compositional problem might cause us to wonder how many unique partitions of the aggregate exist of a particular type. We could answer such a question by applying principles of combinatorics. When we look for or create music to exemplify mathematical constructs, that is mathematics listening to music theory. Music theorists tend to frown upon the latter sort of enterprise when it seems to lack indigenous musical motivation, but really either type of one-way conversation is of limited value. Where mathematical music theory becomes a sum greater than its parts, where it becomes a discipline in its own right rather than a mere semi-permeable membrane, is when lecture gives way to discourse. Music asks questions of mathematics, and mathematics seeks out ways to be experienced as music. It is this kind of productive exchange that is modeled in *Basic Atonal Theory*, as well of many of Rahn's articles and essays. *Basic Atonal Theory* clearly places value on mathematical elegance. As a textbook, it is unique in the degree of mathematical care and precision it asks of the student. It does not compromise in this respect; indeed, it endeavors ultimately to show the student, who works carefully through all of its analyses, definitions, theorems, and exercises, that mathematical elegance ultimately translates into musical elegance. However, it begins not from any disembodied Platonic mathematical premises but with a nine-page "Ear Training: Without Score" of the theme from the second movement of Webern's Op. 21 Symphony, which patiently walks the student through an experiential exploration of the piece, in the course of which she is asked to play the full theme 20 times, plus various short components of the theme. At the end of this, the student is entreated not to neglect the individuality of a piece by virtue of its categorization—e.g. as "serial music": "Every piece of music is unique" he says "with idiosyncratic organizational principles and structures shared with no other pieces of music" (11). The "Ear Training" is followed by a seven-page "Analysis with Score" that requests nine more playings—all of this before any actual theory has been presented at all. Whether or not students diligently follow Rahn's instructions in working through this analysis (nowadays a youtube video might be helpful), the message is clear. The mathematical theory is not worth much unless it is thoughtfully experienced as music at every step of the way. The mathematics is not a predator hunting and devouring pieces of musical prey, but a friend inviting them over for drinks, asking them how they're doing and what they would like. "Not only is any musical activity active and poetic, but so are music perceptions and music analysis . . . mathematics is one such poetic medium" (Rahn 2004, 140).

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