The Fourier Transform and a Theory of Harmony for the Twentieth Century

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Society for Music Theory
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A copy of this talk is available at
people.bu.edu/jyust/

Outline

I. Forte's Project and the DFT

1. A theory of harmony for the 20th century
2. Pc-vectors
3. DFT components and interval content
4. Phase spaces

II. Debussy: "Les sons et les parfums tournent dans l'air du soir"

1. Heptatonic scales and diatonicity
2. Common tones and harmonic qualities

III. Stravinsky and the Octatonic

1. *Rite of Spring*, Introduction and *Augurs*
2. Octatonic scale versus octatonic quality

IV. Feldman, *Palais de Mari*

I. Forte's Project and the DFT

1. A theory of harmony for the 20^{th} century
2. Pc-vectors
3. DFT components and interval content
4. Phase spaces

A Theory of Harmony for the 20th Century

Forte's project:

"It is the intention of the present work to provide a general theoretical framework, with reference to which the processes underlying atonal music may be systematically described."

The Structure of Atonal Music (1973), Preface

A Theory of Harmony for the 20th Century

Forte's project:

General features of harmony that are largely independent of compositional aesthetic:

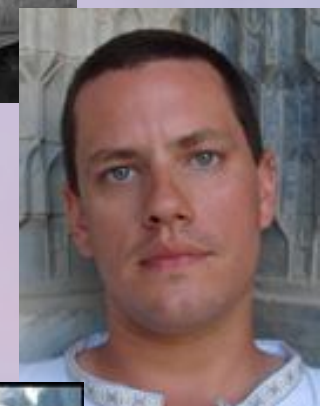
• *Interval content* determines *harmonic quality*

Interval content ↔ DFT components

• *Common pc content* determines *harmonic proximity*

Subset relations ↔ DFT phase spaces

Discrete Fourier Transform on Pcsets



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Amiot, Emmanuel (2013). "The Torii of Phases." *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013* (Springer).

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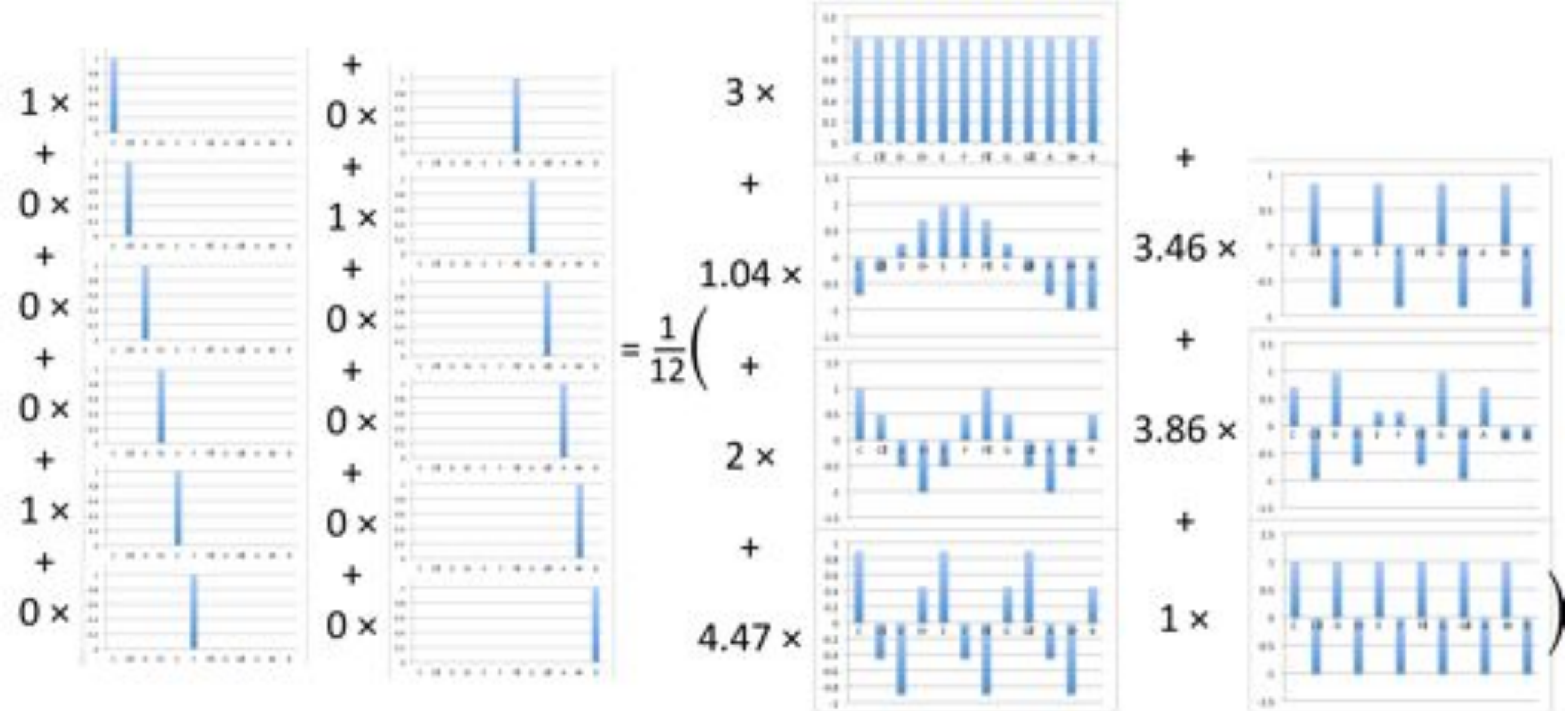
Characteristic Function of a Pcset

The *characteristic function* of a pcset is a **12-place vector** with 1s for each pc and 0s elsewhere:

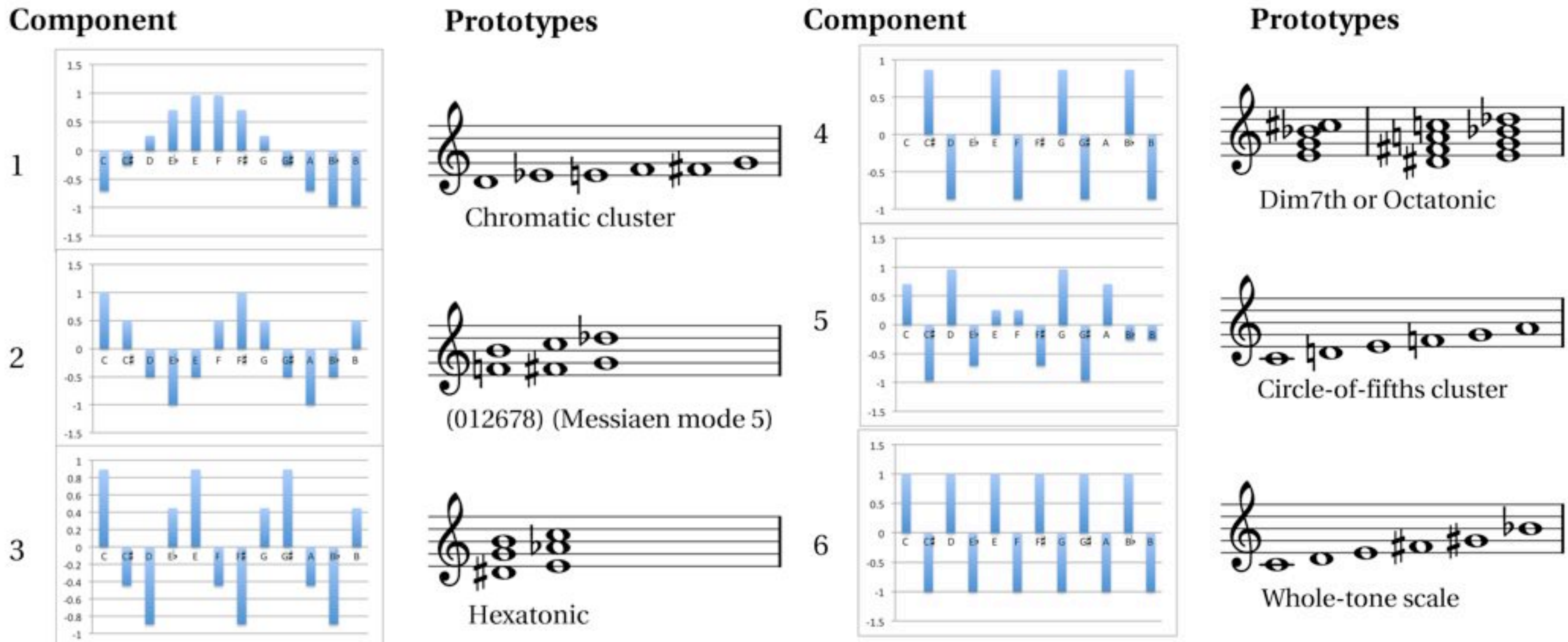
(1,	0,	0,	0,	1,	0,	0,	1,	0,	0,	0,	0)
C	**C♯**	**D**	**E♭**	**E**	**F**	**F♯**	**G**	**G♯**	**A**	**B♭**	**B**

DFT Components



The DFT is a **change of basis** from a sum of pc spikes to a sum of discretized **periodic** (perfectly even) curves.

DFT Components



Quinn's ***generic prototypes*** are pcsets that maximize a given component. **Subsets** and **supersets** of the prototypes are the best representatives of each component

DFT Components

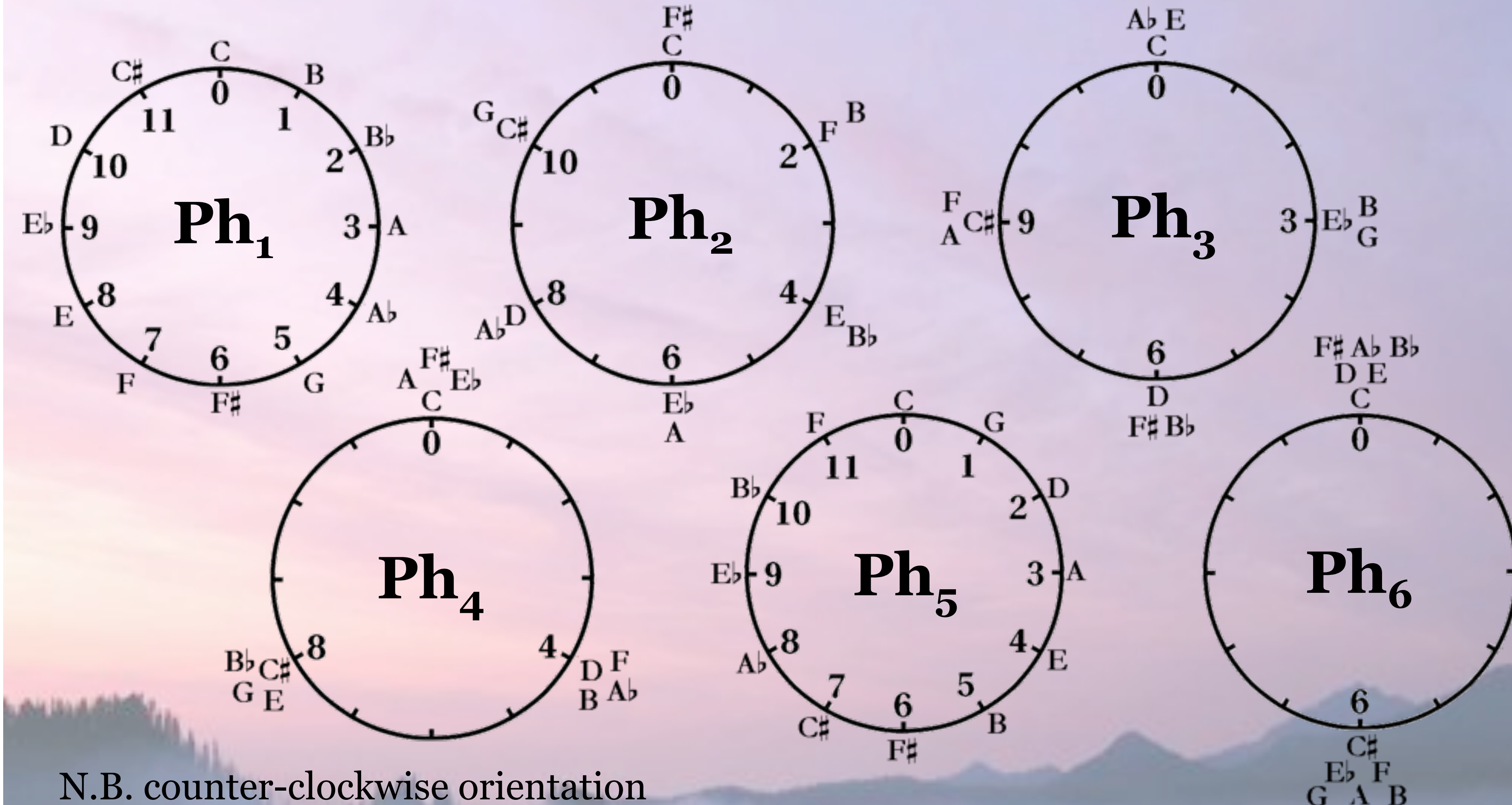
Notation

$\boldsymbol{f_n}$	The *n*th DFT component
$\lvert\boldsymbol{f_n}\rvert$	The *magnitude* of the *n*th component
$\lvert\boldsymbol{f_n}\rvert^2$	Squared magnitude
$\varphi_{\boldsymbol{n}}$	The phase ($0 \leq \varphi_{\boldsymbol{n}} \leq 2\pi$) of the *n*th component
$\mathbf{Ph}_{\boldsymbol{n}}$	Phase normalized to pc-values: ($0 \leq \mathrm{Ph}_{\boldsymbol{n}} \leq 12$)

$$\langle\langle\ (\lvert f_1\rvert^2, \mathrm{Ph}_1), (\lvert f_2\rvert^2, \mathrm{Ph}_2), (\lvert f_3\rvert^2, \mathrm{Ph}_3), (\lvert f_4\rvert^2, \mathrm{Ph}_4), (\lvert f_5\rvert^2, \mathrm{Ph}_5), (\lvert f_6\rvert^2, \mathrm{Ph}_6)\ \rangle\rangle$$

Phase spaces: One dimensional

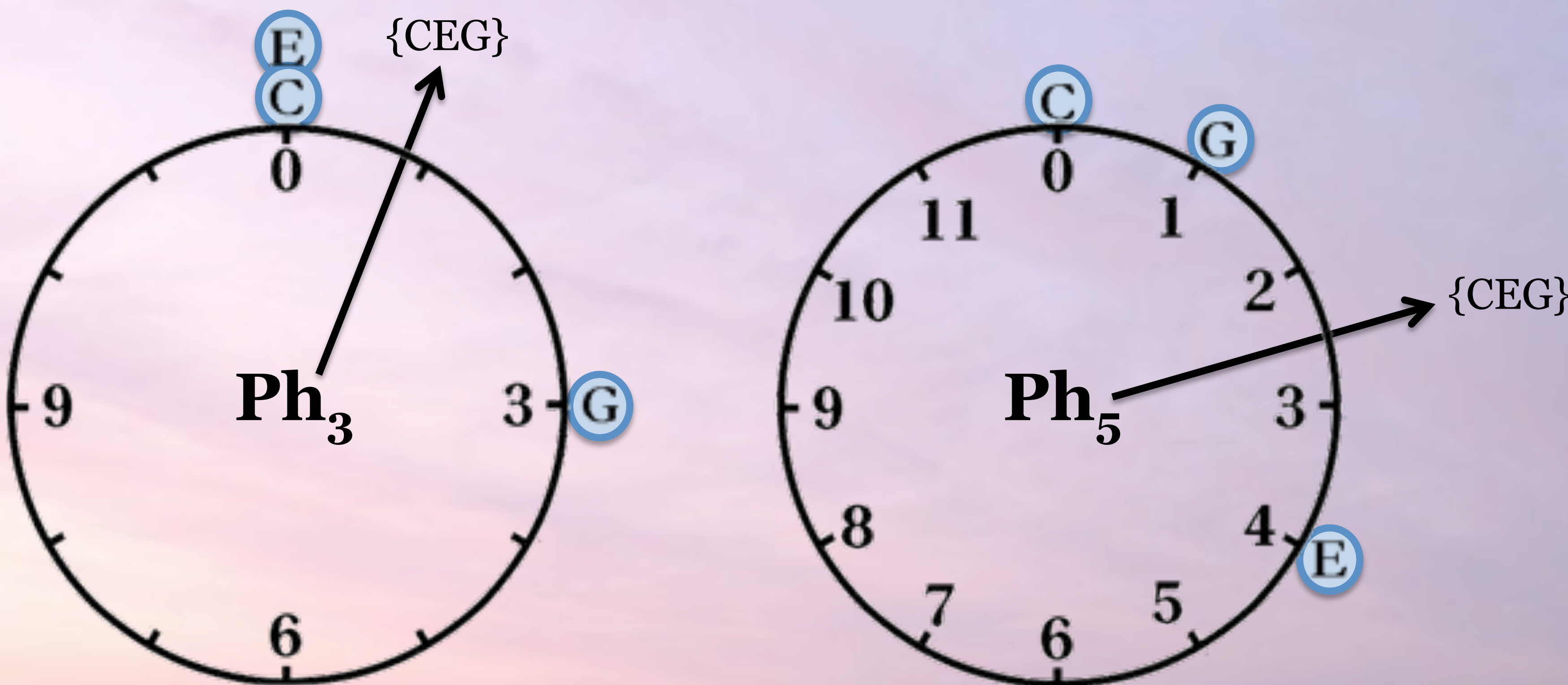
One-dimensional phase spaces are Quinn's *Fourier balances*, superimposed *n*-cycles created by multiplying the pc-circle by *n*.



N.B. counter-clockwise orientation

Phase spaces: One dimensional

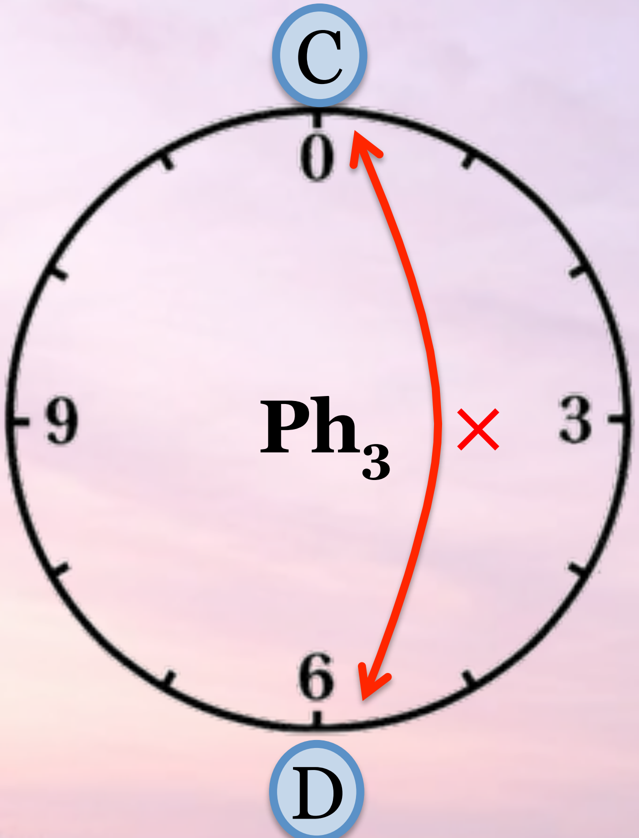
The position of the pcset in the phase space is the circular average of the individual pcs



Phase spaces: One dimensional

Opposite phases cancel one another out.

Therefore pcsets can have undefined phases.



{CD} has undefined Ph_3,

$|f_3| = 0$

This is a kind of **Generalized Complementation:** Complements balance one another in *all* phase spaces.

Phase spaces: One dimensional

An analytical proto-methodology:

Each Fourier component measures an independent musical quality: (1) chromaticism, (2) quartal harmony, (3) triadic harmony, (4) octatonicism, (5) diatonicism, (6) whole-tone balance.

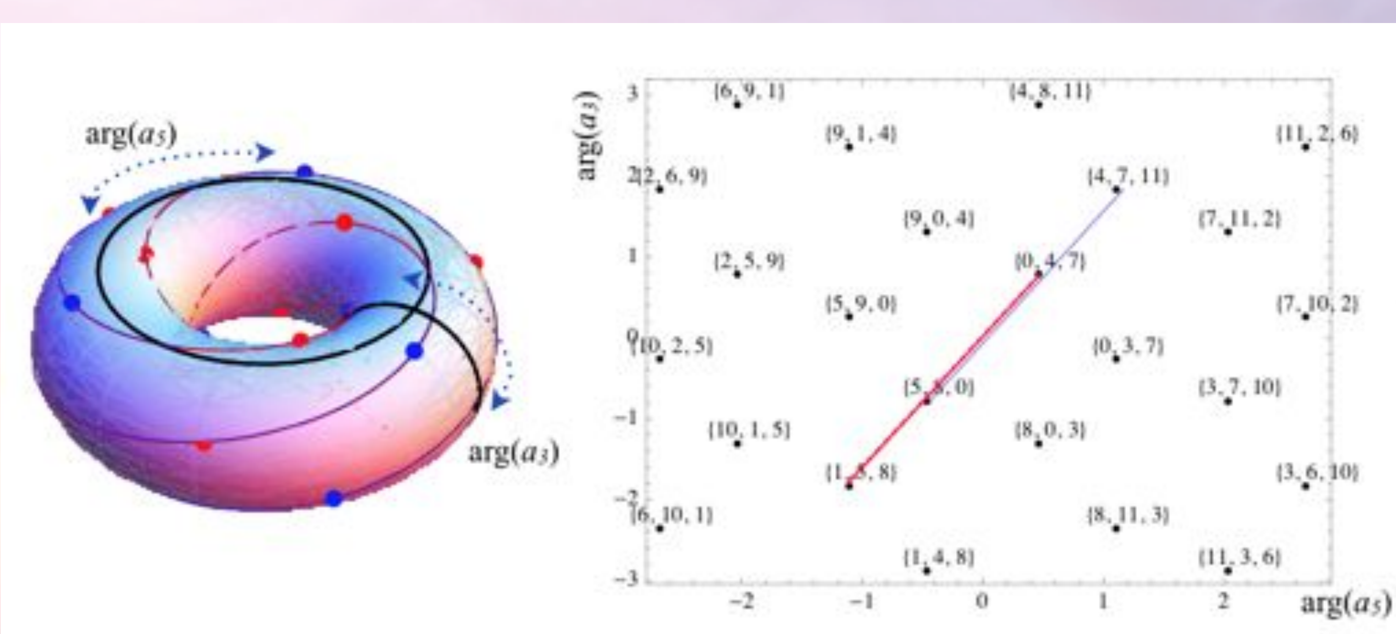
Distances in phase spaces indicate:

- Relatedness of harmonies on the given dimension
- Whether the harmonies **reinforce** one another or **weaken** one another on the given dimension when combined.

Phase Spaces: Two dimensional

A two-dimensional phase space tracks the phases of two components, and is topologically a *torus*.

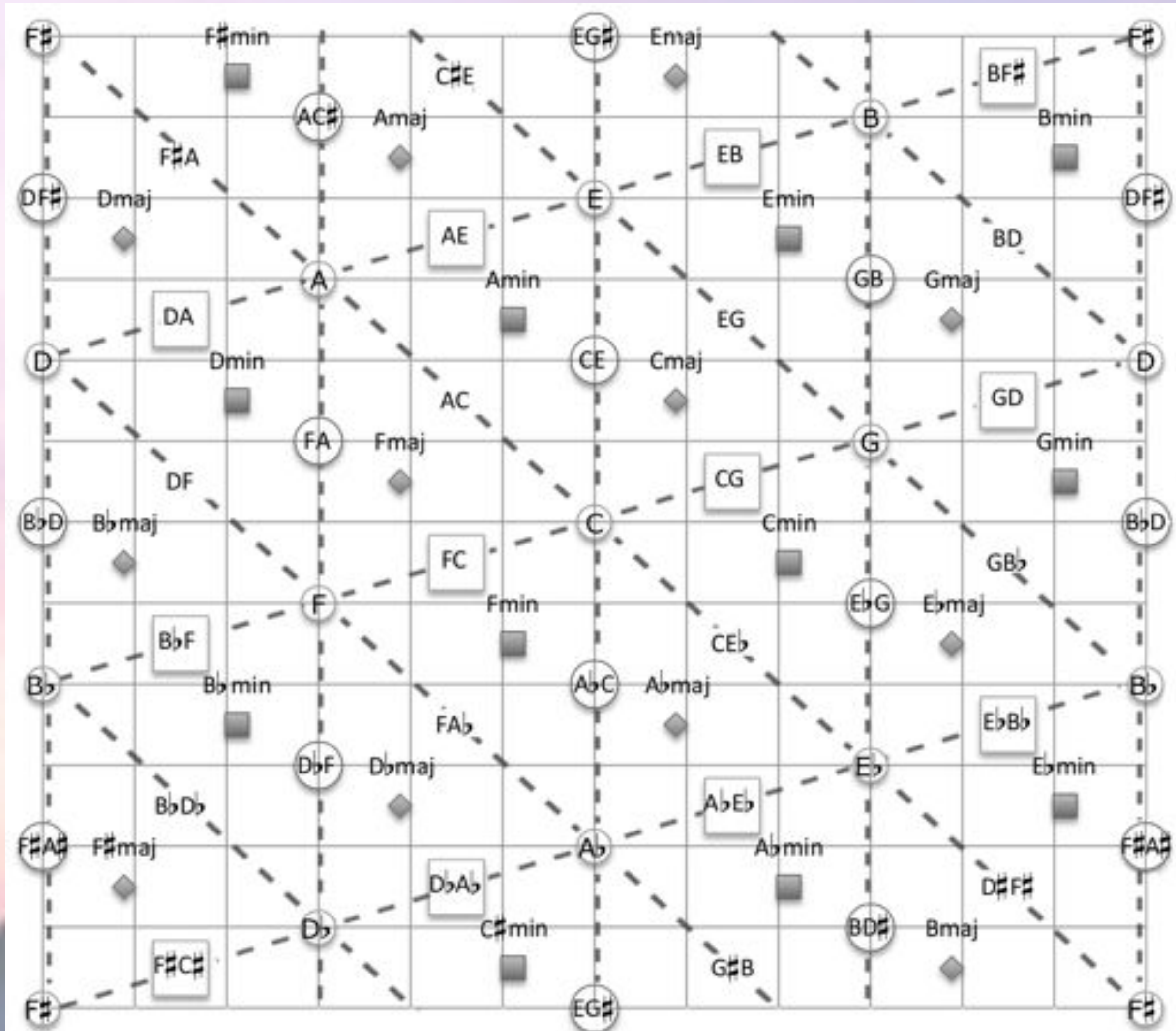
Amiot (2013) and Yust (2015) use Ph_{3-5}-space to describe tonal harmony.



from Amiot, MCM 2013 proceedings

Phase Spaces: Two dimensional

Amiot (2013) and Yust (2015) use $Ph_{3,5}$-space to describe tonal harmony



→ Pcs, consonant dyads and triads, and *Tonnetz* in $Ph_{3,5}$-space, from Yust (2015) (*JMT* 59/1)

II. Debussy, "Les sons et les parfums tournent dans l'air du soir"

1. Heptatonic scales and diatonicity
2. Common tones and harmonic qualities

Scale Theory, Subsets, and Phase Space

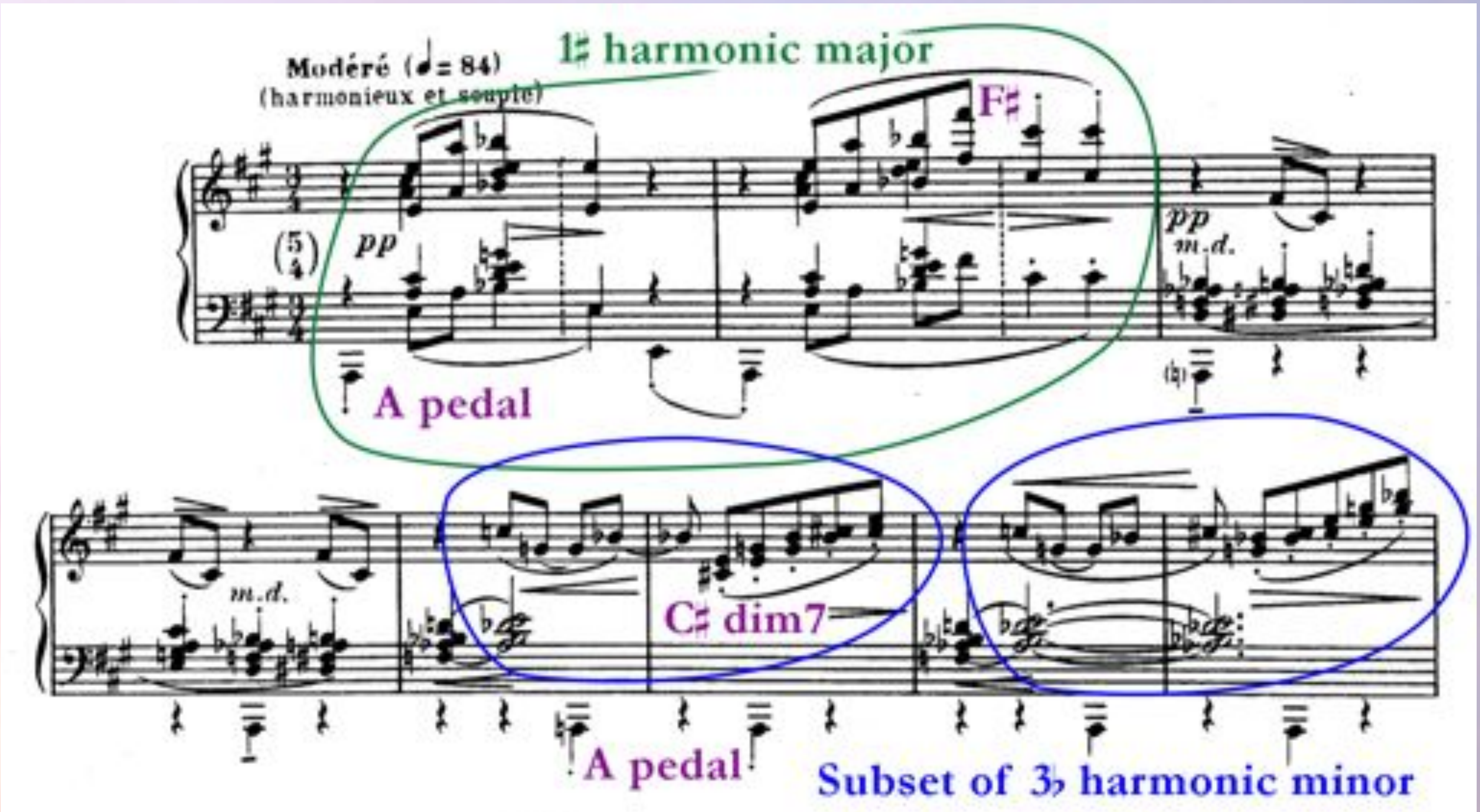
Problems in the analytical application of scale theory:

(1) The status of **subsets** of multiple scales and **supersets** of multiple scales.

(2) The range of variability in what counts as a scalar set.

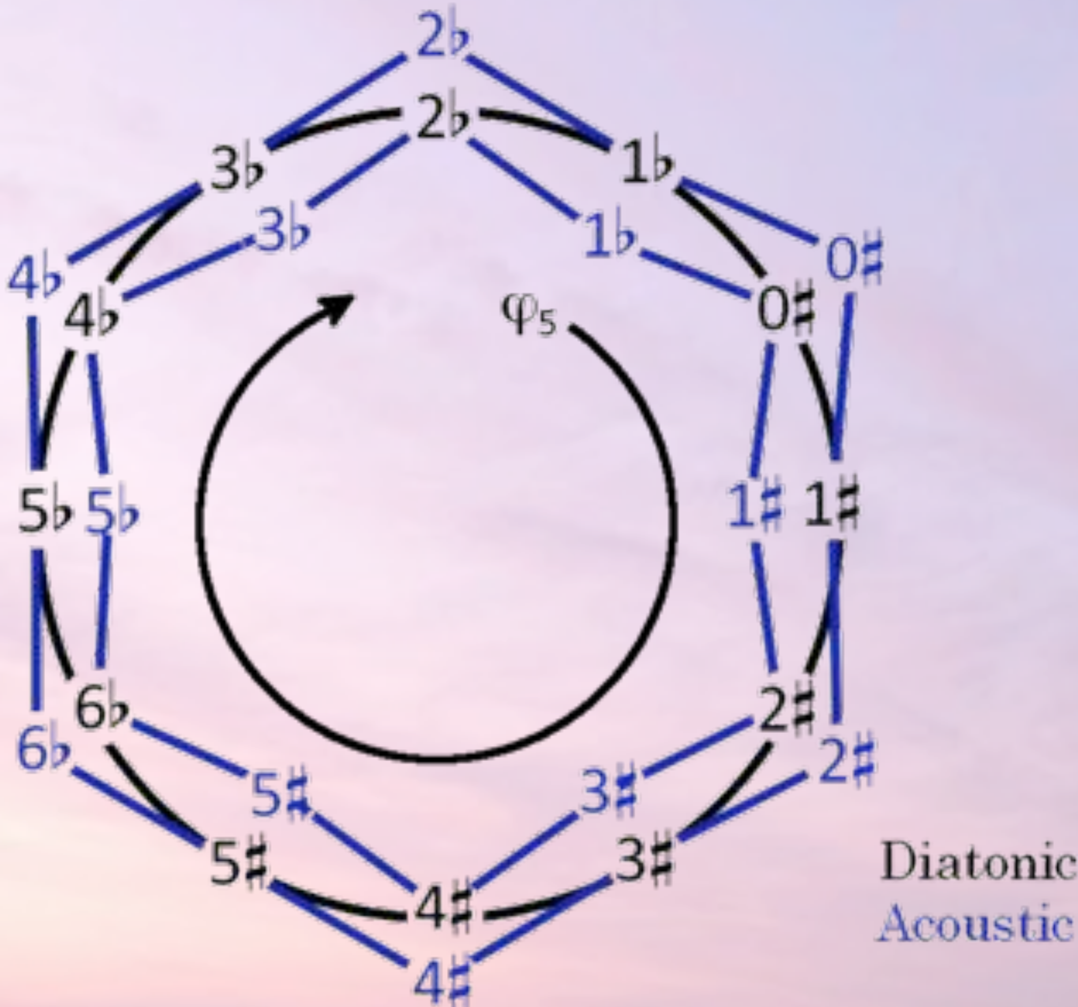
A possible solution: Phase space 5

Debussy and Scale-Network Wormholes



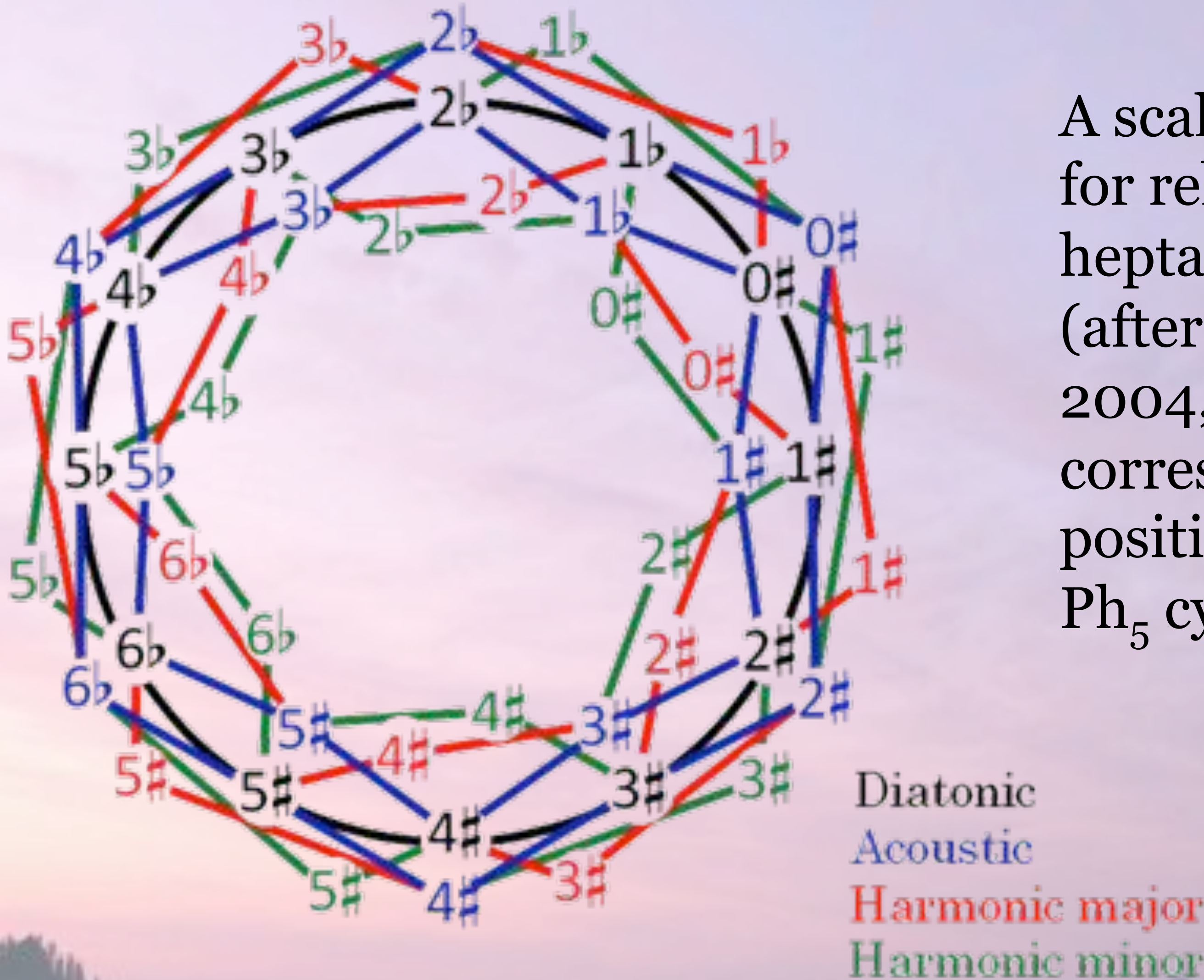
Debussy, *Preludes* I, no. 4, "Les sons et les parfums tournent en l'air de soir"

Debussy and Scale-Network Wormholes



A scalar network for relatively even heptatonic scales (after Tymoczko 2004, 2011) corresponds to positions on a Ph_5 cycle.

Debussy and Scale-Network Wormholes



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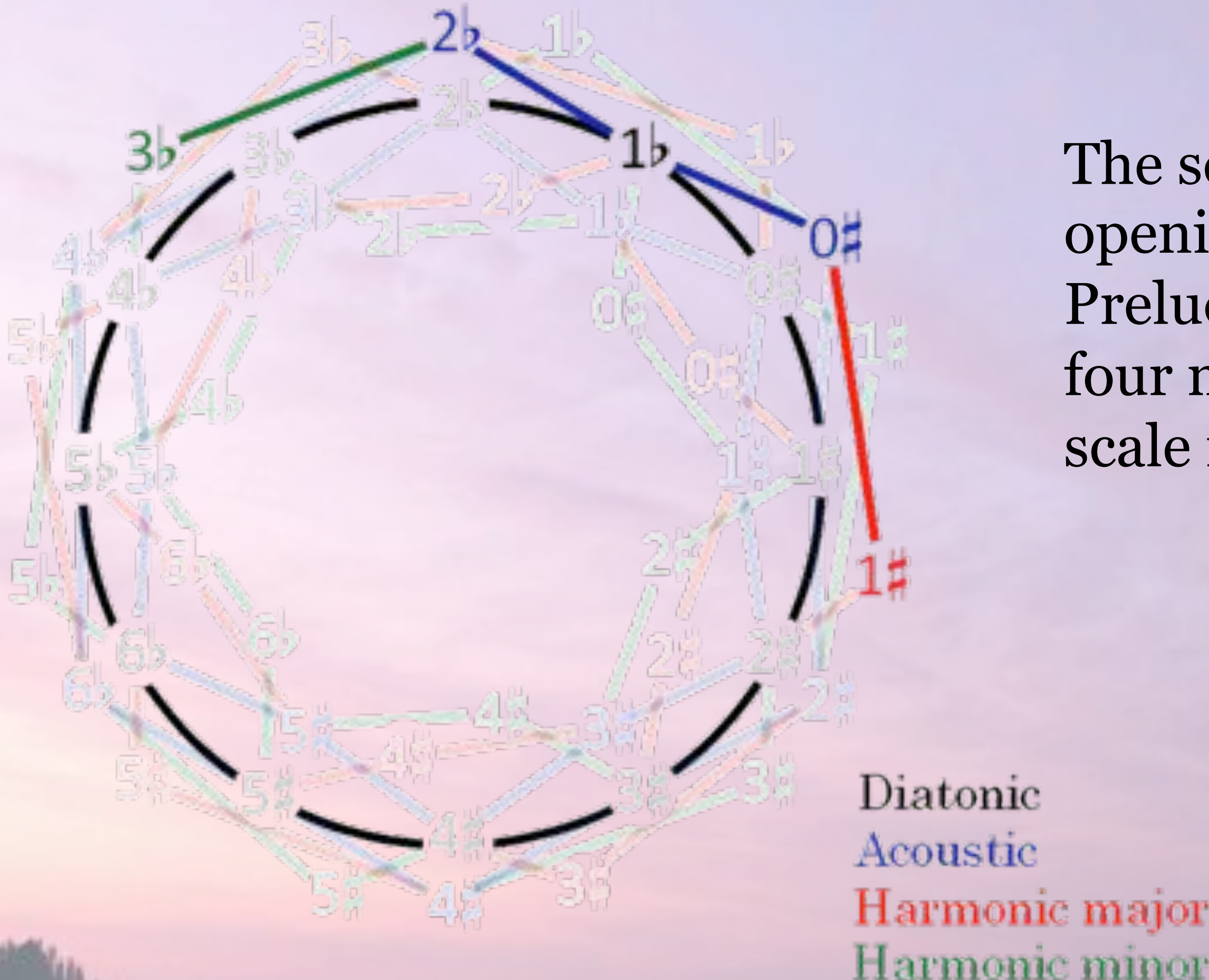
Debussy and Scale-Network Wormholes



The scales in the opening of the Prelude require four moves in the scale network

Debussy and Scale-Network Wormholes

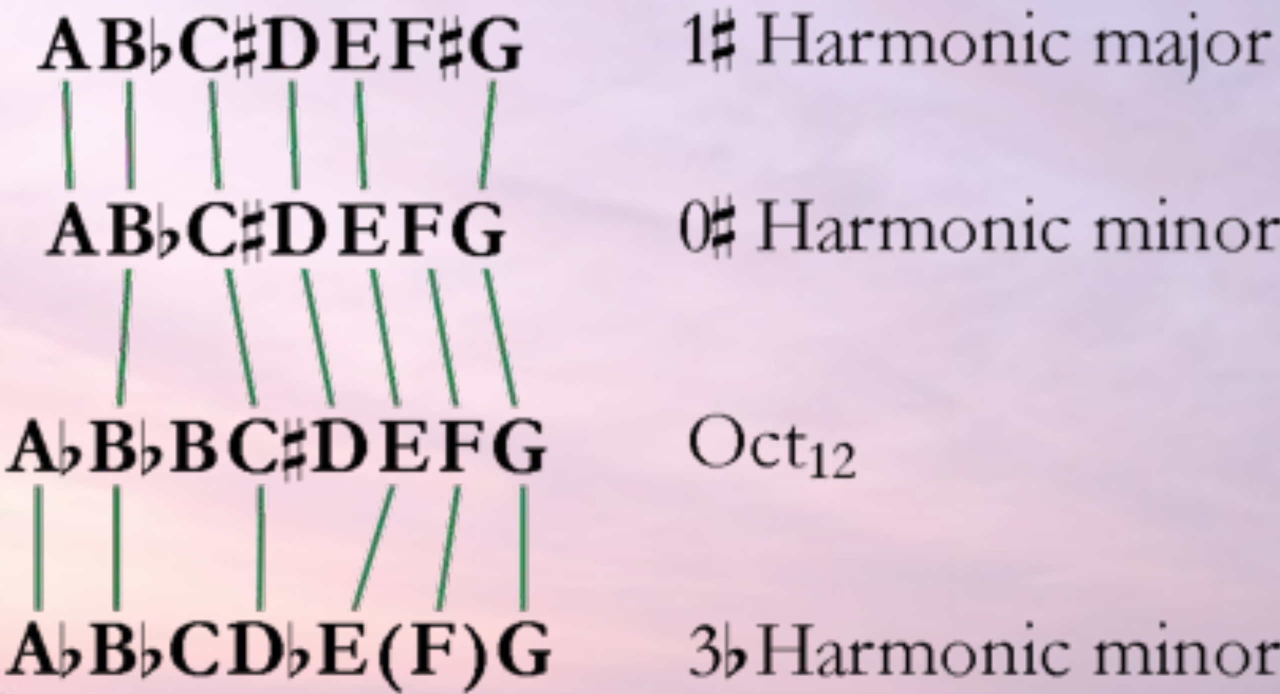
But . . .

These scales can also be connected with just three moves

By using Oct_{01}

Debussy and Scale-Network Wormholes

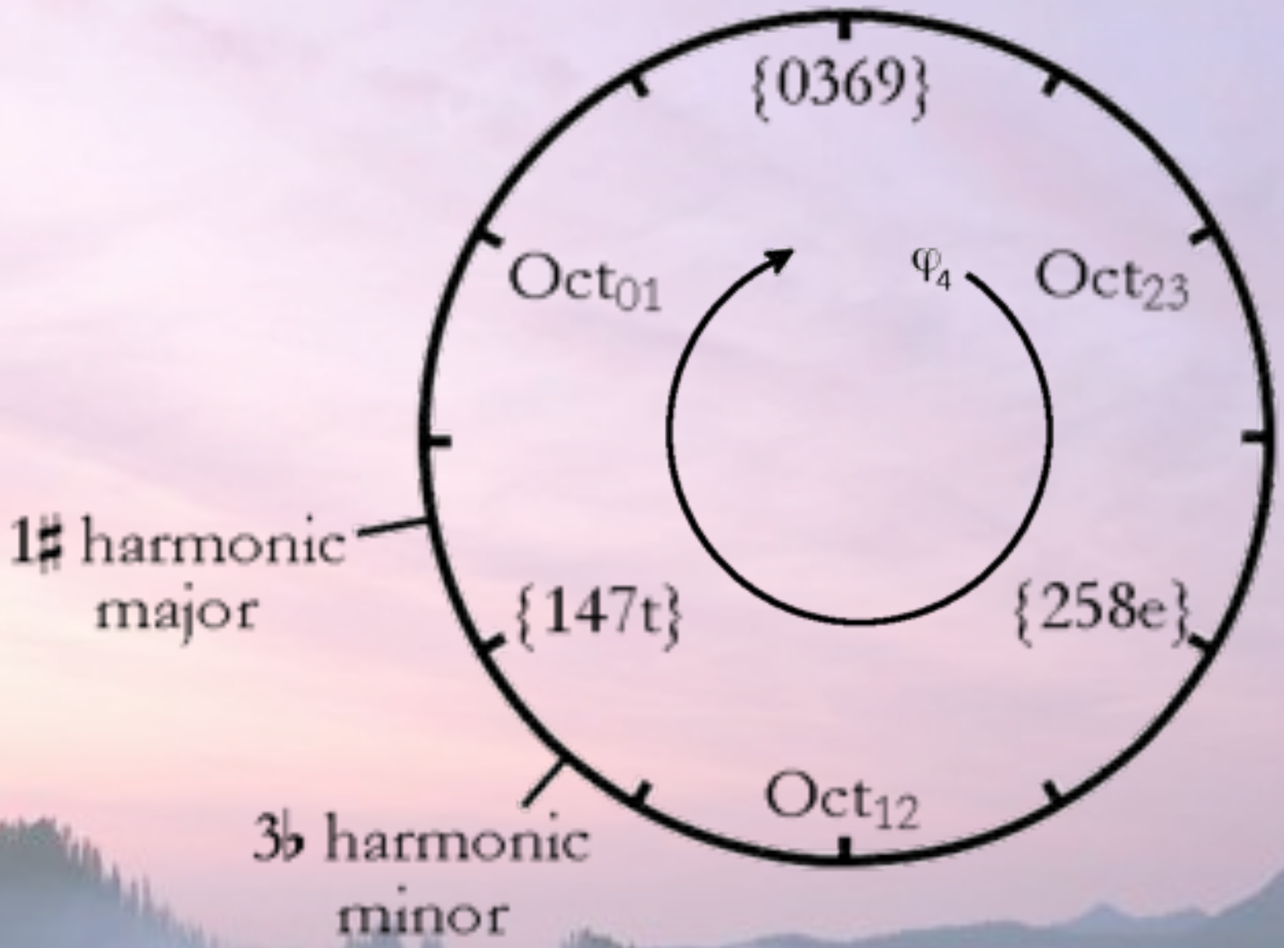
Or . . .
By using Oct_{12}



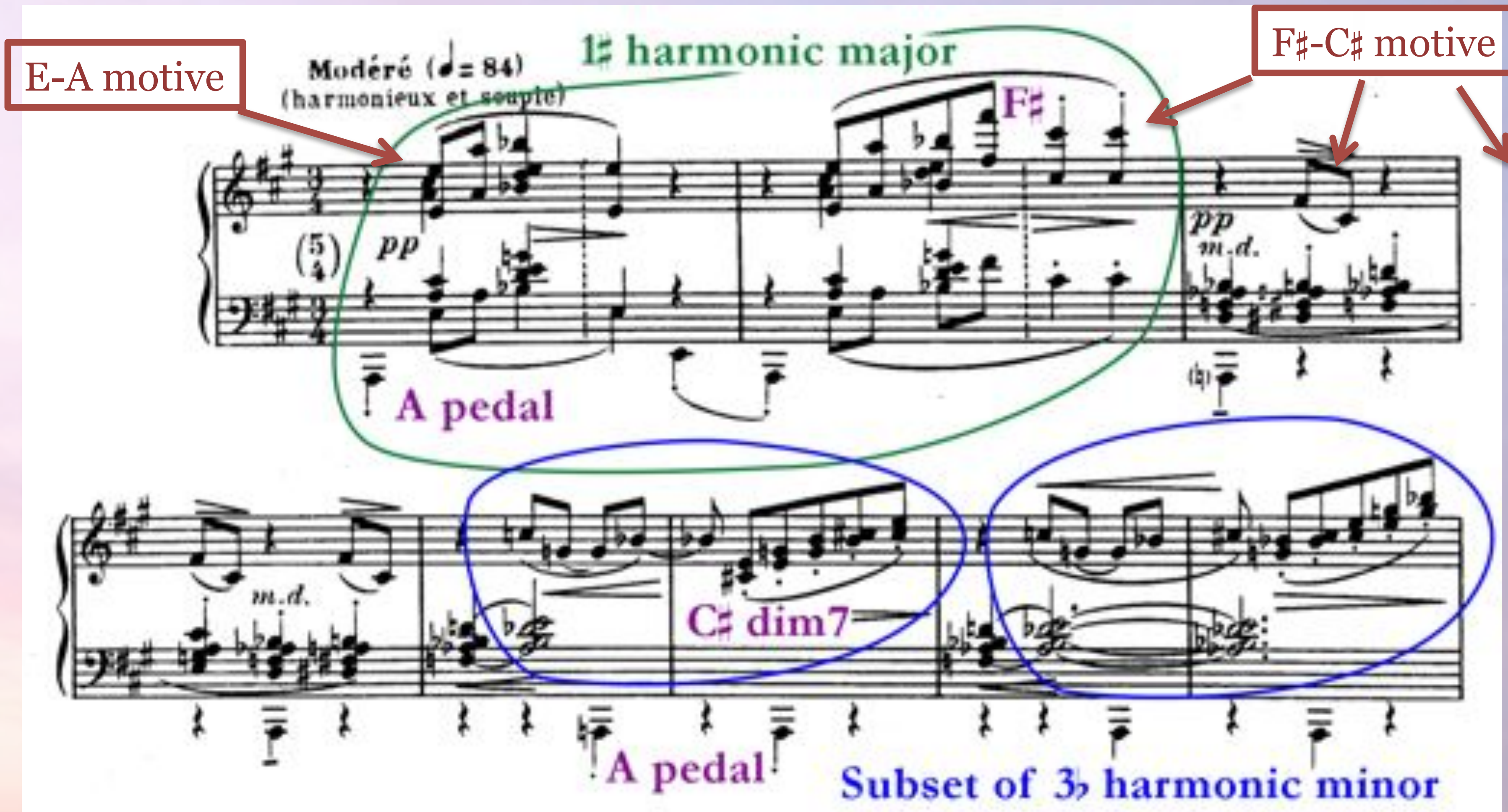
Debussy and Scale-Network Wormholes

Why? Although *far apart* in Ph_5

They are *close together in Ph_4*

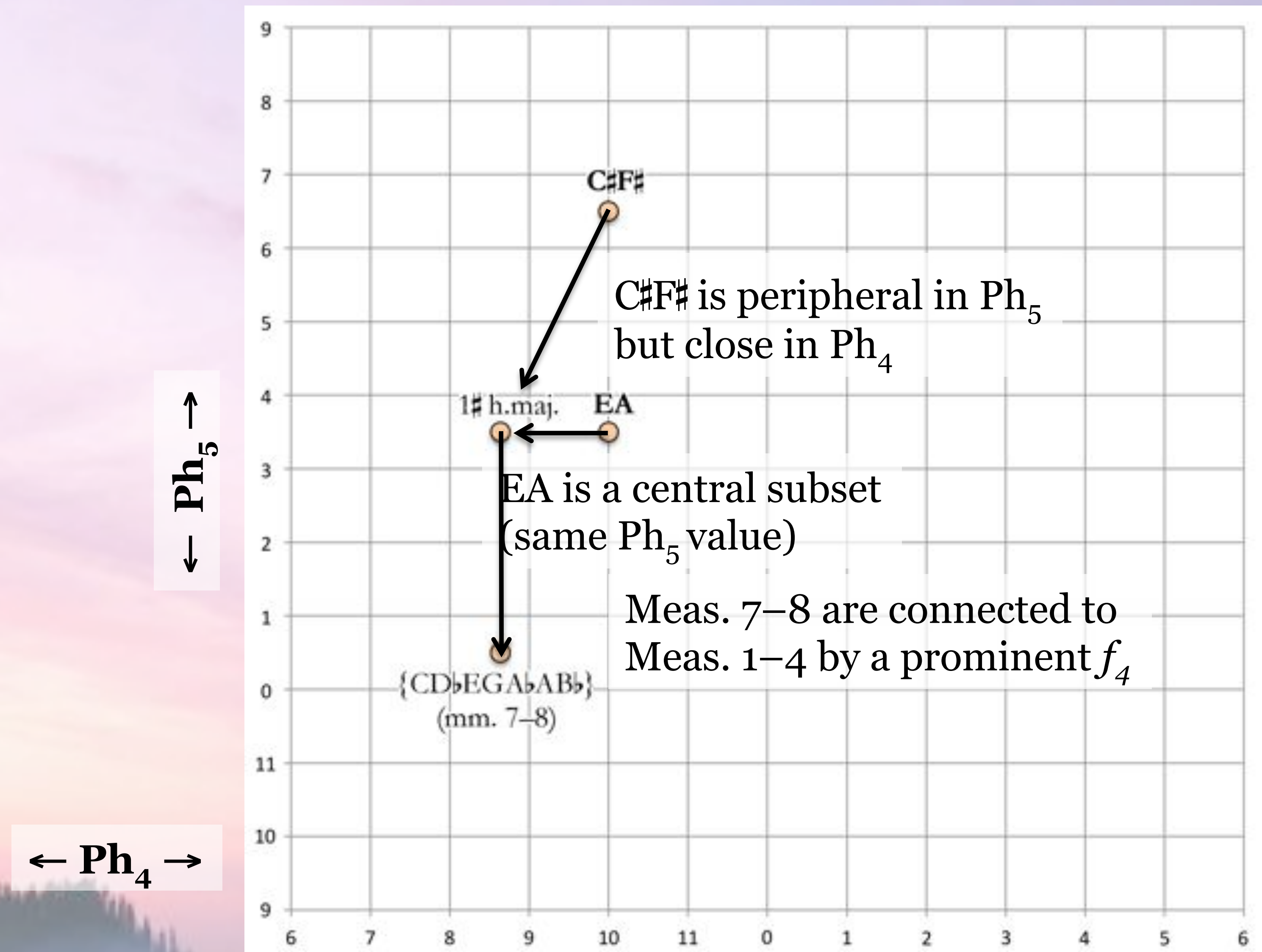


Debussy and Scale-Network Wormholes



Debussy, *Preludes* I, no. 4, "Les sons et les parfums tournent en l'air de soir"

Debussy and Scale-Network Wormholes



Debussy *Airs*: Change of Quality

Debussy *Airs*: Change of Quality

DFT magnitudes^2

	$\lvert f_1\rvert^2$	$\lvert f_2\rvert^2$	$\lvert f_3\rvert^2$	$\lvert f_4\rvert^2$	$\lvert f_5\rvert^2$	$\lvert f_6\rvert^2$
{AB♭CD♭EGA♭}:	⟨⟨ 3.73,	1,	5,	7,	0.27,	1 ⟩⟩
{AC♯D♯E♯}:	⟨⟨ 1,	1,	4,	1,	1,	16 ⟩⟩

Common-Tone Theorem

The number of common tones between sets *X* and *Y* =

$$\frac{1}{12}\sum_{n=0}^{11}|f_n(A)||f_n(B)|\cos(\varphi_n(A)-\varphi_n(B))$$

For each component (sum over components)

Weighted by the component magnitudes

Cosine of the phase difference (ranges from –1 for opposite phases to 1 for same phase)

Common-Tone Theorem

The number of common tones between sets X and Y =

$$\frac{1}{12}\sum_{n=0}^{11}|f_n(A)||f_n(B)|\cos(\varphi_n(A)-\varphi_n(B))$$

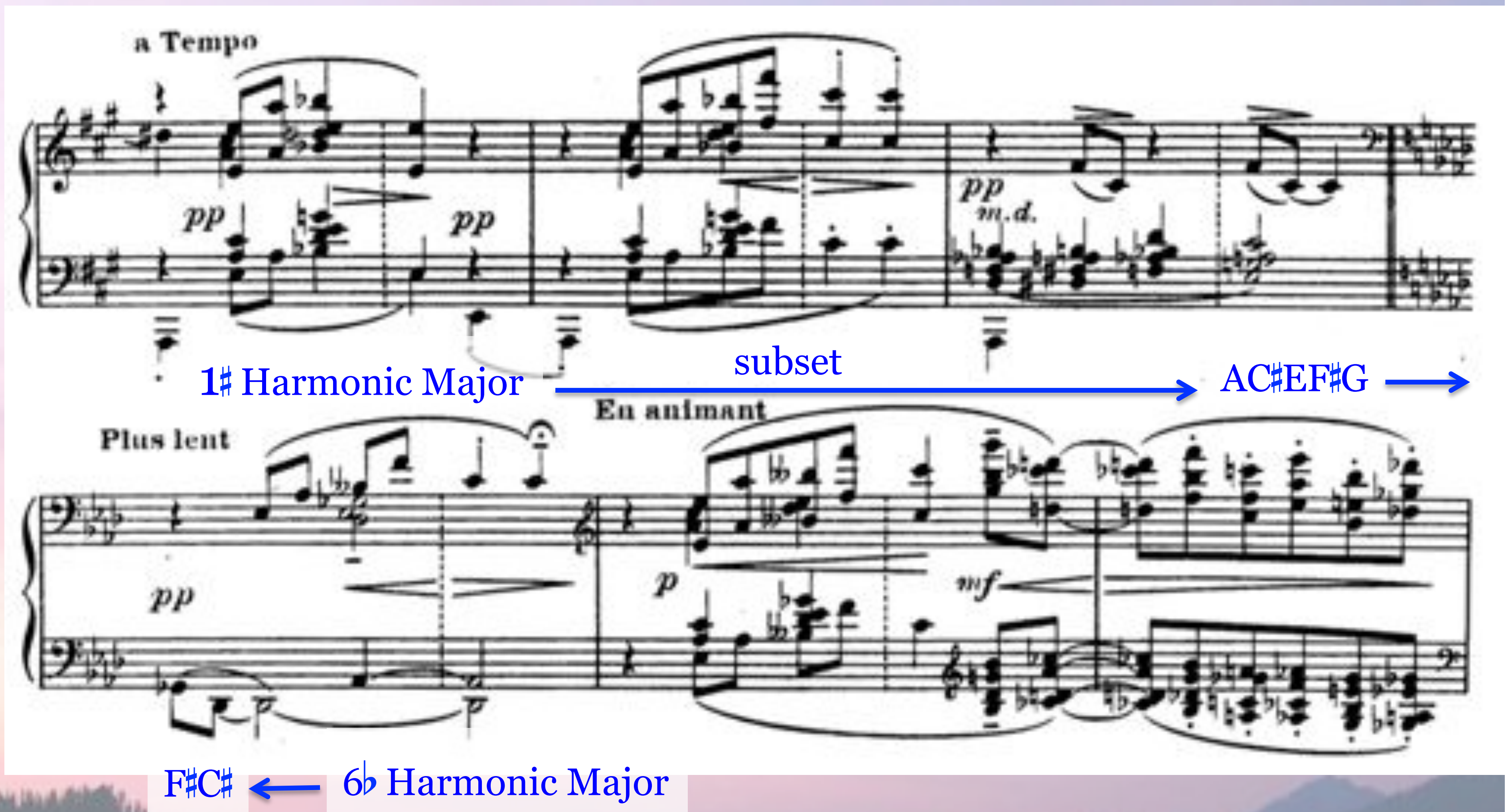
In other words:
Distances in phase space for the *most prominent components* determine the number of common tones

Common-Tone Theorem

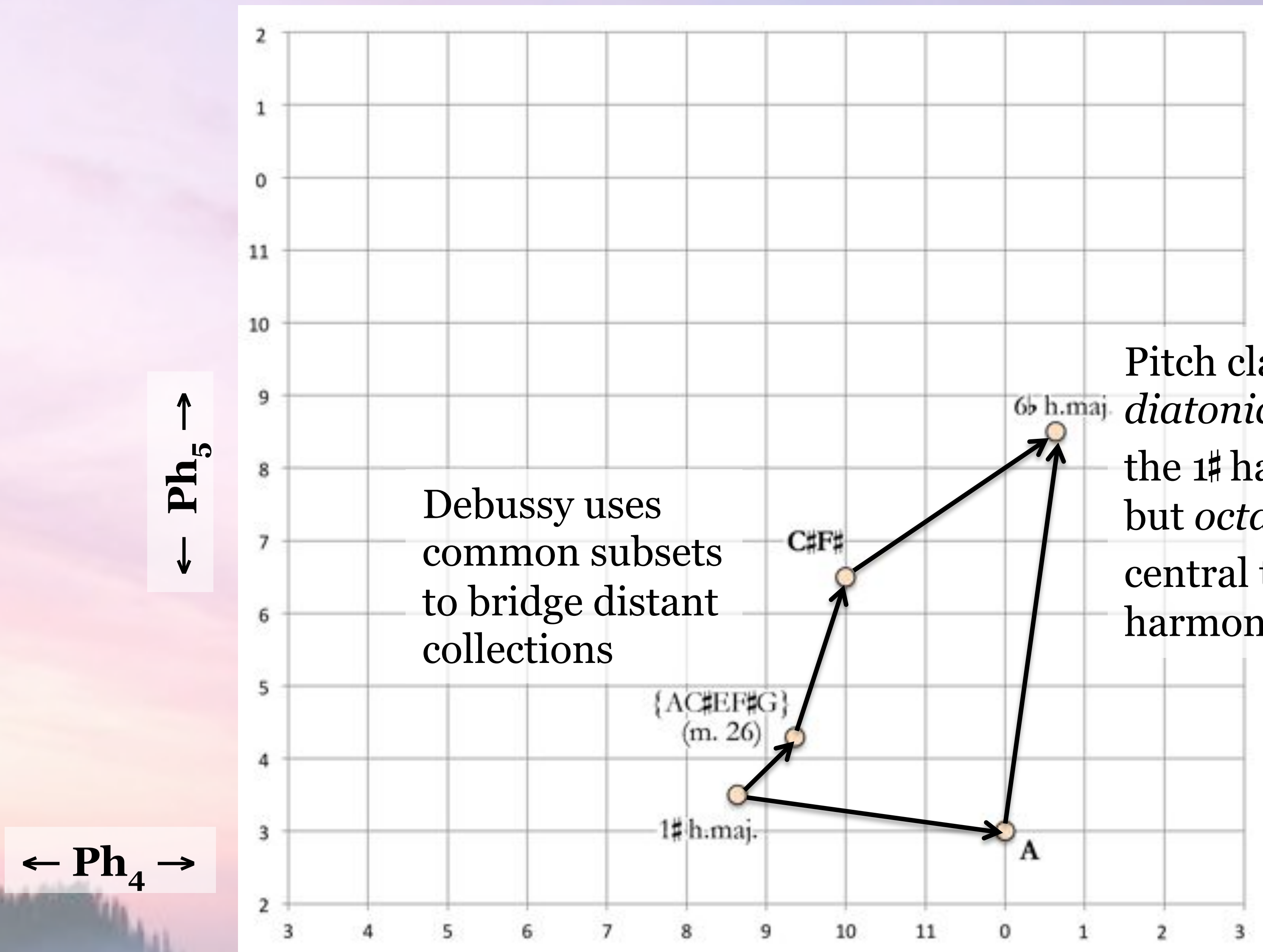
Example:

	$\lvert f_0\rvert$	$(\lvert f_1\rvert, Ph_1)$	$(\lvert f_2\rvert, Ph_2)$	$(\lvert f_3\rvert, Ph_3)$	$(\lvert f_4\rvert, Ph_4)$	$(\lvert f_5\rvert, Ph_5)$	$(\lvert f_6\rvert, Ph_6)$
{AB♭CD♭EGA♭}:	⟨⟨ 7,	(1.93, 2.5)	(1, 8)	(2.24, 11.1),	(2.65, 8.64)	(0.52, 0.5),	(1, 0) ⟩⟩
{AC♯D♯E♯}:	⟨⟨ 4,	(1, 9)	(1, 6)	(2, 9)	(1, 0)	(1, 9)	(4, 6) ⟩⟩
Multiply mag., Cosine of Ph. diff.	28,	(1.9, −1.0)	(1, 0.5)	(4.5, 0.45)	(2.7, −0.19)	(0.52, −0.26)	(4, −1)
Multiply and divide by twelve	2.33 +	−0.31 +	0.08 +	0.33 +	−0.08 +	−0.02 +	−0.33
Sum	= 2 common tones (A and C♯)						

Common-tone Linkage: Debussy



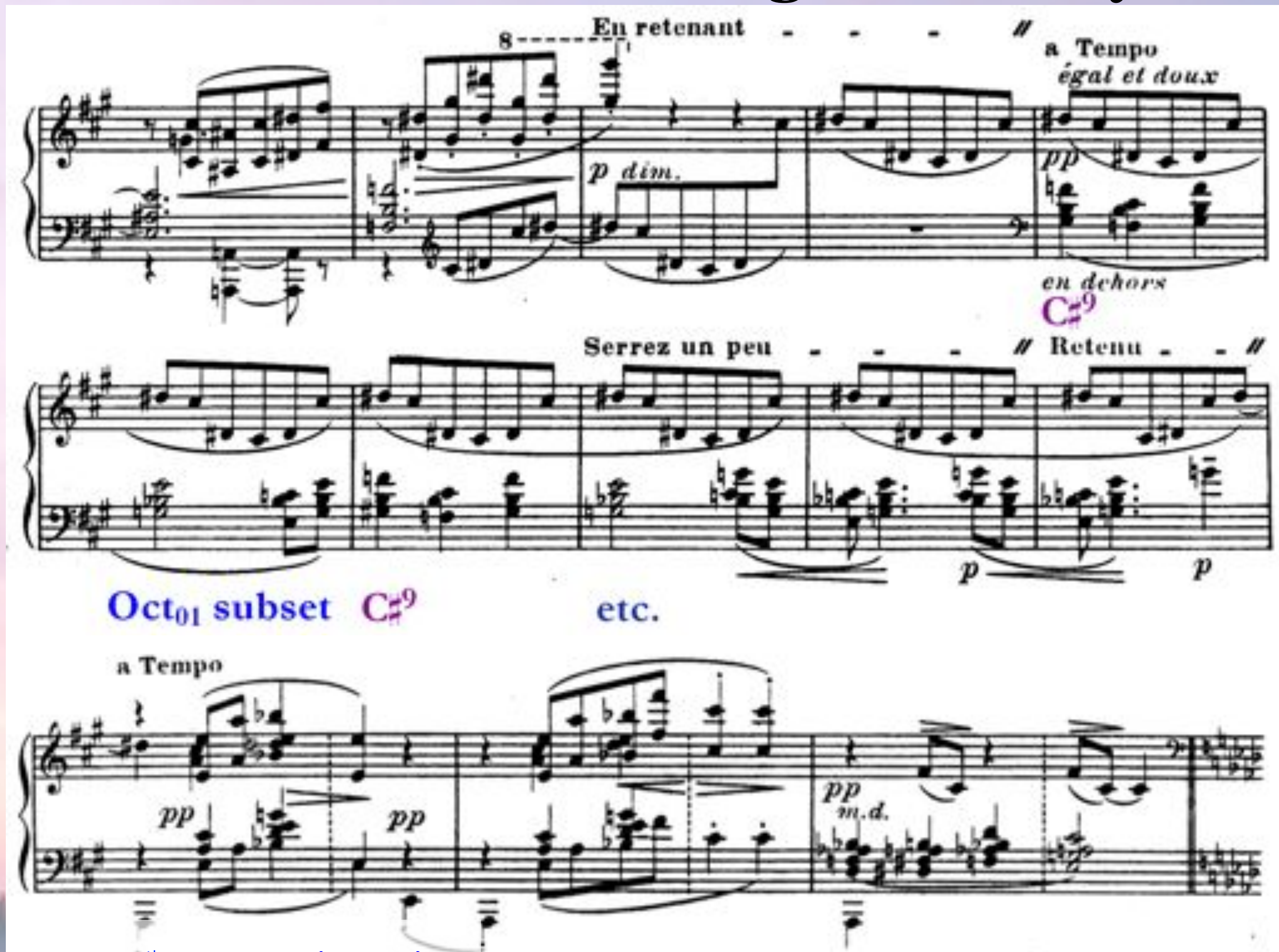
Common-tone Linkage: Debussy



Debussy uses common subsets to bridge distant collections

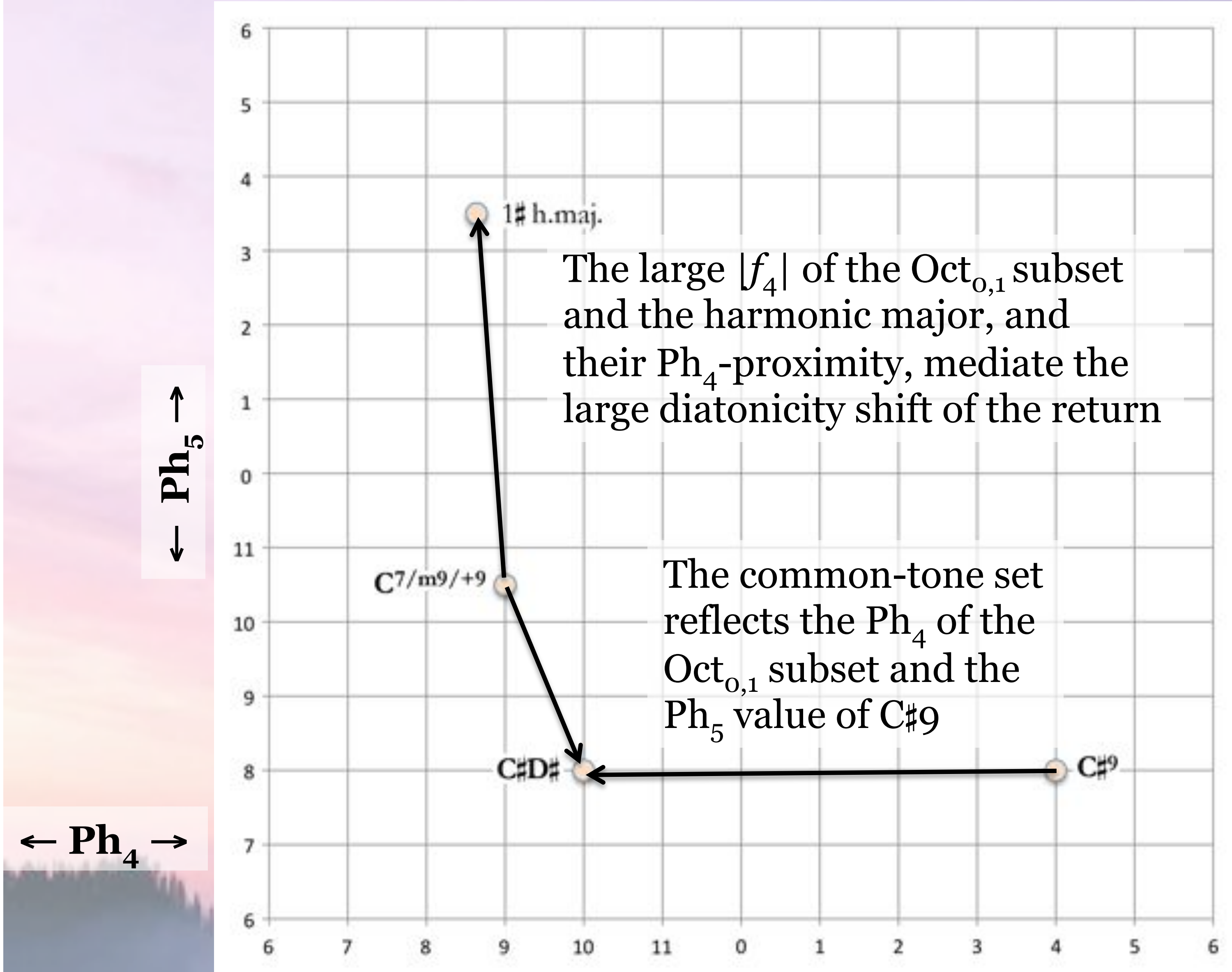
Pitch class A is *diatonically* central to the 1♯ harmonic major but *octatonically* central to the 6♭ harmonic major

Common-tone Linkage: Debussy



1♯ Harmonic Major

Common-tone Linkage: Debussy



The large $|f_4|$ of the $Oct_{0,1}$ subset and the harmonic major, and their Ph_4-proximity, mediate the large diatonicity shift of the return

The common-tone set reflects the Ph_4 of the $Oct_{0,1}$ subset and the Ph_5 value of C♯9

Common-Tone Theorem

Example:

	$\|f_0\|$	$(\|f_1\|, Ph_1)$	$(\|f_2\|, Ph_2)$	$(\|f_3\|, Ph_3)$	$(\|f_4\|, Ph_4)$	$(\|f_5\|, Ph_5)$	$(\|f_6\|, Ph_6)$
$C^{7/m9/+9}$	⟨⟨ 7,	(0.52, 5.5)	(1, 8)	(2.2, 6.9),	(2.6, 8.6)	(1.9, 3.5)	(1, 0) ⟩⟩
1♯ h.maj.	⟨⟨ 6,	(1.4, 10.5)	(0, –)	(1.4, 1.5)	(3.5, 9)	(1.4, 10.5)	(0, –) ⟩⟩
Multiply mag., Cosine of Ph. diff.	42,	(0.7, −0.9)	(0, –)	(3.2, −0.95)	(2.7, −0.19)	(0.52, −0.26)	(0, –)
Multiply and divide by twelve	3.5 +	−0.1 +	0 +	−0.5 +	1.5 +	−0.4 +	0

Sum = 4 common tones (C♯, E, G, B♭)

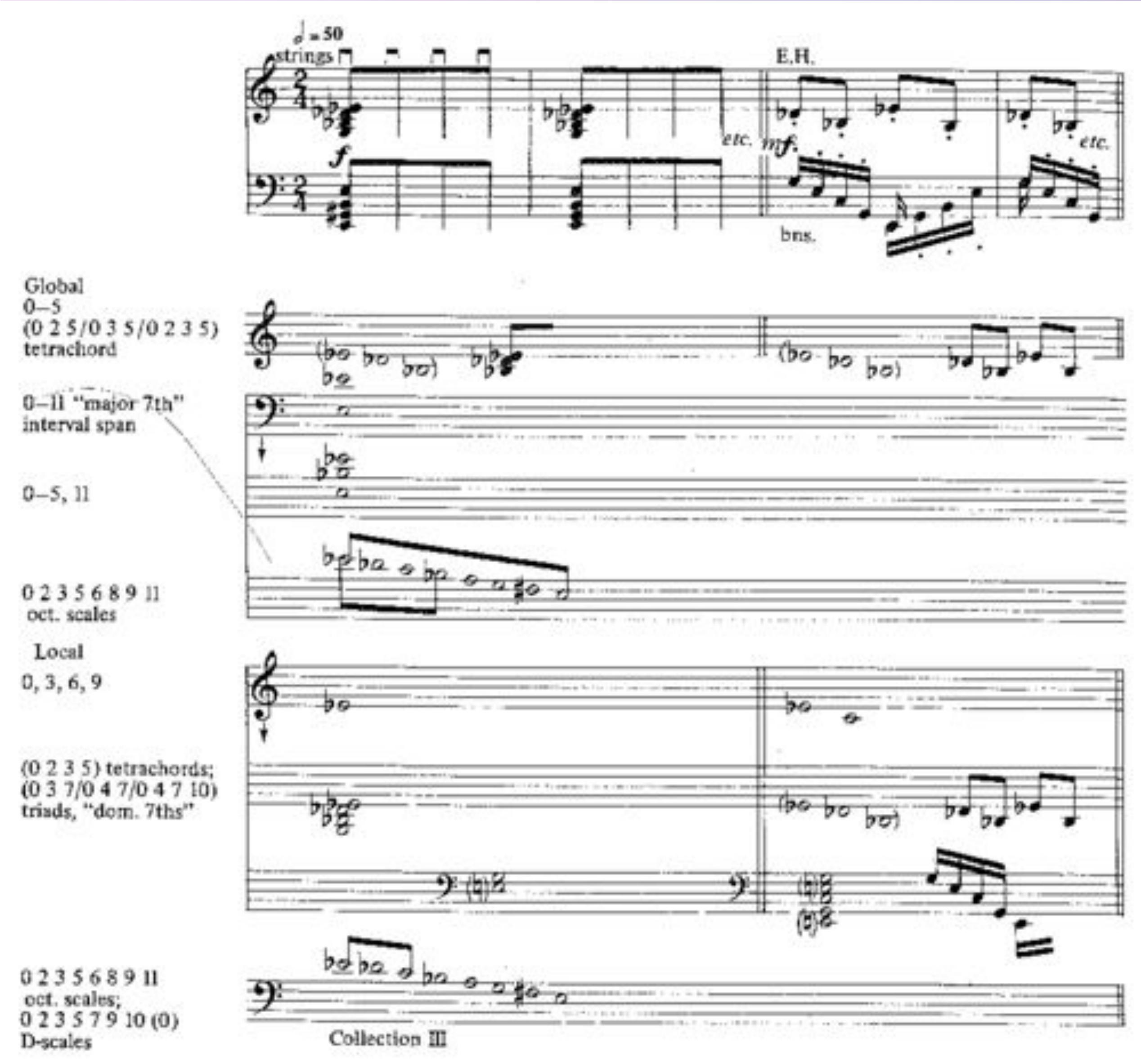
Stravinsky and the Octatonic

1. *Rite of Spring*, Introduction and *Augurs*

Stravinsky and the Octatonic

Is this music octatonic?

Stravinsky and the Octatonic



Van den Toorn:
5 of the 7 notes in the "Augurs" chord and the C triad come from $\text{Oct}_{0,1}$ (E + E♭7).
The G♯ and B reinforce E

Stravinsky and the Octatonic

What are the chances?!?!

100%

All 8-note collections overlap at least one octatonic by six or more pitch classes.

Stravinsky and the Octatonic

Joseph Straus (review of *Music of Stravinsky*):

Van den Toorn "never provides and systematic criteria for determining the presence of the octatonic collection; as a result, a number of his attributions are suspect. Almost any passage containing nine to twelve pitch classes can be discussed as 'diatonic interpenetration' of an octatonic context."

Stravinsky and the Octatonic

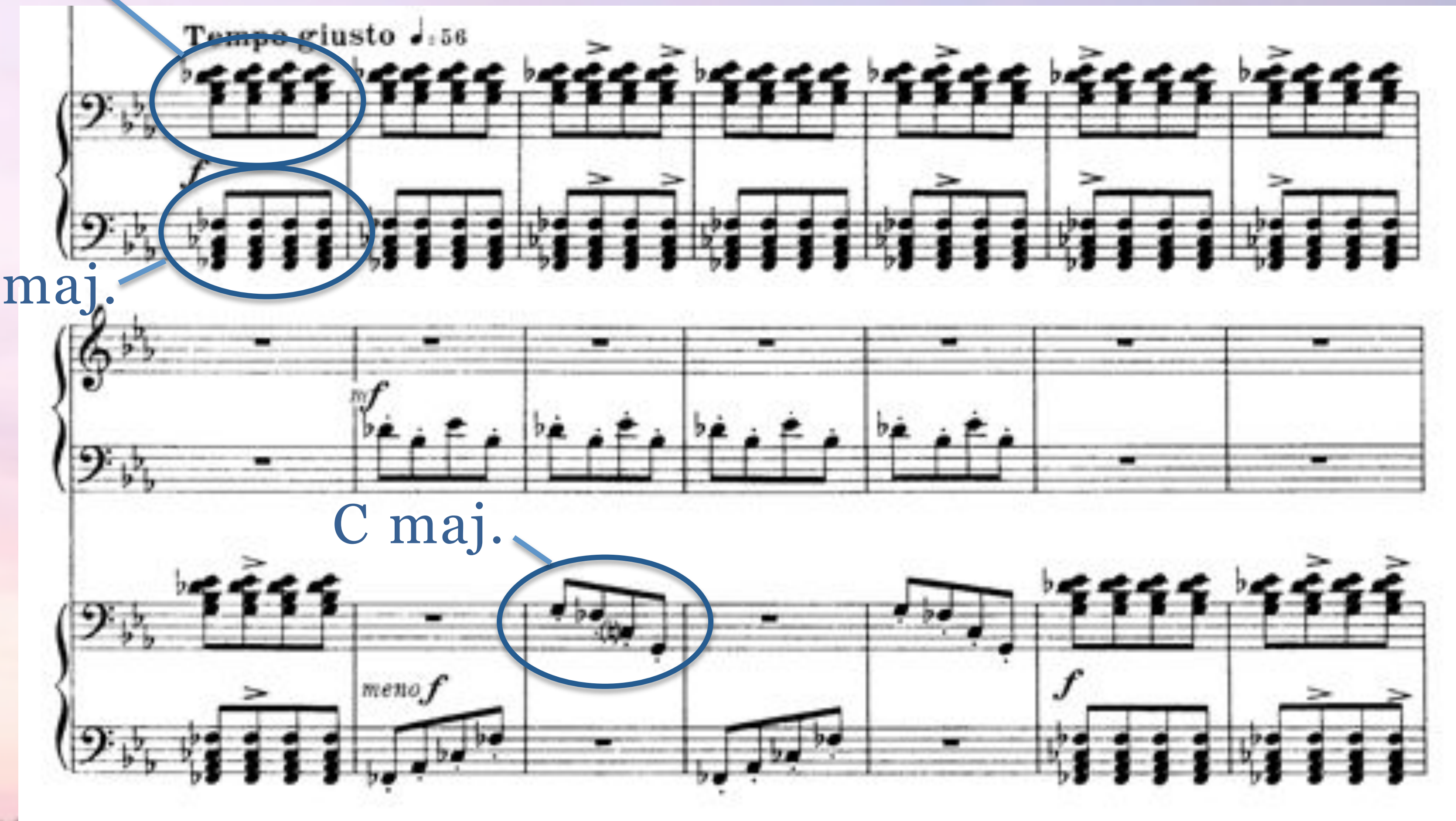
Dmitri Tymoczko (on the analyses in *Music of Stravinsky*):

"If even *these* passages can be understood as the result of 'octatonic-diatonic interpenetration,' then we should rightly ask whether there is any music that *cannot* be understood in this way.

In a sense, there is not: any proper subset can be decomposed into diatonic and octatonic components."

Stravinsky and the Octatonic

Let's try again . . .



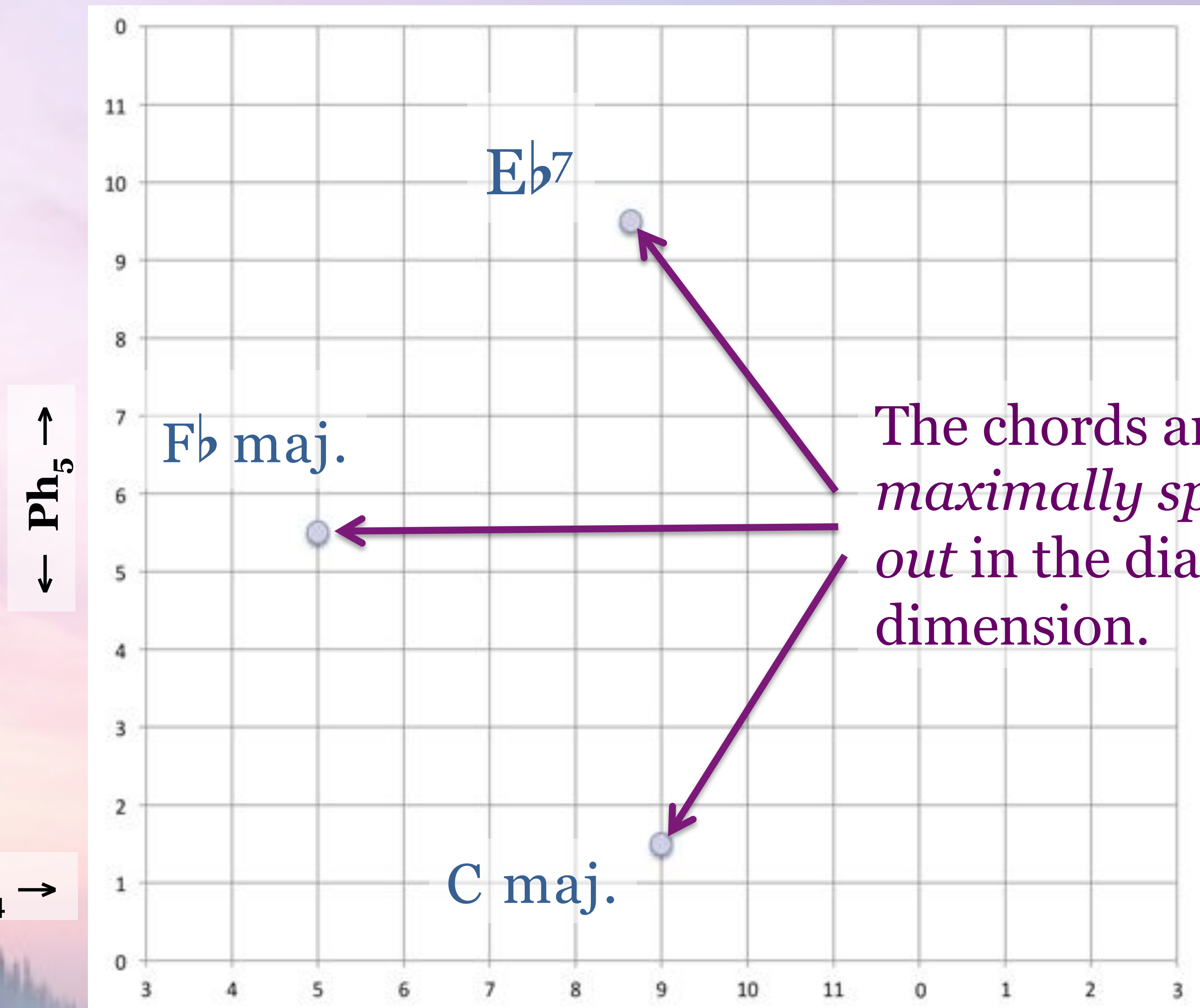
Stravinsky and the Octatonic

Components f_4 and f_5 are the largest across the set-types

DFT magnitudes^2

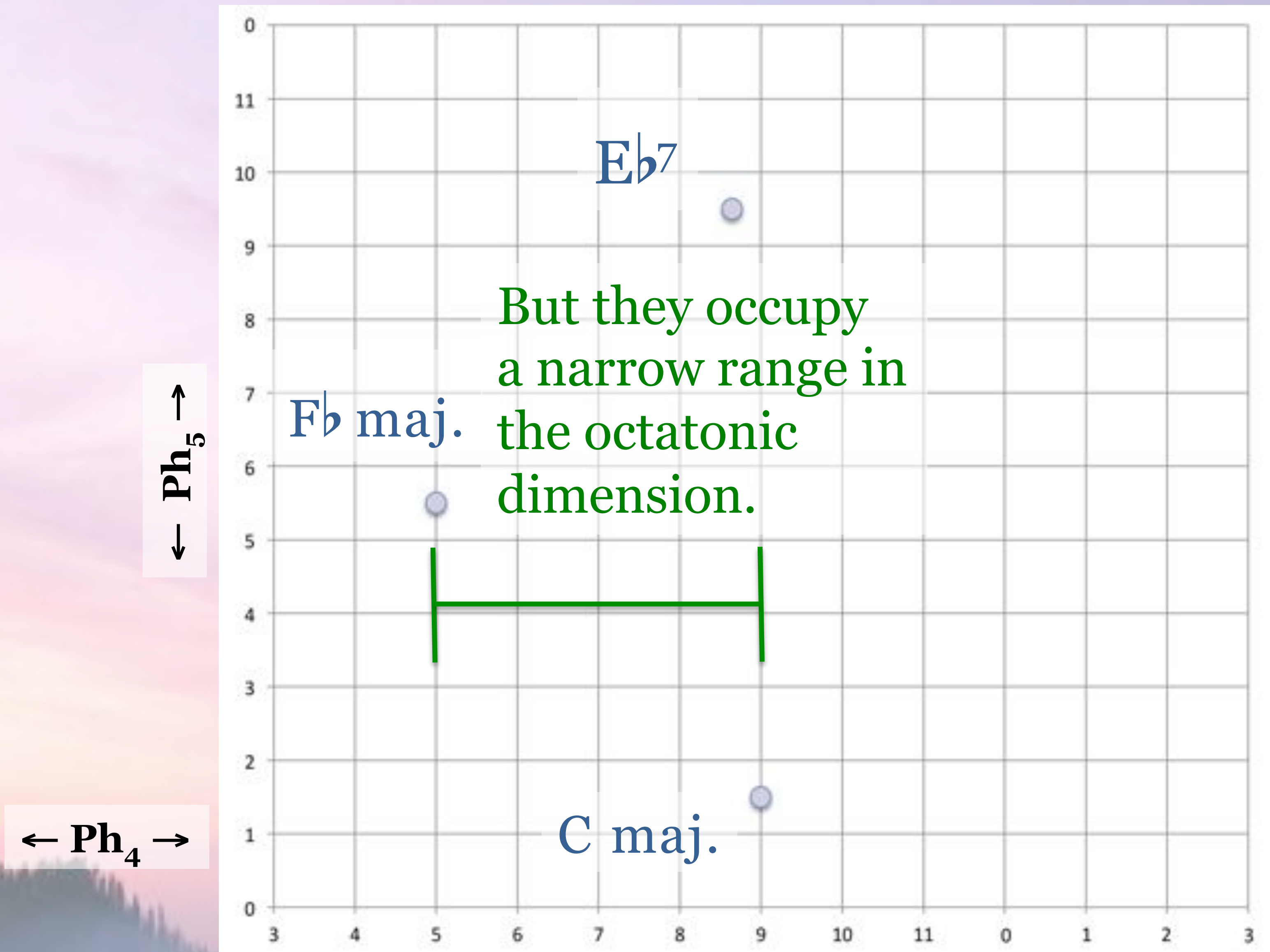
	$\lvert f_1\rvert^2$	$\lvert f_2\rvert^2$	$\lvert f_3\rvert^2$	$\lvert f_4\rvert^2$	$\lvert f_5\rvert^2$	$\lvert f_6\rvert^2$
Major/minor triad:	⟨⟨ 0.27,	1,	5,	3,	3.73,	1 ⟩⟩
Dominant 7^{th}:	⟨⟨ 0.27,	1,	2,	7,	3.73,	4 ⟩⟩

Stravinsky and the Octatonic

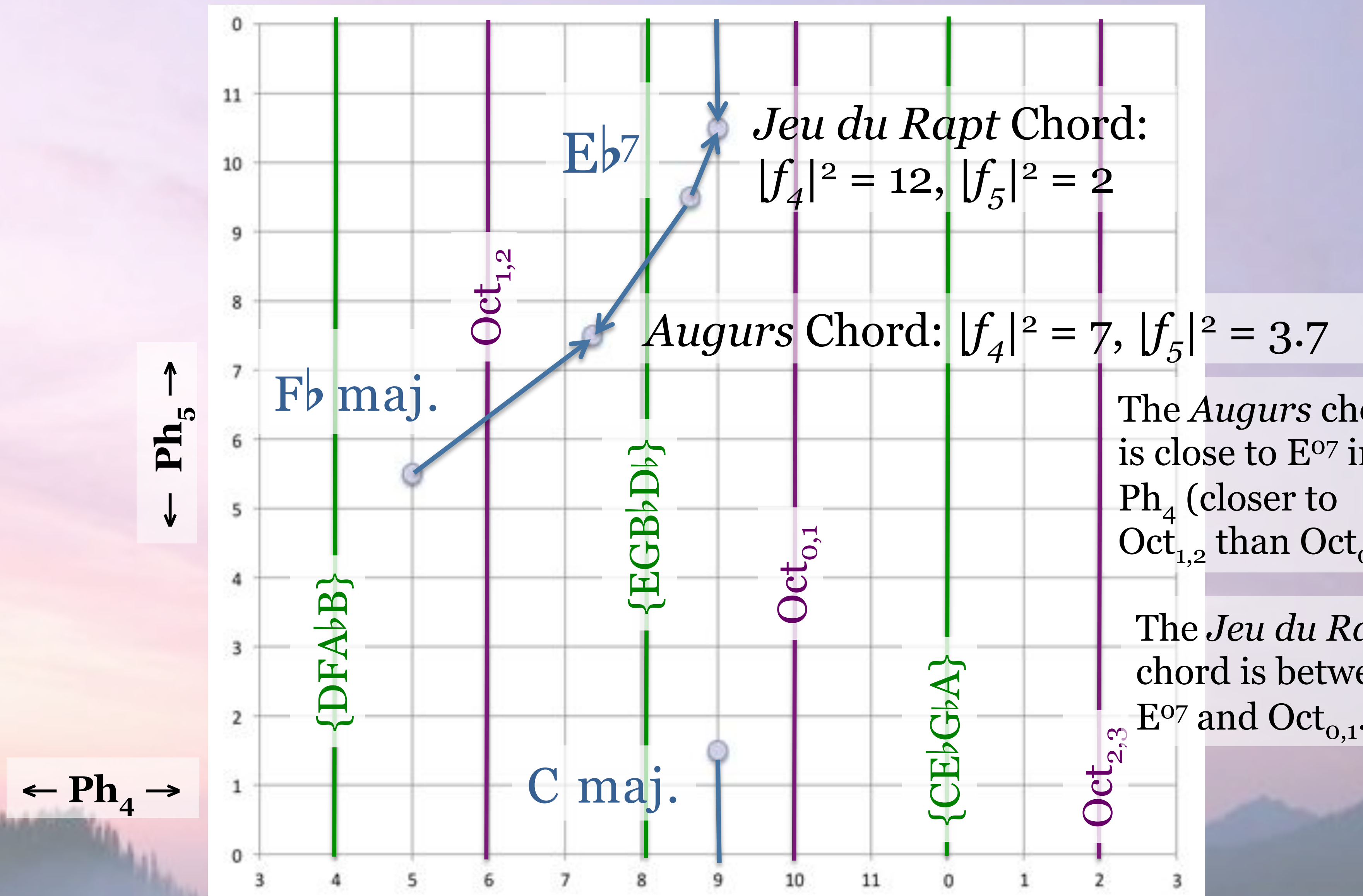


The chords are *maximally spread out* in the diatonic dimension.

Stravinsky and the Octatonic



Stravinsky and the Octatonic

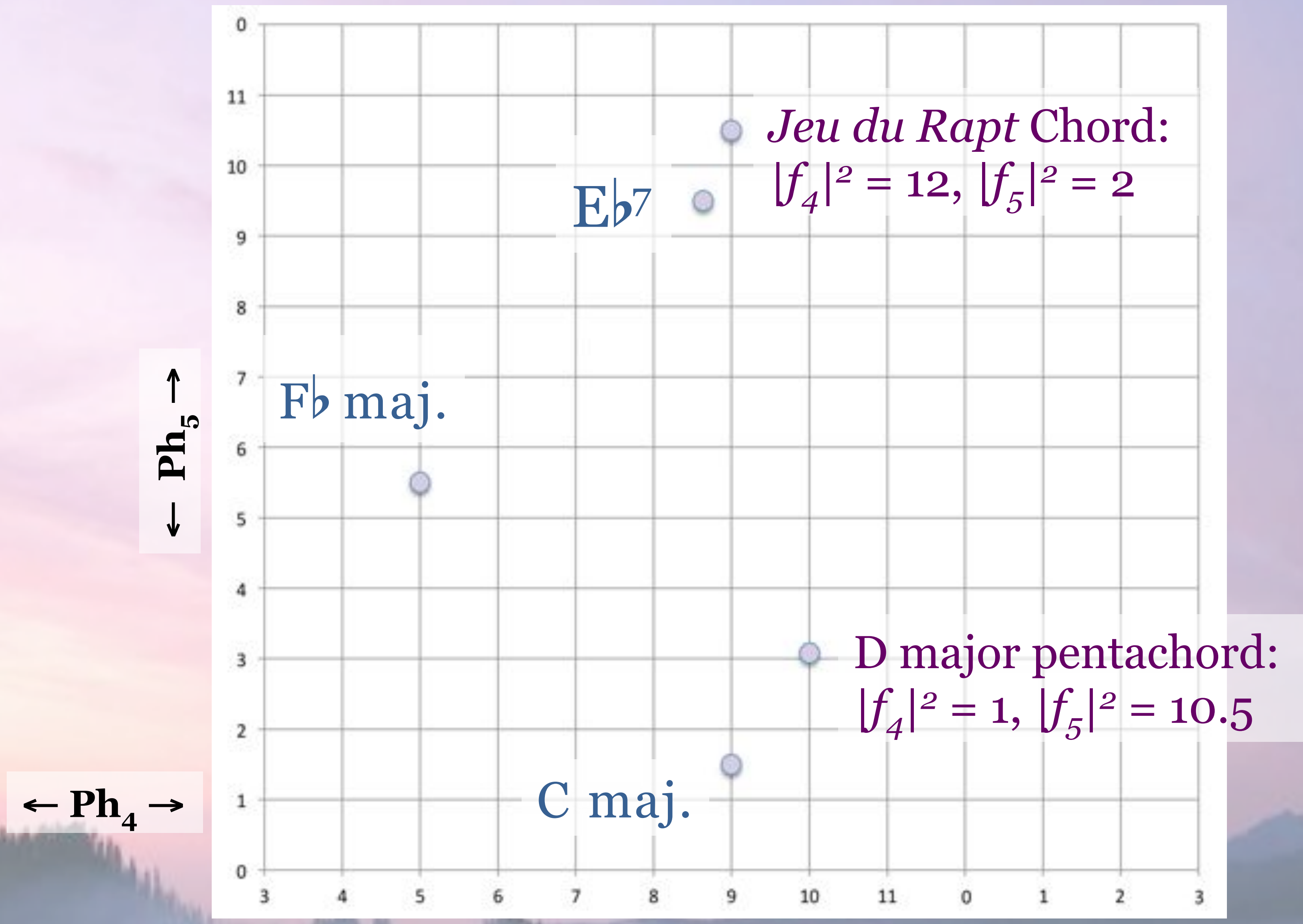


The *Augurs* chord is close to E^{o7} in Ph_4 (closer to $Oct_{1,2}$ than $Oct_{0,1}$!)

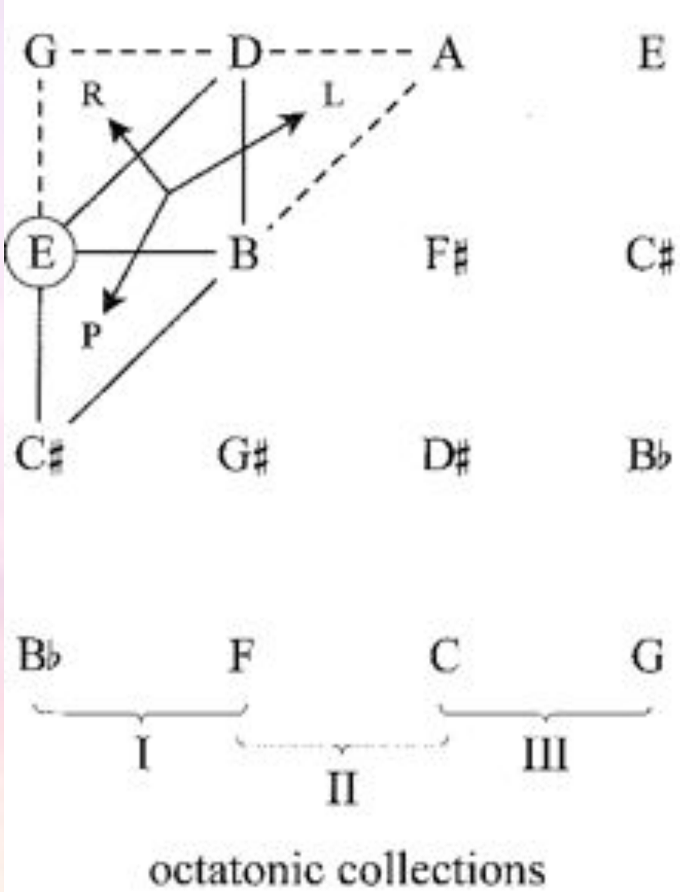
The *Jeu du Rapt* chord is between E^{o7} and $Oct_{0,1}$.

Stravinsky and the Octatonic: *Jeu du Rapt*

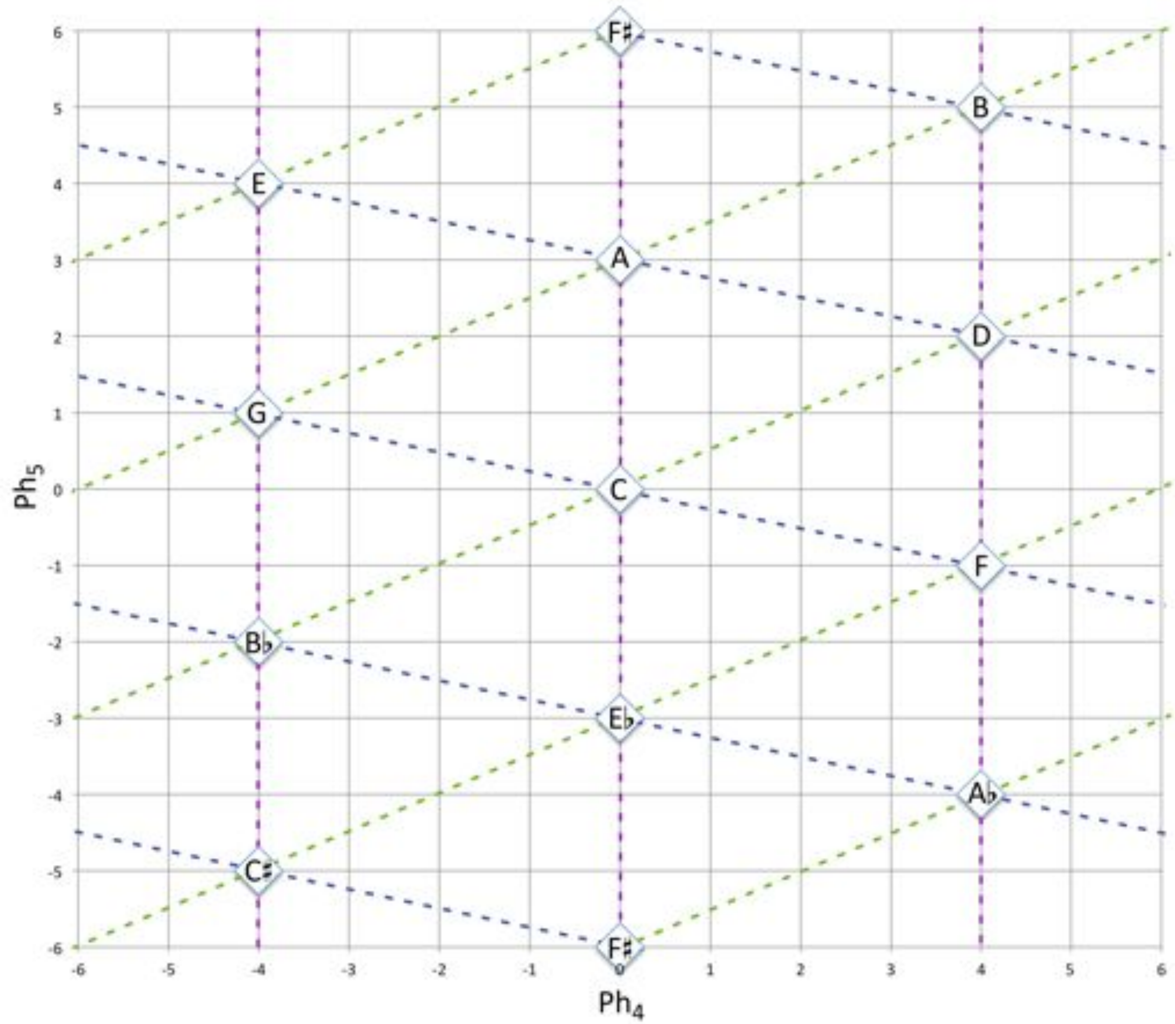
Stravinsky and the Octatonic



Stravinsky and the Octatonic



Octatonic Tonnetz
Van den Toorn and McGuiness 2012



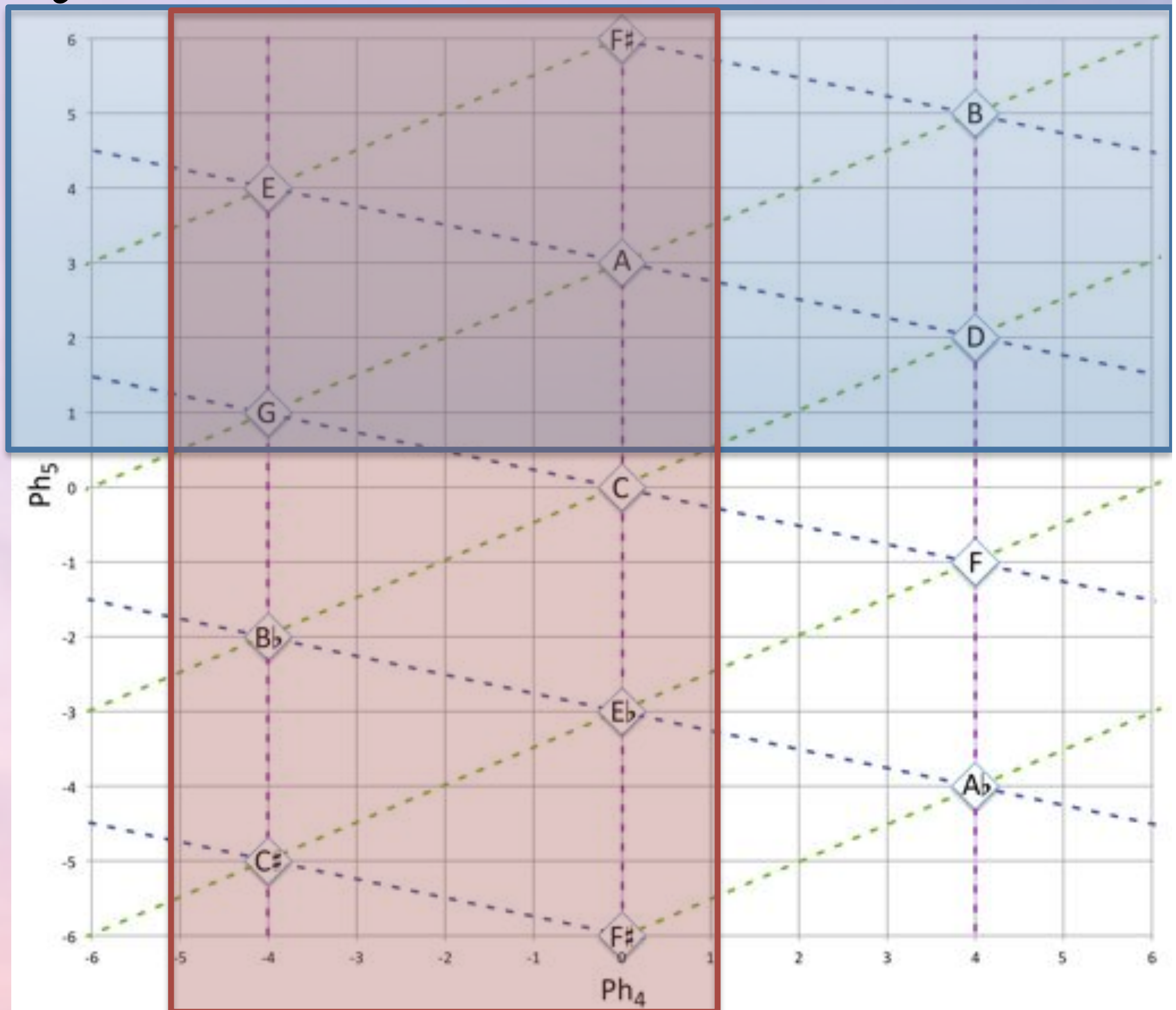
Pitch classes in $Ph_{4,5}$-space

Stravinsky and the Octatonic

Diatonic materials:
D major hexachord

Overlap:
A-dorian tetrachord
(cf. *Jeu du Rapt*)

Octatonic materials:
$Oct_{0,1}$



Pitch classes in $Ph_{4,5}$-space

Feldman, *Palais de Mari*

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Feldman, *Palais de Mari*

Features of the piece:

- Composed in 1986, Feldman's last work for solo piano.
- Long but sparse: the 9-page score takes ca. 25 minutes to play.
- Made up of discrete gestures, frequently repeated and varied (often in subtle ways).
- Pedal is held continuously throughout most of the piece. This blurs the distinction between *successive* and *simultaneous* sounds.
- Extended segmentational analysis in Hanninen, *A Theory of Music Analysis* (2012).

Feldman, *Palais de Mari*

Features of the piece:

- Long sections on the piece tend to dwell on a limited set of gestures, giving the piece a sense of trajectory that is nonetheless non-teleological.
- Composed around the same time as his *Second String Quartet*, which Feldman described as "a dialectic of sorts between such elements as . . . chromaticism/consonance."
- "Reverse Development": Gestures often appear *before* the idea from which they are derived, replacing a process of development with one of *revelation*.

Feldman, *Palais de Mari*

The initial gesture (Hanninen set *A*) stages a chromatic–diatonic conflict

(025)	⟨⟨ **2.27**, 1, 1, 3, **5.73**, 1 ⟩⟩
(012)	⟨⟨ **7.46**, 4, 1, 0, **0.54**, 1 ⟩⟩
(0125)	⟨⟨ **5.73**, 3, 4, 1, **2.27**, 0 ⟩⟩

The first three notes are highly diatonic, but the final E introduces a concentrated chromaticism.

Hanninen: “The contrast between harmonies rich in ics 2 and 5, versus those rich in ic1, resonates throughout the piece.”

Feldman, *Palais de Mari*

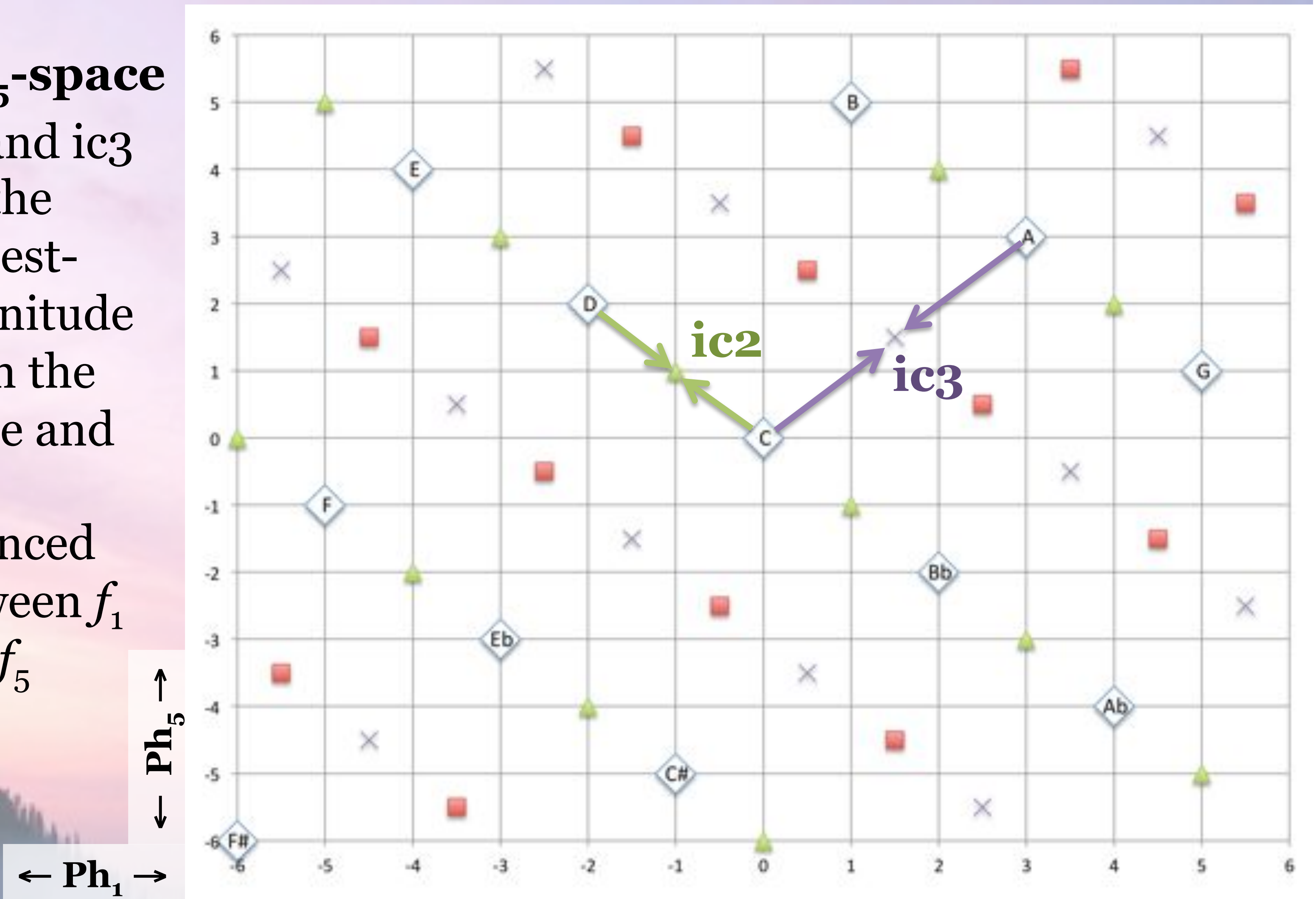
The initial gesture (Hanninen set *A*) stages a chromatic–diatonic conflict

The gesture can also be divided by part into a fourth and a minor second.

Feldman, *Palais de Mari*

$Ph_{1,5}$-space

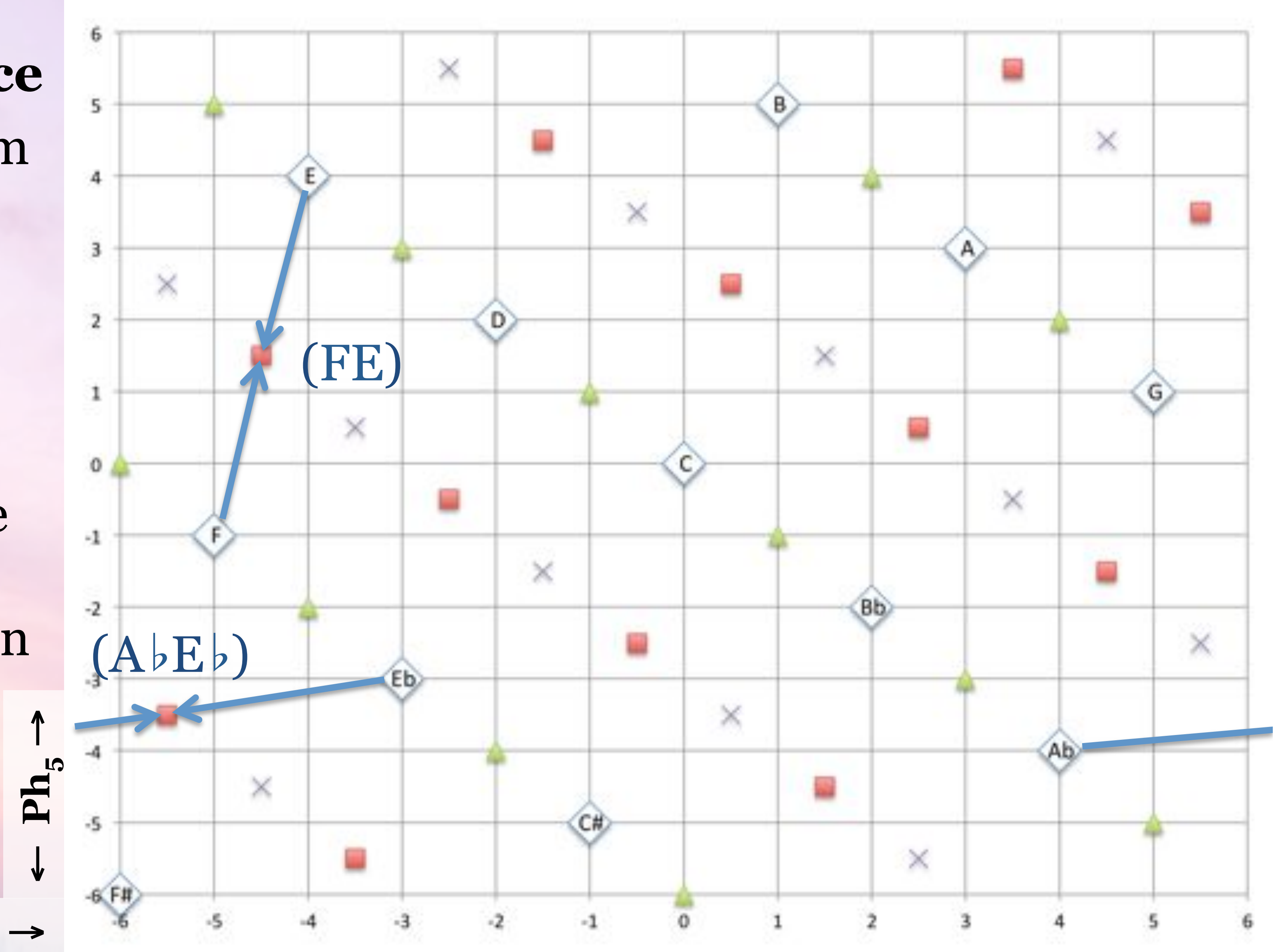
ic2 and ic3 are the highest-magnitude ics in the space and are balanced between f_1 and f_5



Feldman, *Palais de Mari*

$Ph_{1,5}$-space

Dyads from the upper and lower voices of m. 1—ic1 and ic5 occupy the same positions in the space.

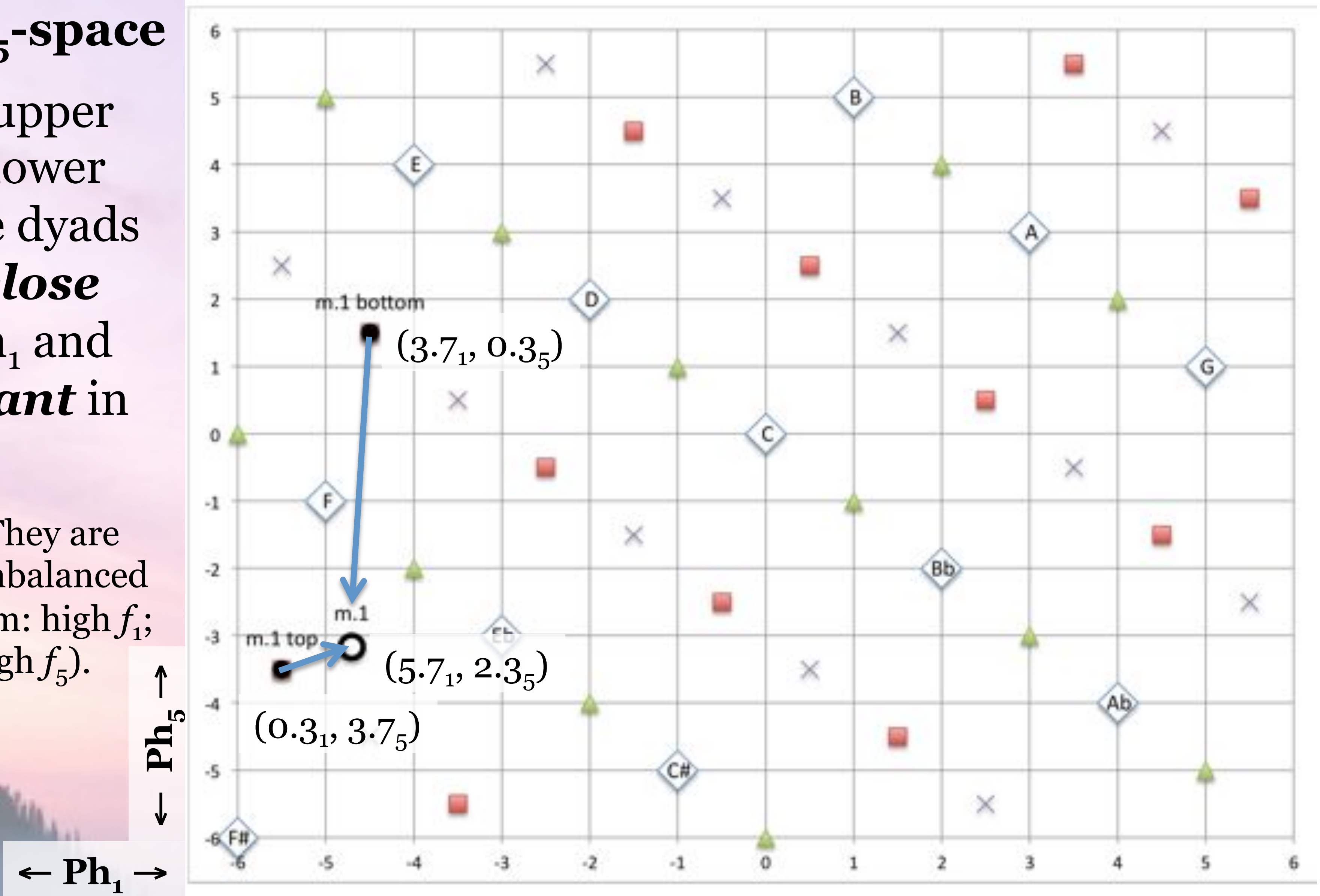


Feldman, *Palais de Mari*

$Ph_{1,5}$-space

The upper and lower voice dyads are ***close*** in Ph_1 and ***distant*** in Ph_5.

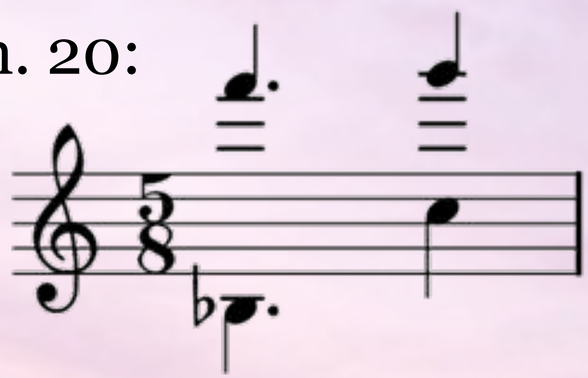
N.B.: They are also imbalanced (bottom: high f_1; top: high f_5).



Feldman, *Palais de Mari*

One of Feldman's basic harmonic techniques is
Transpositional Combination

m. 20:



(B♭F)

T_2

ic5*ic2

m. 41:

(CD)

T_1

ic1*ic2

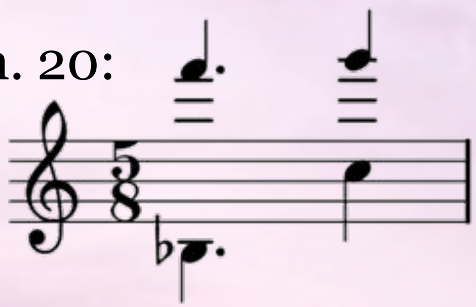
(Hanninen set *C*)

(Hanninen set *E/d*)

Feldman, *Palais de Mari*

Convolution Theorem: Transpositional combination (with doublings retained) is the same as **multiplying DFT magnitudes** and adding the phases.

m. 20:



(B♭F)

T_2

= (t7) ×(02)

$$\begin{array}{l} \quad (t7) \\ \times\ (02) \\ \hline =(t079) \end{array}$$

$$\begin{array}{lllllll} & \langle\langle\ (0.27, 4.5), & (3, 3), & (2, 7.5), & (1, 6), & (3.73, 10.5), & (0, -)\ \rangle\rangle \\ \times\ \langle\langle & (3, 11), & (1, 10), & (0, -), & (1, 2), & (3, 1), & (4, 0)\ \rangle\rangle \\ \hline =\ \langle\langle & (0.8, 3.5), & (3, 1), & (0, -), & (1, 8), & (11.2, 11.5), & (0, -)\ \rangle\rangle \end{array}$$

Feldman, *Palais de Mari*

Transposition of entire gestures by semitone reinforces component 1 and cancels out component 5

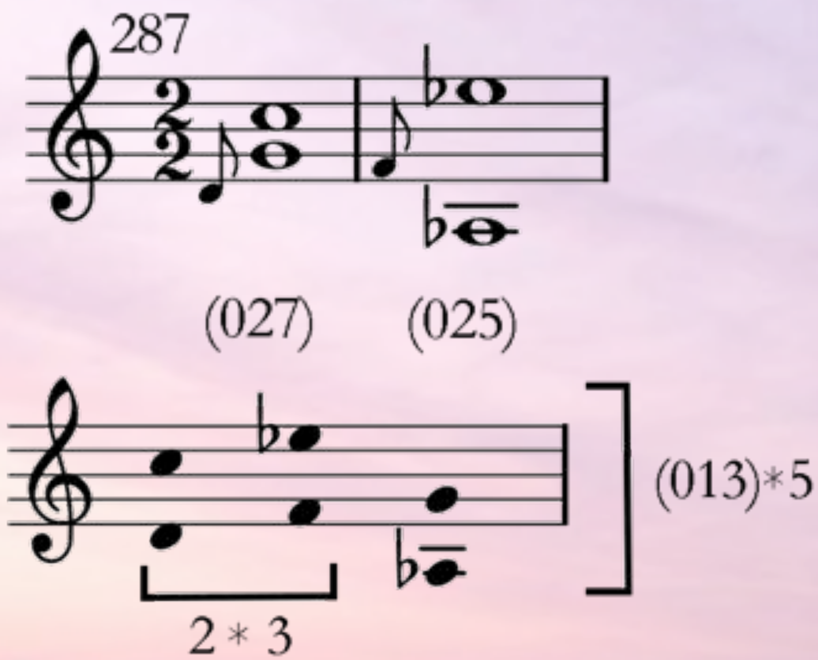
$$\begin{array}{rl} & \mathbf{(46)} \\ \times & \mathbf{(03)} \\ = & \mathbf{(4679)} \end{array} \qquad \begin{array}{rl} & \langle\langle\, (3, 7)_1, \quad (3, 5)_5 \,\rangle\rangle \\ \times & \langle\langle\, (2, 10.5)_1, \ (2, 10.5)_5 \,\rangle\rangle \\ \hline = & \langle\langle\, (\mathbf{6}, 5.5)_1, \ (\mathbf{6}, 3.5)_5 \,\rangle\rangle \end{array}$$

(0235) = 2*3 is balanced between f_1 and f_5, but multiplying by (01) weakens f_5 in favor of f_1

$$\begin{array}{rl} & \mathbf{(4679)} \\ \times & \mathbf{(01)} \\ = & \mathbf{(4567^2 89)} \end{array} \qquad \begin{array}{rl} & \langle\langle\, (6, 5.5)_1, \quad (6, 3.5)_5 \,\rangle\rangle \\ \times & \langle\langle\, (3.73, 11.5)_1, (0.27, 9.5)_5 \,\rangle\rangle \\ \hline = & \langle\langle\, (\mathbf{22.4}, 5)_1, \quad (\mathbf{1.6}, 8)_5 \,\rangle\rangle \end{array}$$

Feldman, *Palais de Mari*

This idea is repeated frequently throughout the last part of the piece.



RH LH

This variant of the idea makes it evident that *both* are products of (013) with ic5 or ic1:

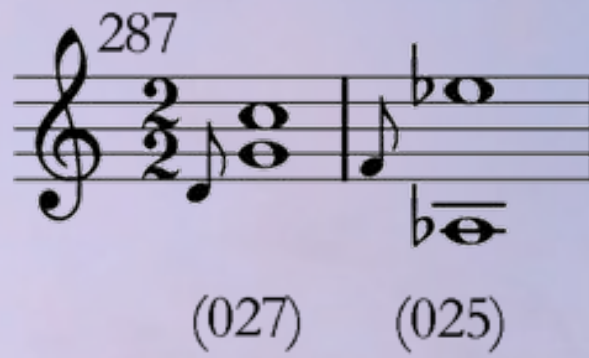
RH LH

(Hanninen *G/a*)

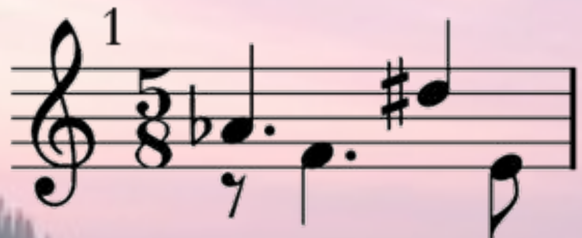
Feldman, *Palais de Mari*

Hanninen:

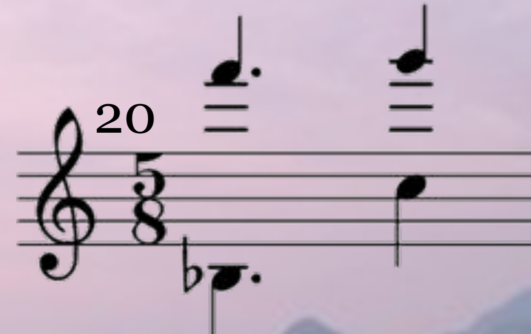
“ The arrival of *G/a*287–88 . . . is the keystone in a remarkable confluence of events. First, it defines the center of subset *G/a*, and also of set *G*. Second, it forms a bridge to set *A*, recalling and rearranging intervals and key pcs of set *A*. Third, it forms a second, and stronger, bridge to set *C*.”



Gesture from set *A*:



Gesture from set *C*:

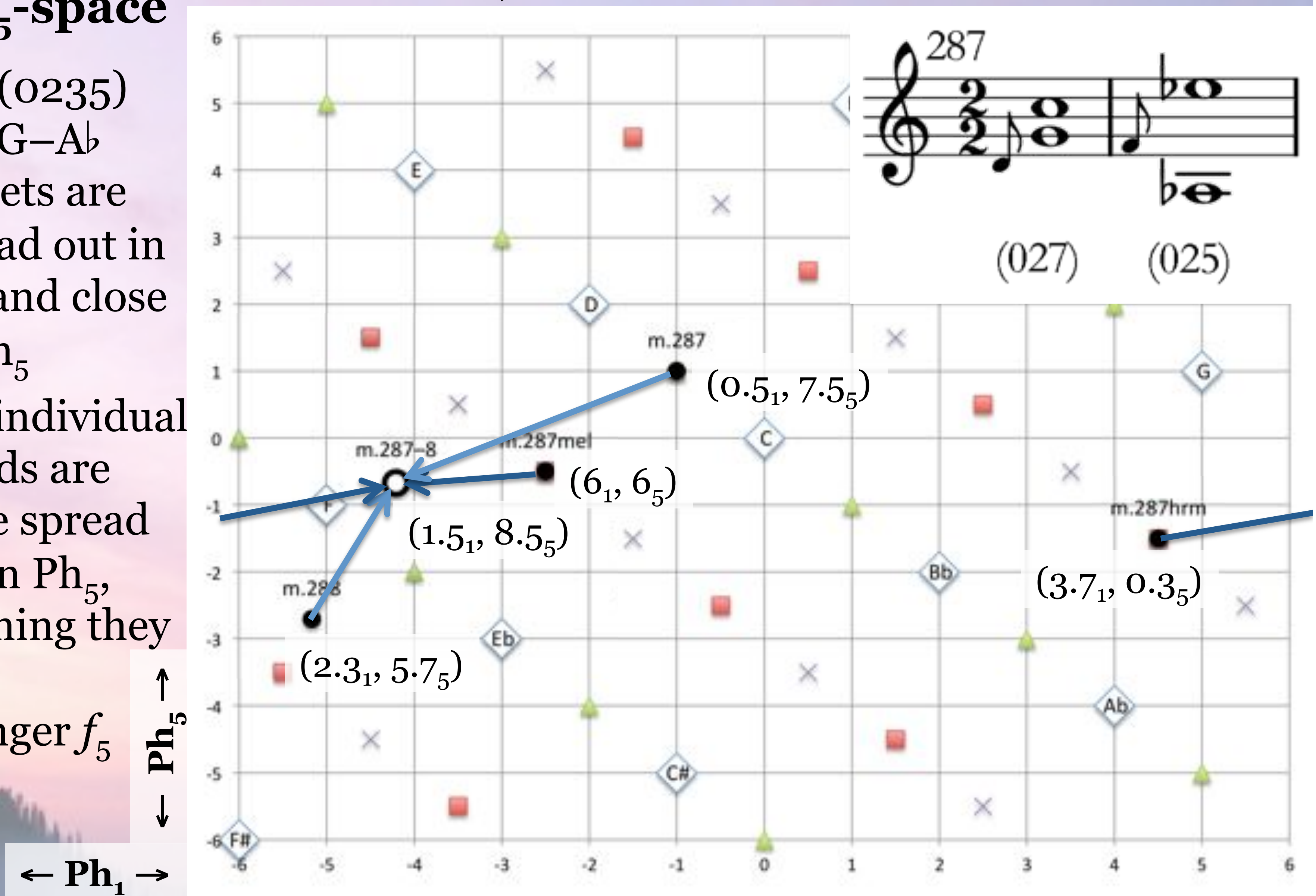


Feldman, *Palais de Mari*

$Ph_{1,5}$-space

The (0235) and G–A♭ subsets are spread out in Ph_1 and close in Ph_5

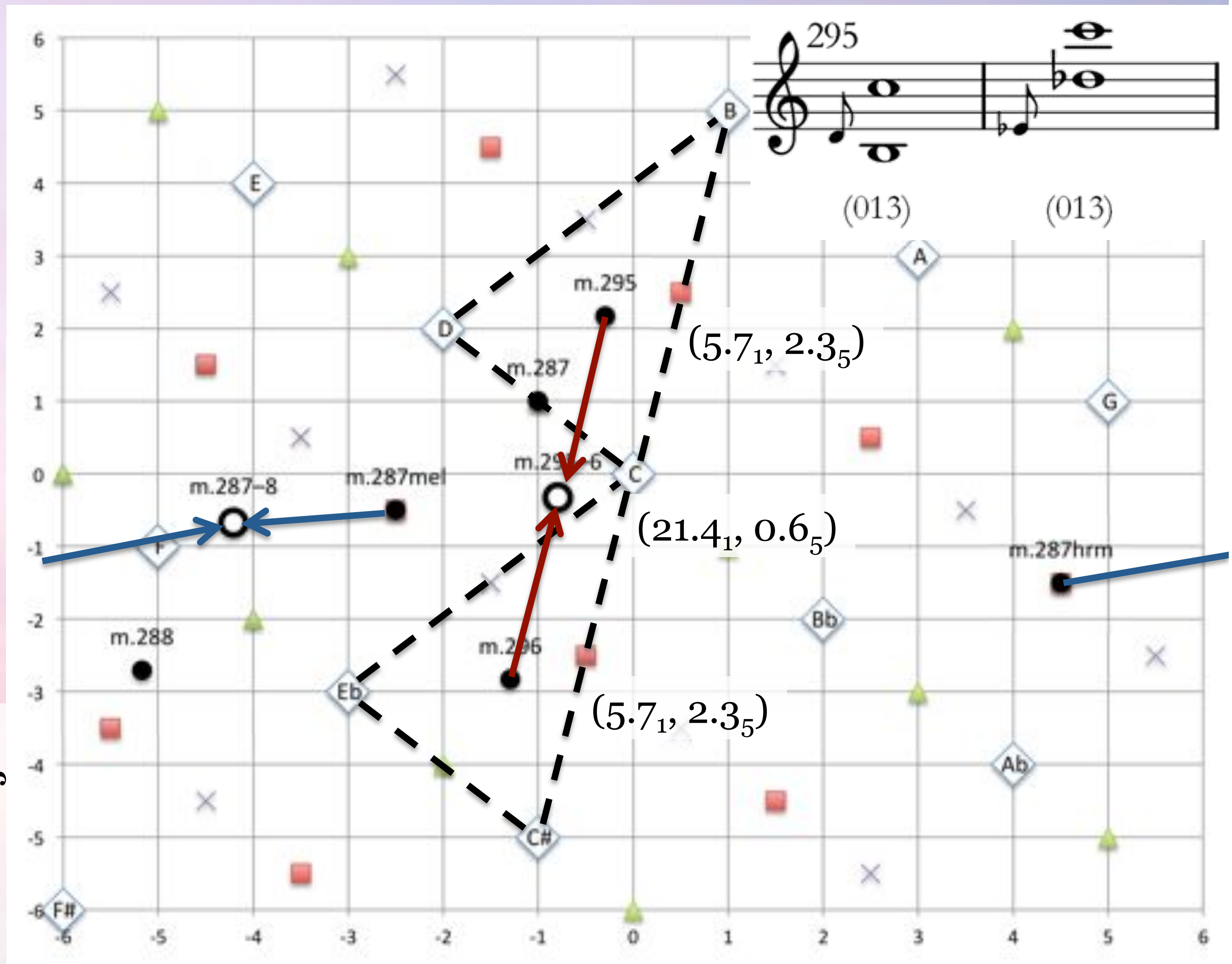
The individual chords are more spread out in Ph_5, meaning they have stronger f_5



Feldman, *Palais de Mari*

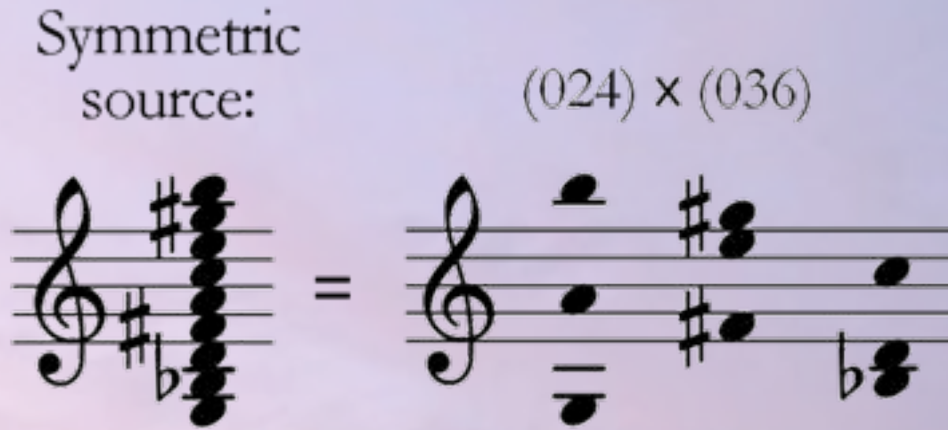
$Ph_{1,5}$-space

The (013)s of the variant gesture are spread out in Ph_5 and close in Ph_1, making the sum strongly chromatic.



Feldman, *Palais de Mari*

One important gesture reveals how Feldman "cripples" symmetries by asymmetrically dividing a symmetric entity



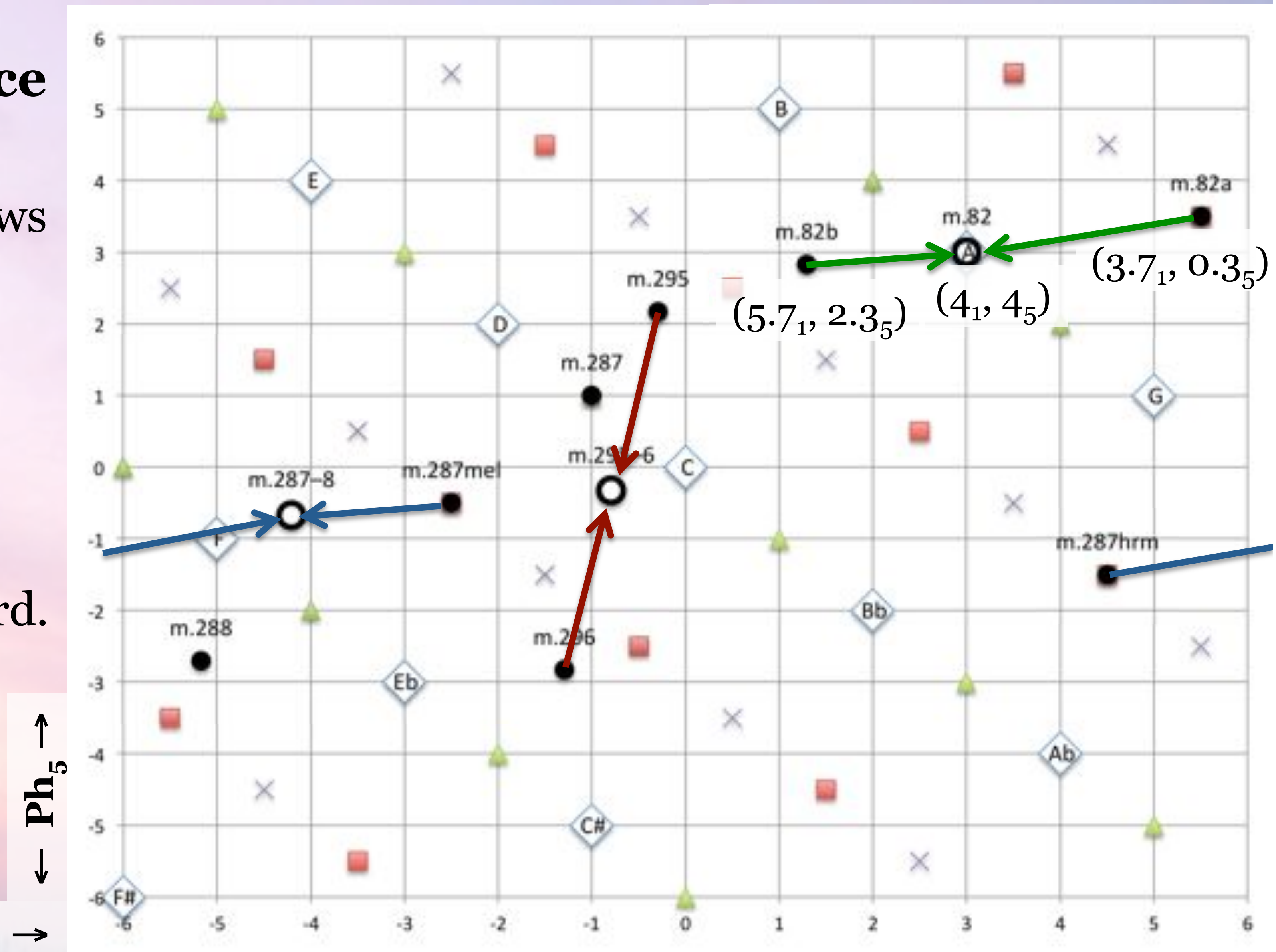
(F♯GG♯D)	$\langle\langle (\mathbf{3.73}, 5.5)_1, (0.27, 3.5)_5 \rangle\rangle$
+ (AB♭BCE)	$\langle\langle (\mathbf{5.73}, 1.3)_1, (2.27, 2.8)_5 \rangle\rangle$
= (F♯GG♯AB♭BCDE)	$\langle\langle (\mathbf{4}, 3)_1, (\mathbf{4}, 3)_5 \rangle\rangle$

The symmetric source chord is balanced between f_1 and f_5, but it is split into more heavily chromatic chords.

Feldman, *Palais de Mari*

$Ph_{1,5}$-space

The Ph_1 spread shows how more chromatic chords are extracted from a balanced parent chord.



Summary

- DFT is a **change of basis** applied to the domain of pc-distributions.
- Each DFT component measures a musically interpretable quality relating to a type of **periodicity**.
- **DFT magnitudes** can replace much of pcset-theory's use of interval content to relate harmonic entities.
- The **fifth Fourier component** measures **diatonicity**, and provides a more systematic approach to reconciling **subsets and supersets** with scale theory.
- The **fourth Fourier component** represents **octatonicity** and is used by composers like Debussy and Stravinsky to relate diatonically distant harmonies.
- Distances in **phase space** provide a common-tone-based measure of relatedness between collections of any size.

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