



The Fourier Transform and a Theory of Harmony for the Twentieth Century

Jason Yust, Boston University

Society for Music Theory
St. Louis, Oct. 29–Nov. 1, 2015

A copy of this talk is available at
people.bu.edu/jyust/

Outline

I. Forte's Project and the DFT

1. A theory of harmony for the 20th century
2. Pc-vectors
3. DFT components and interval content
4. Phase spaces

II. Debussy: “Les sons et les parfums tournent dans l’air du soir”

1. Heptatonic scales and diatonicity
2. Common tones and harmonic qualities

III. Stravinsky and the Octatonic

1. *Rite of Spring*, Introduction and *Augurs*
2. Octatonic scale versus octatonic quality

IV. Feldman, *Palais de Mari*

I. Forte's Project and the DFT

1. A theory of harmony for the 20th century
2. Pc-vectors
3. DFT components and interval content
4. Phase spaces

A Theory of Harmony for the 20th Century

Forte's project:

“It is the intention of the present work to provide a general theoretical framework, with reference to which the processes underlying atonal music may be systematically described.”

The Structure of Atonal Music (1973), Preface



A Theory of Harmony for the 20th Century

Forte's project:

General features of harmony that are largely independent of compositional aesthetic:

- *Interval content* determines *harmonic quality*

Interval content \leftrightarrow DFT components

- *Common pc content* determines *harmonic proximity*

Subset relations \leftrightarrow DFT phase spaces

Discrete Fourier Transform on Pcsets

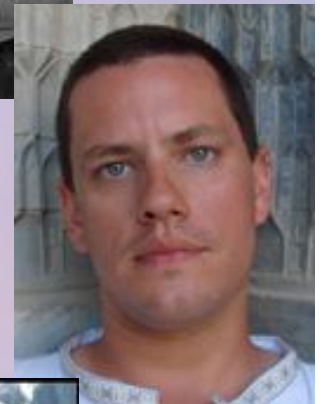
Lewin, David (1959). “Re: Intervallic Relations between Two Collections of Notes,” *JMT* 3/2.

——— (2001). “Special Cases of the Interval Function between Pitch Class Sets X and Y.” *JMT* 45/1.

Quinn, Ian (2006–2007). “General Equal-Tempered Harmony,” *Perspectives of New Music* 44/2–45/1.

Amiot, Emmanuel (2013). “The Torii of Phases.” *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013* (Springer).

Yust, Jason (2015). “Schubert’s Harmonic Language and Fourier Phase Spaces.” *JMT* 59/1.



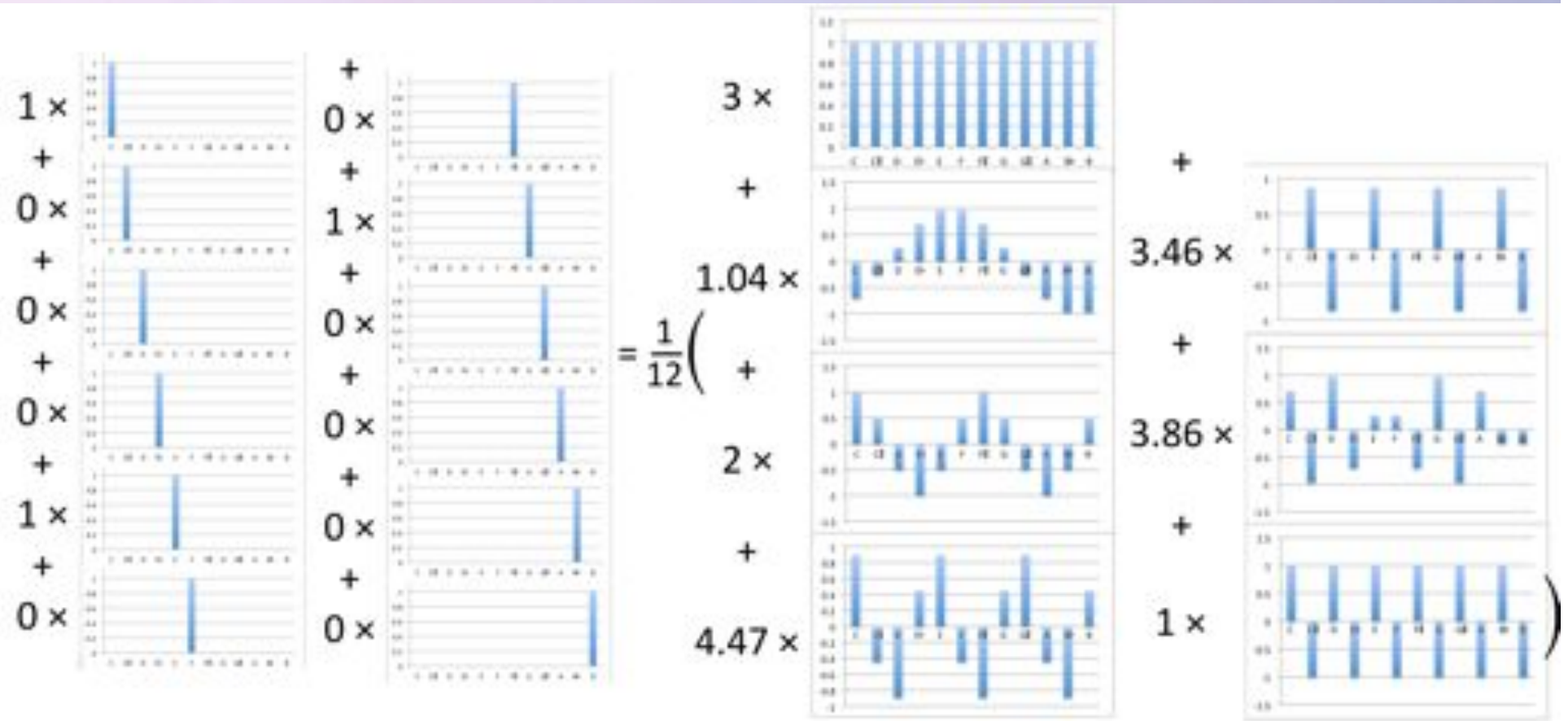
Characteristic Function of a Pcset



The *characteristic function* of a pcset is a **12-place vector** with 1s for each pc and 0s elsewhere:

(1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0)
C C# D E♭ E F F# G G# A B♭ B

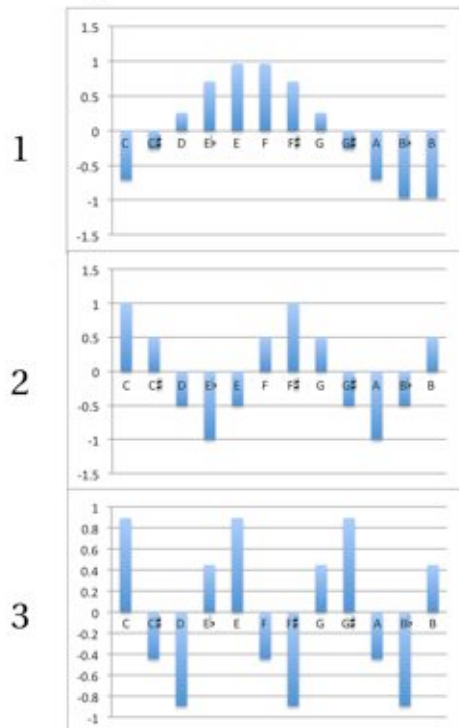
DFT Components



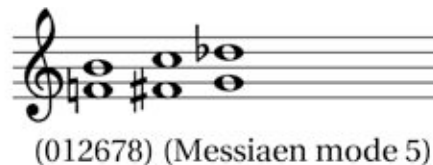
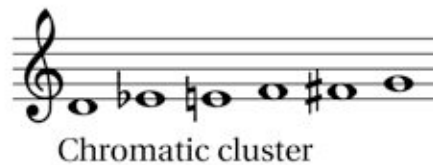
The DFT is a **change of basis** from a sum of pc spikes to a sum of discretized **periodic** (perfectly even) curves.

DFT Components

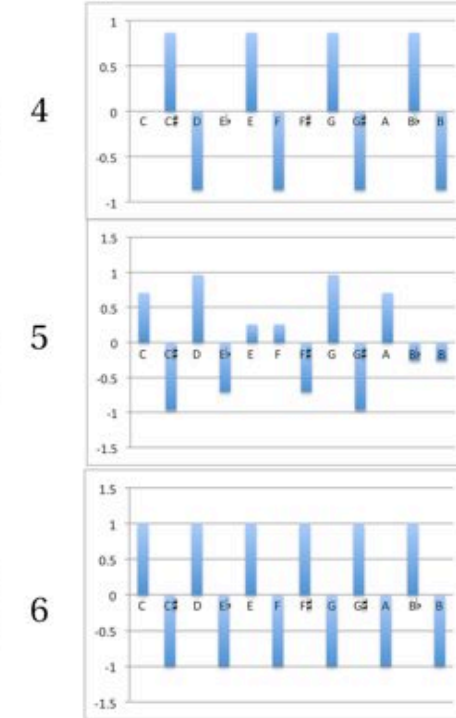
Component



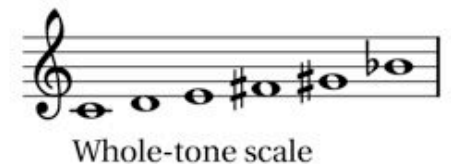
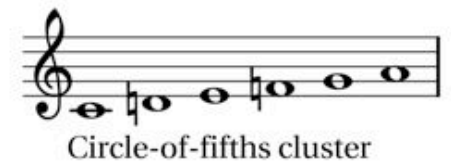
Prototypes



Component



Prototypes



Quinn's **generic prototypes** are pcsets that maximize a given component. **Subsets** and **supersets** of the prototypes are the best representatives of each component

DFT Components

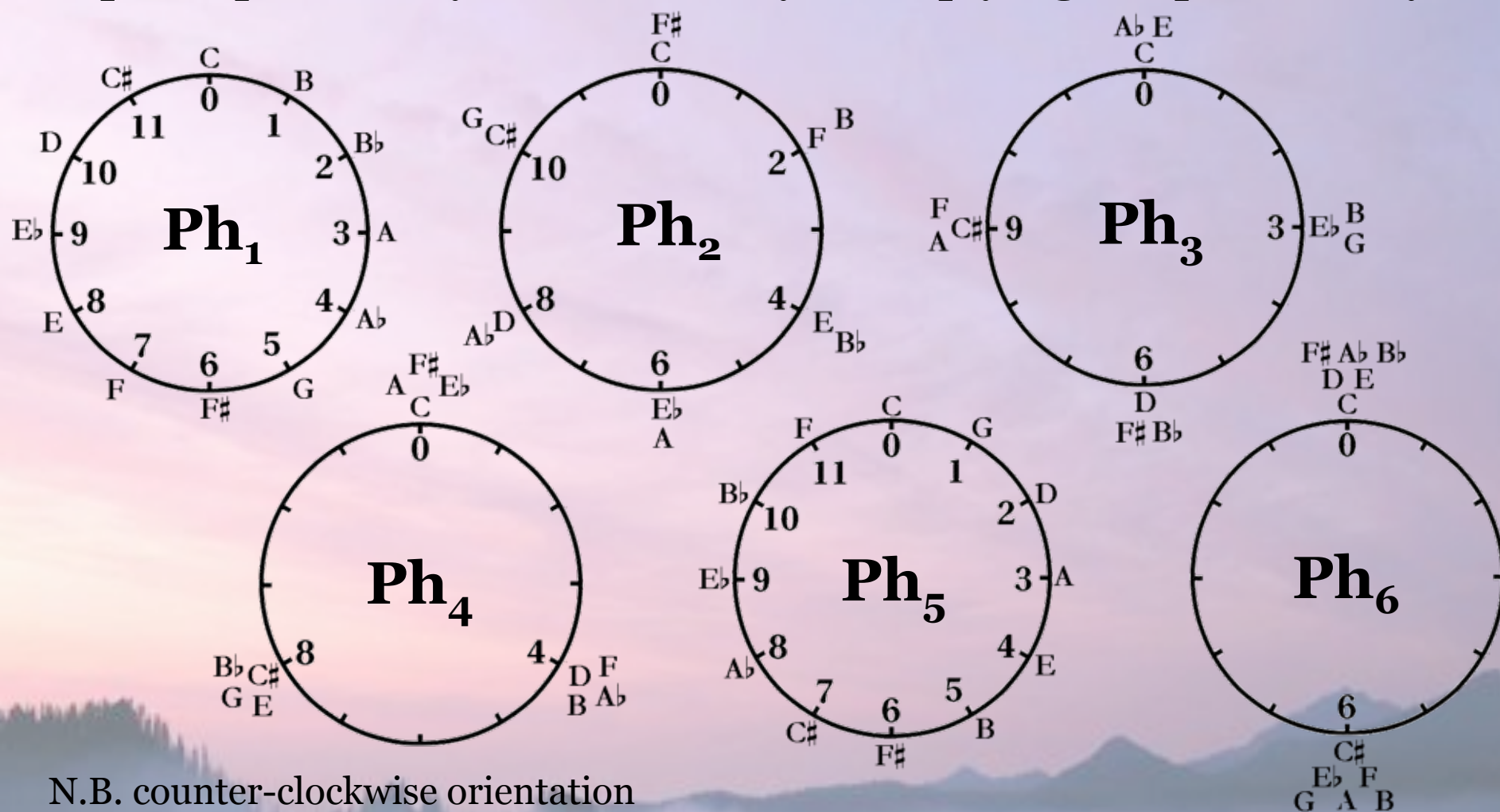
Notation

f_n	The n th DFT component
$ f_n $	The <i>magnitude</i> of the n th component
$ f_n ^2$	Squared magnitude
φ_n	The phase ($0 \leq \varphi_n \leq 2\pi$) of the n th component
\mathbf{Ph}_n	Phase normalized to pc-values: ($0 \leq \mathbf{Ph}_n \leq 12$)

$$\langle\langle (|f_1|^2, \mathbf{Ph}_1), (|f_2|^2, \mathbf{Ph}_2), (|f_3|^2, \mathbf{Ph}_3), (|f_4|^2, \mathbf{Ph}_4), (|f_5|^2, \mathbf{Ph}_5), (|f_6|^2, \mathbf{Ph}_6) \rangle\rangle$$

Phase spaces: One dimensional

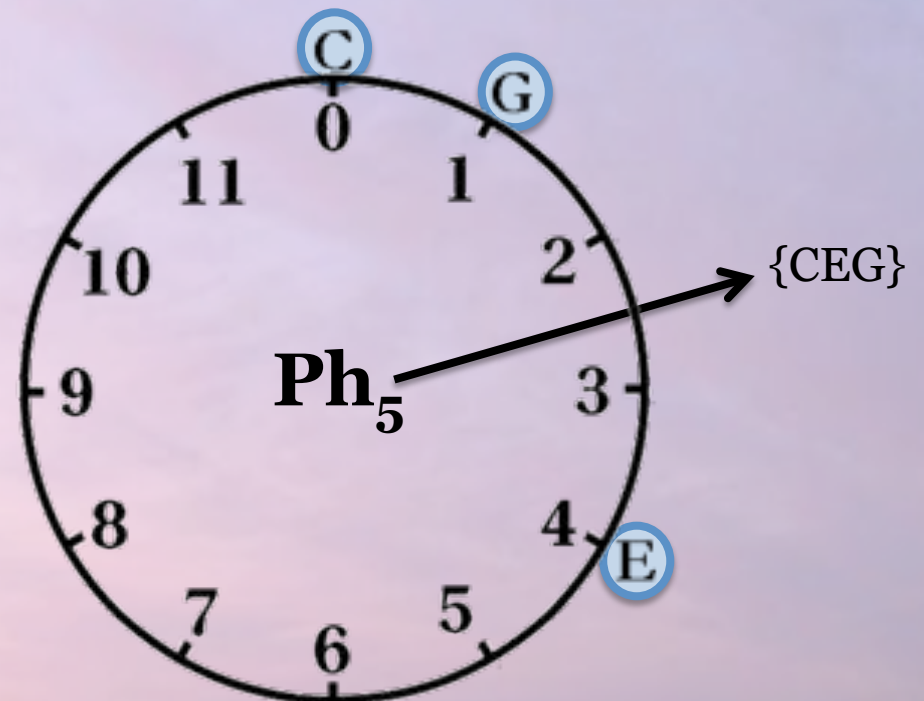
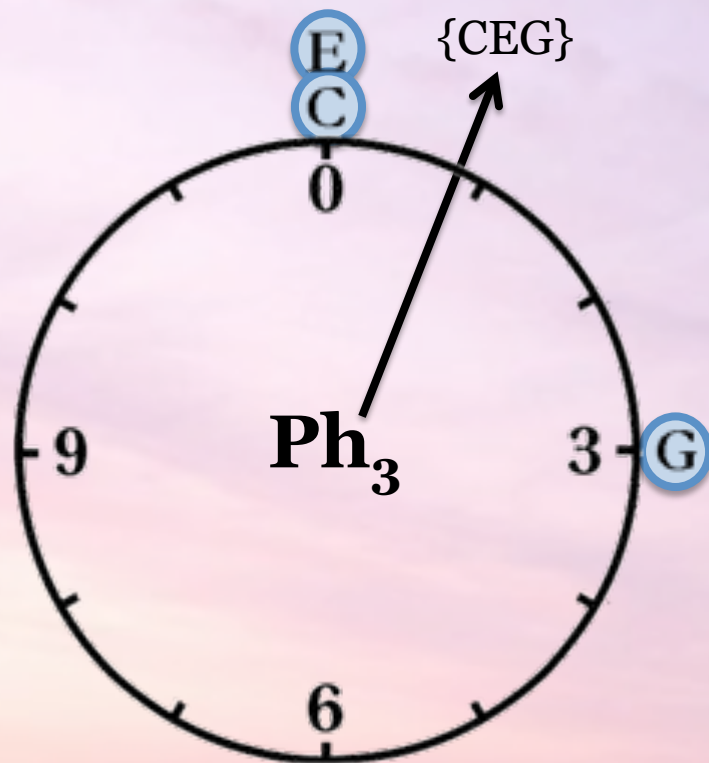
One-dimensional phase spaces are Quinn's *Fourier balances*, superimposed n -cycles created by multiplying the pc-circle by n .



N.B. counter-clockwise orientation

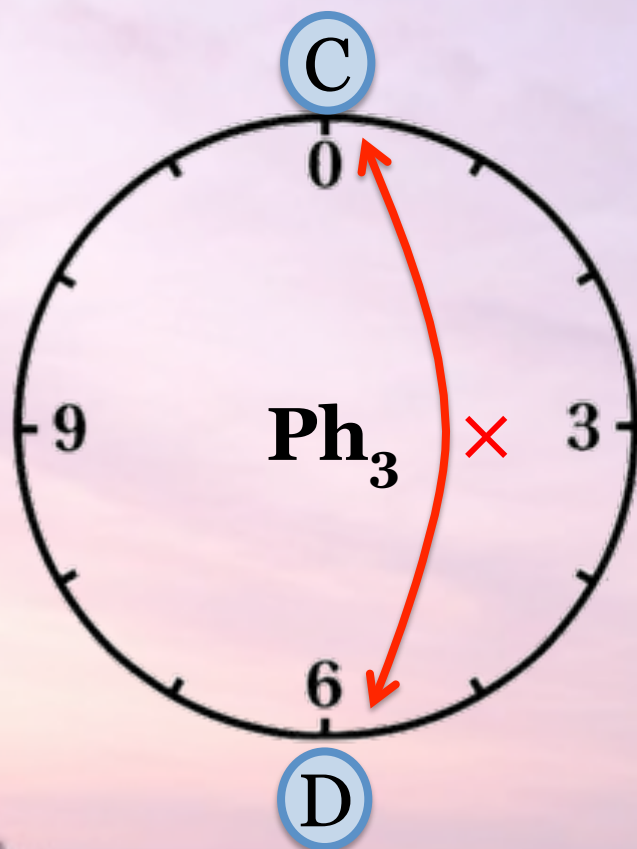
Phase spaces: One dimensional

The position of the pcset in the phase space is the circular average of the individual pcs



Phase spaces: One dimensional

Opposite phases cancel one another out.
Therefore pcsets can have undefined phases.



$\{CD\}$ has undefined Ph_3 ,
 $|f_3| = 0$

This is a kind of
Generalized Complementation:
Complements balance one another
in *all* phase spaces.

Phase spaces: One dimensional

An analytical proto-methodology:

Each Fourier component measures an independent musical quality: (1) chromaticism, (2) quartal harmony, (3) triadic harmony, (4) octatonicism, (5) diatonicism, (6) whole-tone balance.

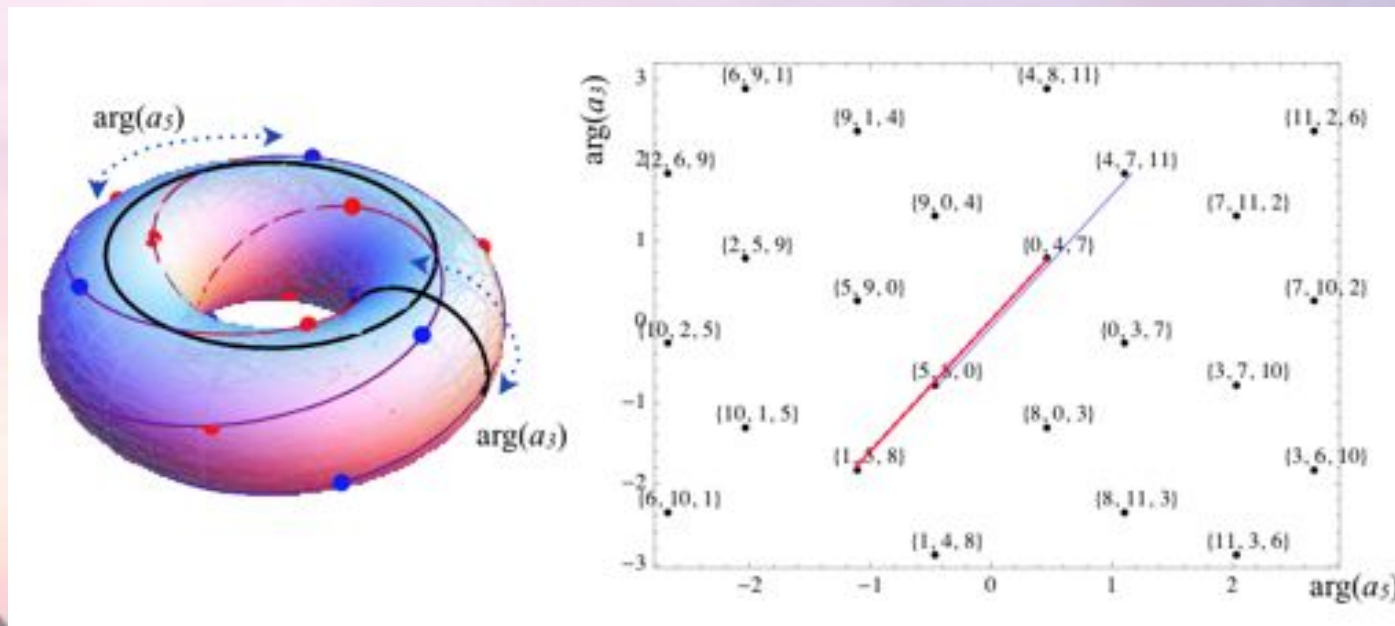
Distances in phase spaces indicate:

- Relatedness of harmonies on the given dimension
- Whether the harmonies **reinforce** one another or **weaken** one another on the given dimension when combined.

Phase Spaces: Two dimensional

A two-dimensional phase space tracks the phases of two components, and is topologically a *torus*.

Amiot (2013) and Yust (2015) use Ph_{3-5} -space to describe tonal harmony.

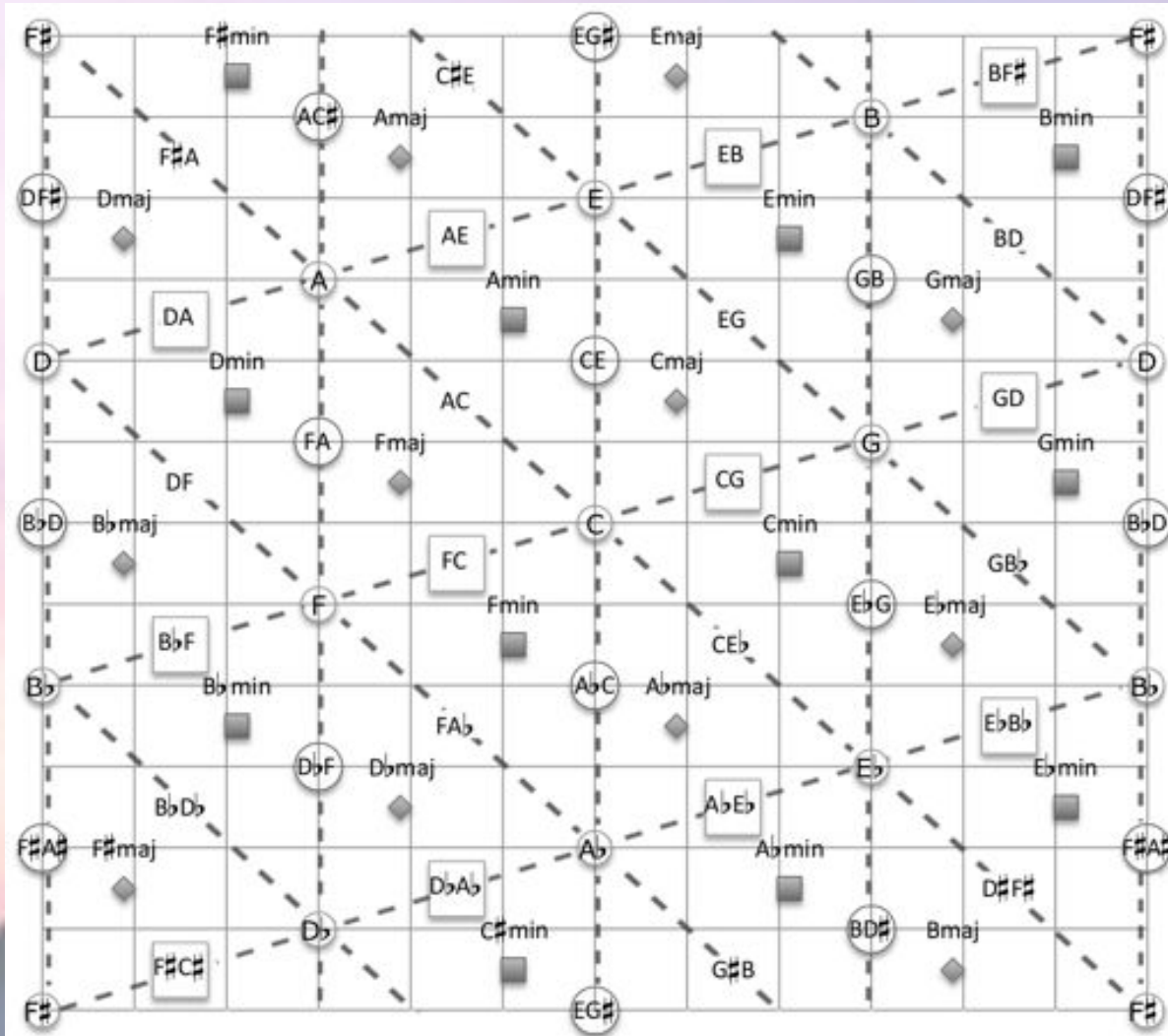


from
Amiot,
MCM 2013
proceedings

Phase Spaces: Two dimensional

Amiot (2013) and Yust (2015) use $\text{Ph}_{3,5}$ -space to describe

tonal harmony



→ Pcs, consonant dyads and triads, and *Tonnetz* in $\text{Ph}_{3,5}$ -space, from Yust (2015) (*JMT* 59/1)



II. Debussy, “Les sons et les parfums tournent dans l’air du soir”

1. Heptatonic scales and diatonicity
2. Common tones and harmonic qualities

Scale Theory, Subsets, and Phase Space

Problems in the analytical application of scale theory:

- (1) The status of **subsets** of multiple scales and **supersets** of multiple scales.
- (2) The range of variability in what counts as a scalar set.

A possible solution: Phase space 5

Debussy and Scale-Network Wormholes

The image displays a musical score for Debussy's *Preludes I, no. 4*, "Les sons et les parfums tournent en l'air de soir". The score is in 3/4 time, marked "Modéré (♩ = 84) (harmonieux et souple)". It features two systems of music, each with a treble and bass staff. The first system is annotated with a green oval labeled "1st harmonic major" and a purple oval labeled "A pedal". The second system is annotated with a blue oval labeled "C[♯] dim7" and a purple oval labeled "A pedal". The score includes various musical notations such as notes, rests, and dynamic markings like "pp" and "m.d.". The background of the slide features a faint, stylized image of a forest or landscape.

Modéré (♩ = 84)
(harmonieux et souple)

1st harmonic major

A pedal

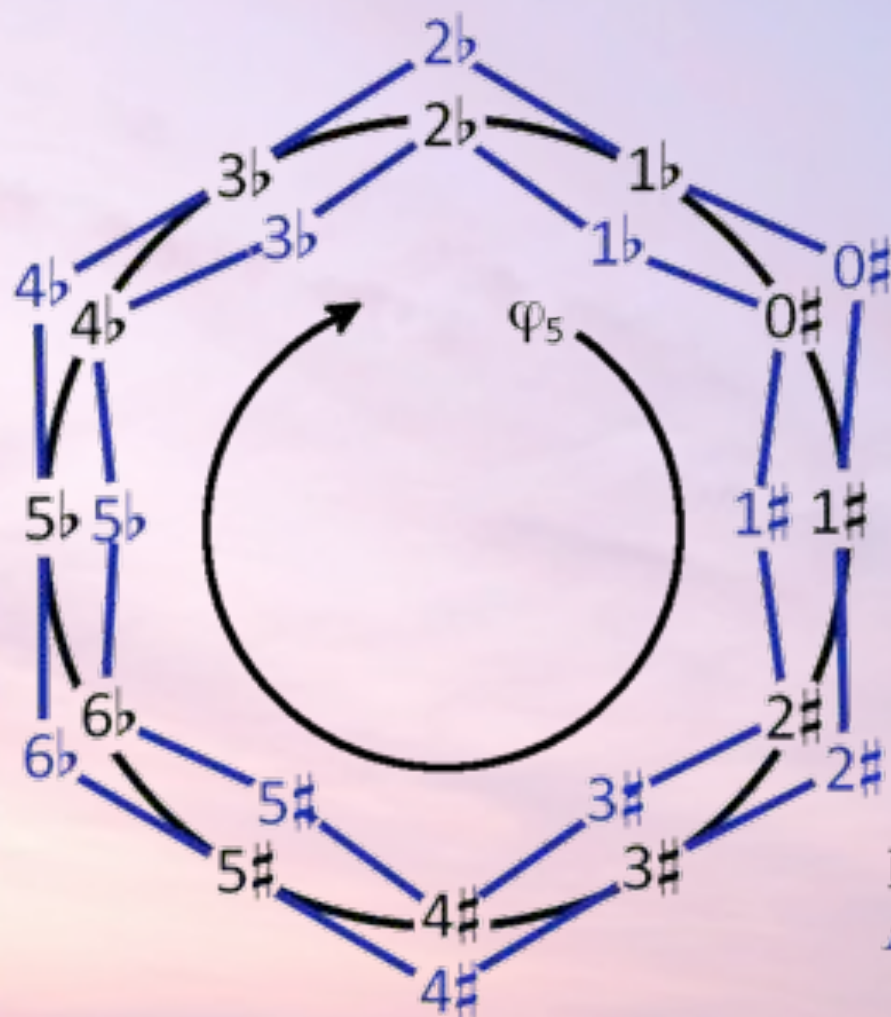
C[♯] dim7

A pedal

Subset of 3rd harmonic minor

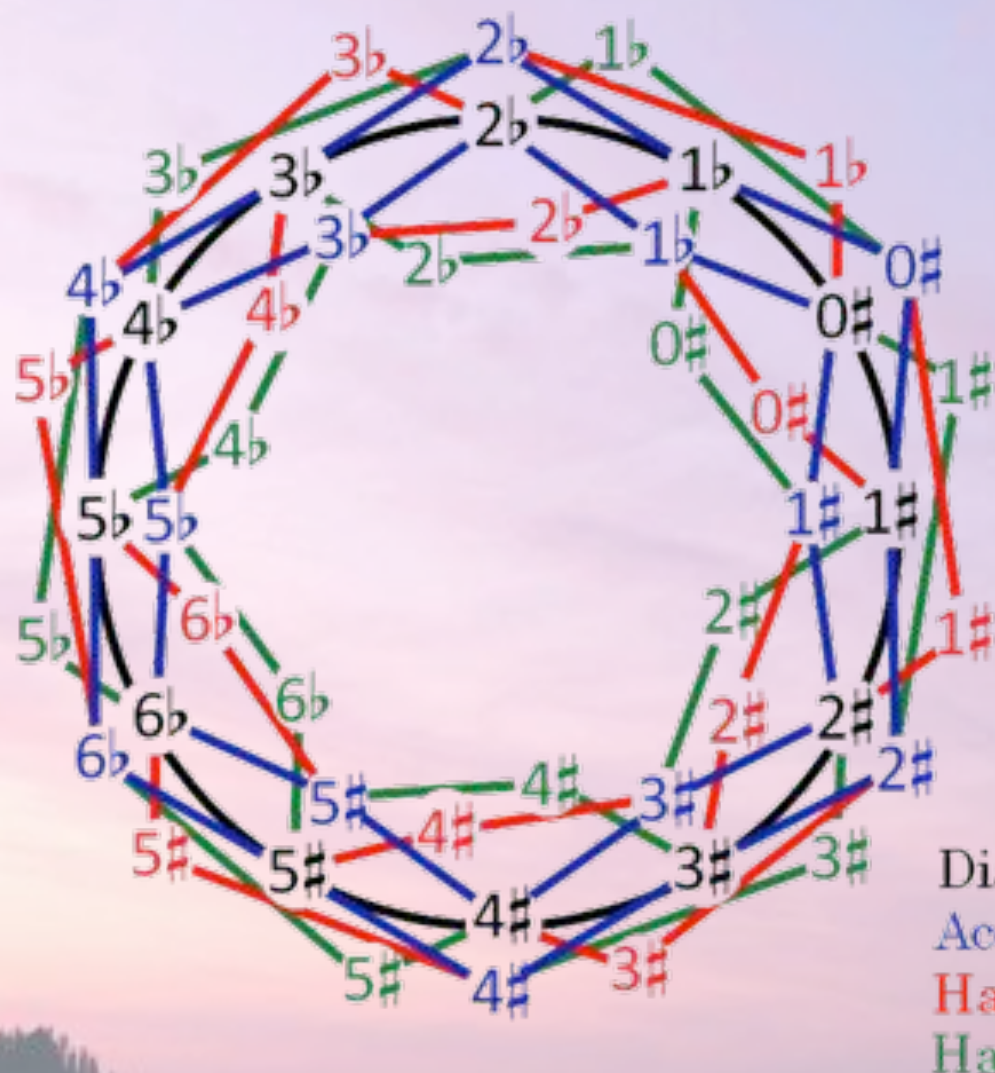
Debussy, *Preludes I*, no. 4, "Les sons et les parfums tournent en l'air de soir"

Debussy and Scale-Network Wormholes



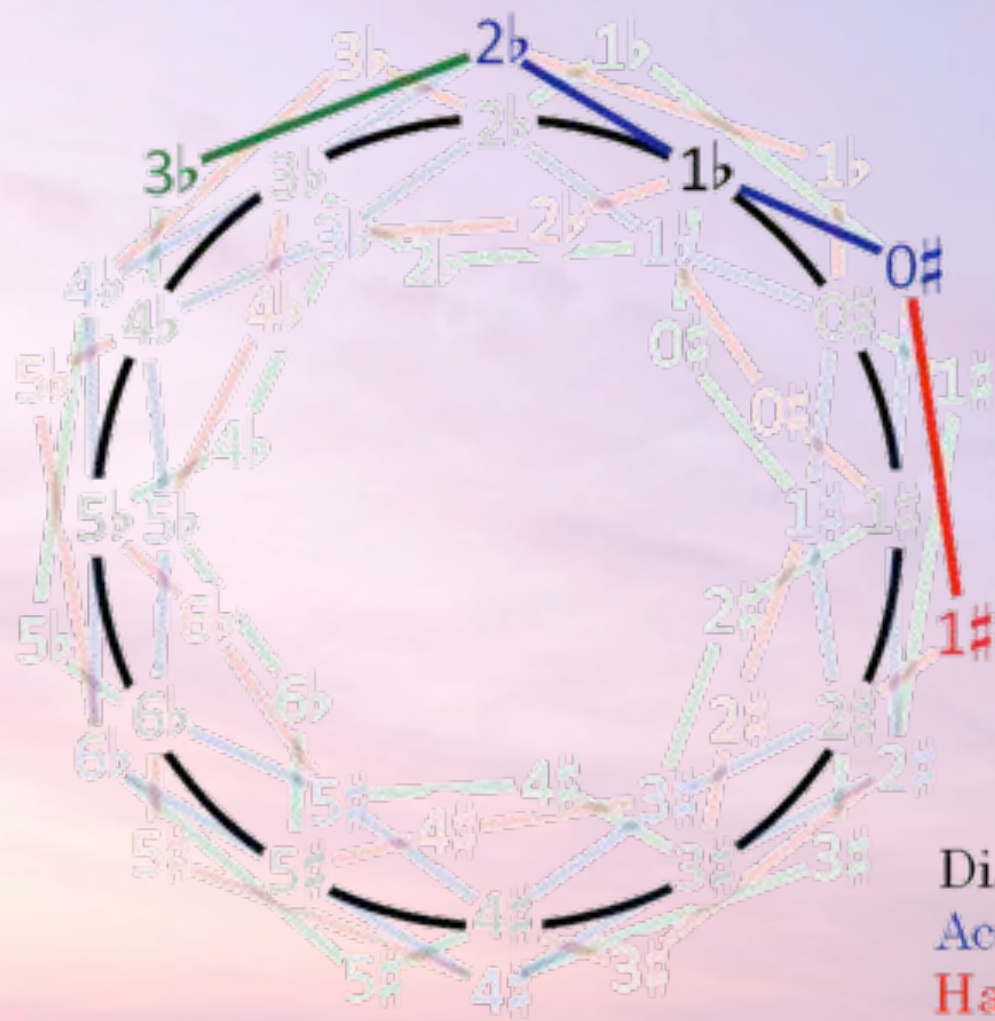
A scalar network for relatively even heptatonic scales (after Tymoczko 2004, 2011) corresponds to positions on a Ph_5 cycle.

Debussy and Scale-Network Wormholes



A scalar network for relatively even heptatonic scales (after Tymoczko 2004, 2011) corresponds to positions on a Ph_5 cycle.

Debussy and Scale-Network Wormholes



The scales in the opening of the Prelude require four moves in the scale network

Diatonic
Acoustic
Harmonic major
Harmonic minor

Debussy and Scale-Network Wormholes

But . . .

These scales can also be connected with just three moves

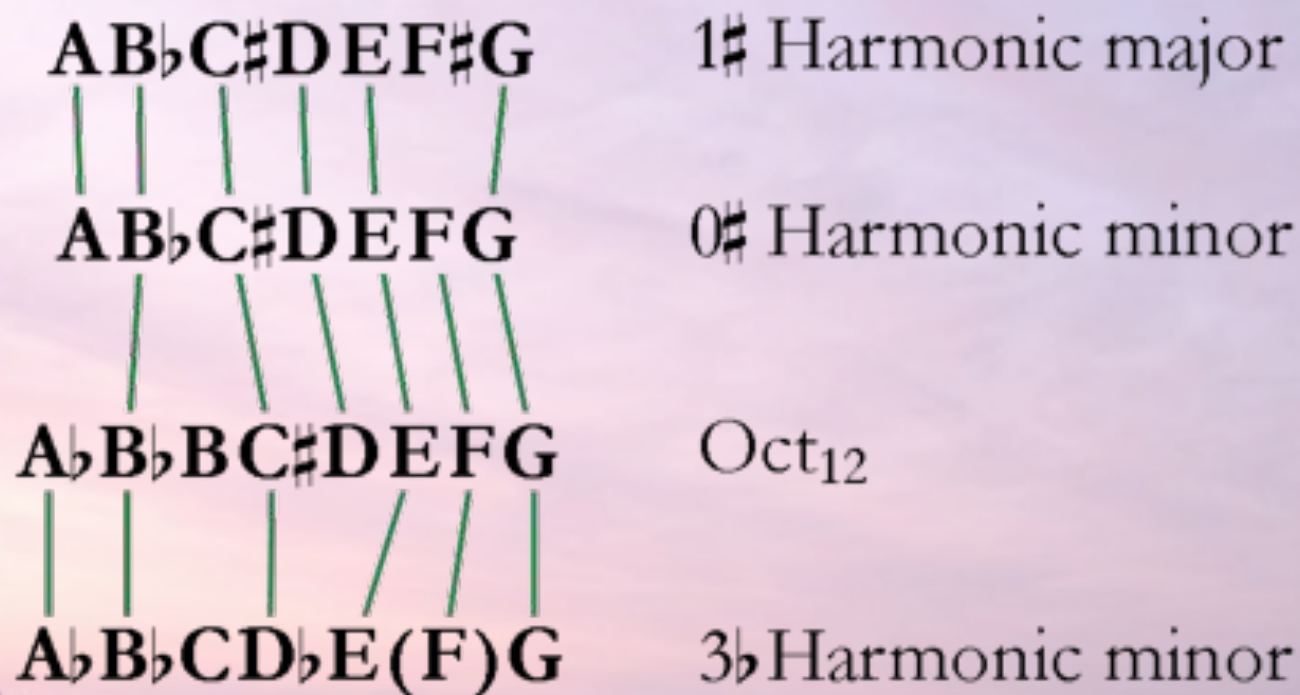
By using Oct_{01}



Debussy and Scale-Network Wormholes

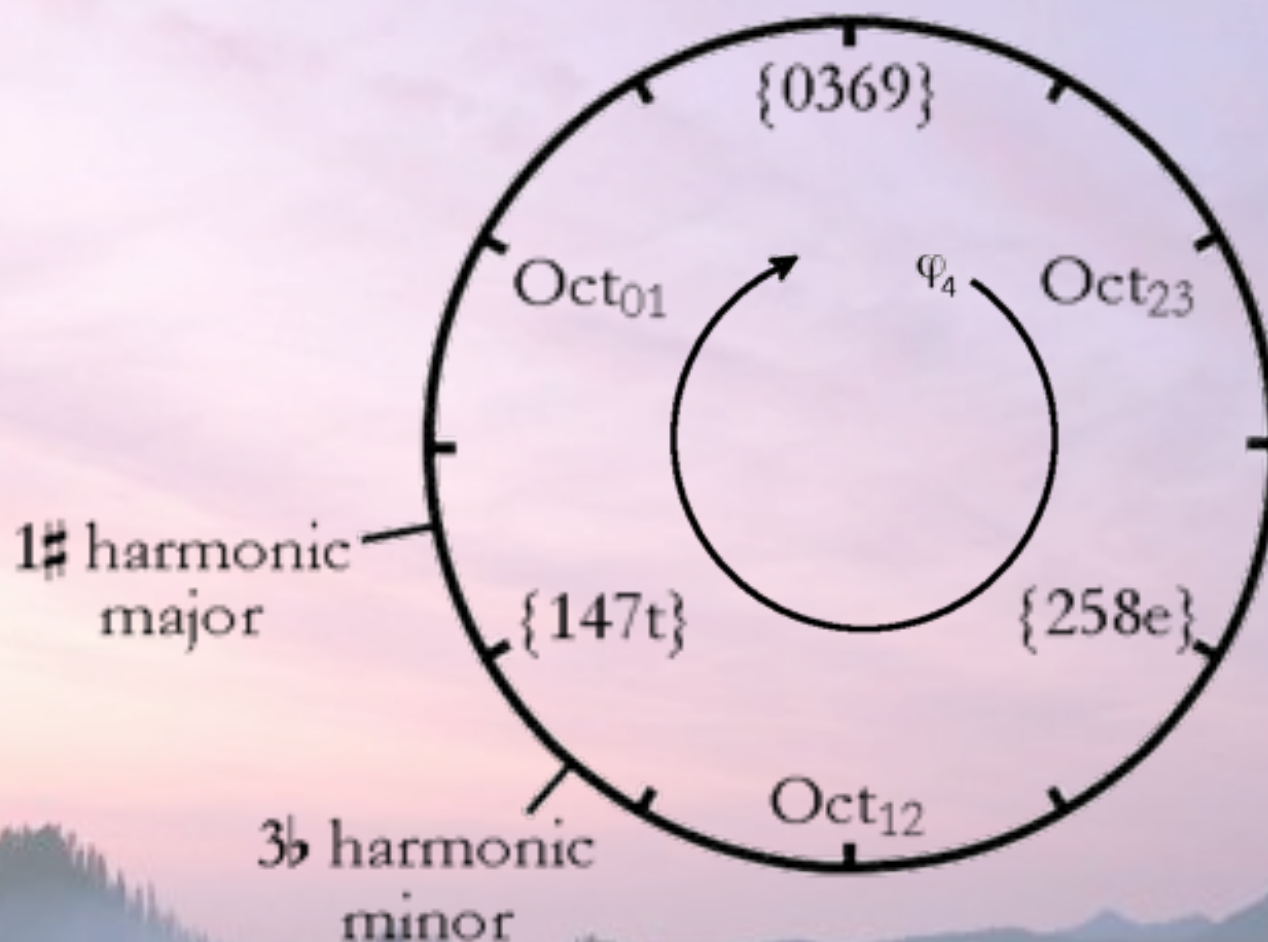
Or . . .

By using Oct₁₂



Debussy and Scale-Network Wormholes

Why? Although *far apart* in Ph_5
They are *close together* in Ph_4



Debussy and Scale-Network Wormholes

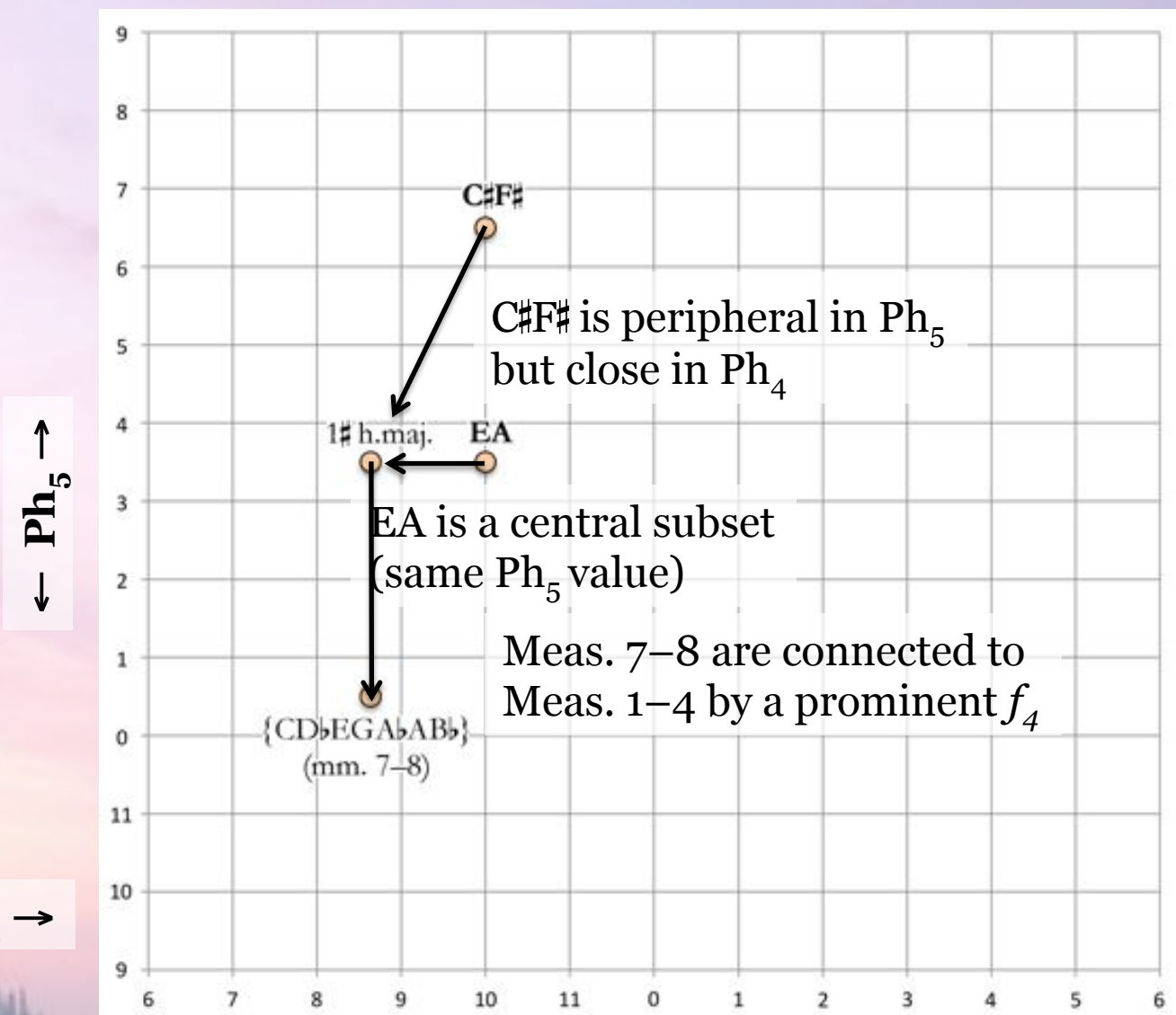
The image displays a musical score for Debussy's *Preludes I, no. 4*, "Les sons et les parfums tournent en l'air de soir". The score is annotated with several key features:

- E-A motive**: A red box with an arrow pointing to the first measure of the first system.
- 1st harmonic major**: A green oval highlights the first system, with the text "1st harmonic major" written above it.
- F#-C# motive**: A red box with an arrow pointing to the second measure of the first system.
- A pedal**: A purple label with an arrow pointing to the first measure of the first system.
- C# dim7**: A purple label with an arrow pointing to the second measure of the first system.
- Subset of 3rd harmonic minor**: A blue oval highlights the second system, with the text "Subset of 3rd harmonic minor" written below it.
- A pedal**: A purple label with an arrow pointing to the first measure of the second system.

The score includes tempo markings: "Modéré (♩ = 84)" and "(harmonieux et souple)". Dynamics include "pp" and "m.d.". The key signature is one sharp (F#).

Debussy, *Preludes I*, no. 4, "Les sons et les parfums tournent en l'air de soir"

Debussy and Scale-Network Wormholes



Debussy *Airs*: Change of Quality

The image displays two systems of musical notation for a piano accompaniment. The first system features a treble and bass staff with various musical notations including notes, rests, and dynamic markings like *m.d.* and *p*. A blue annotation $\{AB, CD, EGA\}$ with an arrow pointing right is positioned to the right of the first system. The second system continues the musical piece, also with treble and bass staves, and includes dynamic markings such as *p*, *m.d.*, and *mf*. A blue annotation $\{AC, D, E\}$ is located below the bass staff of the second system. The text "En animant un peu" is written above the first system of the second system.

Debussy *Airs*: Change of Quality

DFT magnitudes²

$$\begin{array}{c} |f_1|^2 \ |f_2|^2 \ |f_3|^2 \ |f_4|^2 \ |f_5|^2 \ |f_6|^2 \\ \{AB\flat CD\flat EGA\flat\}: \quad \langle\langle \ 3.73, \ 1, \quad 5, \quad 7, \ 0.27, \ 1 \ \rangle\rangle \end{array}$$

$$\{AC\sharp D\sharp E\sharp\}: \quad \langle\langle \ 1, \quad 1, \quad 4, \quad 1, \quad 1, \quad 16 \ \rangle\rangle$$

Common-Tone Theorem

The number of common tones between sets X and $Y =$

$$\frac{1}{12} \sum_{n=0}^{11} |f_n(A)| |f_n(B)| \cos(\varphi_n(A) - \varphi_n(B))$$

For each component
(sum over components)

Cosine of the phase difference
(ranges from -1 for opposite
phases to 1 for same phase)

Weighted by the component
magnitudes

Common-Tone Theorem

The number of common tones between sets X and $Y =$

$$\frac{1}{12} \sum_{n=0}^{11} |f_n(A)| |f_n(B)| \cos(\varphi_n(A) - \varphi_n(B))$$

In other words:

Distances in phase space for the most prominent components determine the number of common tones

Common-Tone Theorem

Example:

$$|f_0| (|f_1|, \text{Ph}_1) (|f_2|, \text{Ph}_2) (|f_3|, \text{Ph}_3) (|f_4|, \text{Ph}_4) (|f_5|, \text{Ph}_5) (|f_6|, \text{Ph}_6)$$

$$\{\text{AB}\flat\text{CD}\flat\text{EGA}\flat\}: \langle\langle 7, (1.93, 2.5) (1, 8) (2.24, 11.1), (2.65, 8.64) (0.52, 0.5), (1, 0) \rangle\rangle$$

$$\{\text{AC}\sharp\text{D}\sharp\text{E}\sharp\}: \langle\langle 4, (1, 9) (1, 6) (2, 9) (1, 0) (1, 9) (4, 6) \rangle\rangle$$

$$\begin{array}{l} \text{Multiply mag.,} \\ \text{Cosine of Ph. diff.} \end{array} \quad 28, (1.9, -1.0) (1, 0.5) (4.5, 0.45) (2.7, -0.19) (0.52, -0.26) (4, -1)$$

$$\begin{array}{l} \text{Multiply and} \\ \text{divide by twelve} \end{array} \quad 2.33 + -0.31 + 0.08 + 0.33 + -0.08 + -0.02 + -0.33$$

$$\text{Sum} = 2 \text{ common tones (A and C}\sharp\text{)}$$



Common-tone Linkage: Debussy

The image displays two staves of music from Debussy's 'Clair de lune'. The top staff is marked 'a Tempo' and the bottom staff is marked 'Plus lent'. The top staff features a 1# Harmonic Major scale (A-B-C#-D-E-F#-G) and the bottom staff features a 6b Harmonic Major scale (F#-G-A-B-C#-D). A blue arrow labeled 'subset' points from the 1# Harmonic Major scale to the AC#EF#G scale, which is then linked to the 6b Harmonic Major scale via a blue arrow labeled 'F#C#'. The music is written in piano (pp) and mezzo-forte (mf) dynamics, with a tempo change indicated by 'En animant'.

a Tempo

1# Harmonic Major

subset

AC#EF#G

Plus lent

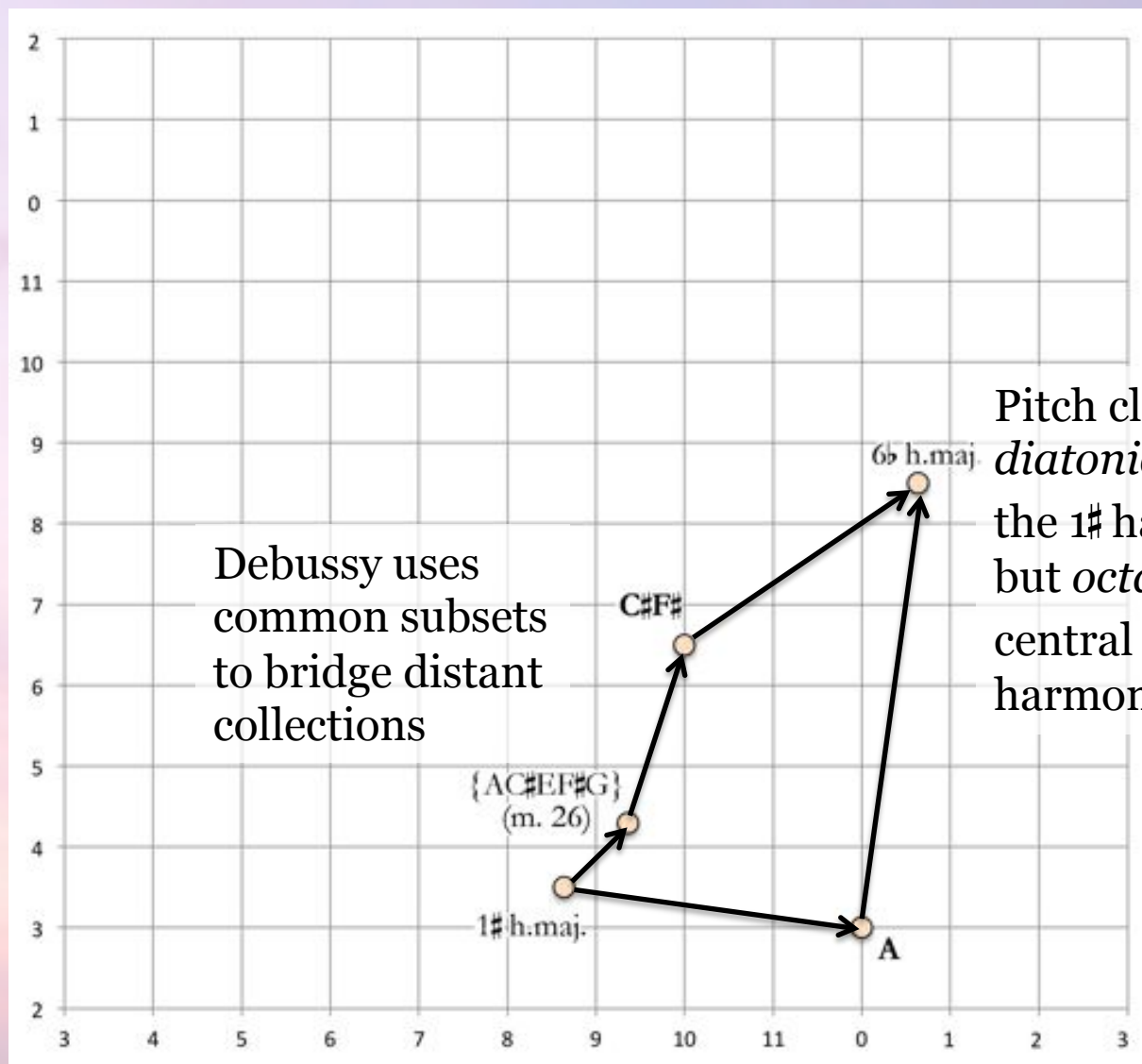
6b Harmonic Major

F#C#

Common-tone Linkage: Debussy

← Ph₅ →

← Ph₄ →





Common-tone Linkage: Debussy

The image displays three systems of musical notation for Debussy's 'En retenant'. The first system is marked 'En retenant' and 'a Tempo égal et doux', featuring a piano (*p*) and a fortissimo (*pp*) section. The second system is marked 'Serrez un peu' and 'Retenu', with a piano (*p*) section. The third system is marked 'a Tempo' and features a fortissimo (*pp*) section. The score illustrates common-tone linkage between the systems, with a specific note (C#9) highlighted in purple. Below the first system, the text 'Oct₀₁ subset C#⁹ etc.' is written. Below the third system, the text '1# Harmonic Major' is written.

En retenant - - - # a Tempo égal et doux

p dim. *pp*

en dehors C#⁹

Serrez un peu - - - # Retenu - - -

p *p*

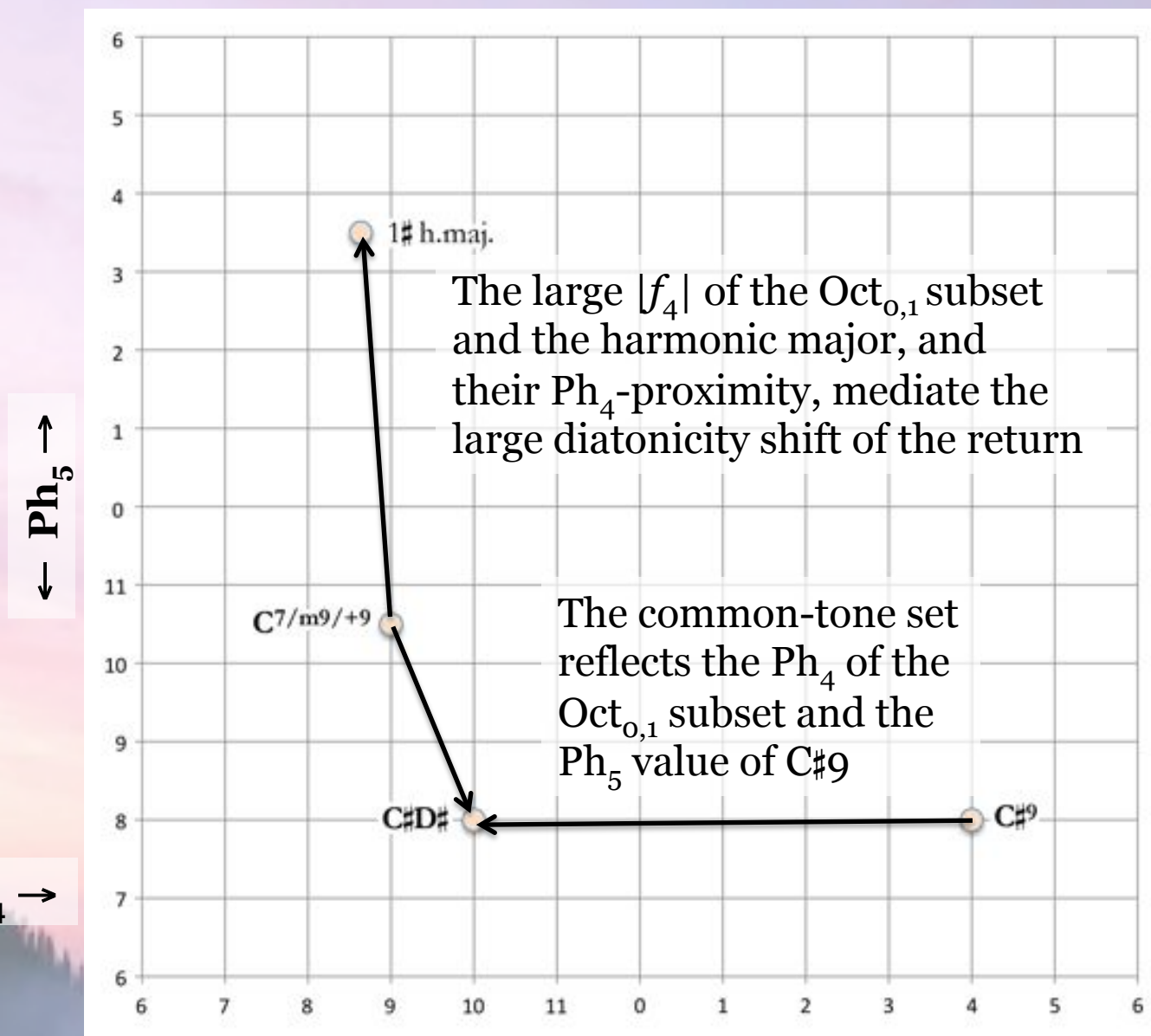
Oct₀₁ subset C#⁹ etc.

a Tempo

pp *pp* *pp m.d.*

1# Harmonic Major

Common-tone Linkage: Debussy



Common-Tone Theorem

Example:

	$ f_0 $	(f_1 , Ph_1)	(f_2 , Ph_2)	(f_3 , Ph_3)	(f_4 , Ph_4)	(f_5 , Ph_5)	(f_6 , Ph_6)
C7/m9/+9	$\langle\langle$	7, (0.52, 5.5)	(1, 8)	(2.2, 6.9)	(2.6, 8.6)	(1.9, 3.5)	(1, 0) $\rangle\rangle$
1^\sharp h.maj.	$\langle\langle$	6, (1.4, 10.5)	(0, -)	(1.4, 1.5)	(3.5, 9)	(1.4, 10.5)	(0, -) $\rangle\rangle$
Multiply mag., Cosine of Ph. diff.	42,	(0.7, -0.9)	(0, -)	(3.2, -0.95)	(2.7, -0.19)	(0.52, -0.26)	(0, -)
Multiply and divide by twelve	3.5	+ -0.1	+ 0	+ -0.5	+ 1.5	+ -0.4	+ 0
Sum	= 4 common tones (C \sharp , E, G, B \flat)						



Stravinsky and the Octatonic

1. *Rite of Spring*, Introduction and *Augurs*

Stravinsky and the Octatonic

Is this music octatonic?

The image displays a musical score for a piano piece, likely from Igor Stravinsky's 'Les Noces'. The score is written for three staves: two grand staves (treble and bass clef) and a single treble staff. The tempo is marked 'Tempo giusto' with a quarter note equal to 58 (♩ = 58). The key signature is B-flat major (two flats). The first system shows a grand staff with a bass line featuring a series of chords and a treble line with a series of chords. The second system shows a grand staff with a bass line featuring a series of chords and a treble line with a series of chords. The third system shows a grand staff with a bass line featuring a series of chords and a treble line with a series of chords. The score includes various musical notations such as notes, rests, and dynamic markings like 'f' (forte) and 'meno f' (meno forte). The background of the slide features a stylized landscape with mountains and a sunset or sunrise sky.

Stravinsky and the Octatonic

Global
0-5
(0 2 5/0 3 5/0 2 3 5)
tetrachord

0-11 "major 7th"
interval span

0-5, 11

0 2 3 5 6 8 9 11
oct. scales

Local
0, 3, 6, 9

(0 2 3 5) tetrachords;
(0 3 7/0 4 7/0 4 7 10)
triads, "dom. 7ths"

0 2 3 5 6 8 9 11
oct. scales;
0 2 3 5 7 9 10 (0)
D-scales

Collection III

Van den Toorn:
5 of the 7 notes in
the "Augurs" chord
and the C triad
come from Oct_{0,1}
(E + E^{b7}).
The G[#] and B
reinforce E

Stravinsky and the Octatonic

What are the chances?!?!?

100%

All 8-note collections overlap at least one octatonic by six or more pitch classes.

Stravinsky and the Octatonic

Joseph Straus (review of *Music of Stravinsky*):

Van den Toorn “never provides and systematic criteria for determining the presence of the octatonic collection; as a result, a number of his attributions are suspect. Almost any passage containing nine to twelve pitch classes can be discussed as ‘diatonic interpenetration’ of an octatonic context.”



Stravinsky and the Octatonic

Dmitri Tymoczko (on the analyses in *Music of Stravinsky*):

“If even *these* passages can be understood as the result of ‘octatonic-diatonic interpenetration,’ then we should rightly ask whether there is any music that *cannot* be understood in this way.

In a sense, there is not: any proper subset can be decomposed into diatonic and octatonic components.”



Stravinsky and the Octatonic

Let's try again . . .

$E\flat 7$

$F\flat$ maj.

C maj.

Tempo giusto 5/8

f

mf

meno f

f

Stravinsky and the Octatonic

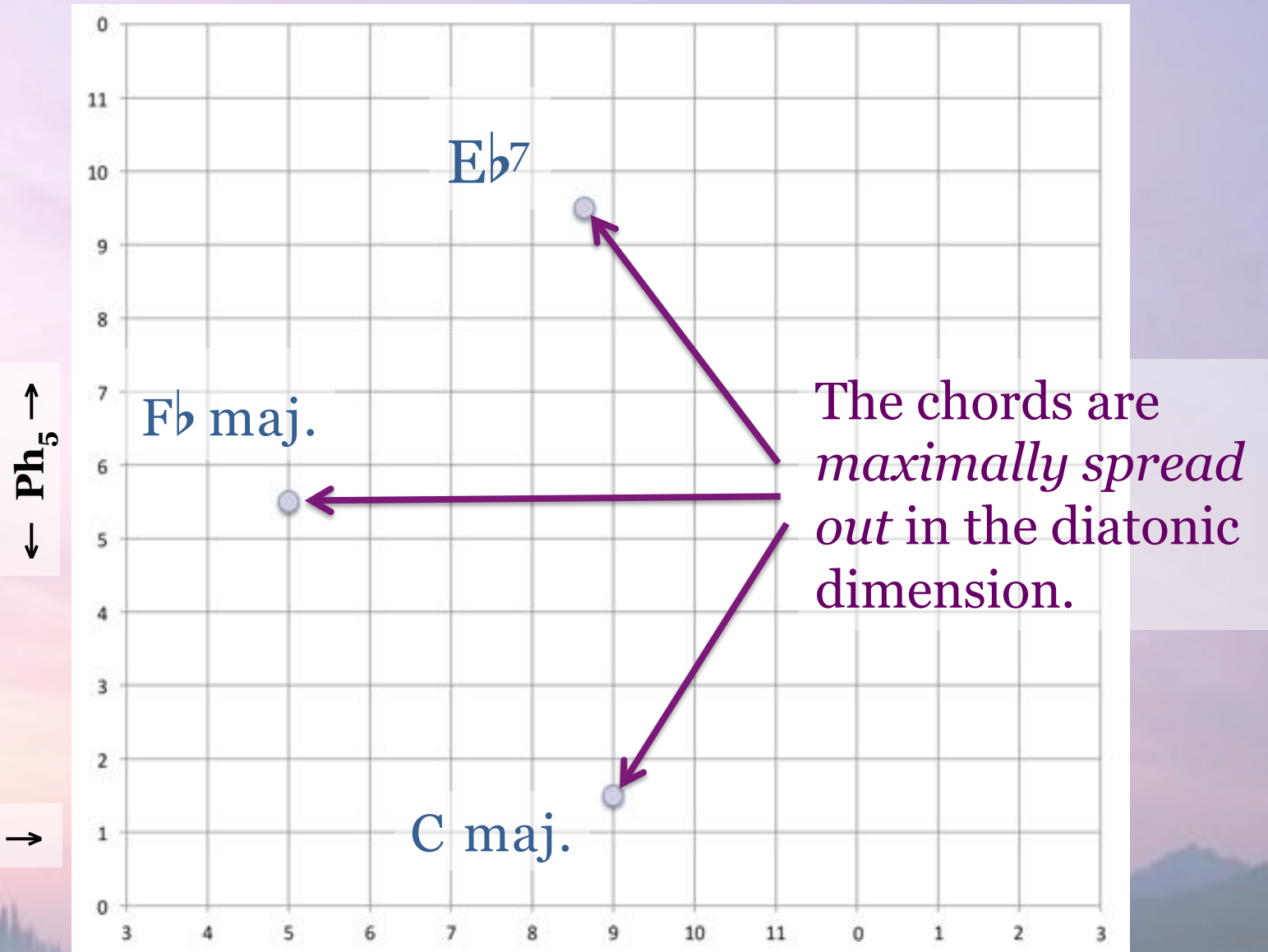
Components f_4 and f_5 are the largest
across the set-types

DFT magnitudes²

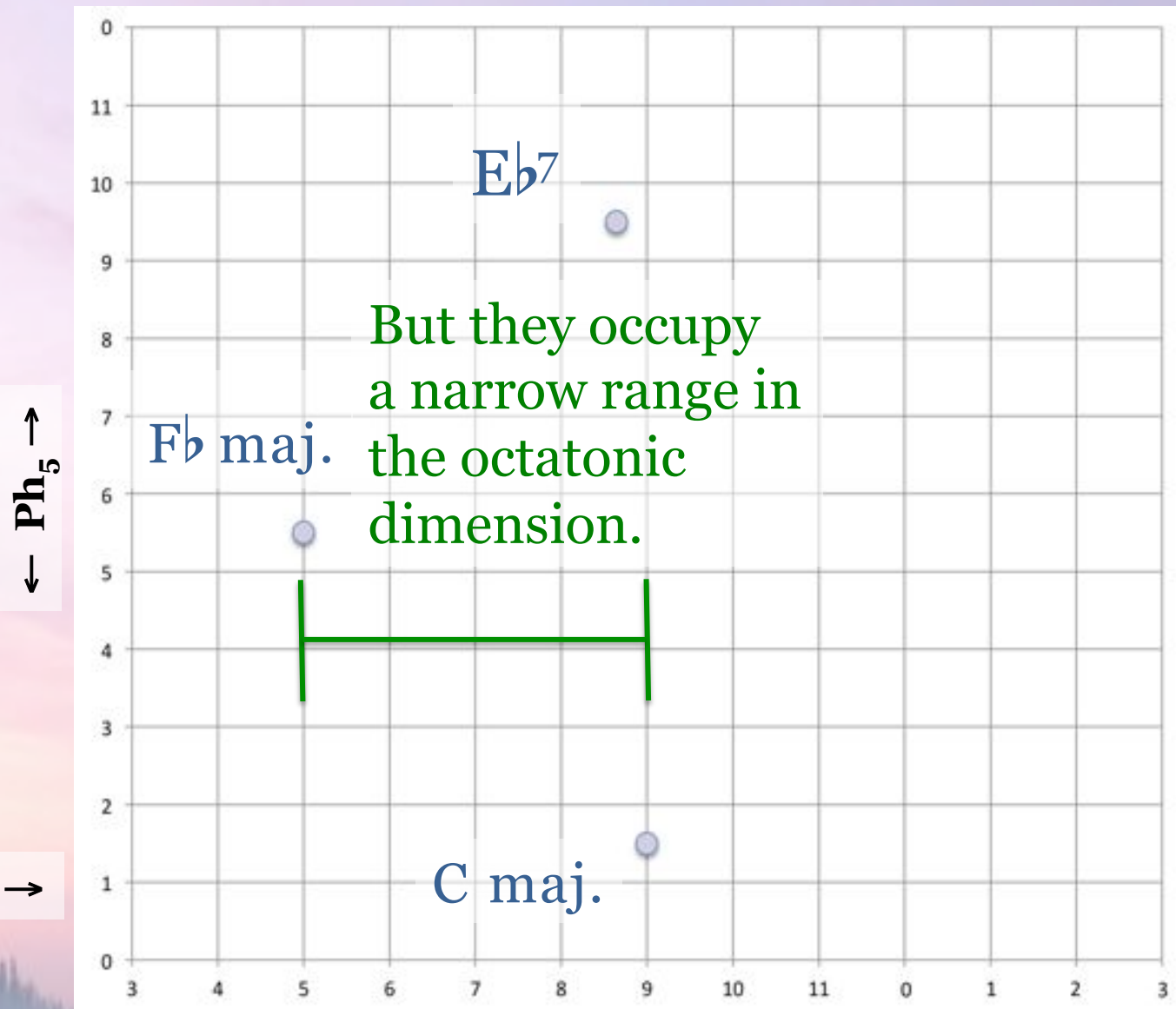
Major/minor triad: $\langle\langle |f_1|^2 \ |f_2|^2 \ |f_3|^2 \ |f_4|^2 \ |f_5|^2 \ |f_6|^2 \rangle\rangle$
 $\langle\langle 0.27, 1, 5, 3, 3.73, 1 \rangle\rangle$

Dominant 7th: $\langle\langle 0.27, 1, 2, 7, 3.73, 4 \rangle\rangle$

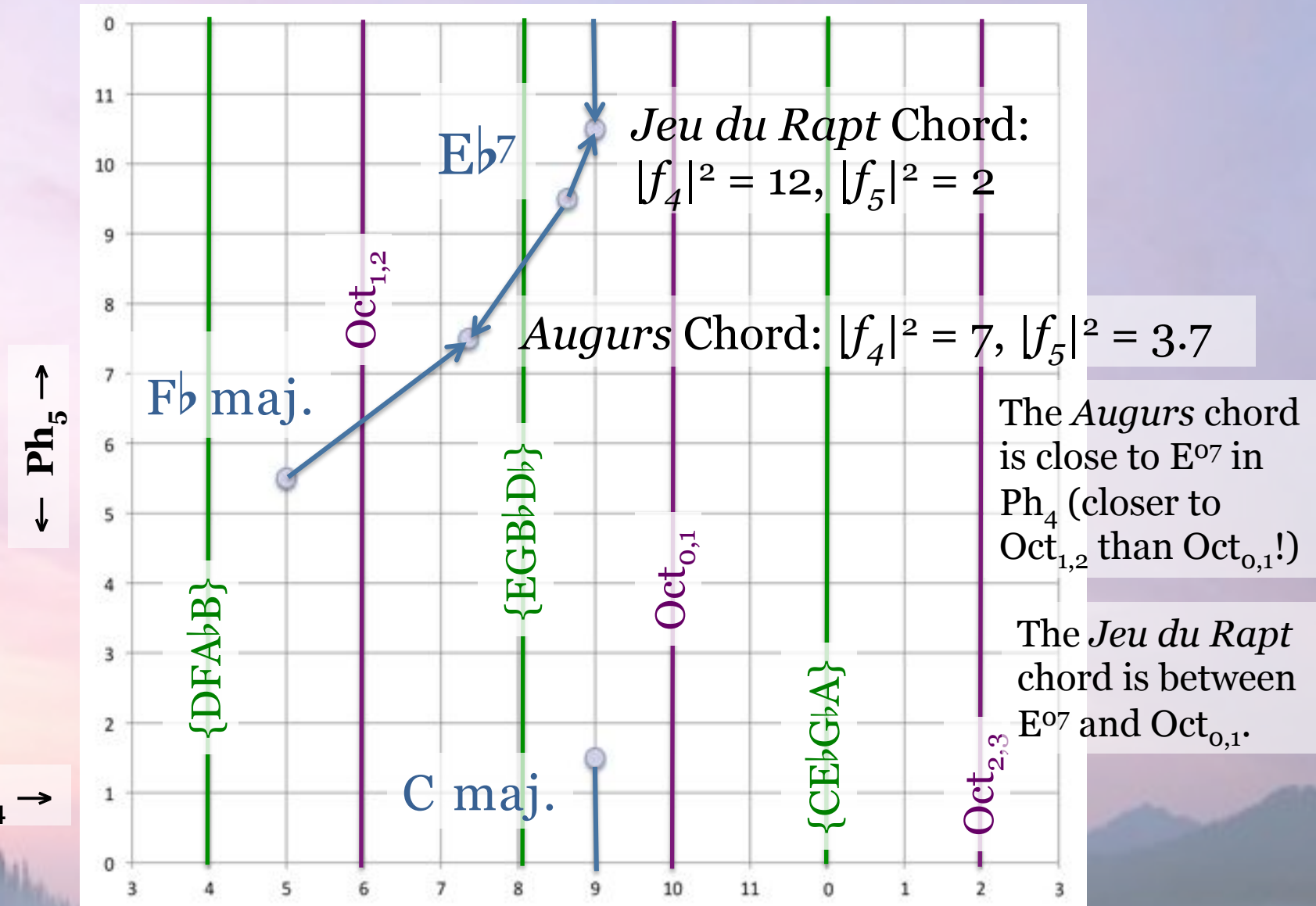
Stravinsky and the Octatonic



Stravinsky and the Octatonic



Stravinsky and the Octatonic



Stravinsky and the Octatonic: *Jeu du Rapt*

Presto ♩ = 132

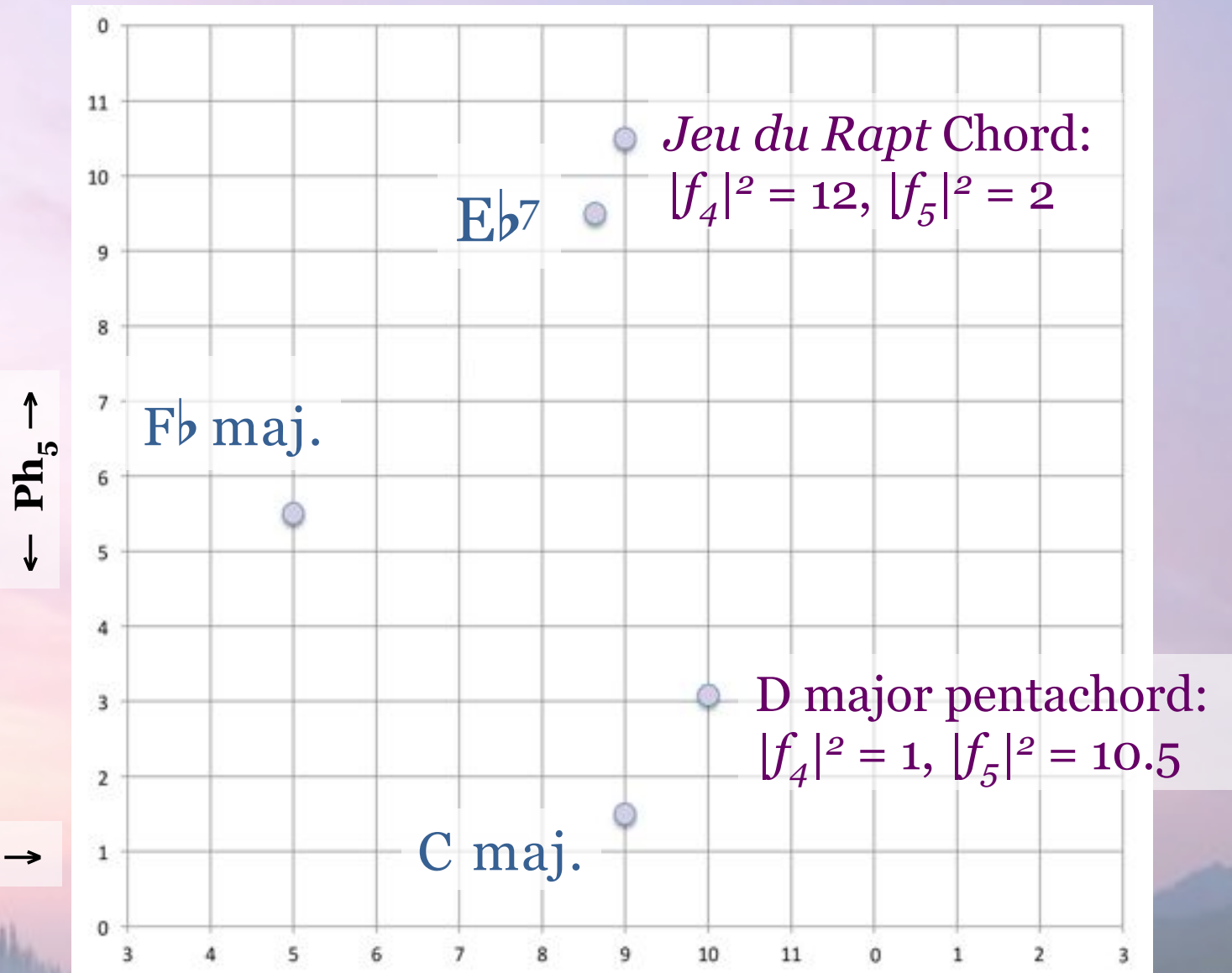
The image shows a page of musical notation for Igor Stravinsky's 'Jeu du Rapt'. It features a piano introduction in 8/8 time, marked 'Presto' with a tempo of 132 beats per minute. The score is written for piano and timpani. The piano part consists of two systems of staves. The first system has a treble and bass staff for the piano, with a timpani part below. The second system also has a treble and bass staff for the piano, with a timpani part below. The timpani part is marked '(Timpani)' and 'f'. The piano part is marked 'f' and 'P'. The score includes various musical notations such as notes, rests, and dynamic markings. There are also annotations in blue text: 'Eb7 + C maj.' and 'D maj. pentachord'.

E \flat 7 + C maj.

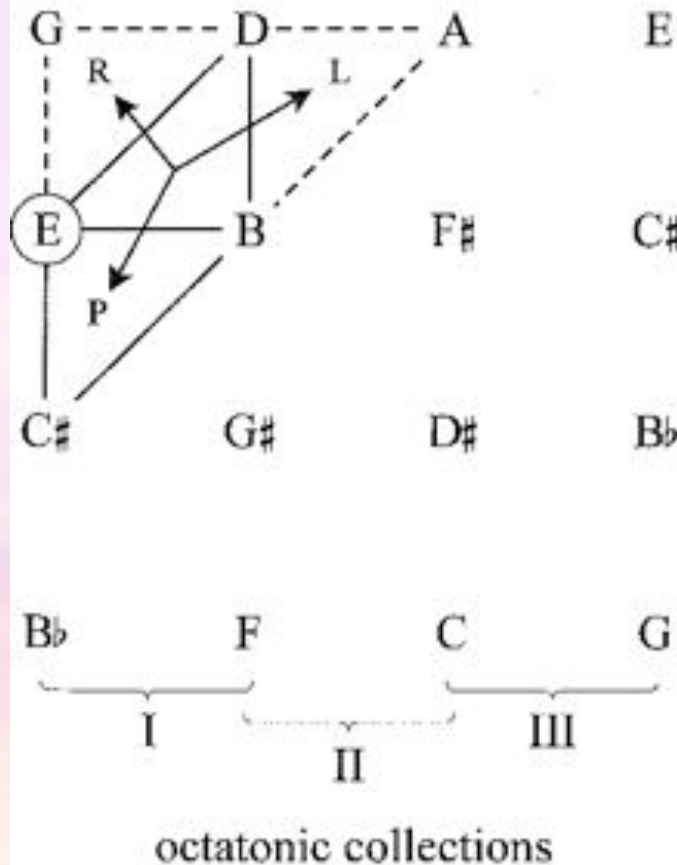
D maj. pentachord

(Timpani)

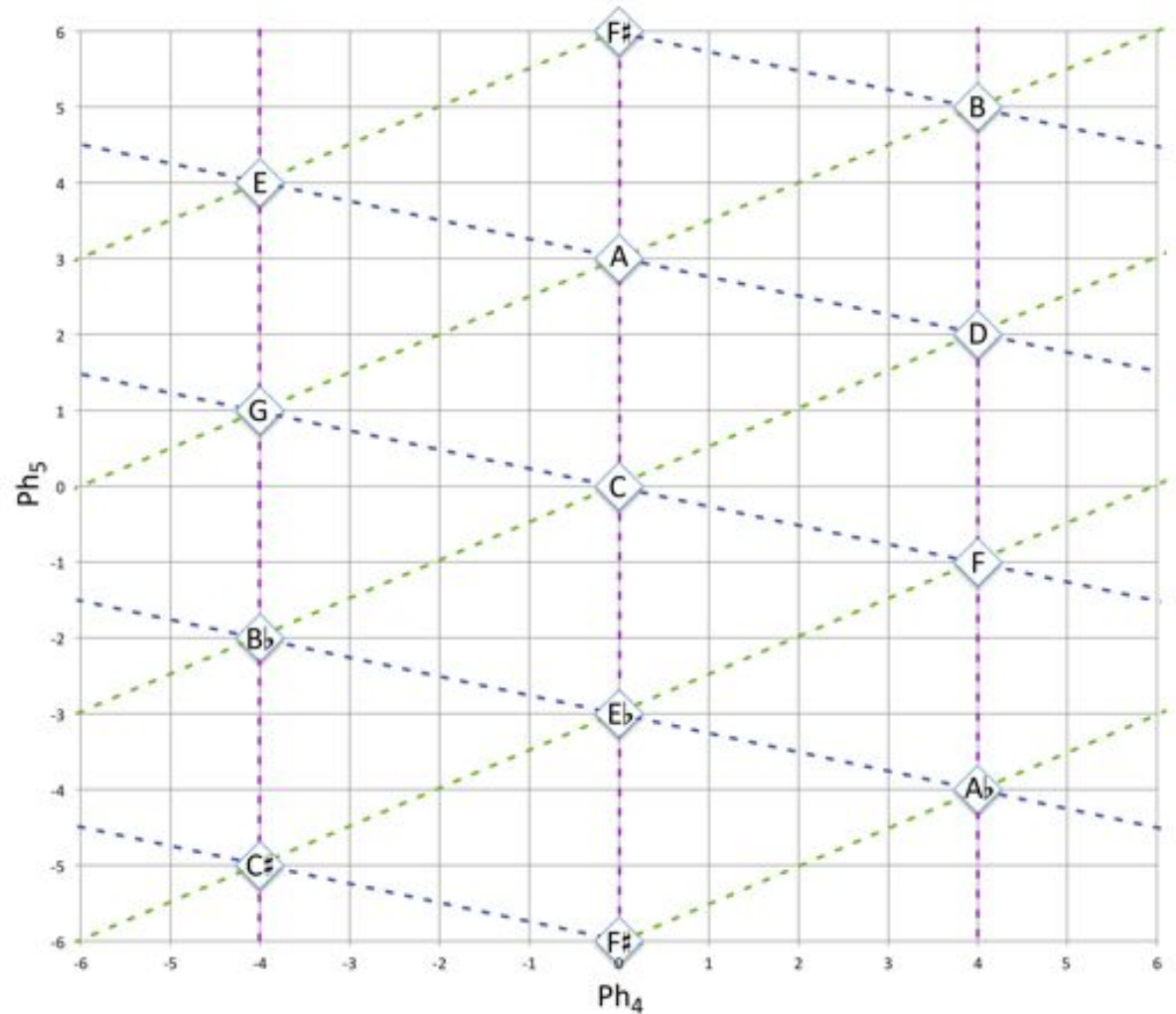
Stravinsky and the Octatonic



Stravinsky and the Octatonic



Octatonic Tonnetz
Van den Toorn and
McGuinness 2012



Pitch classes in $Ph_{4,5}$ -space

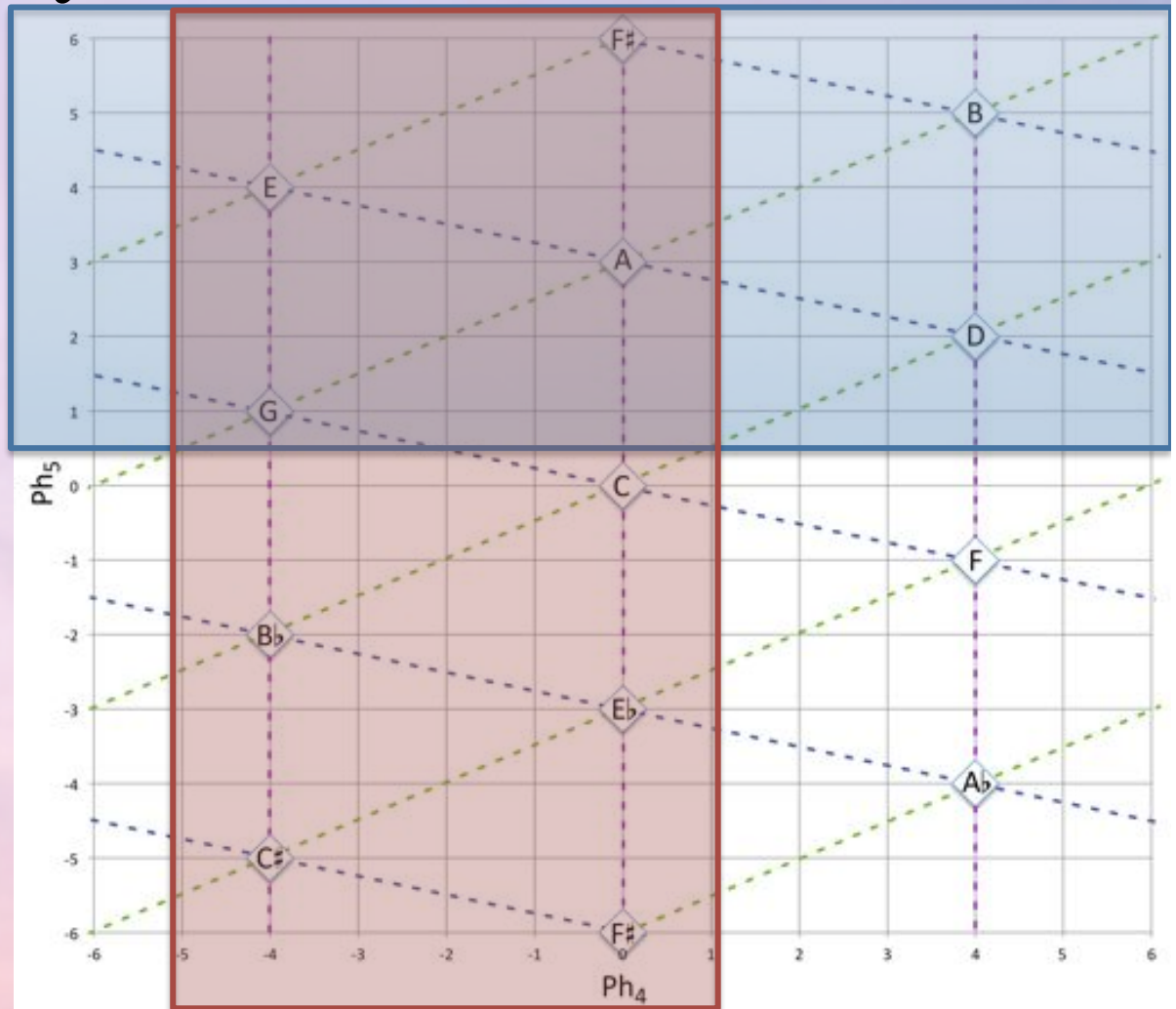
Stravinsky and the Octatonic

Diatonic materials:
D major hexachord

Overlap:

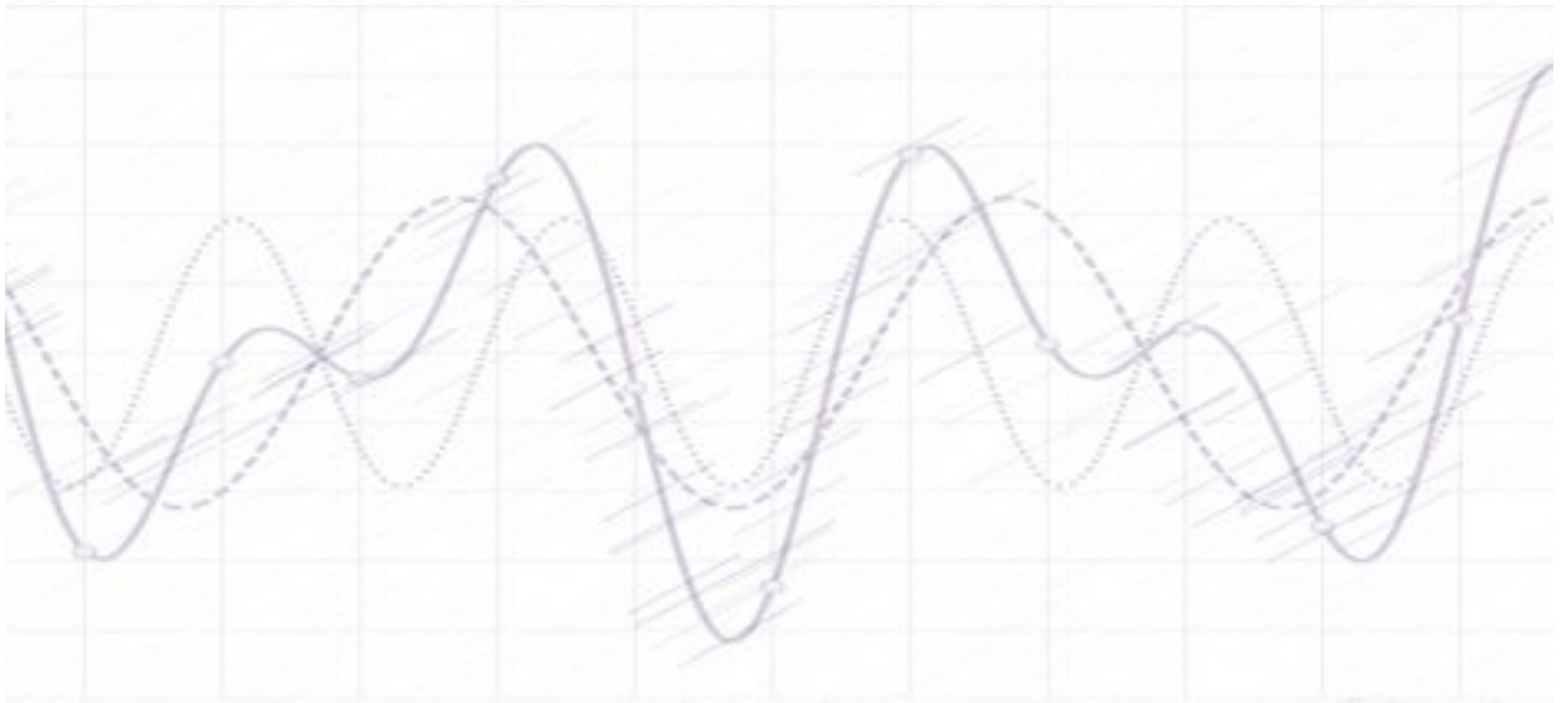
A-dorian tetrachord
(cf. *Jeu du Rapt*)

Octatonic materials:
 $\text{Oct}_{0,1}$



Pitch classes in $\text{Ph}_{4,5}$ -space

Feldman, *Palais de Mari*



Feldman, *Palais de Mari*

Handwritten musical score for Feldman's *Palais de Mari*. The score is written for piano and features a complex, non-functional harmonic language. The tempo is marked $\text{♩} = 63-66$. The music is in 5/8 time and begins with a *ppp* (pianissimo) dynamic. The notation includes various accidentals, ties, and phrasing slurs. Measure numbers 7, 14, and 22 are indicated at the start of their respective systems. The score shows a series of chords and melodic fragments that do not follow traditional harmonic rules, characteristic of Feldman's atonal style.

Feldman, *Palais de Mari*

Features of the piece:

- Composed in 1986, Feldman's last work for solo piano.
- Long but sparse: the 9-page score takes ca. 25 minutes to play.
- Made up of discrete gestures, frequently repeated and varied (often in subtle ways).
- Pedal is held continuously throughout most of the piece. This blurs the distinction between *successive* and *simultaneous* sounds.
- Extended segmentational analysis in Hanninen, *A Theory of Music Analysis* (2012).

Feldman, *Palais de Mari*

Features of the piece:

- Long sections on the piece tend to dwell on a limited set of gestures, giving the piece a sense of trajectory that is nonetheless non-teleological.
- Composed around the same time as his *Second String Quartet*, which Feldman described as “a dialectic of sorts between such elements as . . . chromaticism/consonance.”
- “Reverse Development”: Gestures often appear *before* the idea from which they are derived, replacing a process of development with one of *revelation*.

Feldman, *Palais de Mari*

The initial gesture (Hanninen set A) stages a chromatic–diatonic conflict



(025) $\langle\langle 2.27, 1, 1, 3, 5.73, 1 \rangle\rangle$

(012) $\langle\langle 7.46, 4, 1, 0, 0.54, 1 \rangle\rangle$

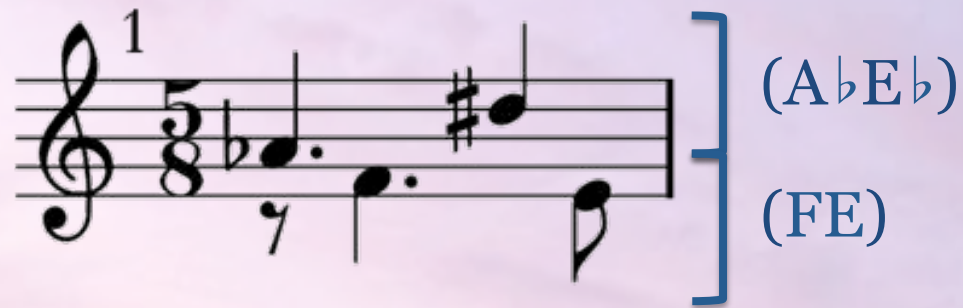
(0125) $\langle\langle 5.73, 3, 4, 1, 2.27, 0 \rangle\rangle$

The first three notes are highly diatonic,
but the final E introduces a concentrated chromaticism.

Hanninen: “The contrast between harmonies rich in ics 2 and 5,
versus those rich in ic1, resonates throughout the piece.”

Feldman, *Palais de Mari*

The initial gesture (Hanninen set A) stages a chromatic–diatonic conflict



The gesture can also be divided by part into a fourth and a minor second.

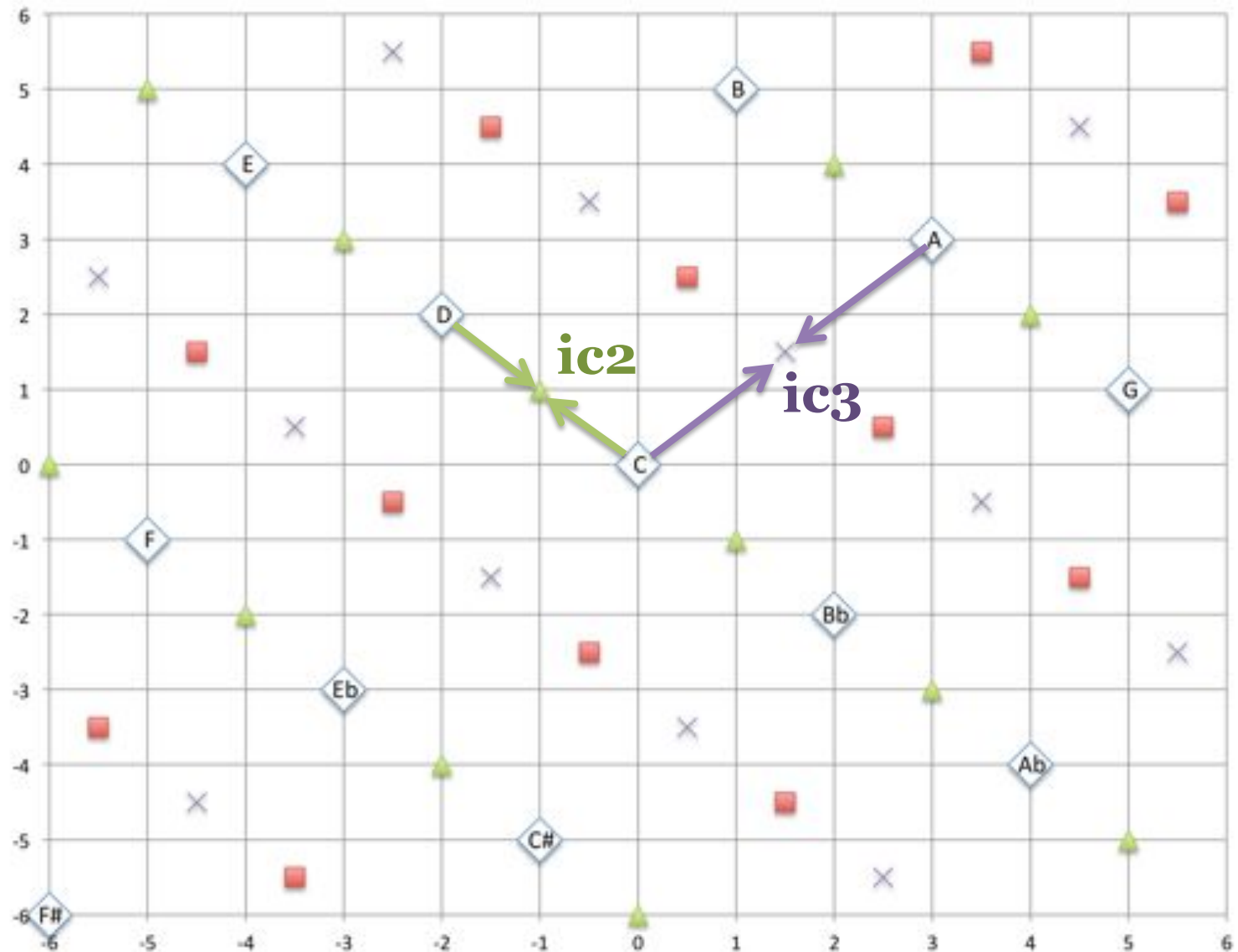
Feldman, *Palais de Mari*

Ph_{1,5}-space

ic2 and ic3 are the highest-magnitude ics in the space and are balanced between f_1 and f_5

← Ph₅ →

← Ph₁ →



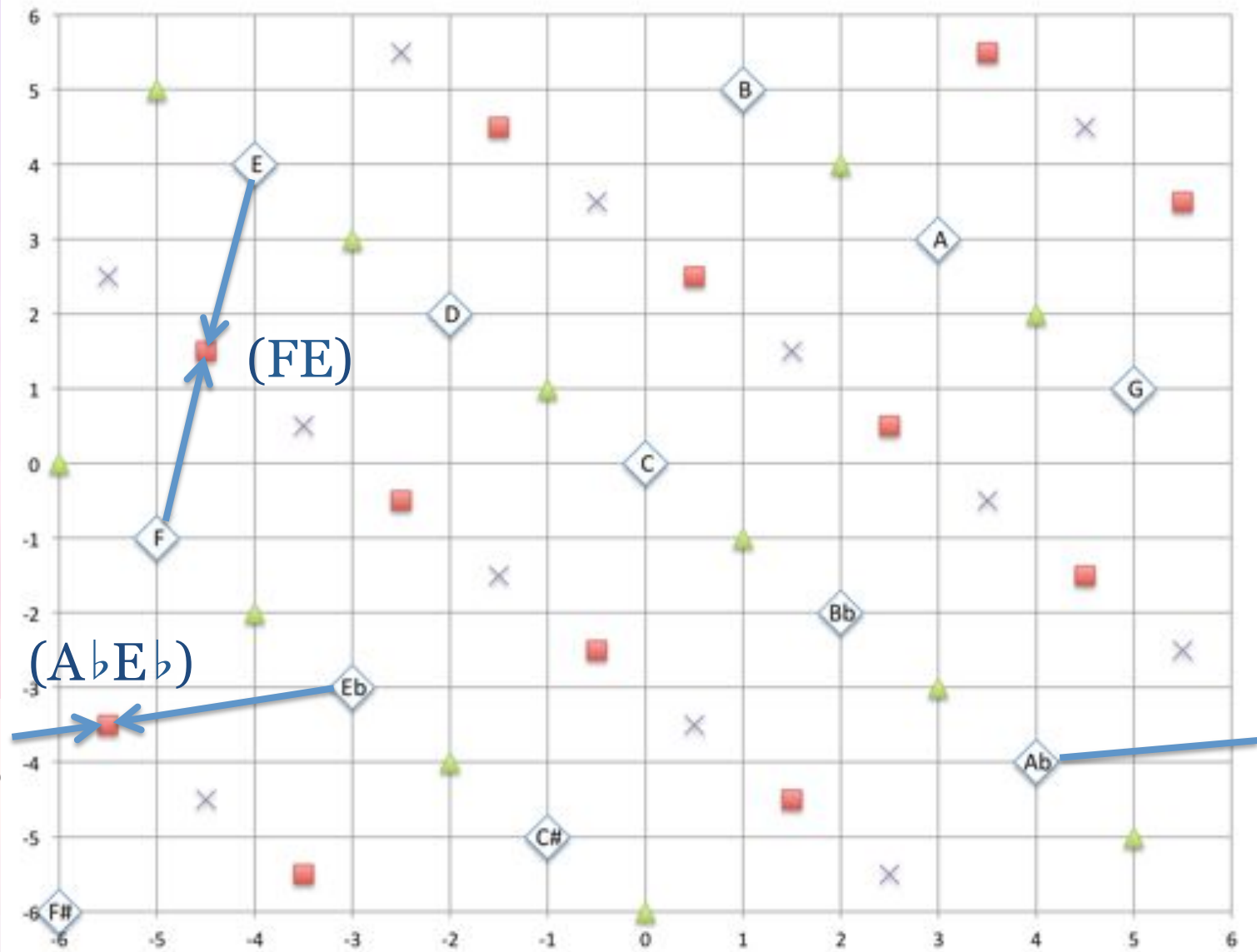
Feldman, *Palais de Mari*

Ph_{1,5}-space

Dyads from the upper and lower voices of m. 1—
ic1 and ic5 occupy the same positions in the space.

← Ph₅ →

← Ph₁ →

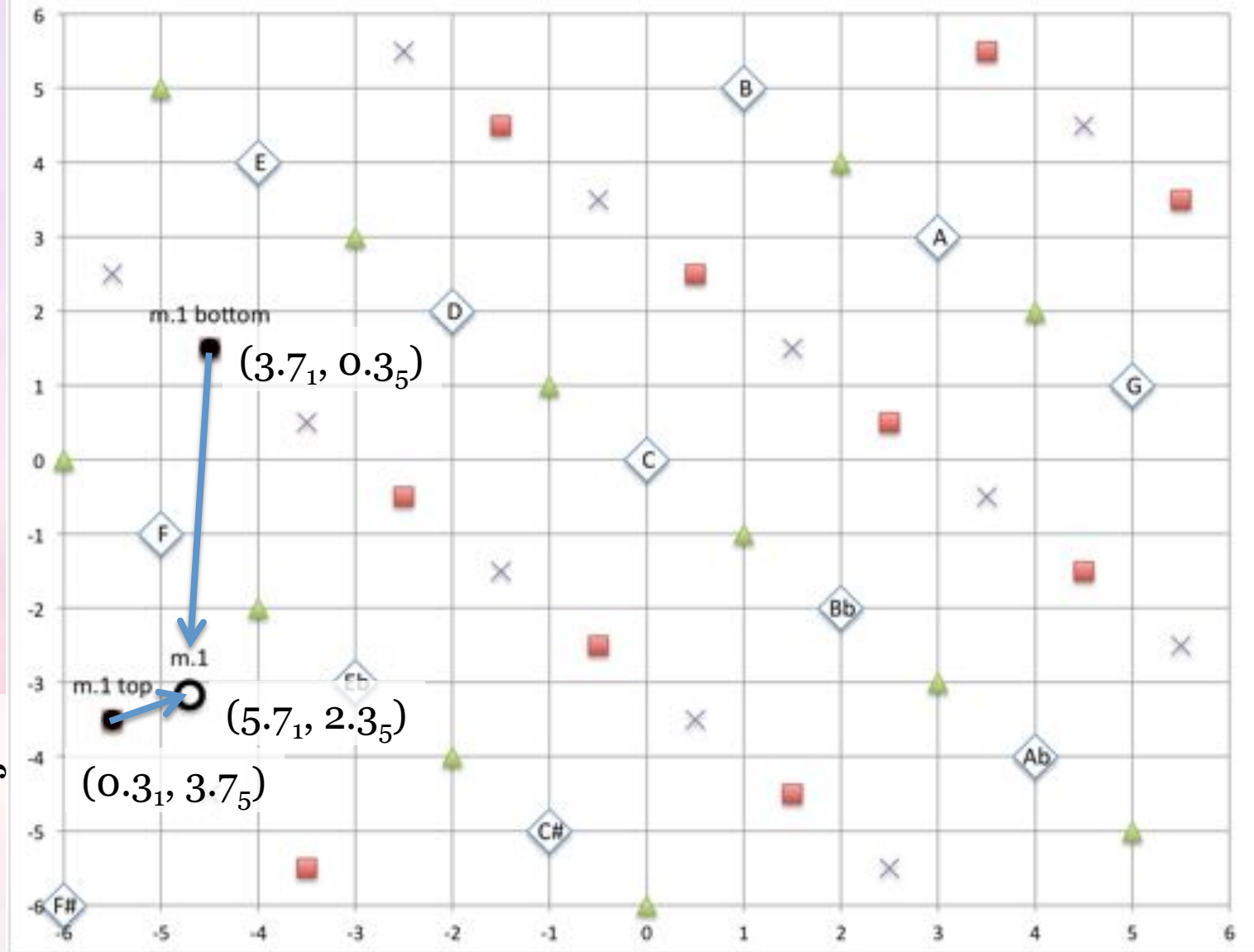


Feldman, *Palais de Mari*

Ph_{1,5}-space

The upper and lower voice dyads are **close** in Ph₁ and **distant** in Ph₅.

N.B.: They are also imbalanced (bottom: high f_1 ; top: high f_5).



Feldman, *Palais de Mari*

One of Feldman's basic harmonic techniques is
Transpositional Combination

m. 20:

(B \flat F)

T_2



(Hanninen set *C*)

m. 41:

(CD)

T_1

$ic5*ic2$

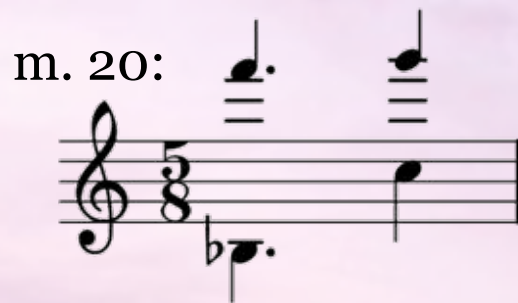
$ic1*ic2$



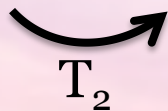
(Hanninen set *E/d*)

Feldman, *Palais de Mari*

Convolution Theorem: Transpositional combination (with doublings retained) is the same as **multiplying DFT magnitudes** and adding the phases.



(B \flat F)



= (t7) \times (02)

$$\begin{array}{r} (t7) \\ \times (02) \\ \hline = (t079) \end{array}$$

$$\begin{array}{l} \langle\langle (0.27, 4.5), (3, 3), (2, 7.5), (1, 6), (3.73, 10.5), (0, -) \rangle\rangle \\ \times \langle\langle (3, 11), (1, 10), (0, -), (1, 2), (3, 1), (4, 0) \rangle\rangle \\ \hline = \langle\langle (0.8, 3.5), (3, 1), (0, -), (1, 8), (11.2, 11.5), (0, -) \rangle\rangle \end{array}$$

Feldman, *Palais de Mari*

Transposition of entire gestures by semitone
reinforces component 1 and cancels out component 5



$(0235) = 2*3$ is balanced
between f_1 and f_5 , but
multiplying by (01)
weakens f_5 in favor of f_1

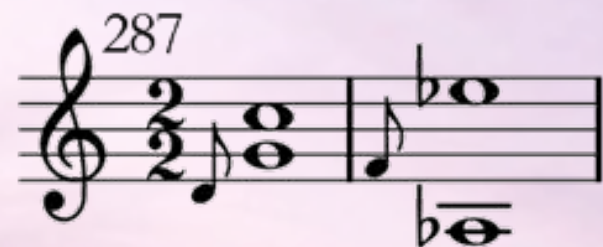
$$\begin{aligned} & (46) \\ & \times (03) \\ & = (4679) \end{aligned}$$

$$\begin{aligned} & \langle\langle (3, 7)_1, (3, 5)_5 \rangle\rangle \\ & \times \langle\langle (2, 10.5)_1, (2, 10.5)_5 \rangle\rangle \\ & \hline & = \langle\langle (6, 5.5)_1, (6, 3.5)_5 \rangle\rangle \end{aligned}$$

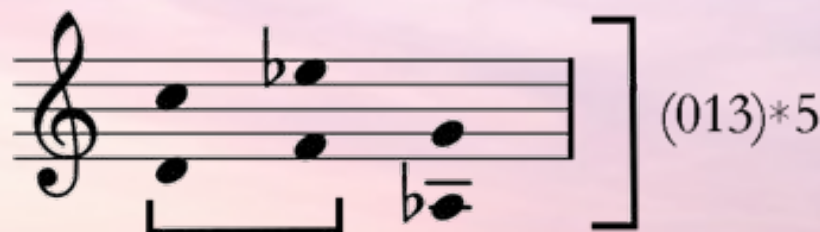
$$\begin{aligned} & (4679) \\ & \times (01) \\ & = (4567^289) \end{aligned} \quad \begin{aligned} & \langle\langle (6, 5.5)_1, (6, 3.5)_5 \rangle\rangle \\ & \times \langle\langle (3.73, 11.5)_1, (0.27, 9.5)_5 \rangle\rangle \\ & \hline & = \langle\langle (22.4, 5)_1, (1.6, 8)_5 \rangle\rangle \end{aligned}$$

Feldman, *Palais de Mari*

This idea is repeated frequently throughout the last part of the piece.



(027) (025)

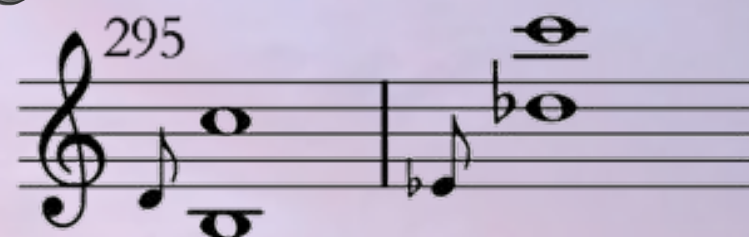


2 * 3

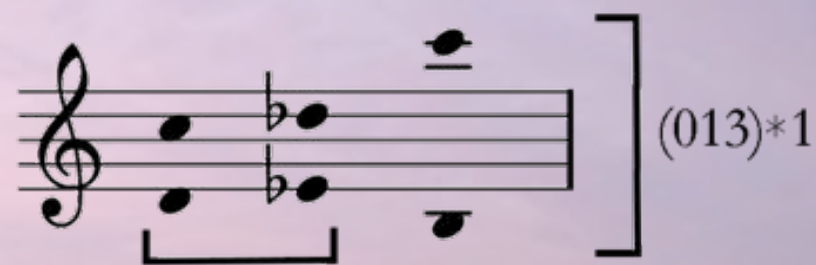
RH

LH

This variant of the idea makes it evident that *both* are products of (013) with ic5 or ic1:



(013) (013)



2 * 1

RH

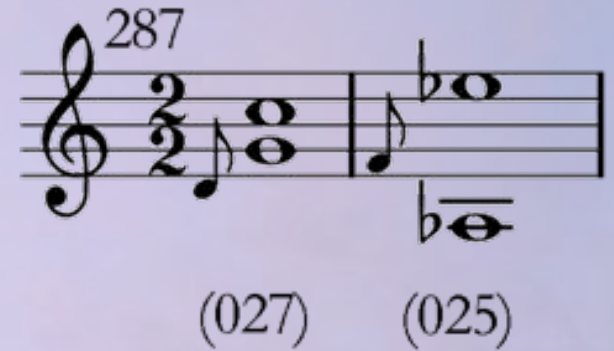
LH

(Hanninen *G/a*)

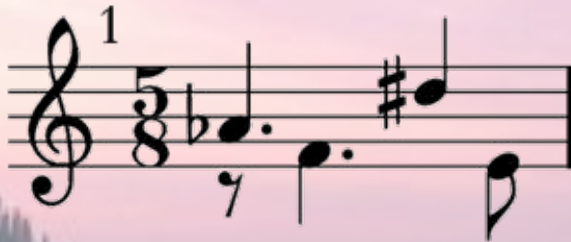
Feldman, *Palais de Mari*

Hanninen:

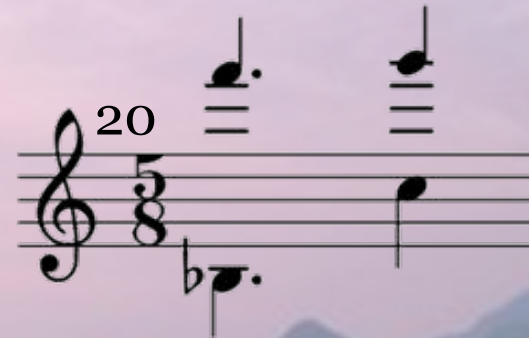
“The arrival of G/a 287–88 . . . is the keystone in a remarkable confluence of events. First, it defines the center of subset G/a , and also of set G . Second, it forms a bridge to set A , recalling and rearranging intervals and key pcs of set A . Third, it forms a second, and stronger, bridge to set C .”



Gesture from set A:



Gesture from set C:



Feldman, *Palais de Mari*

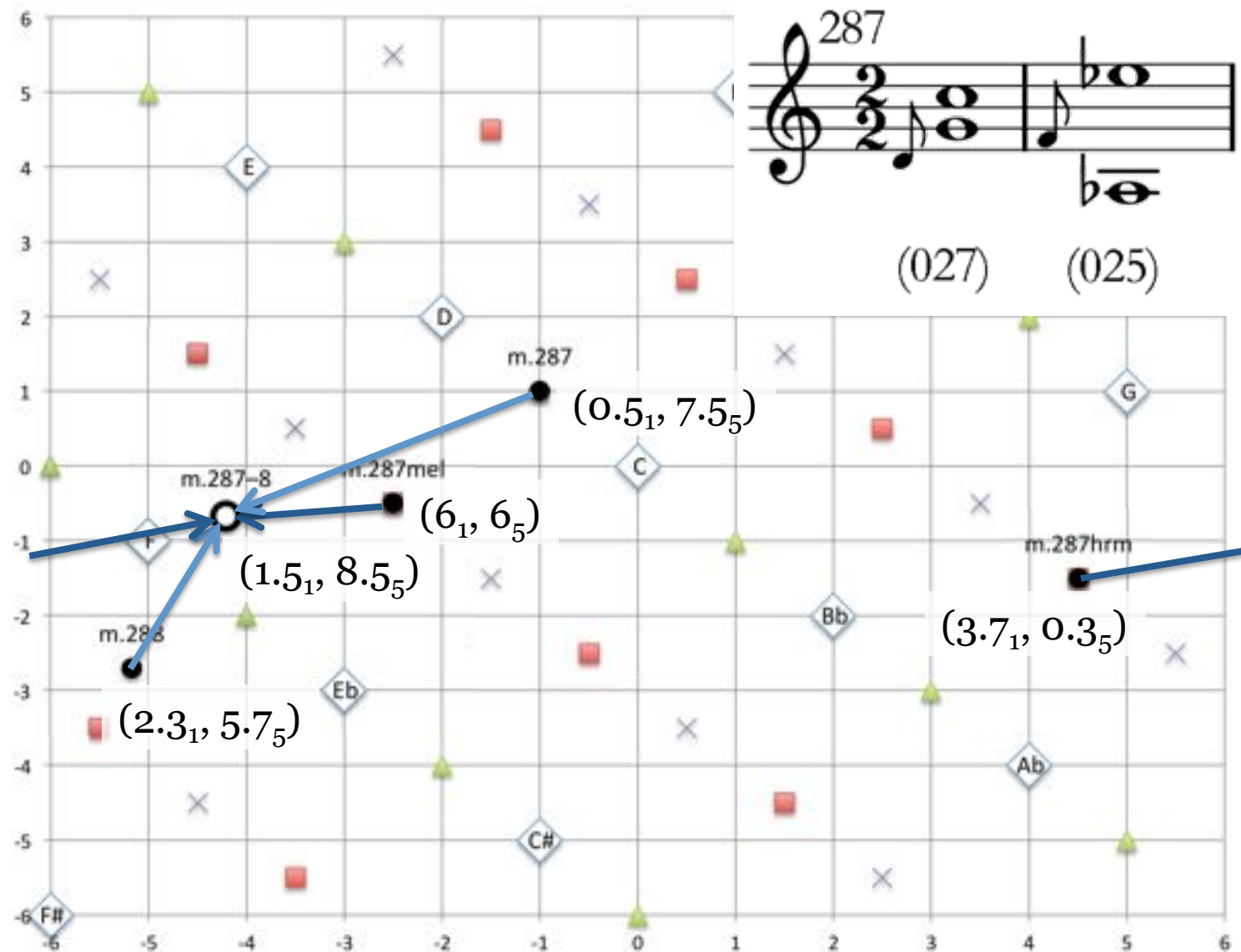
Ph_{1,5}-space

The (0235) and G–A_b subsets are spread out in Ph₁ and close in Ph₅

The individual chords are more spread out in Ph₅, meaning they have stronger f_5

← Ph₅ →

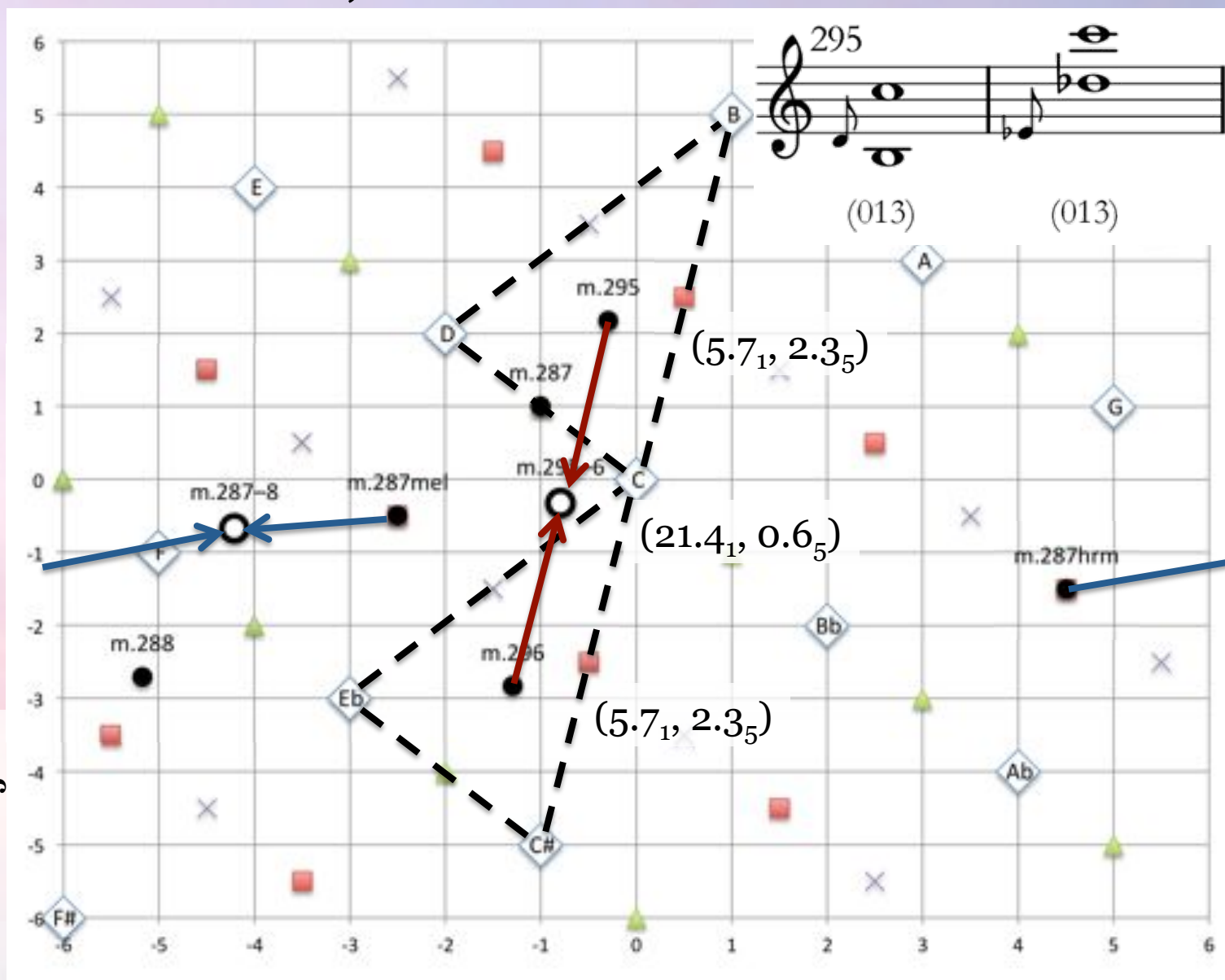
← Ph₁ →



Feldman, *Palais de Mari*

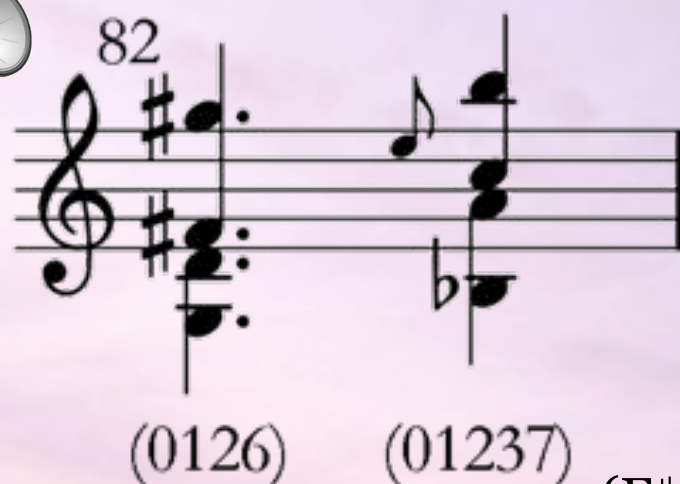
Ph_{1,5}-space

The (013)s of the variant gesture are spread out in Ph₅ and close in Ph₁, making the sum strongly chromatic.



Feldman, *Palais de Mari*

One important gesture reveals how Feldman “cripples” symmetries by asymmetrically dividing a symmetric entity



(0126)

(01237)

(F#GG#D)

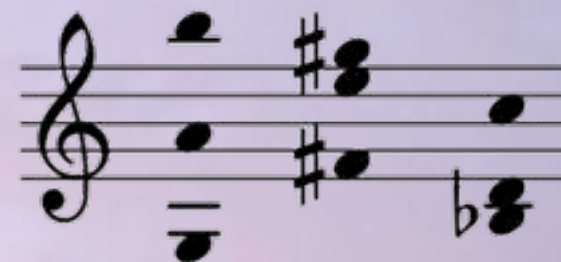
+ (ABbBCE)

= (F#GG#ABbBCDE)

Symmetric
source:



(024) × (036)



$\langle\langle (3.73, 5.5)_1, (0.27, 3.5)_5 \rangle\rangle$

$\langle\langle (5.73, 1.3)_1, (2.27, 2.8)_5 \rangle\rangle$

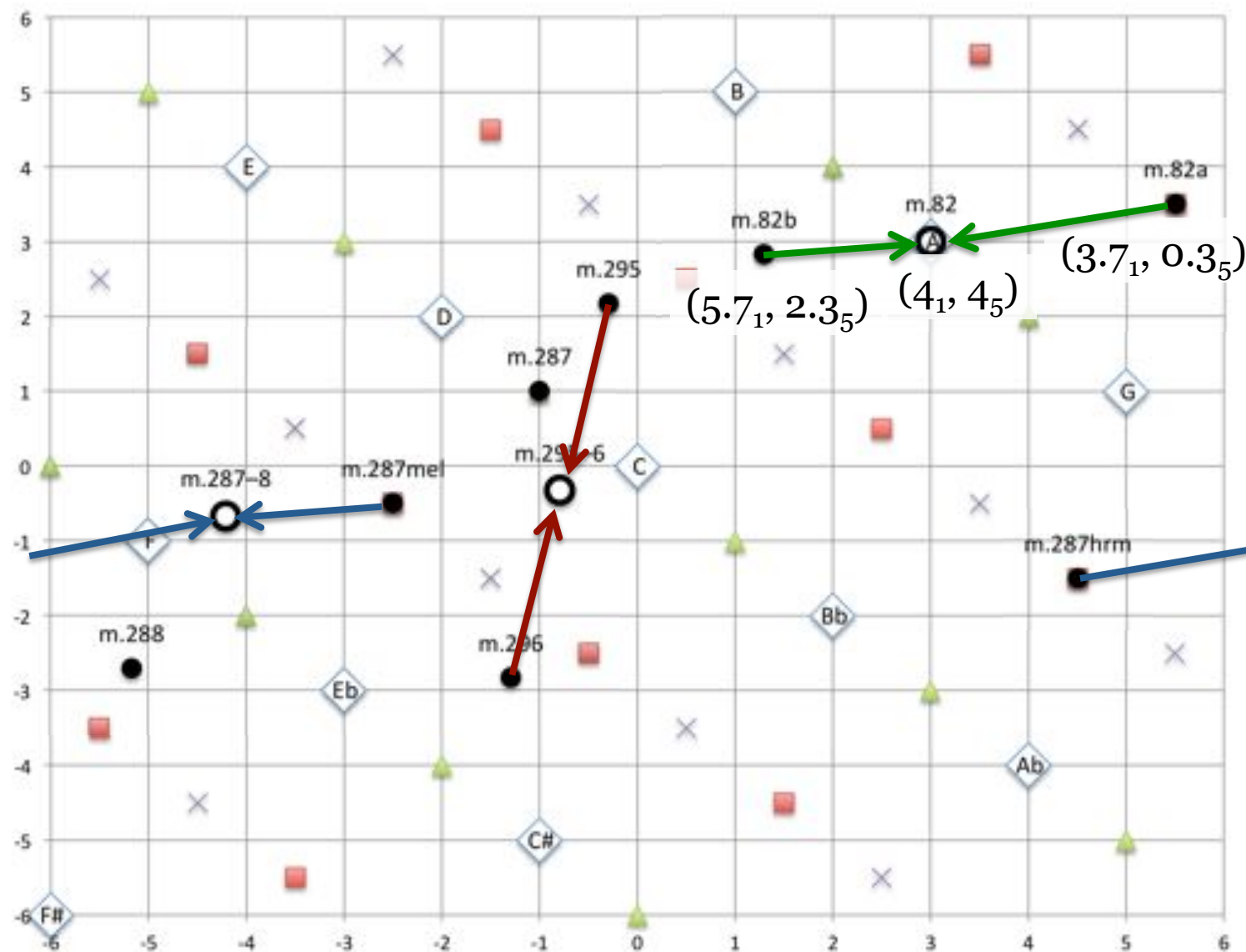
$\langle\langle (4, 3)_1, (4, 3)_5 \rangle\rangle$

The symmetric source chord is balanced between f_1 and f_5 , but it is split into more heavily chromatic chords.

Feldman, *Palais de Mari*

Ph_{1,5}-space

The Ph₁ spread shows how more chromatic chords are extracted from a balanced parent chord.



Summary

- DFT is a **change of basis** applied to the domain of pc-distributions.
- Each DFT component measures a musically interpretable quality relating to a type of **periodicity**.
- **DFT magnitudes** can replace much of pcset-theory's use of interval content to relate harmonic entities.
- The **fifth Fourier component** measures **diatonicity**, and provides a more systematic approach to reconciling **subsets and supersets** with scale theory.
- The **fourth Fourier component** represents **octatonicity** and is used by composers like Debussy and Stravinsky to relate diatonically distant harmonies.
- Distances in **phase space** provide a common-tone-based measure of relatedness between collections of any size.

Bibliography

- Ames, Paula Kopstick. “Piano (1977).” In *The Music of Morton Feldman*, ed. T. Delio (Westport, Conn.: Greenwood Press), 99–146.
- Amiot, Emmanuel. 2007. “David Lewin and Maximally Even Sets.” *Journal of Mathematics and Music* 1: 157–72.
- . 2013. “The Torii of Phases.” *Proceedings of the International Conference for Mathematics and Computation in Music, Montreal, 2013*, ed. J. Yust, J. Wild, and J.A. Burgoyne (Heidelberg: Springer).
- Amiot, Emmanuel, and William Sethares. 2011. “An Algebra for Periodic Rhythms and Scales.” *Journal of Mathematics and Music* 5/3, 149–69.
- Burkhart, Charles. 1980. “The Symmetrical Source of Webern’s Opus 5, No. 4.” In *Music Forum V*, ed. Salzer (New York: Columbia University Press) 317–34.
- Callender, Clifton. 2007. “Continuous Harmonic Spaces.” *Journal of Music Theory* 51/2: 277–332.
- Callender, Clifton, Ian Quinn, and Dmitri Tymoczko. 2008. “Generalized Voice-Leading Spaces.” *Science* 320: 346–8.
- Cohn, Richard. 1988. “Transpositional Combination and Inversional Symmetry in Bartok.” *Music Theory Spectrum* 10: 19–42.
- Delio, Thomas. “Last Pieces #3 (1959)” In *The Music of Morton Feldman*, ed. T. Delio (Westport, Conn.: Greenwood Press), 39–70.
- Feldman, Morton. 2000. *Give my Regards to Eighth Street: Collected Writings of Morton Feldman*, ed. B.H. Friedman. Cambridge, Mass.: Exact Change.

Bibliography

- Forte, Allen. 1964. "A Theory of Set Complexes for Music." *Journal of Music Theory*, 8: 136–83.
- . 1973. *The Structure of Atonal Music*. New Haven: Yale University Press.
- . 1978. *The Harmonic Organization of the Rite of Spring*. Yale University Press.
- . 1986. "Letter to the Editor in Reply to Richard Taruskin from Allen Forte." *Music Analysis* 5/2–3: 321–37.
- Hamman, Michael. "Three Clarinets, Cello and Piano (1971)" In *The Music of Morton Feldman*, ed. T. Delio (Westport, Conn.: Greenwood Press), 71–98.
- Johnson, Steven. 2013. "It Must Mean Something: Narrative in Beckett's *Molloy* and Feldman's *Triadic Memories*." *Contemporary Music Review* 36/2: 639–68.
- Hanninen, Dora. 2012. *A Theory of Music Analysis: On Segmentation and Associative Organization*. Rochester, NY: University of Rochester Press.
- Hook, Julian. 2008. "Signature Transformations." In *Mathematics and Music: Chords, Collections, and Transformations*, ed. Martha Hyde and Charles Smith, 137–60. Rochester: University of Rochester Press.
- . 2011. "Spelled Heptachords." In *Mathematics and Computation in Music: 3rd International Conference, MCM 2011*, ed. Carlos Agon, Moreno Andreatta, Gerard Assayag, Emmanuel Amiot, Jean Bresson, and John Mandereau, 84–97. Heidelberg: Springer.

Bibliography

- Lewin, David. 1959. "Re: Intervallic Relations between Two Collections of Notes." *Journal of Music Theory* 3: 298–301.
- . 2001. Special Cases of the Interval Function between Pitch-Class Sets X and Y. *Journal of Music Theory* 45/1: 1–29.
- . 2007. *Generalized Musical Intervals and Transformations*, 2nd Edition. Oxford University Press.
- Perle, George. *Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg, and Webern*, 6th Edition. Berkeley, Calif.: UC Press.
- Quinn, Ian. 2006. "General Equal-Tempered Harmony" (in two parts). *Perspectives of New Music* 44(2)–45(1): 114–159 and 4–63.
- Van den Toorn, Pieter. 1983. *The Music of Stravinsky*. Yale University Press.
- . 2003. "Colloquy: Stravinsky and the Octatonic: The Sounds of Stravinsky." *Music Theory Spectrum* 25/1: 167–85.
- Van den Toorn, Pieter, and John McGuinness. 2012. *Stravinsky and the Russian Period: Sound and Legacy of a Musical Idiom*. Cambridge University Press.
- Straus, Joseph. 1982. "Stravinsky's 'Tonal Axis.'" *Journal of Music Theory* 26/2: 261–90.
- Taruskin, Richard. 1979. "Review: Allan Forte, *The Harmonic Organization of the Rite of Spring*."
- . 1986. "Letter to the Editor from Richard Taruskin." *Music Analysis* 5/2–3: 313–20.

Bibliography

- Tymoczko, Dmitri. 2002. "Stravinsky and the Octatonic: A Reconsideration." *Music Theory Spectrum* 24/1: 68–102.
- . 2003. "Colloquy: Stravinsky and the Octatonic: Octatonicism Reconsidered Again." *Music Theory Spectrum* 25/1: 185–202.
- . 2004. "Scale Networks and Debussy." *Journal of Music Theory* 44/2: 215–92.
- . 2008. "Set-Class Similarity, Voice Leading, and the Fourier Transform." *Journal of Music Theory* 48/2: 251–72.
- . 2011. *Geometry of Music*. Oxford University Press.
- Yust, Jason. 2015a. "Restoring the Structural Status of Keys through DFT Phase Space." *Proceedings of the International Congress for Music and Mathematics* (forthcoming).
- . 2015b. "Schubert's Harmonic Language and Fourier Phase Space." *Journal of Music Theory* 59/1, 121–181.