Geometrical Realizations of Two- and Three-Dimensional Generalized Tonnetze

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Outline

(1) Two-Dimensional Generalized Tonnetze
(2) Optimizing Spaces and Intervalllic Duplications
(3) Examples of Music Analysis with Non-Triadic Tonnetze
(4) Three-Dimensional (tetrachordal) Tonnetze
(1) Two-Dimensional Generalized Tonnetze
The triadic Tonnetz relates triads via

- Voice-leading efficiency
- Common-tone retention

Tymoczko (JMT 2012) generalizes voice-leading properties of the triadic Tonnetz.
Cohn (JMT 1997) generalizes the common-tone aspect of the triadic Tonnetz.

Previous approaches (Cantazaro JMM 2011, Bigo et al. MCM 2013, Bigo CMJ 2015) have treated it as a network or topology rather than a geometry.
TriadicTonnetz

Perfect 5th

Intervallic Axes of the triadic Tonnetz

A

E

B

C

R

L

G

P

E'

Minor 3rd

Major 3rd
General 2-dimensional *Tonnetz*

Geometric stipulations:
- Each pc is represented by a single distinct point
- Finite number of pcs (universe $\mathbb{Z}_u$)
- Transposability: All transpositions (= translations) of $\mathbb{Z}_u$ correspond to rigid geometric transformations (e.g., translations or rotations).
General 2-dimensional Tonnetz

Definition of a Tonnetz:

• Simplicial decomposition of the space
• Vertices of the simplicial regions include all and only the pc points.
• Transposablility
General 2-dimensional *Tonnetz*

All transpositions of $\mathbb{Z}_u$ are cyclic, therefore the transposability conditions lead naturally to toroidal geometries (by representing transpositions as geometric translations).

Other topologies (e.g., non-orientable) can be made to work in special cases by embedding in higher-dimensional spaces or by folding the torus to equate distinct axes.
Example: Tymoczko’s triadic Tonnetz

- As a two-dimensional space (surface of the figure) it is toroidal. It satisfies transposability through translations.
- Embedded in a three-dimensional space it satisfies transposability through rotations and screw rotations.
- For Tymoczko, the embedding three-dimensional space has an important role in the voice-leading intent of the figure, but to the transposability of vertices and the lattice it is superfluous.
General 2-dimensional Tonnetz

Construction of a two-dimensional Tonnetz:

• Space is $T_2$ with dimensions scaled to $[0, u)$
• Choose $(x, y)$ to represent interval 1 such that $x$, $y$, and $u$ are mutually coprime integers.
• Choose two non-parallel lines through pc 0, add lines parallel to these through all pcs.
• Add parallel lines that bisect the resulting quadrilaterals.
Define the semitone as the vector $(-3, -5)$ in universe $u = 12$. 

Example

\[ \text{Int1} = (-3, -5) \]
Example

Place all pcs by reiterating this interval.
Example

Choose two axes through C.

One corresponds to interval 5 or 7 and cycles through all pcs.

The other corresponds to interval 4 or 8 and goes through three pcs.
Example

Add lines parallel to the major-third axis to cover all pcs.

There are two ways to bisect the resulting parallelograms.
Example

Add a third set of parallel lines bisecting the parallelograms.

The choice of a minor-third axis results in a compact version of the standard triadic Tonnetz.

Geometry of the Generalized Tonnetz

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Example

What happens when a different choice is made for the representative of interval 1?

For example, let $\text{int}1 = (1, 6)$. 
Example

The fifths axis cycles through all pcs, but is also much longer here than in the other space.
Example

Add a minor-third axis.
Example

Add the other parallel minor-third axes.
Example

The resulting parallelograms can be bisected by a major-third axis.
Example

Add the other major-third axes.

This Tonnetz is the same as the previous one skewed. The fit of space to Tonnetz is poor, resulting in non-compact regions.
General 2-dimensional *Tonnetz*

Resulting space:

Equivalent to some *Fourier phase space*  
(Amiot *MCM* 2013, 2016; Yust *JMT* 2015, *JMT* 2016)  
I.e., each dimension represents one of the interval cycles of universe $u$.

Triangulation into $2u$ regions, where triangles represent all instances of a single set class.  
*Any set class is possible* (as long as $u$ is its minimal embedding universe).
(2) Optimizing Spaces and Tonnetze with Intervallic Duplications
One-Dimensional Phase Spaces

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Optimizing Spaces

Available unique two-dimensional spaces ($u = 12$):

- $Ph_1-Ph_1$
- $Ph_2-Ph_2$
- $Ph_3-Ph_3$
- $Ph_4-Ph_4$
- $Ph_5-Ph_5$
- $Ph_6-Ph_6$
- $Ph_1-Ph_2$
- $Ph_1-Ph_3$
- $Ph_1-Ph_4$
- $Ph_1-Ph_5$
- $Ph_1-Ph_6$
- $Ph_2-Ph_3$
- $Ph_2-Ph_4$
- $Ph_2-Ph_5$
- $Ph_2-Ph_6$
- $Ph_3-Ph_4$
- $Ph_3-Ph_5$
- $Ph_3-Ph_6$
- $Ph_4-Ph_5$
- $Ph_4-Ph_6$
Optimizing Spaces

Possible optimizing criteria for a given trichord type:

• Minimize total length of intervallic axes
• Maximize the Fourier coefficients of trichord

These generally agree, with the second method being more sensitive

Note: Trichords with duplicated intervals are special cases!
## Optimizing Spaces

<table>
<thead>
<tr>
<th>Trichord</th>
<th>Best space(s)</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>(012)**</td>
<td>$Ph_1$-$Ph_6$</td>
<td>$Ph_1$-$Ph_5$</td>
</tr>
<tr>
<td>(013)</td>
<td>$Ph_1$-$Ph_4$</td>
<td>$Ph_3$-$Ph_4$</td>
</tr>
<tr>
<td>(014)</td>
<td>$Ph_1$-$Ph_3$</td>
<td></td>
</tr>
<tr>
<td>(015)</td>
<td>$Ph_2$-$Ph_3$</td>
<td></td>
</tr>
<tr>
<td>(016)</td>
<td>$Ph_1$-$Ph_2$, $Ph_2$-$Ph_3$, $Ph_2$-$Ph_5$</td>
<td></td>
</tr>
<tr>
<td>(025)</td>
<td>$Ph_4$-$Ph_5$</td>
<td>$Ph_1$-$Ph_5$</td>
</tr>
<tr>
<td>(027)**</td>
<td>$Ph_5$-$Ph_6$</td>
<td>$Ph_3$-$Ph_4$</td>
</tr>
<tr>
<td>(037)</td>
<td>$Ph_3$-$Ph_5$</td>
<td></td>
</tr>
</tbody>
</table>
Duplicated Intervals

Sets (012) and (027) have duplicated intervals (int1 and int5)

Nonetheless, toroidal (012) and (027) Tonnetze are possible because we can draw multiple distinct axes for the same interval.

The optimization strategy is then to minimize two axes for the same interval, meaning one dimension should minimize the Fourier coefficient for this interval (to spread it out).
$Ph_1$ maximizes Fourier coefficients, but $Ph_6$ minimizes it for int1.

The spread of int1 in the $Ph_6$ dimension allows for two equally good int1-axes (up 6 or down 6).
The two distinct axes may be used to represent distinct forms of the given interval. For instance, here the int1s are spelled as chromatic or diatonic semitones (enharmonic distinctions).
In a scale of 7 equally spaced notes, triads have a duplicated interval (the 3rd).

The Tonnetz therefore has two distinct intervallic axes for 3rds.
The space can be sheared and folded to equate the two kinds of thirds.
7-equal triad Tonnetz

The space can be sheared and folded to equate the two kinds of thirds.

The result is Muzzulini (1995) and Mazzola’s (2002) Möbius strip Tonnetz.
(3) Examples of Music Analysis with Non-Triadic Tonnetze

- (025) Tonnetz, Stravinsky “Owl and the Pussycat”
- (013) Tonnetz, Shostakovich String Quartet 12
- (014) Tonnetz, Beethoven String Quartet Op. 132
Stravinsky: “Owl and the Pussycat”

(025)

\[ P_0: \text{The owl and the pussy-cat went to sea in a beautiful pea green boat.} \]

(025)

\[ P_0: \]

They took some honey and plenty of money, wrapped up in a five pound note.

(025)

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Stravinsky: “Owl and the Pussycat”

(025)s in $P_0$ in blue

Vertical (025)s in gray

$T_3$ cycle (Octatonic)
Stravinsky: “Owl and the Pussycat”

I₀: \[ \begin{align*} & \text{(025)} \quad \text{(025)} \quad \text{(025)} \quad \text{(025)} \\
& \text{The owl looked up to the stars above, and sang to a small guitar.} \end{align*} \]

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Stravinsky: “Owl and the Pussycat”

(025)s in $P_0$ in blue

Vertical (025)s in gray

(025)s in $I_0$ in purple
Stravinsky: “Owl and the Pussycat”

An isolated use of $I_6$ occurs at the pivot in the narrative:

RI$_0$:

$I_6$:

And there in a wood a piggy-wig stood,

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Stravinsky: “Owl and the Pussycat”

(025)s in $l_0$ in purple
(025)s in $l_6$ in green
Vertical (025)s in gray
Shostakovich: String Quartet 12
Beginning of second movement, (013)s:
Shostakovich: String Quartet 12

The melody initially focuses on disjunct and $T_1$ relationships between (013)s.

We can identify cycles ($T_1$ or $T_2$) for the disjunct relationship.
Shostakovich: String Quartet 12

$T_2$ cycle:

$T_1$ cycle:
Shostakovich: String Quartet 12

Beginning of second movement, (013)s:
Shostakovich: String Quartet 12

Meas. 7 extends the previous relationships and also expresses part of a conjunct $T_3$ (octatonic) cycle.
Beethoven, Op. 132 String Quartet

Network of (014)s in the opening of the first movement

\[\text{[FG#A]} \, \text{[EFG#]} \, \text{[G#AC]} \, \text{[G#BC]}\]
Beethoven, Op. 132 String Quartet

Chain of (014)s in optimal space, $Ph_1$-$Ph_3$
Beethoven, Op. 132 String Quartet

Chain of (014)s in non-optimal space, $Ph_3-Ph_5$
(4) Three-dimensional Tonnetze
Tetrachordal Tonnetz

Previous approaches (Gollin *JMT* 1998, Childs *JMT* 1998, Bigo *CMJ* 2015) take it as basic that a Tonnetz involves a single set class.

The result is networks of tetrahedra representing tetrachords, usually (0258)s, that intersect mostly in edges. Geometrically this means that there is empty space between the tetrahedra.

To make the Tonnetz a simplicial decomposition we instead completely fill the space with tetrahedra: this requires *three* set classes intersecting in shared trichords.
Tetrachordal *Tonnetz*: Construction

(1) Choose a three-dimensional phase space and any three non-parallel sets of intervallic axes

This partitions the space into parallelepipeds

Here are axes for intervals 2/10, 3/9, and 4/8 (in $\mathbb{Z}_{12}$)
(2) Add three intervals to bisect the faces

The plane defined by these intervals creates tetrahedra for a single set class separated by octahedral regions. (Like Gollin’s tetrachordal Tonnetz but with no shared faces!)

The new intervals in this example are 1/11, 6, and 5/7. The set class is (0137) (an all-interval tetrachord).
Tetrachordal Tonnetz: Construction

The space is now partitioned by four (sets of) planes, each of which is cut by the others into a two-dimensional Tonnetz based on one of the trichordal subsets of the given tetrachord.

In this example, there are planes with (037), (013), (026) and (016) Tonnetzes.
Tetrachordal *Tonnetz*: Construction

(3) The octahedral regions have three internal quadrilaterals. Bisecting *any two* of these with a single interval completes the partition into tetrachords. There are three possibilities.

The internal quadrilaterals in this example are \((\text{FECD}_b)\), \((\text{EB}_b\text{D}_b\text{G})\), and \((\text{FB}_b\text{CG})\).
Tetrachordal Tonnetz: Construction

The first possibility adds an aug-2\textsuperscript{nd} axis (distinct from the min-3\textsuperscript{rd} axis!) creating (014) and (036) planes.
Tetrachordal *Tonnetz*: Construction

These add new tetrahedral (0147) and (0236) regions
The second possibility adds a distinct perfect-fourth axis creating (027) and (015) planes.
Tetrachordal Tonnetz: Construction

The other tetrahedral regions for this Tonnetz are (0237) and (0157)
The third possibility also adds a distinct minor-thirds axis creating (036) and (025) planes.
Tetrachordal Tonnetz: Construction

The resulting Tonnetz has (0136) and (0258) regions.
Properties and Classes of Three-Dimensional *Tonnetze*

- Three tetrachord types
- Six planes with trichordal *Tonnetze*:
  Each trichord is shared by two tetrachords
- Seven intervallic axes
  – Four are shared by all tetrachords and three trichordal planes
  – Three are shared by two tetrachords and two trichordal planes

In \( \mathbb{Z}_{12} \) there are only six interval classes, so at least one must be duplicated.

*Tonnetze* can be classified according to where (in which group of intervallic axes) duplications occur.
## Classes of 3-D Tonnetz in $\mathbb{Z}_{12}$

<table>
<thead>
<tr>
<th>Class</th>
<th>Properties</th>
<th>Tetrachords</th>
<th>Duplications</th>
<th>Omits</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>One sym. trichord</td>
<td>(0125) (0126) (0146)</td>
<td>ic1</td>
<td></td>
<td>$Ph_{1,2,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0237) (0157) (0137)</td>
<td>ic5</td>
<td></td>
<td>$Ph_{2,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0136) (0236) (0146)</td>
<td>ic3</td>
<td></td>
<td>$Ph_{1,2,4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0147) (0236) (0137)</td>
<td>ic3</td>
<td></td>
<td>$Ph_{2,3,4} / Ph_{3,4,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0147) (0258) (0146)</td>
<td>ic3</td>
<td></td>
<td>$Ph_{2,3,4} / Ph_{3,4,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0136) (0258) (0137)</td>
<td>ic3</td>
<td></td>
<td>$Ph_{2,4,5}$</td>
</tr>
<tr>
<td>B</td>
<td>Two sym. trichords</td>
<td>(0124) (0125) (0135)</td>
<td>ic1, ic2</td>
<td>ic6</td>
<td>$Ph_{1,3,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0135) (0237) (0247)</td>
<td>ic5, ic2</td>
<td>ic6</td>
<td>$Ph_{3,5,6}$</td>
</tr>
<tr>
<td>C</td>
<td>Two sym. and one dup. trichord</td>
<td>(0126) (0127) (0157)</td>
<td>ic1, ic5, (016)</td>
<td>ic3</td>
<td>$Ph_{1,2,6}$ or $Ph_{2,5,6}$</td>
</tr>
</tbody>
</table>
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</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>Dup. tetrachord</td>
<td>(0134), (0236) × 2</td>
<td>ic1, ic3, (013), (014)</td>
<td>ic5</td>
<td>$Ph_{1,4,6}$</td>
</tr>
<tr>
<td></td>
<td>+ one sym. trichord</td>
<td>(0145), (0125) × 2</td>
<td>ic1, ic4, (014), (015)</td>
<td>ic6</td>
<td>$Ph_{1,3,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0156), (0126) × 2</td>
<td>ic1, ic5, (015), (016)</td>
<td>ic3</td>
<td>$Ph_{1,2,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0156), (0157) × 2</td>
<td>ic1, ic5, (015), (016)</td>
<td>ic3</td>
<td>$Ph_{2,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0235), (0135) × 2</td>
<td>ic2, ic3, (013), (025)</td>
<td>ic6</td>
<td>$Ph_{1,3,6}$ or $Ph_{3,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0235), (0136) × 2</td>
<td>ic2, ic3, (013), (025)</td>
<td>ic4</td>
<td>$Ph_{2,3,4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0347), (0147) × 2</td>
<td>ic3, ic4, (014), (037)</td>
<td>ic2</td>
<td>$Ph_{2,3,4}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0158), (0237) × 2</td>
<td>ic4, ic5, (015), (037)</td>
<td>ic6</td>
<td>$Ph_{3,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0358), (0258) × 2</td>
<td>ic3, ic5, (025), (037)</td>
<td>ic1</td>
<td>$Ph_{4,5,6}$</td>
</tr>
</tbody>
</table>
## Classes of 3-D Tonnetz in $\mathbb{Z}_{12}$

<table>
<thead>
<tr>
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<th>Tetrachords</th>
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<th>Omits</th>
<th>Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>Dup. tetrachord &amp; Aug. triad</td>
<td>(0145), (0148) x 2</td>
<td>ic4 (x 3), ic1, (014), (015)</td>
<td>ic2, ic6</td>
<td>$Ph_{1,2,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0347), (0148) x 2</td>
<td>ic4 (x 3), ic3, (014), (037)</td>
<td>ic2, ic6</td>
<td>$Ph_{1,4,6}$ / $Ph_{4,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0158), (0148) x 2</td>
<td>ic4 (x 3), ic5, (015), (037)</td>
<td>ic2, ic6</td>
<td>$Ph_{2,5,6}$</td>
</tr>
<tr>
<td>F</td>
<td>Dup. tetrachord &amp; two sym. trichords</td>
<td>(0134), (0124) x 2</td>
<td>ic1, ic3, ic2, (013), (014)</td>
<td>ic5, ic6</td>
<td>$Ph_{1,3,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0358), (0247) x 2</td>
<td>ic5, ic3, ic2, (025), (037)</td>
<td>ic1, ic6</td>
<td>$Ph_{3,5,6}$</td>
</tr>
<tr>
<td>G</td>
<td>Dup. tetrachord &amp; sym. tetrachord</td>
<td>(0123), (0124) x 2</td>
<td>ic1 (x 3), ic2, (012), (013)</td>
<td>ic5, ic6</td>
<td>$Ph_{1,5,6}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0257), (0247) x 2</td>
<td>ic5 (x 3), ic2, (025), (027)</td>
<td>ic1, ic6</td>
<td>$Ph_{1,5,6}$</td>
</tr>
</tbody>
</table>
Example: Aug-2\textsuperscript{nd} Tonnetz

Class A: (0137), (0147), (0236)

Optimal space: $Ph_2$-$Ph_3$-$Ph_4$ or $Ph_3$-$Ph_4$-$Ph_6$

Duplicated ic3s may be interpreted as min 3\textsuperscript{rd}/aug 2\textsuperscript{nd}

(0137): Diatonic subset, maj. triad + #4\textsuperscript{th} (IV + $\hat{7}$) or
min. triad + b2nd (V\textsuperscript{7(no5th)} + $\hat{3}$)

(0147): Non-diatonic, maj. triad + b2nd (V + b6) or
min. triad + #4\textsuperscript{th} (iv + $\#7$)

(0236): Non-diatonic, harmonic minor scale segment:
b6-$\hat{7}$-$\hat{1}$-$\hat{2}$ or $\hat{4}$-$\hat{5}$-b6-$\hat{7}$
Example: JI scalar Tonnetz

Class D: (0235), (0135) x 2

Optimal space: $Ph_2-Ph_3-Ph_6$

Duplicated ic2s and ic3s may be interpreted as just vs. Pythagorean intervals (10/9 vs. 9/8 and 6/5 vs. 32/27)

$j(0135)$: $\sqrt{s} + \sqrt{p2} + \sqrt{j2}$ or $\sqrt{j2} + \sqrt{p2} + \sqrt{s}$

$p(0135)$: $\sqrt{s} + \sqrt{j2} + \sqrt{p2}$ or $\sqrt{p2} + \sqrt{j2} + \sqrt{s}$

$(0235)$: $\sqrt{j2} + s + \sqrt{p2}$ or $\sqrt{p2} + s + \sqrt{j2}$
Example: Douthett’s Tetrahedral Tonnetz

This can be derived from class D Tonnetz on (0358), (0258) x 2 by equating duplicated ic3 and ic5 axes. The result is cyclic in two dimensions with the (026)-Tonnetz planes forming a boundary in the third dimension.

From unpublished ms., 1997
Summary

- Toroidal topology is central to the Tonnetz idea generally, although it can be modified to reflect other kinds of topology.
- The possible toroidal geometries of any Tonnetz are equivalent to Fourier phase spaces.
- Toroidal spaces can duplicate intervals—i.e., multiple axes representing the same interval.
- Tonnetze may be understood as simplicial decompositions of toroidal spaces. Three-dimensional Tonnetze are then based on three tetrachord types intersecting in shared faces.
- Because three-dimensional Tonnetz have seven intervallic axes, interval duplications are unavoidable in $\mathbb{Z}_{12}$. 
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