Abstract

This Online Appendix to the paper titled “Progressive Learning” presents the proof of Lemma 0, an extension of our equilibrium characterization allowing for mixed strategies, an analysis of the full commitment case, and details for Example 4 showing that path dependence can arise when shocks are ergodic.

JEL Classification Codes: C73, D86

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OA1 Proof of Lemma 0

Proof of part (i). The proof is by strong induction on the cardinality of the support of the principal’s beliefs, $C[h_t]$. Fix an equilibrium $(\sigma, \mu)$, and note that the claim is true for all histories $h_t$ such that $|C[h_t]| = 1$.\(^1\) Suppose next that the claim is true for all histories $h$ with $|C[h]| \leq n - 1$, and consider a history $h_t$ with $|C[h_t]| = n$.

Suppose by contradiction that $V^{(\sigma, \mu)}_{K[h_t]}[h_t, b_t] > 0$. Then, there must exist a history $(h_{t'}, b_{t'})$ with $h_{t'} \succeq h_t$ that arises on the path of play with positive probability at which the principal offers a transfer $T_{t'} > c_{K[h_t]}$ that type $c_{K[h_t]}$ accepts. Note first that, since type $c_{K[h_t]}$ accepts offer $T_{t'}$, all types in the support of $C[h_{t'}]$ must also accept it. Indeed, if this were not true, then there would be a highest type $c_k \in C[h_{t'}]$ that rejects the offer.

By the induction hypothesis, the equilibrium payoff that this type obtains at history $h_{t'}$ is $V^{(\sigma, \mu)}_k[h_{t'}, b_{t'}] = 0$, since this type would be the highest cost of in the support of the principal’s beliefs following a rejection. But this cannot be, since type $c_k$ can get a payoff of at least $T_{t'} - c_k > 0$ by accepting the principal’s offer at time $t'$.

We now construct an alternative strategy profile $\tilde{\sigma}$ that is otherwise identical to $\sigma$ except that at history $(h_{t'}, b_{t'})$ the agent is offered a transfer $\tilde{T} \in (c_{K[h_t]}, T_{t'})$. Specify the principal’s beliefs at history $(h_{t'}, b_{t'})$ as follows: regardless of the agent’s action, the principal’s beliefs at the end of the period are the same as her beliefs at the beginning of the period. At all other histories, the principal’s actions and beliefs are the same as in the original equilibrium. Note that, given these beliefs, at history $h_{t'}$ all agent types in $C[h_{t'}]$ find it strictly optimal to accept the principal’s offer $\tilde{T}$ and take the action. Thus, the principal’s payoff at history $h_{t'}$ is larger than her payoff under the original equilibrium, which cannot be since the original equilibrium was in $\Sigma_K$. \(\square\)

Proof of part (ii). The proof is by induction of the cardinality of $C[h_t]$. Consider first a history $h_t$ such that $|C[h_t]| = 1$. Since $|C[h_t]| = 1 < 2$, the claim is vacuously true.

Suppose next that the result holds for all histories $h$ such that $|C[h]| \leq n - 1$, and consider a history $h_t$ such that $|C[h_t]| = n$. Consider two “adjacent” types $c_i, c_{i+1} \in C[h_t]$. We have two possible cases: (i) with probability 1, types $c_i$ and $c_{i+1}$ take the same action at all histories $(h_{t'}, b_{t'})$ with $h_{t'} \succeq h_t$; (ii) there exists a history $(h_{t'}, b_{t'})$ with $h_{t'} \succeq h_t$ at

\(^1\)Indeed, if $C[h_t] = \{c_i\}$, then in any PBE in $\Sigma_K$ the agent takes action $a = 1$ at time $t' \geq t$ if and only if $b_{t'} \in E_i$, and the principal pays the agent a transfer equal to $c_i$ every time the agent takes the action.
which types $c_i$ and $c_{i+1}$ take different actions. Under case (i),

$$V_i^{(\sigma, \mu)}[h_t, b_t] = E^{(\sigma, \mu)} \left[ \sum_{t'=t}^{\infty} \delta^{t'-t} (1 - \delta)(T_{t'} - c_i) a_{t', i} | h_t, b_t \right]$$

$$= E^{(\sigma, \mu)} \left[ \sum_{t'=t}^{\infty} \delta^{t'-t} (1 - \delta)(T_{t'} - c_{i+1}) a_{t', i+1} | h_t, b_t \right]$$

$$+ E^{(\sigma, \mu)} \left[ \sum_{t'=t}^{\infty} \delta^{t'-t} (1 - \delta)(c_{i+1} - c_i) a_{t', i+1} | h_t, b_t \right]$$

$$= V_i^{(\sigma, \mu)}[h_t, b_t] + A_{i+1}^{(\sigma, \mu)}[h_t, b_t](c_{i+1} - c_i).$$

For case (ii), let $t = \min\{t' \geq t : a_{t', i+1} \neq a_{t', i} \}$ be the first time after $t$ at which types $c_i$ and $c_{i+1}$ take different actions. Let $c_k \in C[h_t]$ be the highest cost type that takes the action at time $t$. The transfer $T_k$ that the principal offers at time $t$ must satisfy $V_k^{(\sigma, \mu)}[h_t, b_t] = (1 - \delta)(T_k - c_k) = V_{k+1}^{(\sigma, \mu)}[h_t, b_t] + A_{k+1}^{(\sigma, \mu)}[h_t, b_t](c_k - c_k), \text{2}$ Note further that $V_{k+1}^{(\sigma, \mu)}[h_t, b_t] \geq (1 - \delta)(T_k - c_{k+1})$, since an agent with cost $c_{k+1}$ can guarantee $(1 - \delta)(T_k - c_{k+1})$ by taking the action at time $t$ and then not taking the action in all future periods. Since $(1 - \delta)(T_k - c_k) = V_{k+1}^{(\sigma, \mu)}[h_t, b_t] + A_{k+1}^{(\sigma, \mu)}[h_t, b_t](c_{k+1} - c_k)$, it follows that $A_{k+1}^{(\sigma, \mu)}[h_t, b_t] \leq 1 - \delta$.

We now show that all types below $c_k$ also take the action at time $t$. That is, we show that all agents in the support of $C[h_t]$ with cost weakly lower than $c_k$ take the action at $t$, and all agents with cost weakly greater than $c_{k+1}$ do not take the action. Note that this implies that $c_i = c_k$ (since types $c_i$ and $c_{i+1}$ take different actions at time $t$). Suppose for the sake of contradiction that this is not true, and let $c_j$ be the highest cost type below $c_k$ that takes does not take the action. The payoff that this agent gets from not taking the action is $V_j^{(\sigma, \mu)}[h_t, b_t] = V_{k+1}^{(\sigma, \mu)}[h_t, b_t] + A_{k+1}^{(\sigma, \mu)}[h_t, b_t](c_{k+1} - c_j)$, which follows since at time $t$ types $c_j$ and $c_{k+1}$ do not take the action and since, by the induction hypothesis, from time $t+1$ onwards the payoff that an agent with cost $c_j$ gets is equal to what this agent would get by mimicking an agent with cost $c_{k+1}$. On the other hand, the payoff

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2The first equality follows since, after time $t$, type $c_k$ is the highest type in the support of the principal’s beliefs if the agent takes action $a = 1$ at time $t$. The second equality follows since we focus on PBE in $\Sigma_K$, so the transfer $T_k$ leaves a $c_k$-agent indifferent between accepting and rejecting.
that agent $c_j$ obtains by taking the action and mimicking type $c_k$ is

$$V_{j \rightarrow k}^{(\sigma, \mu)}[h_t, b_j] = V_k^{(\sigma, \mu)}[h_t, b_j] + A_k^{(\sigma, \mu)}[h_t, b_j](c_k - c_j)$$

$$= (1 - \delta)(T_k - c_j) + A_k^{(\sigma, \mu)}[h_t, b_j](c_k - c_j)$$

$$= V_{k+1}^{(\sigma, \mu)}[h_t, b_j] + A_{k+1}^{(\sigma, \mu)}[h_t, b_j](c_{k+1} - c_k) + A_k^{(\sigma, \mu)}[h_t, b_j](c_k - c_j)$$

$$> V_{k+1}^{(\sigma, \mu)}[h_t, b_j] + A_{k+1}^{(\sigma, \mu)}[h_t, b_j](c_{k+1} - c_j),$$

where the inequality follows since $A_{k+1}^{(\sigma, \mu)}[h_t, b_j] \leq 1 - \delta < A_k^{(\sigma, \mu)}[h_t, b_j].$ Hence, type $j$ strictly prefers to take the action, a contradiction. Therefore, all types below $c_k$ take the action at time $t$, and so $c_i = c_k$.

By the arguments above, the payoff that type $c_i = c_k$ obtains at time $t$ is

$$V_i^{(\sigma, \mu)}[h_t, b_j] = (1 - \delta)(T_k - c_i) = V_{i+1}^{(\sigma, \mu)}[h_t, b_j] + A_{i+1}^{(\sigma, \mu)}[h_t, b_j](c_{i+1} - c_i),$$

since transfer that the principal offers at time $t$ satisfies $(1 - \delta)(T_k - c_i) = V_{i+1}^{(\sigma, \mu)}[h_t, b_j] + A_{i+1}^{(\sigma, \mu)}[h_t, b_j](c_{i+1} - c_i)$. Moreover,

$$V_i^{(\sigma, \mu)}[h_t, b_j] = \mathbb{E}^{(\sigma, \mu)} \left[ \sum_{t'=t}^{t-1} \delta^{t'-t} (1 - \delta)(T_{t'} - c_i)a_{t',i} + \delta^{t-t} V_i^{(\sigma, \mu)}[h_t, b_j]|h_t, b_j] \right]$$

$$\quad = \mathbb{E}^{(\sigma, \mu)} \left[ \sum_{t'=t}^{t-1} \delta^{t'-t} ((1 - \delta)(T_{t'} - c_{i+1})a_{t',i+1} + (1 - \delta)(c_{i+1} - c_i)a_{t',i+1}) |h_t, b_j] \right]$$

$$\quad + \mathbb{E}^{(\sigma, \mu)} \left[ \delta^{t-t} \left( V_{i+1}^{(\sigma, \mu)}[h_t, b_j] + A_{i+1}^{(\sigma, \mu)}[h_t, b_j](c_{i+1} - c_i) \right) |h_t, b_j] \right]$$

$$\quad = V_{i+1}^{(\sigma, \mu)}[h_t, b_j] + A_{i+1}^{(\sigma, \mu)}[h_t, b_j](c_{i+1} - c_i),$$

where the second equality follows since $a_{t',i} = a_{t',i+1}$ for all $t' \in \{t, ..., t-1\}$. Hence, the result also holds for histories $h_t$ with $|C[h_t]| = n$. $\square$

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3Recall that, for all $(h_t, b_j)$, $A_k^{(\sigma, \mu)}[h_t, b_j] = (1 - \delta)\mathbb{E}^{(\mu, \sigma)}[\sum_{t'=t}^{\infty} \delta^{t'-t} a_{t',k}|h_t, b_j]$. By assumption, an agent with type $c_k$ takes action $a = 1$ at time $t$, so $a_{t,k} = 1$. Moreover, it is easy to show that an agent with cost $c_k$ will take action $a = 1$ with positive probability at some date $t > t$. Therefore, $A_k^{(\sigma, \mu)}[h_t, b_j] > 1 - \delta$. 

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This appendix extends the results in the main text to allow for mixed strategies. In particular, we show that the equilibrium we characterize in Theorem 1 remains the unique PBE that is sequentially optimal for the principal among all finitely revealing PBE; i.e., among all PBE in which, along any history, the principal updates her beliefs a finite number of periods.

Fix a PBE \((\sigma, \mu)\), with \(\sigma = (\tau, \{\alpha_k\}_{k=1}^K)\). For any history \((h_t, b_t)\), we say that period \(t\) is a period of revelation if (a) \(\mu[h_t] \notin S_1\) (i.e., if the principal is uncertain about the agent’s type) and (b) there exists \(c_i, c_j \in C[h_t]\) such that \(\alpha_i(h_t, b_t) \neq \alpha_j(h_t, b_t)\) (i.e., there exists at least two types in the support of the principal’s beliefs that play different –possibly mixed– actions at history \((h_t, b_t)\)). We say that an equilibrium \((\sigma, \mu)\) is \(T\)-revealing if, for any \(t\) and along any history \(h_t\), the number of periods of revelation \(t' < t\) is not greater than \(T\).

Three things are worth noting about \(T\)-revealing PBE. First, a \(T\)-revealing strategy does not put any bound on the occurrence of the last period of revelation. Hence, information may be revealed at any point during the game. Second, a \(T\)-revealing strategy does not require the agent to reveal her information fully. Third, since the set of possible types of the agent is finite, any pure strategy PBE is \(T\)-revealing for some \(T\).

Let \(\Sigma^M_0\) denote the set of PBE that are finitely revealing (i.e., the set of PBE that are \(T\)-revealing for some finite \(T\)). For all \(k = 1, ..., K\), we define the sets \(\Sigma^M_k\) recursively as follows:

\[
\Sigma^M_k := \left\{ (\sigma, \mu) \in \Sigma^M_{k-1} : \forall (h_t, b_t) \text{ with } \mu[h_t] \in S_k \text{ and } \forall (\sigma', \mu') \in \Sigma^M_{k-1} \right. \\
\left. U(\sigma, \mu)[h_t, b_t] \geq W(\sigma', \mu')[\mu[h_t], b_t] \right\}.
\]

Let \((\sigma^P, \mu^P)\) denote the PBE characterized in Theorem 1, and note that \((\sigma^P, \mu^P) \in \Sigma^M_0 \). The following theorem shows that \((\sigma^P, \mu^P)\) belongs to the set \(\Sigma^M_K\). Note that this implies that any PBE in \(\Sigma^M_K\) gives the principal the same payoff as \((\sigma^P, \mu^P)\) at every history. Moreover, as the proof the theorem clarifies, any equilibrium \((\sigma, \mu) \in \Sigma^M_K\) induces the same outcome as \((\sigma^P, \mu^P)\).

**Theorem OA1.** \((\sigma^P, \mu^P) \in \Sigma^M_K\).

**Proof.** Fix a finitely revealing equilibrium \((\sigma, \mu) \in \Sigma^M_K\), and let \(T\) be the upper bound on the periods of revelation under \((\sigma, \mu)\). We start by showing that, at histories at which

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\(^4\)This definition is borrowed from Peski (2008).
there have already been \( T \) periods of information revelation, players’ behavior under \((\sigma, \mu)\) must coincide with their behavior under \((\sigma^P, \mu^P)\).

Consider a history \((h_t, b_t)\) at which there have already been \( T \) periods of information revelation. Hence, \(\mu[h_t] = \mu[h_{t+s}]\) for all \( s \geq 0 \) and all histories \( h_{t+s} \) that follow history \( h_t \). This implies that

\[
U^{(\sigma, \mu)}[h_t, b_t] \leq (1 - \delta) \mathbb{E} \left[ \sum_{s=0}^{\infty} \delta^s \mathbf{1}_{\{b_{t+s} \in E_{\mathbb{F}[h_t]}\}} (b_{t+s} - c_{\mathbb{F}[h_t]}|b_t) \right],
\]

where \( U^{(\sigma, \mu)}[h_t, b_t] \) is the principal’s continuation payoff at history \((h_t, b_t)\). To see why the inequality holds, note that all agent types in the support of \(\mu[h_t]\) use the same strategy at all periods after time \( t \). Moreover, since an agent of type \( c_{\mathbb{F}[h_t]} \) gets a continuation payoff of 0 at all histories, she only takes the action at time \( \tau \geq t \) if \( T_\tau = c_{\mathbb{F}[h_t]} \).

These two observations together imply the bound in equation (OA1). Since the principal’s continuation payoff at history \((h_t, b_t)\) under equilibrium \((\sigma^P, \mu^P)\) is weakly larger than the right-hand side (OA1), it follows that players’ behavior under \((\sigma, \mu)\) must coincide with their behavior under \((\sigma^P, \mu^P)\) at all histories after information revelation has stopped.

Next, consider a history \( h_t \) with the property that, for all histories \( h_{t+s} \) with \( s \geq 1 \) that follow history \((h_t, b_t)\), players’ behavior under \((\sigma, \mu) \in \Sigma^M_K\) coincides with their behavior under \((\sigma^P, \mu^P)\). We now show that, at such a history \((h_t, b_t)\), the players’ behavior under \((\sigma, \mu) \in \Sigma^M_K\) coincides with their behavior under \((\sigma^P, \mu^P)\). Before presenting its proof, we note that this result and the result above together establish Theorem OA1.

To see why the result is true, we consider two separate cases: (i) \( b_t \) such that \( X(b_t, E_{\mathbb{F}[h_t]}) > 1 - \delta \), and (ii) \( b_t \) such that \( X(b_t, E_{\mathbb{F}[h_t]}) \leq 1 - \delta \).

**Case (i).** Let \( T_t \) be the principal’s offer at history \((h_t, b_t)\) and note that \( T_t \leq c_{\mathbb{F}[h_t]} \) (see footnote 5). We start by showing that if \( T_t \leq c_{\mathbb{F}[h_t]} \) is such that an agent with type \( c_{\mathbb{F}[h_t]} \) rejects the offer with probability 1, then all agents types also reject the offer with probability 1. Suppose by contradiction that the set of types that accept offer \( T_t \) with positive probability is non-empty. Let \( c_i < c_{\mathbb{F}[h_t]} \) be the highest cost of a type that accepts \( T_t \) with positive probability. The payoff that type \( c_i \) obtains by accepting the offer is 

\[
(1 - \delta)(T_t - c_i) + \delta \times 0 \leq (1 - \delta)(c_{\mathbb{F}[h_t]} - c_i),
\]

since from \( t + 1 \) onwards type \( c_i \) would be the

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5In any PBE in \( \Sigma^M_K \), the principal never makes an offer \( T_t \) that is larger than the highest cost in the support of her beliefs. Indeed, if \( T_t > c_{\mathbb{F}[h_t]} \) for some history \((h_t, b_t)\), we can construct an alternative finitely-revealing equilibrium in \( \Sigma^M_{k-1} \) (where \( k = |C[h_t]| \)) that gives the principal strictly more profits than \((\sigma, \mu)\), which would contradict \((\sigma, \mu) \in \Sigma^M_k \).
highest type in the support of the principal’s beliefs following an acceptance, and since equilibrium \((\sigma, \mu)\) coincides with \((\sigma^P, \mu^P)\) at all histories that follow history \((h_t, b_t)\). In contrast, the payoff that type \(c_i\) gets by rejecting the offer and mimicking type \(c_{\bar{E}[h]}\) at all times \(\tau > t\) is \(X(b_t, E_{\bar{E}[h]})(c_{\bar{E}[h]} - c_i) > (1 - \delta)(c_{\bar{E}[h]} - c_i)\), a contradiction. Hence, if \(T_t \leq c_{\bar{E}[h]}\) is such that an agent with type \(c_{\bar{E}[h]}\) rejects the offer with probability 1, then all agent types also reject the offer with probability 1.

There are two subcases to consider: (ia) \(b_t \in E_{\bar{E}[h]}\); and (ib) \(b_t \notin E_{\bar{E}[h]}\). Consider case (ia). We show that, in this case, the principal makes offer \(T_t = c_{\bar{E}[h]}\), and that this offer is accepted by all types with probability 1 (so behavior under equilibrium \((\sigma, \mu)\) coincides with behavior under \((\sigma^P, \mu^P)\)). As a first step, we show that the principal makes offer \(T_t = c_{\bar{E}[h]}\), and that this offer is accepted by an agent of type \(c_{\bar{E}[h]}\) with positive probability. Indeed, if this was not the case, then by the arguments above no agent type would accept offer \(T_t\), so \(\mu[h_{t+1}] = \mu[h_t]\). But then we would be able to construct an alternative finitely revealing equilibrium in \(\Sigma^M_{k-1}\) (where \(k = |C[h_t]|\)) that gives the principal strictly more profits than \((\sigma, \mu)\), which would contradict \((\sigma, \mu) \in \Sigma^M_k\). To see how, consider an equilibrium in which players’ behavior is identical to their behavior under \((\sigma, \mu)\) at every history except for history \((h_t, b_t)\). At history \((h_t, b_t)\), the principal makes offer \(T_t = c_{\bar{E}[h]}\) and every type accepts this offer with probability 1. The principal’s beliefs at \(t + 1\) are identical to \(\mu[h_t]\) regardless of whether the agent accepts or not the offer. One can check that this modified strategy profile is a PBE in finitely revealing strategies that lies in \(\Sigma^M_{k-1}\). Moreover, it delivers the principal a strictly larger payoff at history \((h_t, b_t)\) than \((\sigma, \mu)\), which contradicts \((\sigma, \mu) \in \Sigma^M_k\).

Next, we show that offer \(T_t = c_{\bar{E}[h]}\) is accepted with probability 1 by all agent types \(c_i < c_{\bar{E}[h]}\). Towards a contradiction, let \(c_i\) be the highest cost type below \(c_{\bar{E}[h]}\) that rejects the offer. The payoff that this type obtains by rejecting is at most \(X(b_t, E_{\bar{E}[h]})(c_{\bar{E}[h]} - c_i)\), since either type \(c_i\) will be the second highest cost in the support of \(\mu[h_{t+1}]\) (and type \(c_{\bar{E}[h]}\) will be the highest), or type \(c_i\) will be the highest cost in the support of \(\mu[h_{t+1}]\). In contrast, by accepting the offer and then mimicking type \(c_{\bar{E}[h]}\), she obtains \((1 - \delta + X(b_t, E_{\bar{E}[h]}))(c_{\bar{E}[h]} - c_i)\), which cannot be. Hence, offer \(T_t = c_{\bar{E}[h]}\) is accepted with probability 1 by all agent types \(c_i < c_{\bar{E}[h]}\).

Finally, we show that \(T_t = c_{\bar{E}[h]}\) is accepted by an agent with cost \(c_{\bar{E}[h]}\) with probability 1. Suppose by contradiction this is not true, and consider an alternative finitely revealing equilibrium such that players’ behavior coincides with their behavior under \((\sigma, \mu)\) at all histories except \((h_t, b_t)\). At such a history, the principal makes offer \(T_t = c_{\bar{E}[h]}\), and this offer is accepted by all types of the agent with probability 1 (the principal’s beliefs after
remain equal to $\mu[h_i]$ regardless of the agent’s action). One can check that this is a PBE in $\Sigma_{k-1}$, and that this PBE gives the principal a strictly larger profit than the original equilibrium $(\sigma, \mu)$, a contradiction. Hence, offer $T_t = c_{k[h_i]}$ is accepted by all agent types with probability 1.

Consider next case (ib). We show that, in this case, the principal makes an offer $T_t < c_{k[h_i]}$ that all agent types reject. From our arguments above, if $T_t \leq c_{k[h_i]}$ is rejected by an agent of type $c_{k[h_i]}$ with probability 1, then the offer is rejected by all agent types $c_i < c_{k[h_i]}$ with probability 1. This implies that any offer $T_t < c_{k[h_i]}$ is rejected by every agent type with probability 1. Note that in an equilibrium $(\sigma, \mu) \in \Sigma^M_K$, at such a history the principal would never make an offer $T_t = c_{k[h_i]}$ that is accepted by an agent of type $c_{k[h_i]}$ with positive probability. If this were the case, and by the same arguments used in case (1a), such an offer would be accepted by all types $c_i < c_{k[h_i]}$ with probability 1. Since $b_t < c_{k[h_i]}$, the principal would be strictly better off by making an offer $T_t < c_{k[h_i]}$ that is rejected by all types with probability 1.6

**Case (ii).** Consider next histories $(h_t, b_t)$ with $X(b_t, E_{k[h_i]}) \leq 1 - \delta$. We show that, in this case, there exists a threshold $c_k \in C[h_i]$ such that types in $C^- = \{c \in C[h_i] : c < c_k\}$ accept with probability 1, and that types in $C^+ = \{c \in C[h_i] : c \geq c_k\}$ reject with probability 1. When $C^-$ is non-empty, the principal offers transfer $T_t$ in equation (*) in the main text.

We start by showing that, at such a history $(h_t, b_t)$, type $c_{k[h_i]}$ takes the action with probability 0. Suppose to the contrary that type $c_{k[h_i]}$ takes the action with positive probability, so that $T_t = c_{k[h_i]}$. If this is so, then all types $c_i < c_{k[h_i]}$ must take the action with probability 1. To see why, suppose this is not true, and let $c_i$ be the highest type below $c_{k[h_i]}$ that does not take the action with probability 1. Since equilibrium behavior under $(\sigma, \mu)$ coincides with equilibrium behavior under $(\sigma^P, \mu^P)$ at all times $\tau \geq t + 1$, the payoff that type $c_i$ obtains by rejecting the offer is at most $X(b_t, E_{k[h_i]})(c_{k[h_i]} - c_i)$. However, type $c_i$ can guarantee herself a payoff of $(1 - \delta + X(b_t, E_{k[h_i]}))(c_{k[h_i]} - c_i)$ by accepting the offer today and then mimicking type $c_{k[h_i]}$ at all times $\tau \geq t + 1$, a contradiction. Since $c_{k[h_i]} < b_t$, then the principal would be strictly better off under an equilibrium in $\Sigma^M_{k-1}$ that

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6Indeed, starting from $t + 1$ equilibrium behavior under $(\sigma, \mu)$ coincides with equilibrium behavior under $(\sigma^P, \mu^P)$. As a result, the profits that the principal obtains from each type of agent $c_i < c_{k[h_i]}$ from $t + 1$ onwards do not depend on the relative likelihood that she assigns to type $c_{k[h_i]}$. Moreover, the profits that she extracts from type $c_{k[h_i]}$ from $t + 1$ onwards are the same regardless of whether this type accepts or not. These two observations imply that, at time $t$, the principal is better off making an offer that every type of agent rejects.
is identical to $(\sigma, \mu)$, except that at history $(h_t, b_t)$ the principal makes offer $T_t = c_{\bar{K}[h_t]}$ which is rejected by type $c_{\bar{K}[h_t]}$ and accepted by all types $c_i < c_{\bar{K}[h_t]}$. This contradicts $(\sigma, \mu) \in \Sigma^M_K$. Hence, at history $(h_t, b_t)$ type $c_{\bar{K}[h_t]}$ takes the action with probability 0.

Next, we show that at history $(h_t, b_t)$, there exists a threshold $c_{k^*} \in C[h_t]$ such that types in $C^- = \{ c \in C[h_t] : c < c_{k^*} \}$ accept with probability 1, and that types in $C^+ = \{ c \in C[h_t] : c \geq c_{k^*} \}$ reject with probability 1. The statement is true if all types reject the offer with probability 1. Suppose the set of types in $C[h_t]$ that accept the offer with positive probability is non-empty. Let $c_{k^*}$ be the lowest type below $c_{j^*}$ be the highest type in this set. Since equilibrium behavior at times $\tau \geq t + 1$ coincides with $(\sigma^P, \mu^P)$, type $c_{j^*}$ obtains a payoff of $(1 - \delta)(T_t - c_{j^*}) + \delta \times 0$. Let $c_{k^*}$ be the lowest type in $\{ c \in C[h_t] : c > c_{j^*} \}$. Note that the offer that the principal makes must satisfy $(*)$ in the main text:

$$(1 - \delta)(T_t - c_{j^*}) = V_k^{(\sigma^P, \mu^P)}[h_t, b_t] + A_k^{(\sigma^P, \mu^P)}[h_t, b_t] (c_{k^*} - c_{j^*}).$$

Indeed, this transfer leaves type $c_{j^*}$ indifferent between accepting the offer and rejecting it. Since type $c_{k^*}$ rejects the offer with probability 1, it must be that

$$V_k^{(\sigma^P, \mu^P)}[h_t, b_t] \geq (1 - \delta)(T_t - c_{k^*}) \iff 1 - \delta \geq A_k^{(\sigma^P, \mu^P)}[h_t, b_t].$$  (OA2)

We now show that type $c_{j^*}$ accepts with probability 1. Indeed, the payoff that the principal obtains from type $c_{j^*}$ from $t + 1$ onwards if this type accepts the offer is $(1 - \delta)\mathbb{E}[(\sum_{s=1}^{\infty} \delta^s 1_{b_{t+s} \in E_{j^*}} (b_{t+s} - c_{j^*})[b_t])], which is the efficient payoff and is clearly higher than what she would obtain from this type if the type rejects the offer.\footnote{Moreover, if some types $c_i < c_{j^*}$ were to reject the offer, the continuation payoff that the principal would get from them would be weakly higher if type $c_{j^*}$ were to accept offer $T_t$ with probability 1, than if type $c_{j^*}$ were to reject the offer with positive probability. Indeed, if type $c_{j^*}$ is not in the support of the principal’s beliefs at time $t + 1$, then types $c_i < c_{j^*}$ get smaller informational rents.}

Next, we show that all types in $C[h_t]$ with $c_i < c_{j^*}$ accept offer $T_t$ with probability 1. Towards a contradiction, let $c_i$ be the highest type below $c_j$ that rejects $T_t$ with positive probability. Since equilibrium behavior from $t + 1$ onwards under $(\sigma, \mu)$ coincides with equilibrium behavior under $(\sigma^P, \mu^P)$, type $c_i$ obtains payoff $V_k^{(\sigma^P, \mu^P)}[h_t, b_t] + A_k^{(\sigma^P, \mu^P)}[h_t, b_t] (c_{k^*} - c_i)$ from rejecting offer $T_t$. In contrast, the payoff that type $c_i$ would obtain from accepting offer $T_t$ and mimicking type $c_{j^*}$ from time $t + 1$ onwards
is \((1 - \delta)(T_t - c_i) + X(b_t, E_{j*})(c_{j*} - c_i)\). Note that

\[
(1 - \delta)(T_t - c_i) + X(b_t, E_{j*})(c_{j*} - c_i) - V_{k*}^{(\sigma_P, \mu_P)}[h_t, b_t] - A_{k*}^{(\sigma_P, \mu_P)}[h_t, b_t](c_{k*} - c_i)
\]

\[
= (c_{j*} - c_i) \left( 1 - \delta + X(b_t, E_{j*}) - A_{k*}^{(\sigma_P, \mu_P)}[h_t, b_t] \right) > 0,
\]

where we used equation (OA2).

The arguments above show that, at histories \((h_t, b_t)\) with \(X(b_t, E_{k*}[h_t]) \leq 1\), there exists a threshold \(c_{k*} \in C[h_t]\) such that types in \(C^- = \{c \in C[h_t] : c < c_{k*}\}\) accept with probability 1, and that types in \(C^+ = \{c \in C[h_t] : c \geq c_{k*}\}\) reject with probability 1. Since the threshold \(c_{k*}\) is chosen optimally under equilibrium \((\sigma_P, \mu_P)\), under equilibrium \((\sigma, \mu)\) the principal would choose the same cutoff. Hence, at history \((h_t, b_t)\), players’ behavior under \((\sigma, \mu) \in \Sigma_M^K\) coincides with their behavior under \((\sigma_P, \mu_P)\).

**OA3 Full Commitment**

This appendix studies the problem of a principal who has full commitment power. For conciseness, we focus on the case in which there are two types of agents: \(C = \{c_1, c_2\}\), with \(c_1 < c_2\). Let \(\mu \in (0, 1)\) be the probability that the agent’s cost is \(c_2\).

The principal’s problem is to choose processes \(\{a_{i,t}, T_{i,t}\}\) for \(i = 1, 2\), with \(a_{i,t} \in \{0, 1\}\) and \(T_{i,t} \in \mathbb{R}\), to solve

\[
U^{FC}(b) = \max_{\{a_{i,t}, T_{i,t}\} = 1, 2} (1 - \delta) \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t ((1 - \mu)(a_{1,t}b_t - T_{1,t}) + \mu(a_{2,t}b_t - T_{2,t})) | b_0 = b \right]
\]

subject to

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t (T_{i,t} - a_{i,t}c_i) | b_0 = b \right] \geq 0 \text{ for } i = 1, 2
\]

and

\[
\mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t (T_{i,t} - a_{i,t}c_i) | b_0 = b \right] \geq \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t (T_{j,t} - a_{j,t}c_j) | b_0 = b \right] \text{ for } i, j = 1, 2.
\]

By familiar arguments, the participation constraint of type \(c_1\) and the incentive compatibility constraint of type \(c_2\) do not bind. The participation constraint of type \(c_2\) and the incentive compatibility constraint of type \(c_1\) hold with equality at the solution to (OA3). Using these two constraints to solve for the expected discounted transfers
The solution to problem (OA4) has: 

Proposition OA1. Let $C = \{c_1, c_2\}$, and assume there exists $b \in E_2 \setminus E_1$ with $X(b, E_2) = \epsilon < 1 - \delta$. Then, at histories $(h_t, b_t)$ with $C[h_t] = \{c_1, c_2\}$ and $b_t = b$,

$$U^{FC}(b_t) - U^{\sigma, \mu}[h_t, b_t] \leq (1 - \mu)(c_2 - c_1)\epsilon.$$  

Proof. Note that, at such a history, the principal can make a separating offer $T$ with $(1 - \delta)(T - c_1) = X(b, E_1)(c_2 - c_1)$ that only low types accept. Conditional on the agent being a low type, the principal’s profits are

$$(1 - \delta)\mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t (1 - \mu)a_{1,t}(b_t - c_1) + \mu a_{2,t} \left( b_t - c_2 - \frac{(1 - \mu)}{\mu}(c_2 - c_1) \right) \right] | b_0 = b.$$  

(OA4)

The principal’s expected payoff at history $(h_t, b_t)$ from making offer $T$ is then

$$U^{\sigma, \mu}[h_t, b_t] = (1 - \delta)\mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \left( (1 - \mu)1_{b_t \in E_1}(b_t - c_1) + \mu 1_{b_t \in E_2} \left( b_t - c_2 - \frac{(1 - \mu)}{\mu}(c_2 - c_1) \right) \right) \right] | b_t.$$  

(OA5)

The principal’s full commitment payoffs are

$$U^{FC}(b_t) = (1 - \delta)\mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \left( (1 - \mu)1_{b_t \in E_1}(b_t - c_1) + \mu 1_{b_t \in E_2} \left( b_t - c_2 - \frac{(1 - \mu)}{\mu}(c_2 - c_1) \right) \right) \right],$$  

(OA6)
where \( \hat{E}_2 = \{ b \in B : b_t \geq c_2 + (1 - \mu)(c_2 - c_1)/\mu \} \subset E_2 \). Using (OA5) and (OA6),

\[
U^{FC}(b_t) - U^{\sigma}(b_t) = -(1 - \delta)\mu E \sum_{t=0}^{\infty} \delta^{t-t} \mathbf{1}_{b_{t-r} \in E_2 \setminus \hat{E}_2} \left( b_t - c_2 - \frac{1 - \mu}{\mu}(c_2 - c_1) \right) |b_t|
\]

\[
\leq (1 - \delta)\mu E \sum_{t=0}^{\infty} \delta^{t-t} \mathbf{1}_{b_{t-r} \in E_2 \setminus \hat{E}_2} \left( \frac{1 - \mu}{\mu}(c_2 - c_1) \right) |b_t|
\]

\[
= (1 - \mu)(c_2 - c_1)X(b_t, E_2 \setminus \hat{E}_2)
\]

\[
\leq (1 - \mu)(c_2 - c_1)\epsilon,
\]

where the first inequality follows since \( b_r \geq c_2 \) for all \( b_r \in E_2 \), and the second inequality
follows since \( X(b_t, E_2 \setminus \hat{E}_2) \leq X(b_t, E_2) = \epsilon. \)

\[ \square \]

**OA4  Path Dependence when Shocks are Ergodic**

In this appendix, we show by example that the equilibrium may exhibit long-run path dependence when the shock process is ergodic. Let \( B = \{ b_L, b_{ML}, b_{MH}, b_H \} \), with \( b_L < b_{ML} < b_{MH} < b_H \) and \( C = \{ c_1, c_2, c_3 \} \). Assume that the efficiency sets are \( E_1 = E_2 = \{ b_{ML}, b_{MH}, b_H \} \) and \( E_3 = \{ b_H \} \).

**Proposition OA2.** Suppose that the transition matrix \([Q_{b,b'}]\) satisfies:

(a) \( Q_{b,b'} > 0 \) for all \( b, b' \in B \);

(b) \( X(b_{MH}, \{ b_H \}) > 1 - \delta \), \( X(b, \{ b_H \}) < 1 - \delta \) for \( b = b_{ML}, b_L \), and \( X(b_{ML}, \{ b_{ML} \}) > 1 - \delta \)

Then, there exists \( \epsilon_1 > 0, \epsilon_2 > 0, \Delta_1 > 0 \) and \( \Delta_2 > 0 \) such that, if \( Q_{b,b_L} < \epsilon_1 \) for all \( b \in B \setminus \{ b_L \} \) and \( Q_{b,b_{ML}} < \epsilon_2 \) for all \( b \in B \setminus \{ b_{ML} \} \), and if \( |b_L - c_1| < \Delta_1 \) and \( |b_L - c_2| > \Delta_2 \), the unique equilibrium satisfies:

(i) For histories \( h_t \) such that \( C[h_t] = \{ c_1, c_2 \} \), \( \mu[h_r] = \mu[h_t] \) for all \( h_r \preceq h_t \) (i.e., there is no more learning by the principal from time \( t \) onwards);

(ii) For histories \( h_t \) such that \( C[h_t] = \{ c_2, c_3 \} \): if \( b_t = b_L \) or \( b_t = b_{MH} \), types \( c_2 \) and \( c_3 \) take action \( a = 0 \); if \( b_t = b_{ML} \), type \( c_2 \) takes action \( a = 1 \) and type \( c_3 \) takes action \( a = 0 \); and if \( b_t = b_H \), types \( c_2 \) and \( c_3 \) take action \( a = 1 \);
(iii) For histories $h_t$ such that $C[h_t] = \{c_1, c_2, c_3\}$: if $b_t = b_L$, type $c_1$ takes action $a = 1$ while types $c_2$ and $c_3$ take action $a = 0$; if $b_t = b_{ML}$, types $c_1$ and $c_2$ take action $a = 1$ and type $c_3$ takes action $a = 0$; if $b_t = b_{MH}$, all agent types take action $a = 0$; and if $b_t = b_H$, all agent types take action $a = 1$.

We prove the three properties in Proposition OA2 separately.

**Proof of Property (i).** Note first that, by Theorem 1, after such a history the principal makes a pooling offer $T = c_2$ that both types accept if $b_t \in E_2 = \{b_{ML}, b_{MH}, b_H\}$. To establish the result, we show that if $b_t = b_L$, types $c_1$ and $c_2$ take action $a = 0$ after history $h_t$. If the principal makes a separating offer that only a $c_1$ agent accepts, she pays a transfer $T_t = c_1 + \frac{1}{1-\delta} X(b_L, E_2)(c_2 - c_1)$ that compensates the low cost agent for revealing his type. The principal’s payoff from making such an offer, conditional on the agent being type $c_1$, is

$$
\tilde{U}_{sc}[c_1] = (1 - \delta)(b_L - T_t) + \mathbb{E}\left[\sum_{t' > t} \delta^{t' - t}(1 - \delta)\mathbf{1}_{b_t \in E_1}(b_{t'} - c_1)\big| b_t = b_L\right]
$$

$$
= (1 - \delta)(b_L - c_1) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})[b - c_2].
$$

Her payoff from making that offer conditional on the agent’s type being $c_2$ is $\tilde{U}_{sc}[c_2] = \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})[b - c_2]$. If she doesn’t make a separating offer when $b_t = b_L$, she never learns the agent’s type and gets a payoff $\tilde{U}_{ns} = \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})[b - c_2]$. Since $b_L - c_1 < 0$ by assumption, $\tilde{U}_{ns} > \mu[h_t][c_1]\tilde{U}_{sc}[c_1] + \mu[h_t][c_2]\tilde{U}_{sc}[c_2]$, and therefore the principal does not make a separating offer.

**Proof of Property (ii).** Theorem 1 implies that, after such a history, the principal makes a pooling offer $T = c_3$ that both types accept if $b_t \in E_3 = \{b_H\}$. Theorem 1 also implies that, if $b_t = b_{MH}$, then after such a history the principal makes an offer that both types reject (since $X(b_{MH}, \{b_H\}) > 1 - \delta$ by assumption). So it remains to show that, after history $h_t$, the principal makes an offer that a $c_2$ agent accepts and a $c_3$ agent rejects if $b_t = b_{ML}$, and that the principal makes an offer that both types reject if $b_t = b_L$.

Suppose $b_t = b_{ML}$. Let $U[c_i]$ be the principal’s value at history $(h_t, b_t = b_{ML})$ conditional on the agent’s type being $c_i \in \{c_2, c_3\}$, and let $V_i$ be the value of an agent of type $c_i$ at history $(h_t, b_t = b_{ML})$. Note that $U[c_2] + V_2 \leq (1 - \delta)(b_{ML} - c_2) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_{ML}, \{b\})[b - c_2]$, since the right-hand side of this equation corresponds
to the efficient total payoff when the agent is of type $c_2$ (i.e., the agent taking the action if and only if the state is in $E_2$.) Note also that incentive compatibility implies $V_2 \geq X(b_{ML}, \{b_H\})(c_2 - c_3)$, since a $c_2$-agent can mimic a $c_3$-agent forever and obtain $X(b_{ML}, \{b_H\})(c_2 - c_3)$. It thus follows that $U[c_2] \leq (1 - \delta)(b_{ML} - c_2) + X(b_{ML}, \{b_H\})[b_H - c_3] + \sum_{s \in \{b_{ML}, b_{MH}\}} X(b_{ML}, \{b\})[b - c_2].$

If when $b_t = b_{ML}$ the principal makes an offer that only a $c_2$ agent accepts, the offer must satisfy $T_t = c_2 + \frac{1}{1 - \delta} X(b_{ML}, \{b_H\})(c_3 - c_2) < c_3$. The principal’s payoff from making such an offer when the agent’s type is $c_2$ is

$$U[c_2] = (1 - \delta)(b_{ML} - T_t) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_{ML}, \{b\})[b - c_2]$$

$$= (1 - \delta)(b_{ML} - c_2) + X(b_{ML}, \{b_H\})[b_H - c_3] + \sum_{b \in \{b_{ML}, b_{MH}\}} X(b_{ML}, \{b\})[b - c_2],$$

which, from the arguments in the previous paragraph, is the highest payoff that the principal can ever get from a $c_2$ agent after history $(h_t, b_t = b_{ML})$. Hence, it is optimal for the principal to make such a separating offer.8

Suppose next that $b_t = b_L$. If the principal makes an offer that a $c_2$-agent accepts and a $c_3$-agent rejects, she pays a transfer $T_t = c_2 + \frac{1}{1 - \delta} X(b_L, E_3)(c_3 - c_2)$. Thus, the principal’s payoff from making such an offer, conditional on the agent being type $c_2$, is

$$\bar{U}^{sc}[c_2] = (1 - \delta)(b_L - T_t) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})[b - c_2]$$

$$= (1 - \delta)(b_L - c_2) + X(b_L, \{b_H\})[b_H - c_3] + \sum_{b \in \{b_{ML}, b_{MH}\}} X(b_L, \{b\})[b - c_2].$$

If the principal makes an offer that both types reject when $b_t = b_L$, then by the arguments above she learns the agent’s type the first time at which shock $b_{ML}$ is reached. Let $\tilde{t}$ be the random variable that indicates the next date at which shock $b_{ML}$ is realized. Then, conditional on the agent’s type being $c_2$, the principal’s payoff from making an offer that

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8Indeed, the principal’s payoff from making an offer equal to $T_t$ when the agent’s type is $c_3$ is $X(2, \{4\})[b(4) - c_3]$, which is also the most that she can extract from an agent of type $c_3$.  

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both types reject when \( b_t = b_{L} \) is

\[
\tilde{U}^{nsc}[c_2] = (1 - \delta)E \left[ \sum_{t' = t+1}^{t-1} \delta^{t'-t}1_{b_{H}=b_H}(b_{H} - c_3)b_t = b_{L} \right] \\
+ E \left[ \delta^{t-t} \left( (1 - \delta)(b_{ML} - T_t) + \sum_{b \in \{b_{ML}, b_{MH}, b_{H}\}} X(b_{ML}, \{b\})[b - c_2] \right) | b_t = b_{L} \right].
\]

The offer \( T_t \) that the principal makes at time \( t \) satisfies \( T_t = c_2 + \frac{1}{1 - \delta}X(b_{ML}, \{b_{H}\})(c_3 - c_2) \).

Using this in the equation above,

\[
\tilde{U}^{nsc}[c_2] = X(b_{L}, \{b_{H}\})[b_{H} - c_3] + X(b_{L}, \{b_{ML}\})[b_{ML} - c_2] + E[\delta^{t-t}|b_t = b_{L}]X(b_{ML}, \{b_{MH}\})[b_{MH} - c_2].
\]

Then, we have

\[
\tilde{U}^{nsc}[c_2] - \tilde{U}^{sc}[c_2] = -(1 - \delta)[b_{L} - c_2] - \left[ X(b_{L}, \{b_{MH}\}) - E[\delta^{t-t}|b_t = b_{L}]X(b_{ML}, \{b_{MH}\}) \right] [b_{MH} - c_2].
\]

Since \( b_{L} < c_2 \) by assumption, there exists \( \Delta_1^1 > 0 \) such that, if \( (1 - \delta)(c_2 - b_{L}) > \Delta_1^1 \), the expression above is positive. Since the principal’s payoff conditional on the agent’s type being \( c_3 \) is the same regardless of whether she makes a separating offer or not when \( b_t = b_{L} \) (i.e., in either case the principal earns \( X(b_{L}, \{b_{H}\})(b_{H} - c_3) \)), when this condition holds the principal chooses not to make an offer that \( c_2 \) accepts and \( c_3 \) rejects when \( b_t = b_{L} \). \( \square \)

**Proof of Property (iii).** Suppose \( C[h_t] = \{c_1, c_2, c_3\} \). Theorem 1 implies that all agent types take action \( a = 1 \) if \( b_t = b_{H} \), and all agent types take action \( a = 0 \) if \( b_t = b_{ML} \). (this last claim follows since \( X(b_{MH}, \{b_{H}\}) > 1 - \delta \)).

Suppose next that \( C[h_t] = \{c_1, c_2, c_3\} \) and \( b_t = b_{ML} \). Note that, by Lemma A.1, an agent with type \( c_3 \) takes action \( a = 0 \) if \( b_t = b_{ML} \notin E_3 = \{b_{H}\} \). We first claim that if the principal makes an offer that only a subset of types accept at state \( b_{ML} \), then this offer must be such that types in \( \{c_1, c_2\} \) take action \( a = 1 \) and type \( c_3 \) takes action \( a = 0 \). To see this, suppose that she instead makes an offer that only an agent with type \( c_1 \) accepts, and that agents with types in \( \{c_2, c_3\} \) reject. The offer that she makes in this case satisfies \( (1 - \delta)(T_t - c_1) = V_2^{(\sigma, \mu)}[h_t, b_t] + A_2^{(\sigma, \mu)}[h_t, b_t](c_2 - c_1) \). By property (ii) above, under this proposed equilibrium a \( c_2 \)-agent will from period \( t + 1 \) onwards take the action at all times \( t' > t \) such that \( b_{t'} = b_{ML} \). Therefore, \( A_2^{(\sigma, \mu)}[h_t, b_t] \geq \ldots \)

\[\ldots \quad \text{[additional details]} \]

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9Under the proposed equilibrium, if the offer is rejected the principal learns that the agent’s type is in \( \{c_2, c_3\} \). By property (ii), if the agent’s type is \( c_2 \), the principal will learn the agent’s type the next
\[X(b_{ML}, \{b_{ML}\}) > 1 - \delta,\] where the last inequality follows by assumption. The payoff that an agent of type \(c_2\) obtains by accepting offer \(T_t\) at time \(t\) is bounded below by 
\[(1 - \delta)(T_t - c_2) = (1 - \delta)(c_1 - c_2) + \sum_{h_i, b_i} A_{2}^{(\sigma, \mu)}[h_i, b_i](c_2 - c_1) > V_{2}^{(\sigma, \mu)}[h_i, b_i],\] where the inequality follows since \(A_{2}^{(\sigma, \mu)}[h_i, b_i] > 1 - \delta\). Thus, type \(c_2\) strictly prefers to accept the offer, a contradiction. Therefore, when \(C[h_i] = \{c_1, c_2, c_3\}\) and \(b_i = b_{ML}\), either the principal makes an offer that only types in \(\{c_1, c_2\}\) accept, or she makes an offer that all types reject.

We now show that, under the conditions in the Lemma, the principal makes an offer that types in \(\{c_1, c_2\}\) accept and type \(c_3\) rejects when \(b_i = b_{ML}\) and \(C[h_i] = \{c_1, c_2, c_3\}\). If she makes an offer that agents with cost in \(\{c_1, c_2\}\) accept and a \(c_3\)-agent rejects, then she pays a transfer \(T_t = c_2 + \frac{1}{1 - \delta}X(b_{ML}, \{b_H\})(c_3 - c_2)\). Note then that, by property (i) above, when the agent’s cost is in \(\{c_1, c_2\}\), the principal stops learning: for all times \(t' > t\) the principal makes an offer \(T_{t'} = c_2\) that both types accept when \(b_{t'} \in E_2\), and she makes a low offer \(T_{t'} = 0\) that both types reject when \(b_{t'} \notin E_2\). Therefore, conditional on the agent’s type being either \(c_1\) or \(c_2\), the principal’s payoff from making at time \(t\) an offer \(T_t\) that agents with cost in \(\{c_1, c_2\}\) accept and a \(c_3\)-agent rejects is

\[
\hat{U}^{sc}[\{c_1, c_2\}] = (1 - \delta)(b_{ML} - T_t) + \sum_{b \in \{b_{ML}, b_{MH, b_H}\}} X(b_{ML}, \{b\})[b - c_2] = (1 - \delta)(b_{ML} - c_2) + \sum_{b \in \{b_{ML}, b_{MH}\}} X(b_{ML}, \{b\})[b - c_2].
\]

On the other hand, if she does not make an offer that a subset of types accept when \(b_i = b_{ML}\), then the principal’s payoffs conditional on the agent being of type \(c_i \in \{c_1, c_2\}\) is bounded above by

\[
\hat{U}^{nsc}[c_i] = \mathbb{E}
\left[
\sum_{t' = t}^{i-1} \delta^{t'-t}(1 - \delta)\textbf{1}_{b_{t'} = b_H}(b_H - c_3) + \delta^{i-t'} \sum_{b \in E_i} X(b_L, \{b\})(b - c_i)[b_i = b_{ML}]
\right].
\]

---

\(t\) time the shock is \(b_{ML}\) (because at that time type \(c_2\) takes the action, while type \(c_3\) doesn’t), and from that point onwards the agent will take the action when the shock is in \(E_2 = \{b_{ML}, b_{MH}, b_H\}\).
where $\hat{t}$ denotes the next period that state $b_L$ is realized.\(^\text{10}\) Note that there exists $\epsilon_1 > 0$ small such that, if $Q_{b,b_L} < \epsilon_1$ for all $b \neq b_L$, then $\hat{U}^{sc}[\{c_1, c_2\}] > \hat{U}^{sc}[c_i]$ for $i = 1, 2$. Finally, note that the payoff that the principal obtains from an agent of type $c_3$ at history $h_t$ when $b_t = b_{ML}$ is $X(b_{ML}, \{b_H\})(b_H - c_3)$, regardless of the principal’s offer. Therefore, if $Q_{b,b_L} < \epsilon_1$ for all $b \neq b_L$, when $C[h_t] = \{c_1, c_2, c_3\}$ and $b_t = b_{ML}$ the principal makes an offer $T_t$ that only types in $\{c_1, c_2\}$ accept.

Finally, we show that when $C[h_t] = \{c_1, c_2, c_3\}$ and $b_t = b_L$, the principal makes an offer that only type $c_1$ accepts. Let $\hat{t}$ be the random variable that indicates the next date at which state $b_{ML}$ is realized. If the principal makes an offer $T_t$ that only a $c_1$-agent accepts, this offer satisfies

$$(1 - \delta)(T_t - c_1) = V_2^{(\sigma, \mu)}[h_t, b_L] + A_2^{(\sigma, \mu)}[h_t, b_L](c_2 - c_1)$$

$$= X(b_L, \{b_H\})(c_3 - c_1) + (X(b_L, \{b_{ML}\}) + \mathbb{E}[\delta^{\hat{t} - t} | b_t = b_L] X(b_{ML}, \{b_{MH}\}))(c_2 - c_1) \quad (\text{OA7})$$

where the second equality follows since $V_2^{(\sigma, \mu)}[h_t, b_L] = A_3^{(\sigma, \mu)}[h_t, b_L](c_3 - c_2) = X(b_L, \{b_H\})(c_3 - c_2)$ and since, by property (ii), when the support of the principal’s beliefs is $\{c_2, c_3\}$ and the agent’s type is $c_2$, the principal learns the agent’s type at time $\hat{t}$.\(^\text{11}\) Therefore, conditional on the agent’s type being $c_1$, the principal’s equilibrium payoff from making an offer that only an agent with cost $c_1$ accepts at state $b_L$ is

$$\hat{U}^{sc}[c_1] = (1 - \delta)(b_L - T_t) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})[b - c_1]$$

$$= (1 - \delta)(b_L - c_1) + X(b_L, \{b_H\})(b_H - c_3) + X(b_L, \{b_{MH}\})[b_{MH} - c_1] + X(b_L, \{b_{ML}\})[b_{ML} - c_2] - \mathbb{E}[\delta^{\hat{t} - t} | b_t = b_L] X(b_{ML}, \{b_{MH}\})(c_2 - c_1)$$

\(^\text{10}\)To see why, note that if no type of agent takes the productive action when $C[h_t] = \{c_1, c_2, c_3\}$ and $b_t = b_{ML}$, then the principal can only learn the agent’s type when state $b_L$ is realized (i.e., at time $\hat{t}$). At times before $\hat{t}$, all agent types take the action if the shock is $b_H$ (and the principal pays transfer $T = c_3$), and no agent type takes the action at states $b_{ML}$ or $b_{MH}$. After time $\hat{t}$, the payoff that the principal gets from type $c_i$ is bounded above by her first-best payoff $\sum_{b \in E} X(b_L, \{b\})(b - c_i)$.

\(^\text{11}\)The fact that the principal learns the agent’s type at time $\hat{t}$ implies that

$$A_2^{(\sigma, \mu)}[h_t, b_L] = (1 - \delta) \mathbb{E} \left[ \sum_{t' = \hat{t}}^{\hat{t} - 1} \delta^{\hat{t} - t} \mathbb{1}_{b_{t'} = b_H} + \delta^{\hat{t} - t} \sum_{t' = \hat{t}}^{\infty} \delta^{t' - t} \mathbb{1}_{b_{t'} \in E_2} | b_t = b_L \right]$$

$$= X(b_L, \{b_H\}) + X(b_L, \{b_{ML}\}) + \mathbb{E} \left[ \delta^{\hat{t} - t} X(b_{ML}, \{b_{MH}\}) | b_t = b_L \right].$$
where the second line follows from substituting the transfer in (OA7). On the other hand, the principal’s payoff from making such an offer at state \(b_L\), conditional on the agent’s type being \(c_2\), is

\[
\tilde{U}^{sc}[c_2] = (1 - \delta) \mathbb{E} \left[ \sum_{t'=l}^{l-1} \delta^{t'-l} \mathbf{1}_{b_{t'} = b_H} (b_H - c_3) | b_t = b_L \right] + (1 - \delta) \mathbb{E} \left[ \delta^{l-t} \left( (b_{ML} - c_2) - \frac{X(b_{ML}, \{b_H\}) (c_3 - c_2)}{1 - \delta} \right) + \sum_{t'=l+1}^{\infty} \delta^{t'-t} \mathbf{1}_{b_{t'} \in E_2} (b_{t'} - c_2) | b_t = b_L \right]
\]

\[
= X(b_L, \{b_H\})(b_H - c_3) + X(b_L, \{b_{ML}\})(b_{ML} - c_2) + \mathbb{E} \left[ \delta^{l-t} X(b_{ML}, \{b_{MH}\}) | b_t = b_L \right] (b_{MH} - c_2),
\]

where we used the fact that, when the support of her beliefs is \(\{c_2, c_3\}\), the principal makes an offer that only a \(c_2\)-agent accepts when the state is \(b_{ML}\) (the offer that she makes at that point is \(T = c_2 + \frac{1}{1-\delta} X(b_{ML}, \{b_H\})(c_3 - c_2)\)).

Alternatively, suppose the principal makes an offer that both \(c_1\) and \(c_2\) accept but \(c_3\) rejects. Then she pays a transfer \(T_t = c_2 + \frac{1}{1-\delta} X(b_L, \{b_H\})(c_3 - c_2)\); thus, her payoff from learning that the agent’s type is in \(\{c_1, c_2\}\) in state \(b_L\) is

\[
\tilde{U}^{sc}[\{c_1, c_2\}] = (1 - \delta)(b_L - T_t) + \sum_{b \in \{b_{ML}, b_{MH}, b_H\}} X(b_L, \{b\})(b - c_2)
\]

\[
= (1 - \delta)(b_L - c_2) + X(b_L, \{b_H\})(b_H - c_3)
\]

\[
+ X(b_L, \{b_{ML}\})(b_{ML} - c_2) + X(b_L, \{b_{MH}\})(b_{MH} - c_2),
\]

where we used the fact that the principal never learns anything more about the agent’s type when the support of her beliefs is \(\{c_1, c_2\}\) (see property (i) above). Note that there exists \(\epsilon_2 > 0\) and \(\Delta_2^2 > 0\) such that, if \(Q_{b_{b_{ML}}} < \epsilon_2\) for all \(b \neq b_{ML}\) and if \(c_2 - b_L > \Delta_2 = \max\{\Delta_1^2, \Delta_2^2\}\), then the following two inequalities hold:

\[
\tilde{U}^{sc}[c_1] - \tilde{U}^{sc}[\{c_1, c_2\}] = \left[ 1 - \delta + X(b_L, \{b_{MH}\}) - \mathbb{E}[\delta^{\hat{t}-t} | b_t = b_L] X(b_{ML}, \{b_{MH}\}) \right] (c_2 - c_1) > 0
\]

\[
\tilde{U}^{sc}[c_2] - \tilde{U}^{sc}[\{c_1, c_2\}] = \left[ \mathbb{E} \left[ \delta^{\hat{t}-t} X(b_{ML}, \{b_{MH}\}) | b_t = b_L \right] - X(b_L, \{b_{MH}\}) \right] (b_{MH} - c_2) - (1 - \delta)(b_L - c_2) > 0.
\]

Therefore, under these conditions, at state \(b_L\) the principal strictly prefers to make an offer that a \(c_1\)-agent accepts and agents with cost \(c \in \{c_2, c_3\}\) reject than to make an offer that agents with cost in \(\{c_1, c_2\}\) accept and a \(c_3\)-agent rejects.
However, the principal may choose to make an offer that all agent types reject when \( b_t = b_L \) and \( C[h_t] = \{c_1, c_2, c_3\} \). In this case, by the arguments above, the next time the state is equal to \( b_{ML} \) the principal will make an offer that only types in \( \{c_1, c_2\} \) accept. The offer that she makes in this case is such that \((1 - \delta)(T - c_2) = X(b_{ML}, \{b_H\})(c_3 - c_2)\). Then, from that point onwards, she will never learn more (by property (i) above). In this case, the principal’s payoff conditional on the agent’s type being \( \{c_1, c_2\} \) is

\[
\bar{U}^{ns_c} = (1 - \delta) \mathbb{E} \left[ \sum_{\tau=t}^{t-1} 1_{b_\tau = b_H}(b_\tau - c_3) | b_t = b_L \right] \\
\quad + \mathbb{E} \left[ \delta^{t-t}(1 - \delta)(b_{ML} - T) + \sum_{b \in E_2} X(b_{ML}, \{b\})(b - c_2) | b_t = b_L \right] \\
\quad = X(b_L, \{b_H\})(b_H - c_3) + X(b_L, \{b_{ML}\})(b_{ML} - c_2) + \mathbb{E}[\delta^{t-t}|b_t = b_L]X(b_{ML}, \{b_{MH}\})(b_{MH} - c_2),
\]

where \( t \) be the random variable that indicates the next date at which state \( b_{ML} \) is realized. Note that there exists \( \epsilon_2 > 0 \) and \( \Delta_1 > 0 \) such that, if \( Q_{b,b_{ML}} < \epsilon_2 \) for all \( b \neq b_{ML} \), and if \( b_L - c_1 > -\Delta_1 \), then the following hold:

\[
\bar{U}^{sc}[c_1] - \bar{U}^{ns_c} = (1 - \delta)(b_L - c_1) + \left[ X(b_L, \{b_{MH}\}) - \mathbb{E}[\delta^{t-t}|b_t = b_L]X(b_{ML}, \{b_{MH}\}) \right] [b_{MH} - c_1] > 0 \\
\bar{U}^{sc}[c_2] - \bar{U}^{ns_c} = 0.
\]

Therefore, under these conditions, the principal makes an offer that type \( c_1 \) accepts and types in \( \{c_2, c_3\} \) reject when \( C[h_t] = \{c_1, c_2, c_3\} \) and \( b_t = b_L \).

Properties (i)-(iii) in Proposition OA2 imply that the equilibrium exhibits long-run path dependence. Suppose that the agent’s type is \( c_1 \). Then, properties (i)-(iii) imply that the principal eventually learns the agent’s type if and only if \( t(b_L) := \min\{t \geq 0 : b_t = b_L\} < t(b_{ML}) := \min\{t \geq 0 : b_t = b_{ML}\} \) (i.e., if state \( b_L \) is visited before state \( b_{ML} \)). Indeed, if \( b_L \) is visited before \( b_{ML} \), at time \( t(b_L) \) the principal will learn that the agent’s type is \( c_1 \) (see property (iii)). From that point onwards, the agent will take the productive action at all periods \( t > t(b_L) \) such that \( b_t \in E_1 \) at cost \( c_1 \) for the principal.

In contrast, if \( b_{ML} \) is visited before \( b_L \), at time \( t(b_{ML}) \) the principal will learn that the agent’s type is in \( \{c_1, c_2\} \) (see property (iii)). From that point onwards there will be no more learning (property (i)). As a consequence, the agent will take the productive action at all periods \( t > t(b_{ML}) \) such that \( b_t \in E_2 = E_1 \) at cost \( c_2 \) for the principal (this follows from Theorem 1(i)).
References