Robust Screens for Non-Competitive Bidding in Procurement Auctions

Sylvain Chassang  Kei Kawai  Jun Nakabayashi
Princeton University  U.C. Berkeley  Kindai University
Juan Ortner∗†
Boston University
October 29, 2020

Abstract

We document a novel bidding pattern observed in procurement auctions from Japan: winning bids tend to be isolated, and there is a missing mass of close losing bids. This pattern is suspicious in the following sense: its extreme forms are inconsistent with competitive behavior under arbitrary information structures. Building on this observation, we develop systematic tests of competitive behavior in procurement auctions that hold under minimal assumptions on the information structure and unobserved heterogeneity. We provide an empirical exploration of our tests, and show they can help identify other suspicious patterns in the data.

Keywords: missing bids, collusion, antitrust, procurement.

∗Contact information, Chassang: chassang@princeton.edu, Kawai: kei@berkeley.edu, Nakabayashi: nakabayashi.1@eco.kindai.ac.jp, Ortner: jortner@bu.edu.
†We are especially indebted to Hiroaki Kaido and Steve Tadelis for detailed feedback. The paper benefited from discussions with Pierpaolo Battigali, Eric Budish, Yeon-Koo Che, Francesco Decarolis, Rieko Ishii, Emir Kamenica, Ulrich Müller, Roger Myerson, Ariel Pakes, Wolfgang Pesendorfer, Andrea Prat, Michael Riordan, Jozsef Sakovics, Larry Samuelson, Andy Skrzypacz, Paulo Somaini, as well as comments from seminar participants at the 2018 ASSA meeting, the 2018 Canadian Economic Theory Conference, Bocconi, the 2017 Berkeley-Sorbonne workshop on Organizational Economics, Boston University, Columbia University, EIEF, the 2018 ESSET meeting at Gerzens, FGV Rio, Toulouse, Harvard, HKUST, Johns Hopkins, MIT, NYU, the 2017 NYU CRATE conference on theory and econometrics, Penn State, Princeton, PUC Rio, Queen’s University, Rochester, the 2018 Stanford SITE conference, Stanford GSB, the 2018 Transparency in Procurement Conference, the Triangle Microeconomics Conference, the University of Arizona, the University of Chicago, the University of Copenhagen, the University of Illinois at Urbana-Champaign, UC Berkeley, the University of Tokyo, the University of Virginia, UCLA, Warwick, and Yale. The study has been supported by JSPS KAKENHI (Grant Number JP18H03643)
1 Introduction

One of the key functions of antitrust authorities is to detect and punish collusion. Although concrete evidence is required for successful prosecution, screening devices that flag suspicious firms help regulators identify collusion, and encourage members of existing cartels to apply for leniency programs.\textsuperscript{1} Correspondingly, an active research agenda has sought to build methods to detect cartels using naturally occurring market data (e.g. Porter, 1983, Porter and Zona, 1993, 1999, Ellison, 1994, Bajari and Ye, 2003, Harrington, 2008). This paper seeks to make progress on this research agenda by developing systematic tests of competitive behavior in procurement auctions under minimal assumptions on the environment.

We begin by documenting a suspicious bidding pattern observed in first-price sealed-bid procurement auctions in Japan: the density of the bid distribution just above the winning bid is very low; there is a missing mass of close losing bids. These missing bids are related to bidding patterns observed among collusive firms in Hungary (Tóth et al., 2014), Switzerland (Imhof et al., 2018), and Canada (Clark et al., 2020). We establish that extreme forms of this pattern are inconsistent with competitive behavior under a general class of asymmetric information structures. Indeed, when winning bids are isolated, bidders can profitably deviate by increasing their bids. Expanding on this observation, we propose general tests of competitive behavior in procurement auctions that are robust, in the sense of holding under minimal assumptions on the information structure and arbitrary unobserved heterogeneity.

Our data come from two sets of public works procurement auctions in Japan. The first dataset contains information on roughly 7,000 city-level auctions held between 2004 and 2018 by 14 different municipalities in Ibaraki prefecture and the Tohoku region of Japan. The second dataset, analyzed by Kawai and Nakabayashi (2018), contains data on approxi-\textsuperscript{2}

\textsuperscript{1}A growing number of agencies are adopting algorithm-based screens that analyze bidding data from public procurement auctions to flag suspicious behavior including those in Brazil, South Korea, Switzerland and United Kingdom, and most recently, the United States. For example, the U.S. Department of Justice announced the formation of a procurement collusion strike force whose goal includes bolstering “data analytics employment to identify signs of potential anticompetitive, criminal collusion.” (Announcement of the Antitrust Division’s Procurement Collusion Strike Force, November 22, 2019.)
mately 78,000 national-level auctions held between 2001 and 2006 by the Ministry of Land, Infrastructure and Transportation. We are interested in the distribution of bidders’ margins of victory (or defeat). For every (bidder, auction) pair, we compute \( \Delta \equiv \frac{\text{own bid} - \min(\text{other bids})}{\text{reserve}} \), the difference between the bidder’s own bid and the most competitive bid among this bidder’s opponents, divided by the reserve price. When \( \Delta < 0 \), the bidder won the auction. When \( \Delta > 0 \) the bidder lost. For both the municipal and national datasets, we document a missing mass in the distribution of \( \Delta \) around \( \Delta = 0 \). Our results clarify the sense in which this missing mass of close losing bids is suspicious, and help us identify other patterns in the data that are inconsistent with competition.

We analyze our data within a fairly general framework. A group of firms repeatedly participates in first-price procurement auctions. Players can observe arbitrary signals about one another, and bidders’ costs and types can be arbitrarily correlated within and across periods. Importantly, intertemporal linkages between actions and payoffs are ruled out. Behavior is called competitive if it is stage-game optimal under the players’ information.

Our first set of results establishes that, in its more extreme forms, the pattern of missing bids is not consistent with competitive behavior under any information structure. We exploit the fact that in any competitive equilibrium, firms must not find it profitable in expectation to increase their bids. This incentive constraint implies that with high probability the elasticity of firms’ sample residual demand (i.e., the empirical probability of winning an auction at any given bid) must be bounded above by -1. This condition is not satisfied in portions of our data: because winning bids are isolated, the elasticity of sample residual demand is close to zero. In addition, we are able to derive bounds on the minimum number of histories at which non-competitive bidding must happen.

Our second set of results generalizes this test. In particular, we show how to exploit weak equilibrium conditions to derive sharper bounds on the extent of non-competitive behavior in our data. The bounds that we propose hold under minimal assumptions on the information structure, contrasting with existing approaches that rely on specific assumptions such as
independent private values (e.g. Bajari and Ye, 2003). As we show in our companion paper Ortner et al. (2020), antitrust policy based on tests that are robust to information structure cannot be exploited by cartels to enhance collusion. This addresses the concern articulated by Cyrenne (1999) and Harrington (2004) that data driven antitrust policies may end up facilitating collusion by cartel members.

Our third set of results takes our tests to the data. We delineate how different moment conditions (i.e. different deviations) uncover different non-competitive patterns. While missing bids suggest that a small increase in bids is attractive, we show that a moderate-sized drop in bids (on the order of 2%) may be attractive to bidders: it yields large increases in demand. In addition, we show that downward deviations tend to be more informative about the competitiveness of auctions than upward deviations, and that upward and downward deviations can be more informative together than separately. Finally, although failing our tests does not necessarily imply bidder collusion, we show that the outcomes of our tests are consistent with other proxy evidence for competitiveness and collusion. Histories in which bids are high relative to the reserve price are more likely to fail our tests than histories in which bids are low. Histories before an industry is investigated for collusion are more likely to fail our tests than histories after it is investigated for collusion. Altogether this suggests that, although our tests are conservative, they still have bite in practice.


---

in bids across stages to detect collusion. Marmer et al. (2016) and Schurter (2017) design tests of collusion for English auctions and for first-price sealed bid auctions focusing on partial cartels. The tests that we propose relax assumptions imposed in previous work such as symmetry, independence and private values (at the cost of reduced power), and can be used to detect both all-inclusive cartels and partial cartels.

More broadly, our paper relates to prior work that seeks to test for competitive behavior in other (non auction) markets. Sullivan (1985) and Ashenfelter and Sullivan (1987) propose tests of whether firms behave as a perfect cartel, and apply these tests to the cigarette industry. Bresnahan (1987) and Nevo (2001) test for competition in the automobile and ready-to-eat cereal industries.

Our tests are also related to revealed preference tests seeking to quantify violations of choice theoretic axioms. Afriat (1967), Varian (1990), and Echenique et al. (2011) propose tests to quantify the extent to which a given consumption dataset violates GARP. More closely related, Carvajal et al. (2013) propose a revealed preference test of the Cournot model.

Finally, our paper makes an indirect contribution to the literature on the internal organization of cartels. Asker (2010) studies stamp auctions, and analyses the effect of a particular collusive scheme on non-cartel bidders and sellers. Pesendorfer (2000) studies the bidding patterns for school milk contracts and compares the collusive schemes used by cartels that used transfers and those that did not. Clark and Houde (2013) document the collusive strategies used by the retail gasoline cartel in Quebec. Clark et al. (2018, 2020) study the effect of an investigation on firms’ bidding behavior. We add to this literature by documenting a puzzling bidding pattern, and by showing that this bidding pattern is non-competitive.

---

3Kawai and Nakabayashi (2018) study the subset of auctions with rebids while the current analysis applies to all auctions.

4Also related are Porter (1983) and Ellison (1994), who exploit dynamic patterns of play predicted by the theory of repeated games (Green and Porter, 1984, Rotemberg and Saloner, 1986) to test for tacit collusion.

5See Chambers and Echenique (2016) for a recent review of the literature on revealed preferences.
2 Motivating Facts

Our first dataset consists of roughly 7,100 auctions for public works contracts held between 2004 and 2018 by municipalities located in the Tohoku region and Ibaraki prefecture of Japan. The auctions are sealed-bid first-price auctions with a publicly announced reserve price. The top panel of Table 1 reports summary statistics. The mean reserve price is 23.2 million yen, or about 230,000 USD, and the mean winning bid is 21.5 million yen. The mean number of bidders is 7.4. On average, a bidder in the dataset participates in 23.3 auctions and wins 3.1 auctions.

For any given firm $i$ participating in auction $a$ with reserve price $r$, we denote by $b_{i,a}$ the bid of firm $i$ in auction $a$, and by $b_{-i,a}$ the profile of bids by bidders other than $i$. We investigate the distribution of

$$\Delta_{i,a} = \frac{b_{i,a} - \wedge b_{-i,a}}{r}$$

aggregated over firms $i$, and auctions $a$, where $\wedge$ denotes the minimum operator. The value $\Delta_{i,a}$ represents the margin by which bidder $i$ wins or loses auction $a$. If $\Delta_{i,a} < 0$ the bidder won, if $\Delta_{i,a} > 0$ she lost. Figure 1(a) plots the histogram and the density estimate of bid differences $\Delta$ aggregating over all firms and auctions in the sample. The mass of missing bids around 0 is clearly noticeable.

Our second dataset, studied in Kawai and Nakabayashi (2018), consists of roughly 78,000 auctions for construction projects held between 2001 and 2006 by the Ministry of Land, Infrastructure and Transportation in Japan (the Ministry). The auctions are sealed-bid first-price auctions with a secret reserve price. The average reserve price is 105.1 million yen, or about 1 million USD and the mean lowest bid is 101.9 million yen, which is 97.0% of

6Our city-level dataset combines two datasets. The first dataset contains auctions held by municipalities in the Tohoku region in Japan. For the current analysis, we restrict attention to municipalities using a sealed-bid first-price auction with a public reserve price. The second dataset, studied in Chassang and Ortner (2019), contains auctions held by municipalities in the prefecture of Ibaraki. For the current analysis, we use data from the city of Tsuchiura during 2007-2009, when the city was using sealed-bid first-price auctions with a public reserve price.
(a) city auctions  

(b) national auctions

Figure 1: Distribution of bid-differences $\Delta$ over (bidder, auction) pairs.

The dotted curves correspond to local (6th order) polynomial density estimates with bandwidth set to 0.0075.

the reserve price. Because the reserve price is secret, the lowest bid may be higher than the reserve price in which case there is rebidding. In that event, the reserve price remains secret to the bidders, but the lowest bid from the initial round is announced. There are at most two rounds of rebidding. If none of the bids are below the reserve price at the end of the second round of rebidding, the lowest bidder from the last round enters into a bilateral negotiation with the buyer. The auction concludes in the initial round of bidding about 75% of the time. The auction concludes after one round of rebidding in more than 97% of auctions and concludes after two rounds of rebidding in more than 99% of auctions. The mean number of participants is 9.9. For both datasets, all of the bids become public information after the auction. Figure 1(b) illustrates the distribution of bid-differences $\Delta$ for national auctions, where $\Delta$ is defined using first-round bids. The missing mass of bids around $\Delta = 0$ is stark.

We point out another important (though less visually striking) feature of the densities plotted in Figure 1: the tails of the distribution taper off rapidly. This implies that much of the mass of $\Delta$ is concentrated within a relatively small interval around 0. For example, $\Delta$ lies between 0 and 0.02 for 50.0% of the losing bids in the city auctions and 25.6% of the losing bids in the national auctions. This implies that a drop in bids of 2% increases demand
considerably (by 349% and 307%, respectively).\textsuperscript{7} Hence, while the missing bid pattern in Figure 1 suggests small increases in bids are profitable, the relatively large concentration of mass around $\Delta = 0$ suggests that a small reduction in bids is also attractive.

<table>
<thead>
<tr>
<th>City Auctions</th>
<th>Mean</th>
<th>S.D.</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>By Auctions</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>reserve price (mil. Yen)</td>
<td>23.189</td>
<td>91.32</td>
<td>7,111</td>
</tr>
<tr>
<td>lowest bid (mil. Yen)</td>
<td>21.500</td>
<td>85.10</td>
<td>7,111</td>
</tr>
<tr>
<td>lowest bid / reserve</td>
<td>0.938</td>
<td>0.06</td>
<td>7,111</td>
</tr>
<tr>
<td>#bidders</td>
<td>7.425</td>
<td>3.77</td>
<td>7,111</td>
</tr>
<tr>
<td>By Bidders</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>participation</td>
<td>23.29</td>
<td>43.58</td>
<td>2,267</td>
</tr>
<tr>
<td>number of times lowest bidder</td>
<td>3.14</td>
<td>6.22</td>
<td>2,267</td>
</tr>
</tbody>
</table>

| National Auctions              |        |        |      |
| By Auctions                     |        |        |      |
| reserve price (mil. Yen)        | 105.121| 259.58 | 78,272|
| lowest initial bid (mil. Yen)   | 101.909| 252.30 | 78,272|
| winning bid (mil. Yen)          | 100.338| 252.30 | 78,272|
| lowest bid / reserve            | 0.970  | 0.10   | 78,272|
| winning bid / reserve           | 0.946  | 0.10   | 78,272|
| ends in one round of bidding    | 0.752  | 0.43   | 78,272|
| ends after one rebidding        | 0.971  | 0.17   | 78,272|
| ends after two rebidding        | 0.996  | 0.06   | 78,272|
| #bidders                       | 9.883  | 2.27   | 78,272|
| By Bidders                      |        |        |      |
| participation                   | 26.40  | 94.61  | 29,670|
| number of times lowest bidder   | 2.64   | 10.57  | 29,670|

Table 1: Sample Statistics – City and National Level Data

Our analysis shows the extent to which the bidding patterns in Figure 1 are inconsistent with competitive behavior under arbitrary information structure. While non-competitive

\textsuperscript{7}Figure OD.1 (Online Appendix OD.1) illustrates this point by plotting firms’ sample residual demand for our two datasets.
behavior need not be collusive, we note that missing bids are in fact correlated with plausible indicators of collusion, as we discuss next.

**Correlation with indicators of collusion.** Since the goal of collusion is to elevate prices, we would expect to see suspicious bidding patterns in auctions with high bids. Figure 2 breaks down the auctions in Figure 1(b) by bid level: the figure plots the distribution of $\Delta_{i,a}$ for normalized bids $\frac{b_{i,a}}{r}$ below .8 (Panel (a)) and above .9 (Panel (b)). The mass of missing bids is considerably reduced in Panel (a). The tails of the distribution taper off more gradually in Panel (a).

![Figure 2: Distribution of bid-difference $\Delta$ – national data.](image)

(a) Bids below 80% of reserve price  
(b) Bids above 90% of reserve price

The dotted curves correspond to local (6th order) polynomial density estimates with bandwidth set to 0.0075.

Figure 3 plots the distribution of $\Delta_{i,a}$ for participants of auctions held by the Ministry that were implicated by the Japanese Fair Trade Commission (JFTC). The JFTC implicated four bidding rings participating in auctions in our data: (i) firms installing electric traffic signs (Electric); (ii) builders of bridge upper structures (Bridge); (iii) pre-stressed concrete providers (PSC); and (iv) floodgate builders (Flood). The left panels in Figure 3 plot the distribution of $\Delta$ for auctions that were run before the JFTC started its investigation, and the right panels plot the distribution in the after period. In all cases except case (iii), the
pattern of missing bids disappears after the JFTC launched its investigation. Interestingly, firms in case (iii) initially denied the charges against them (unlike firms in the other three
cases), and seem to have continued colluding for some time (see Kawai and Nakabayashi (2018) for a more detailed account of these collusion cases).

**What does not explain this pattern.** We end this section by arguing that missing bids are not explained by either the granularity of bids, or ex post renegotiation.

Figures 2 and 3 show that the pattern of missing bids in Figure 1 is not a mechanical consequence of the granularity of bids. If this was the case, we should see similar patterns across all bid levels, or before and after the JFTC investigations. In addition, Figure OD.2 in Online Appendix OD.1 plots the distribution of $\Delta^2$, the differences between the bids after the lowest bid is excluded. The figure shows that the distribution of $\Delta^2$ has no corresponding missing mass at 0.

Renegotiation could potentially account for missing bids by making apparent incentive compatibility issues irrelevant. However, in the auctions we study, contracts signed between the awarder and the awardee include a renegotiation provision which stipulates that renegotiated prices should be anchored to the initial bid. Specifically, if the project is estimated to cost $y$ more than initially thought, the renegotiated price is increased by $\frac{\text{initial bid}}{\text{reserve price}} \times y$. This implies that a missing mass of $\Delta$ makes a small increase in bids profitable even taking into account the possibility of renegotiation. We provide further evidence on this point in Online Appendix OD.1.

**3 Framework**

**3.1 The Stage Game**

We consider a dynamic setting in which, at each period $t \in \mathbb{N}$, a buyer needs to procure a single project. In the main body of the paper, we assume that the auction format is a sealed-bid first-price auction with a *public* reserve price $r$, which we normalize to $r = 1$. Appendix OB extends the analysis to auctions with *secret* reserve prices and re-bidding (as
In each period $t$, a state $\theta_t \in \Theta$ captures all relevant past information about the environment. State $\theta_t$ may be unknown to the bidders at the time of bidding, but is revealed to bidders by the end of period $t$. Importantly, $\theta_t$ need not be observed by the econometrician. We assume that $\theta_t$ is a Markov chain (i.e. given any event $E$ anterior to time $t$, $\theta_t|\{\theta_{t-1}, E\} \sim \theta_t|\{\theta_{t-1}\}$), but do not assume that there are finitely many states, that the chain is irreducible, ergodic. The important assumption here is that by the end of each period $t$, bidders observe a sufficient statistic $\theta_t$ of future environments. Since the distribution of $\theta_t$ depends solely on $\theta_{t-1}$, we rule out intertemporal linkages between actions and payoffs.\(^8\) In practice, state $\theta_t$ may include distances between the project site and each of the firms, public knowledge about each firm’s capabilities, the specifications laid out in the construction plan, etc. Once again, we emphasize that we do not assume that this information is observed by the econometrician.

In each period $t \in \mathbb{N}$, a set $\hat{N}_t \subset N$ of bidders is able to participate in the auction, where $N$ is the overall set of bidders. We think of this set of participating firms as those eligible to produce in the current period.\(^9\) The distribution of the set of eligible bidders $\hat{N}_t$ can vary over time, but depends only on state $\theta_{t-1}$. Participants discount future payoffs using a common discount factor $\delta < 1$.

Realized costs of production for eligible bidders $i \in \hat{N}_t$ are denoted by $c_t = (c_{i,t})_{i \in \hat{N}_t}$. Each bidder $i \in \hat{N}_t$ submits a bid $b_{i,t}$. Profiles of bids are denoted by $b_t = (b_{i,t})_{i \in \hat{N}_t}$. We let $b_{-i,t} \equiv (b_{j,t})_{j \neq i}$ denote bids from firms other than firm $i$, and define $\land b_{-i,t} \equiv \min_{j \neq i} b_{j,t}$ to be the lowest bid among $i$’s competitors at time $t$. The procurement contract is allocated to the bidder submitting the lowest bid, at a price equal to her bid. Ties are broken randomly.

\(^8\)This rules out dynamic considerations such as capacity constraints or learning by doing. In our companion paper, Kawai et al. (2020), we propose tests of competition that hold in the presence of intertemporal links between bids and costs.

\(^9\)For simplicity, we take the set of participating bidders as exogenous. In practice, the set of participants may well be endogenous (see the Online Appendix of Chassang and Ortner (2019) for a treatment of endogenous participation by cartel members). This does not affect our analysis: bids would still have to satisfy the optimality conditions we rely on for inference.
Costs. The profile of costs $c_t = (c_{i,t})_{i \in \tilde{N}_t}$ may exhibit correlation across players and over time, but its distribution depends only on state $\theta_t$. All costs are assumed to be positive.

In period $t$, bidder $i \in \tilde{N}_t$ obtains profits

$$\pi_{i,t} = x_{i,t} \times (b_{i,t} - c_{i,t}),$$

where $x_{i,t} \in [0, 1]$ is the probability with which $i$ wins the auction at time $t$. Note that costs include both the direct costs of production and the opportunity cost of backlog.

Information. In each period $t$, bidder $i$ gets a signal $z_{i,t}$ prior to bidding. The distribution of the profile of signals $(z_{i,t})_{i \in \tilde{N}_t}$ depends only on $(\theta_t, (c_{i,t})_{i \in \tilde{N}_t})$. We stress that signals $(z_{i,t})_{i \in \tilde{N}_t}$ are arbitrary, and may reveal information about current state $\theta_t$, or the realized costs $c_{j,t}$ of other players. This allows our model to nest many informational environments, including private and common values, correlated values, asymmetric bidders and asymmetric information. The model also allows for general forms of unobserved heterogeneity, nesting models with scalar unobserved heterogeneity such as that of Krasnokutskaya (2011). A bidder’s costs and signals may be correlated to other bidders’ costs and signals. Bids $b_t$ are publicly observable at the end of the auction.

3.2 Solution Concepts

A public history $h^0_t$ in period $t$ takes the form $h^0_t = (\theta_{s-1}, b_{s-1})_{s \leq t}$. We let $\mathcal{H}^0$ denote the set of all public histories. Our solution concept is perfect public Bayesian equilibrium (Athey

---

10Krasnokutskaya (2011) considers an auction model in which costs depend multiplicatively on a scalar unobservable that is common to all bidders and is independent of bidders’ private costs. Our model allows for scalar or vector unobservables as well as unobservables that are independent of, or correlated with bidders’ private costs. The unobservables in our environment do not have to affect costs multiplicatively or in any pre-specified way.

11Bids are publicly reported in the auctions we study. The assumption that bidders (rather than just the econometrician) observe bids can be relaxed. However, echoing Fershtman and Pakes (2012), this makes it more plausible that bidding behavior approaches equilibrium and satisfies the weak optimality conditions used in our identification strategy.
and Bagwell, 2008). Because state $\theta_t$ is revealed by the end of each period, past play conveys no information about the private types of other players. As a result we do not need to specify out-of-equilibrium beliefs. A perfect public Bayesian equilibrium consists only of a strategy profile $\sigma = (\sigma_i)_{i \in N}$, such that for all $i \in N$, $\sigma_i$ maps public histories and payoff relevant private signals to bids

$$\sigma_i : h^0_{i,t}, z_{i,t} \mapsto b_{i,t}.$$  

Because our framework doesn’t allow for intertemporal linkages between past actions and future payoffs, we can identify the class of competitive equilibria with the class of Markov perfect equilibria (Maskin and Tirole, 2001).

**Definition 1** (competitive strategy). We say that $\sigma$ is Markov perfect if and only if $\forall i \in N$ and $\forall h^0_i \in H^0$, $\sigma_i(h^0_i, z_{i,t})$ depends only on $(\theta_{t-1}, z_{i,t})$.

We say that a strategy profile $\sigma$ is a competitive equilibrium if it is a perfect public Bayesian equilibrium in Markov perfect strategies.

In a competitive equilibrium, firms must be playing a stage-game Nash equilibrium at every period; i.e. firms play a static best-reply to the actions of their opponents.

**Competitive histories.** Our datasets involve many firms, interacting over an extensive timeframe. Realistically, an equilibrium may include periods in which (a subset of) firms collude and periods in which firms compete: we allow for both full and partial cartels. This leads us to define competitiveness at the history level.

**Definition 2** (competitive histories). Fix a common knowledge profile of play $\sigma$ and a history $h_{i,t} = (h^0_{i,t}, z_{i,t})$ of player $i$. We say that player $i$ is competitive at history $h_{i,t}$ if play at $h_{i,t}$ is stage-game optimal for firm $i$ given the behavior of other firms $\sigma_{-i}$.
4 Beliefs and Realized Demand in Equilibrium

We make few assumptions on players’ information, so that players’ belief process may be quite general. Still, in equilibrium the difference between realized demand and beliefs regarding demand is a martingale, which means that we can apply analogs of the central limit theorem for martingales to obtain probabilistic constraints on players’ beliefs given realized demand. We highlight that our approach: (i) allows the residual demand at history $h_{i,t}$ to depend on the bidder’s signal $z_{i,t}$ in an arbitrary way; (ii) allows for arbitrary forms of unobserved heterogeneity; and (iii) avoids ergodicity assumptions on the underlying state $\theta_t$.

Fix a perfect public Bayesian equilibrium $\sigma$. For all histories $h_{i,t} = (h^0_t, z_{i,t})$ and all bids $b' \in [0, 1]$, player $i$’s residual demand at $h_{i,t}$ is

$$D_i(b'|h_{i,t}) \equiv \text{prob}_\sigma(\wedge b_{-i,t} > b'|h_{i,t}).$$

In words, bidder $i$’s residual demand at history $h_{i,t}$ represents the probability with which the bidder expects to win the auction at $t$, for each possible bid $b'$ she may place.

Take as given a finite set of histories $H$, and a scalar $\rho \in (-1, \infty)$. We denote by $\overline{D}(\rho|H)$ average residual demand for histories in $H$ and by $\hat{D}(\rho|H)$ its sample equivalent:

$$\overline{D}(\rho|H) \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} D_i((1 + \rho)b_{i,t}|h_{i,t}),$$

$$\hat{D}(\rho|H) \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} 1_{\wedge b_{-i,t} > (1+\rho)b_{i,t}}.$$  

**Definition 3.** We say that a set of histories $H$ is adapted to the players’ information if and only if the event $h_{i,t} \in H$ is measurable with respect to player $i$’s information at time $t$, prior

\[12\text{Note that there are alternative ways to specify weights assigned to observations in (1) and (2). For our empirical application, we choose the weighting to optimize the standard error of estimates. See Online Appendix OA for details.}\]
to bidding.

In words, we say that an event is adapted, if it depends only on the information available to individual bidders at the time of bidding. For instance, the entire set of histories for a specific industry or location is adapted – a bidder knows its industry, and its location. In contrast, the set of histories in which a specific bidder wins is not.\textsuperscript{13} The ability to legitimately vary the conditioning set $H$ lets us explore the competitiveness of auctions in particular settings of interest, across industries, or time periods.

It is necessary for us to focus on adapted sets of histories to link realized and expected demand (and payoffs). If we select a subset of histories $H$ of interest, we are effectively evaluating outcomes under the conditioning event $h_{i,t} \in H$. When this event is in the information set of bidders at the time of decision making, then even conditional on this event, the bidders’ expectations are an unbiased predictor of realized demand and realized payoffs. This allows us to link realized outcomes to bidders’ expectations, and therefore put constraints on bidders’ expected demand and payoffs using realized data. This link disappears if we focus on a set of histories that is not adapted: a bidder’s expectation at the time of bidding is no longer an unbiased predictor of realized outcomes. For instance, if we focus on the set of histories such that a given bidder wins, then the bidder’s realized demand under this conditioning event is 1, whereas the bidder’s expected demand at the time of bidding will likely have been strictly below 1. Let $N_{\text{max}}$ denote an upper bound on the number of participants in any auction.\textsuperscript{14}

\textbf{Proposition 1.} Consider an adapted set of histories $H$. Under any perfect public Bayesian equilibrium $\sigma$, for any $\nu > 0$,

$$\text{prob}(\left| \hat{D}(\rho|H) - \overline{D}(\rho|H) \right| \leq \nu) \geq 1 - 2 \exp(-\nu^2 |H|/2N_{\text{max}}).$$

\textsuperscript{13}Note that the set of competitive histories itself is adapted (a bidder knows whether its bid is subjectively competitive), though unobserved by the econometrician.

\textsuperscript{14}It is sufficient for $N_{\text{max}}$ to be a bound on the number of participants with histories in $H$ in each auction.
In particular, with probability 1, \( \hat{D}(\rho|H) - \overline{D}(\rho|H) \to 0 \) as \( |H| \to \infty \).

In words, in equilibrium, the sample residual demand conditional on an adapted set of histories converges to the true average residual demand. The restriction to adapted histories implies that

\[
\mathbb{E}_\sigma [\text{prob}_\sigma(\land b_{i,s} > (1 + \rho)b_{i,s}|h_{i,s}) - 1_{\land b_{i,s} > (1 + \rho)b_{i,s}}|h_t^0, \{h_{i,s} \in H\}] = 0 \quad (3)
\]

for all histories \( h_t^0, h_{i,s} \in H \) with \( t < s \). Hence, difference \( |H| \left( \hat{D}(\rho|H) - \overline{D}(\rho|H) \right) \) evolves like a Martingale as \( |H| \) grows. The Azuma-Hoeffding Inequality yields Proposition 1. Condition (3) clarifies the role of the equilibrium assumption in the proof of Proposition 1. We note that since bidders observe past bids, Proposition 1 is also implied by conditions weaker than equilibrium. For instance, it would hold if participants used data-driven predictors of demand satisfying no-regret (see Hart and Mas-Colell, 2000).

Proposition 1 relies on non-asymptotic concentration bounds that are simple to interpret but conservative. We use tighter asymptotic bounds relying on the Central Limit Theorem for renormalized sums of martingale increments (Billingsley, 1995) in our empirical analysis. Because these asymptotic results are standard but notationally cumbersome, we express our main results using the conservative bound of Proposition 1, and delay our treatment of asymptotic bounds to Online Appendix OA.

Our empirical application varies set \( H \). We consider the set of all auctions, all auctions from a given city, auctions in which a given firm participates, etc. In each case, the set \( H \) that we consider is quite coarse, pooling across many different auctions. However, Proposition 1 ensures that, as long as \( H \) is adapted, \( \hat{D}(\rho|H) \) is a consistent estimator of \( \overline{D}(\rho|H) \) regardless of observed and unobserved auction and bidder heterogeneity.
5 Testing for Competition

5.1 Missing Bids are Inconsistent with Competition

Our first main result shows that \textit{extreme forms} of the pattern of bids illustrated in Figure 1 are inconsistent with competitive behavior.

\textbf{Proposition 2.} Let $\sigma$ be a competitive equilibrium. Then,

\begin{align*}
\forall h_i, \quad & \frac{\partial \log D_i(b'|h_i)}{\partial \log b'} \bigg|_{b'=\sigma_i(h_i)+} \leq -1, \quad (4) \\
\forall H, \quad & \frac{\partial \log D(\rho|H)}{\partial \rho} \bigg|_{\rho=0+} \leq -1. \quad (5)
\end{align*}

In words, under any competitive equilibrium, the elasticity of a bidder’s residual demand must be less than -1 at every history, and the inequality aggregates to sets of histories. Proposition 2 extends to first-price auctions with \textit{secret} reserve prices (see Online Appendix OB).

\textbf{Proof.} Consider a competitive equilibrium $\sigma$. Let

$$V(h_{i,t}) \equiv \mathbb{E}_\sigma \left( \sum_{s \geq t} \delta^{s-t} (b_{i,s} - c_{i,s}) 1_{b_{i,s} < b_{i-1,s}} \big| h_{i,t} \right)$$

denote player $i$’s discounted expected payoff at history $h_{i,t}$. Let $b$ denote the bid that bidder $i$ places at history $h_{i,t}$. Since $b$ is an equilibrium bid, it must be that for all bids $b' > b$,

$$\mathbb{E}_\sigma \left[ (b - c_{i,t}) 1_{b_{i-1,t} > b} + \delta V(h_{i,t+1}) \big| h_{i,t}, b_{i,t} = b \right] \geq \mathbb{E}_\sigma \left[ (b' - c_{i,t}) 1_{b_{i-1,t} > b'} + \delta V(h_{i,t+1}) \big| h_{i,t}, b_{i,t} = b \right]$$
Since $\sigma$ is competitive, $E[\sigma(V(h_{i,t+1})|h_{i,t}, b_{i,t} = b) = E[\sigma(V(h_{i,t+1})|h_{i,t}, b_{i,t} = b')]$. Hence,

$$bD_i(b|h_{i,t}) - b'D_i(b'|h_{i,t}) = E[\sigma(b1_{\Delta b_{i,t} > b} - b'1_{\Delta b_{i,t} > b'}|h_{i,t})]$$

$$\geq E[\sigma(c_{i,t}1_{\Delta b_{i,t} > b} - 1_{\Delta b_{i,t} > b'}|h_{i,t})] \geq 0,$$

where the last inequality uses the assumption that $c_{i,t} \geq 0$. This implies that for all $b' > b$,

$$bD_i(b|h_{i,t}) \geq b'D_i(b'|h_{i,t}) \iff \log b + \log D_i(b|h_{i,t}) \geq \log b' + \log D_i(b'|h_{i,t})$$

$$\iff \frac{\log D_i(b'|h_{i,t}) - \log D_i(b|h_{i,t})}{\log b' - \log b} \leq -1.$$ 

Inequality (4) follows from taking the limit as $b' \to b$. Inequality (5) follows from a similar argument: for all $h_{i,t}$, $b_{i,t}$ and $\rho > 0$, we have that

$$b_{i,t}D_i(b_{i,t}|h_{i,t}) \geq (1 + \rho)b_{i,t}D_i((1 + \rho)b_{i,t}|h_{i,t}) \iff D_i(b_{i,t}|h_{i,t}) \geq (1 + \rho)D_i((1 + \rho)b_{i,t}|h_{i,t})$$

Averaging over histories $h_{i,t} \in H$, this implies that

$$\mathcal{D}(0|H) \geq (1 + \rho)\overline{D}(\rho|H) \iff \frac{\log \overline{D}(\rho|H) - \log \overline{D}(0|H)}{\log(1 + \rho)} \leq -1.$$

Taking $\rho$ to 0 yields inequality (5). ■

Proposition 2 extends the standard result that an oligopolistic competitor must price in the elastic part of her residual demand curve to settings with arbitrary incomplete information. It can be tested by replacing the true average residual demand with its sample average, using Proposition 1. Extreme forms of missing bids contradict Proposition 2: when the density of $\Delta$ at 0 is close to 0, the elasticity of demand is approximately zero.

As the proof highlights, this result exploits the fact that in procurement auctions, zero is
a natural lower bound for costs. In contrast, for auctions where bidders are purchasing a good with positive value, there is no corresponding natural upper bound to valuations. One would need to impose an upper bound on values to establish similar results.

Proposition 2 yields a simple test of whether an adapted set of histories $H$ can be generated by a competitive equilibrium or not. Our test holds under arbitrary information structures, and hence strengthens existing approaches that make specific assumptions such as symmetry, independent values and private values (see for instance Bajari and Ye, 2003).

We now refine this test to obtain bounds on the minimum share of non-competitive histories needed to rationalize the data. We begin with a loose bound and then propose a more sophisticated program resulting in tighter bounds. Estimating the share of non-competitive histories, rather than offering a binary test, provides a sense of the prevalence of non-competitive behavior. This potentially helps regulators gauge the magnitude of potential cartels and target investigations efficiently. In addition, establishing that failures of non-competitive behavior are not rare clarifies that bidders have plenty of opportunities to learn how to improve their bids. This suggests that failures to optimize stage-game profits are not merely errors.

5.2 Estimating the share of competitive histories

It follows from Proposition 2 that extreme forms of missing bids cannot be explained in a model of competitive bidding. We now establish that competitive behavior must fail at a significant number of histories in order to explain isolated winning bids. This implies that bidders have frequent opportunities to learn that their bids are not optimal.

Fix a perfect public Bayesian equilibrium $\sigma$ and an adapted set of histories $H$. Recall from Definition 2 that a history $h_{i,t} = (h^0_t, z_{i,t}) \in H$ is competitive if bidder $i$ is playing a stage-game best-response at time $t$. Let $H^{\text{comp}} \subset H$ be the set of competitive histories in $H$.

15Dynamic considerations such as learning by doing might imply that firms incur a negative net cost of winning an auction. As we argue in Section 7, such dynamic considerations are unlikely to explain the bidding patterns in our data.
Define $s_{\text{comp}} \equiv \frac{|H^\text{comp}|}{|H|}$, where $|H|$ denotes the cardinality of set $H$, the fraction of competitive histories in $H$.

For all histories $h_{i,t} = (h_i^0, z_{i,t})$ and all bids $b' \geq 0$, player $i$’s residual revenue at $h_{i,t}$ is

$$R_{i}(b'|h_{i,t}) \equiv b'D_{i}(b'|h_{i,t}).$$

For any finite set of histories $H$ and any scalar $\rho \in (-1, \infty)$, let

$$\overline{R}(\rho|H) \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} (1 + \rho)b_{i,t}D_{i}((1 + \rho)b_{i,t}|h_{i,t})$$

denote the average residual revenue for histories in $H$. An extension of Proposition 1 shows that the sample residual revenue $\hat{R}(\rho|H) \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} (1 + \rho)b_{i,t}1_{b_{i,t} > (1 + \rho)b_{i,t}}$ is a consistent estimator of $\overline{R}(\rho|H)$, whenever set $H$ is adapted.

Our next result builds on Proposition 2 to derive a bound on $s_{\text{comp}}$.

**Proposition 3.** The share $s_{\text{comp}}$ of competitive histories is such that

$$s_{\text{comp}} \leq 1 - \sup_{\rho > 0} \frac{\overline{R}(\rho|H) - \overline{R}(0|H)}{\rho}.$$ 

**Proof.** Let $H^{-\text{comp}} = H \setminus H^\text{comp}$ be the set of non-competitive histories in $H$. For any $\rho > 0$,

$$\frac{1}{\rho}[\overline{R}(\rho|H) - \overline{R}(0|H)] = s_{\text{comp}} \frac{1}{\rho} [\overline{R}(\rho|H^\text{comp}) - \overline{R}(0|H^\text{comp})] + (1 - s_{\text{comp}}) \frac{1}{\rho} [\overline{R}(\rho|H^{-\text{comp}}) - \overline{R}(0|H^{-\text{comp}})].$$

Summing inequality (6) over histories implies that $\overline{R}(\rho|H^\text{comp}) - \overline{R}(0|H^\text{comp}) \leq 0$. In addition, for all histories $h_{i,t}$, we have that $(1 + \rho)b_{i,t}D_{i}((1 + \rho)b_{i,t}|h_{i,t}) \leq (1 + \rho)b_{i,t}D_{i}(b_{i,t}|h_{i,t})$. Summing up over histories, this implies that $\overline{R}(\rho|H^{-\text{comp}}) \leq (1 + \rho)\overline{R}(0|H^{-\text{comp}})$. Hence, recalling that
reserve price \( r \) is normalized to 1,

\[
\frac{1}{\rho} \left[ \bar{R}(\rho|H^{-\text{comp}}) - \bar{R}(0|H^{-\text{comp}}) \right] \leq \bar{R}(0|H^{-\text{comp}}) \leq r = 1.
\]

Altogether, this implies that \( \frac{1}{\rho} \left[ \bar{R}(\rho|H) - \bar{R}(0|H) \right] \leq 1 - s_{\text{comp}} \), which concludes the proof. 

In words, if average residual revenue for histories \( H \) increases by more than \( \kappa \times \rho \) when bids are multiplied by \((1 + \rho)\), the share of competitive histories in \( H \) is bounded above by \( 1 - \kappa \). In the extreme case where the density of competing bids is zero just above winning bids, we have that \( \bar{R}(\rho|H) - \bar{R}(0|H) \simeq \rho \bar{R}(0|H) \) for \( \rho \) small. This implies that \( s_{\text{comp}} \leq 1 - \bar{R}(0|H) \).

Together, estimator \( \hat{R} \) of \( \bar{R} \) and Proposition 3 let us compute a probabilistic upper-bound to the share of competitive histories. Because we place few restrictions on the environment, Proposition 3 is necessarily conservative. Still, Section 6 shows that this approach yields non-trivial bounds on the share of competitive histories for certain firms.

The next section derives a tighter bound on the share of competitive histories by exploiting a greater set of incentive compatibility constraints.

### 5.3 A Tighter Bound on Competitive Histories

The bound in Proposition 3 exploits upward deviations in bids combined with the restriction that costs are non-negative. A test that only uses a lower bound of zero for costs may be too conservative in many settings. Our tighter bound exploits the informational content of both upward and downward deviations combined with additional restrictions on costs in the form of markup constraints. For expository purposes, we assume private values and address the case of common values in Online Appendix OC.

Take as given an adapted set of histories \( H \). Take also as given scalars \( \rho_n \in (-1, \infty) \) for \( n \in \mathcal{M} = \{ -n, \ldots, n \} \), such that \( \rho_0 = 0 \) and \( \rho_n < \rho_{n'} \) for all \( n' > n \). For each history \( h_{i,t} \in H \) and for each \( n \in \mathcal{M} \), let \( d_{h_{i,t},n} \equiv D_i((1 + \rho_n)b_{h_{i,t}}|h_{i,t}) \). That is, \( d_{h_{i,t},n} \) is
firm $i$’s residual demand at history $h_{i,t}$, when applying a coefficient $1 + \rho_n$ to its original bid. For any history $h \in H$, let $\omega_h = ((d_{h,n})_{n \in M}, c_h)$ be the demand and cost of the firm associated with history $h$. Let $\omega_H = (\omega_h)_{h \in H}$ denote the profile of demands and costs across all histories in $H$. In the following, we consider whether or not there exists a profile $\omega_H$ that rationalizes observed data while simultaneously satisfying the restrictions implied by competitive equilibrium.

For each set of adapted histories $H$, each deviation $n$, and each profile $\omega_H = (\omega_h)_{h \in H}$, let

$$D_n(\omega_H, H) \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} d_{h_{i,t}, n}$$

be the average residual demand when firms’ demands and costs are given by $\omega_H$. Recall that, for each $n \in M$

$$\hat{D}((\rho_n | H) = \frac{1}{|H|} \sum_{h_{i,t} \in H} 1_{(1+\rho_n)b_{h_{i,t}} < \Lambda_{-i,h_{i,t}}}$$

is the sample counterpart of average residual demand. Proposition 1 implies that uniformly over possible demand and cost profiles $\omega_H$, estimate $\hat{D}(\rho_n | H)$ and true demand $D_n(\omega_H, H)$ must be close with probability approaching one.$^{16}$

Recall that at every competitive history, firms must be playing a stage-game best response (Definition 2). Hence, under the private values assumption, at every competitive history $h \in H$, cost $c_h$ and beliefs $d_h = (d_{h,n})_{n \in M}$ must satisfy the following:

$$\text{feasibility: } c_h \geq 0; \quad \forall n, \quad d_{h,n} \in [0, 1]; \quad \forall n, n' > n, \quad d_{h,n} \geq d_{h,n'} \quad \text{(F)}$$

$$\text{incentive compatibility: } \forall n, \quad [(1 + \rho_n)b_h - c_h] d_{h,n} \leq [(1 + \rho_0)b_h - c_h] d_{h,0} \quad \text{(IC)}$$

$^{16}$Proposition OA.1 in Online Appendix OA provides tighter asymptotic confidence intervals. We use these tighter bounds in our empirical implementation.
We allow the analyst or econometrician to include markup constraints of the form
\[
\forall h, \quad \frac{b_h}{c_h} \in [1 + m, 1 + M] \tag{MKP}
\]
where \(m \geq 0\) and \(M \in (m, +\infty]\) are minimum and maximum markups.\(^{17}\) Constraint (MKP) provides what we think is a transparent way for regulators to express minimal subjective beliefs over the environment without making further assumptions of the sort embedded in a Bayesian prior.\(^{18,19}\) We provide a detailed discussion of the impact of constraints (MKP) when we implement our test in Section 6.

For each profile \(\omega_H\) of costs and demand, define
\[
H_{\text{comp}}(\omega_H) \equiv \{h \in H \text{ s.t. } (d_h, c_h) \text{ satisfy (F), (IC) and (MKP)}\},
\]
to be the set of histories in \(H\) that satisfy markup constraint (MKP) and are rationalizable as competitive under \(\omega_H\).

Let \(K > 0\) denote a tolerance margin used to define confidence intervals. We define inference problem (P) and its solution \(\hat{s}\) by
\[
\hat{s} = \max_{\omega_H} \frac{|H_{\text{comp}}(\omega_H)|}{|H|} \tag{P}
\]
s.t. \(\forall n \in \mathcal{M}, \quad D_n(\omega_H, H) \in \left[\hat{D}(\rho_n|H) - K, \hat{D}(\rho_n|H) + K\right]. \tag{CR}
\]

\(^{17}\)In Chassang et al. (2019), we discuss plausibility constraints on the informativeness of signals.

\(^{18}\)A partially sophisticated regulator may express probabilistic constraints without being willing to commit to a full prior. For instance: “90% of the time, margins are greater than 5%”. Our existing framework lets us test whether at least 90% of histories can be deemed competitive when markups are greater than 5%. Alternatively, we could accommodate probabilistic restrictions by specifying that (MKP) must hold for a minimum share of histories.

\(^{19}\)We note that the minimum markup constraint in (MKP) can be microfounded with a model in which bidders incur a small bid preparation cost \(\kappa > 0\) if they participate in an auction. In such a model, bidder \(i\) will only participate at time \(t\) and bid \(b_{i,t}\) if \(D_i(b_{i,t}|h_{i,t})(b_{i,t} - c_{i,t}) \geq \kappa\). Since \(D_i(b_{i,t}|h_{i,t}) \leq 1\) and since \(c_{i,t} \leq r = 1\), if bidder \(i\) chooses to participate, it must be that \(\frac{b_{i,t}}{c_{i,t}} \geq 1 + \frac{\kappa}{c_{i,t}} \geq 1 + \kappa\). Previous papers (e.g., Krasnokutskaya and Seim, 2011) have estimated that bid preparation costs can be substantial, between 2.2 and 3.9% of the engineer’s cost estimate.

24
Program (P) finds demands and costs $\omega_H = ((d_{h,n})_{n \in M}, c_h)_{h \in H}$ that maximize the share of histories in $H$ that can be rationalized as competitive, across all profiles $\omega_H$ that are consistent with the data, in the sense that average residual demands $(D_n(\omega_H, H))_{n \in M}$ are close to their sample counterparts $(\hat{D}(\rho_n|H))_{n \in M}$. Note that our program treats demands and costs $((d_{h,n})_{n \in M}, c_h)$ as nuisance parameters rather than as primitives to be consistently estimated. Given that we allow for general forms of unobserved heterogeneity and information structure, $((d_{h,n})_{n \in M}, c_h)$ at any given history $h$ cannot be identified.

Conditions (F), (IC) and $(\hat{CR})$ exploit some but not all the informational content of equilibrium. We clarify in Appendix OC that we would be exploiting all of the empirical content of equilibrium if we imposed demand consistency requirements $(\hat{CR})$ conditional on all different values of bids and costs $c$ (corresponding to the bidder’s private information at the time of bidding).²⁰

Our next result shows that estimator $\hat{s}$ provides a conservative upper bound to the share of competitive histories in $H$. Recall that $|M|$ denotes the number of deviations in $M$.

**Proposition 4.** Assume costs satisfy (MKP), and denote by $s_{\text{comp}}$ the true share of competitive histories in $H$. With probability at least $1 - 2|M| \exp(-K^2|H|/2N_{\text{max}})$, $\hat{s} \geq s_{\text{comp}}$.

Proposition 4 lets us define conservative tests of non-competitive behavior. For any threshold fraction $s_0 \in (0, 1]$ of competitive histories, define $\tau \equiv 1_{\hat{s} < s_0}$. Test $\tau$ rejects the null that $s_{\text{comp}} \geq s_0$ whenever $\hat{s}$ is strictly lower than $s_0$. By Proposition 4, for any $s_0 \in (0, 1]$, the probability that test $\tau$ rejects the null when the null is true is bounded above by $2|M| \exp(-K^2|H|/2N_{\text{max}})$, permitting conservative inference.²¹ As $|H|$ grows large, test $\tau$ accepts data generated by competing firms with probability 1. By varying the set $H$ of adapted histories, we can apply test $\tau$ to a single firm, to firms in a given industry, etc.

²⁰In practice, this stretches both the limits of our data, and of bidder sophistication. Relying on a weaker set of optimality conditions makes our estimates more robust to partial failures of optimization, consistent with the critique of Fershtman and Pakes (2012).

²¹This testing strategy follows the lines of calibrated projection (Kaido et al., 2019): the coverage of an underlying parameter (here demand) is calibrated to achieve a specific coverage of a function of the parameter (here, the share of competitive histories).
We end this section by noting that upward and downward deviations in Program (P) are complementary from the perspective of inference. For any history \( h \in H \), an upward deviation \( \rho_n > 0 \) is least attractive when cost \( c_h \) is low, while a downward deviation \( \rho_n < 0 \) is least attractive when \( c_h \) is large. As a result, considering upward and downward deviations jointly can lead to tighter bounds on the share of competitive histories than either upward or downward deviations alone. We illustrate this complementarity in the context of our data in Section 6 and establish a formal result in Online Appendix OC.

6 Empirical Evaluation

In this section, we estimate upper bounds on the share of competitive histories for the city-level and national-level auctions described in Section 2. Before presenting our findings, we clarify adjustments made to (P) that reflect limits of our data and computational power.

Rebidding. City-level auctions use public reservation prices, and the results of Section 5 apply directly. National-level auctions use secret reserve prices, and dealing with rebidding requires theoretical adjustments. As we explain in Online Appendix OB, incentive compatibility constraints for bid increases are essentially unchanged. For bid reductions, we need to assess losses in continuation values when the bid reduction changes the lowest bid reported to bidders if the lowest bid is greater than the reserve price.\(^{22}\) We report bounds on the share of competitive histories computed under the assumption that changing the reported minimum bid reduces a bidder’s continuation value by at most 50%.\(^{23}\)

Computation. As is, Problem (P) does not lend itself to computational implementation: a solution \( \omega_H \) is a real vector of dimension \(|H| \times (|\mathcal{M}| + 1)\), and the objective function is not concave. Appendix A shows that (P) admits a tight convex relaxation in which a solution is

\(^{22}\)Recall that the lowest bid in each round of bidding is announced to all of the bidders before the next round of bidding.

\(^{23}\)Note that the reported bid is above the fixed reserve price which bidders must beat to win the auction.
a distribution $\mu$ over demand vectors for different deviations. This relaxation can be stated as an infinite dimensional linear optimization problem whose complexity is independent of sample size $|H|$. Importantly, this infinite dimensional problem can be arbitrarily well approximated by finite dimensional linear problems in which the support of distribution $\mu$ is constrained to a finite sample of points in $\mathbb{R}^{|M|}$.

**Small samples.** Inference problem (P) and Proposition 4 allow for transparent interpretation but are highly conservative. Our empirical implementation makes adjustments to (P) that do not change inference in large samples, and improve power in small samples. First, we use an asymptotic version on the Central Limit Theorem for martingales rather than the non-asymptotic version used in Proposition 1 (see Appendix OA). Second, we impose consistency requirements on selected linear combinations of demand at different deviations, allowing us to replace two-sided consistency requirement ($\hat{C} \hat{R}$) with one-sided constraints.

### 6.1 A Case Study

We first illustrate the mechanics of inference using bidding data from the city of Tsuchiura, located in Ibaraki prefecture. We do so by pooling across histories associated with all auctions from the city prior to October 2009, which constitute an adapted set. We select this city for two reasons: first, Chassang and Ortner (2019) provide evidence that there was collusion in auctions held prior to October 2009; second, the data turns out to be well suited to illustrate the information content of different incentive compatibility conditions.

We consider different combinations of deviations $\rho \in \{-0.02, 0, 0.0008\}$, where by convention, $\rho = 0$ is an infinitesimal downward deviation that amounts to breaking ties. The distribution of $\Delta$ and the deviations we consider (in dashed lines) are illustrated in Figure 4. The deviations are selected to deliver crisp illustrative results. We are specifically interested in illustrating the empirical content of individual deviations, as well as complementarities.
between upward and downward deviations.

**A single upward deviation.** We first consider a single small upward deviation $\rho_1 = 0.0008$. Under inference problem (P), at every history $h_{i,t} = h$ we seek demand $d_{h,0} \geq d_{h,1}$ and a cost $c_h$ such that the following IC constraint is satisfied:

$$d_{h,0} (b_h - c_h) \geq d_{h,1} ((1 + \rho_1) b_h - c_h) \iff d_{h,0} - (1 + \rho_1) d_{h,1} \geq (d_{h,0} - d_{h,1}) \frac{c_h}{b_h}.$$ 

Since $d_{h,0} \geq d_{h,1}$, this is most easily satisfied by setting $c_h/b_h = 1/(1 + M)$. An upward deviation is least profitable (and so the data is best explained) when costs are low. Aggregating across all histories, the IC constraints imply the following:

$$D_0(\omega_H, H) - (1 + \rho_1) D_1(\omega_H, H) \geq \frac{1}{1 + M} (D_0(\omega_H, H) - D_1(\omega_H, H)).$$

Hence, if

$$\hat{D}(0|H) - (1 + \rho_1) \hat{D}(\rho_1|H) < \frac{1}{1 + M} (\hat{D}(0|H) - \hat{D}(\rho_1|H)), \quad (7)$$

28
then for a tolerance $K$ small enough, (IC), (F), and $(\widehat{CR})$ cannot be solved together for all histories $h \in H$. In auctions from Tsuchiura, a small upward deviation does not change a bidder’s demand: $\widehat{D}(0|H) \simeq \widehat{D}(\rho_1|H)$. This implies that (7) holds regardless of the value of $M$, and at least some fraction of histories in $H$ must be deemed non-competitive.

The dotted line in Figure 5 corresponds to our estimate of the upper bound on the share of competitive histories based on Proposition 4 as a function of minimum markup $m$, using single upward deviation $\rho_1 = .0008$. For these estimates and all the estimates we present below, we use Proposition OA.1 (Online Appendix OA) to construct confidence bounds and set tolerance $K$ so that our estimate is an upper bound for the true share of competitive histories with 95% confidence. We set $M = 0.5$. Since bounds based only on upward deviations do not depend on $m$, the dotted line in Figure 5 is constant at 0.86.

![Figure 5: Share of competitive histories, Tsuchiura.](image)

Deviations $\{-0.02, 0, 0.001\}$; maximum markup 0.5.

A single upward deviation and tied bids. We now consider combining the upward deviation with an infinitesimal downward deviation ($\rho \in \{0, .0008\}$). Any mass of tied

\footnote{As Figure OD.7 in the Online Appendix shows, our results are not highly sensitive to the choice of $M$.}
bids is inherently non-competitive since they create a meaningful benefit from reducing bids by the smallest possible amount. We note that tied bids are present in the data, but that their mass is small. Combining the upward deviation with an infinitesimal downward deviation, we estimate a 95% confidence bound on the share of competitive histories to be 0.85. This is illustrated as the horizontal dashed line in Figure 5. As the figure shows, the presence of ties has a very small impact on our estimate of the share of non-competitive histories.

A single downward deviation. We now consider the implications from a single downward deviation $\rho = -0.02$. At every history $h_{i,t} = h$ we seek residual demands $d_{h,-1} \geq d_{h,0}$ and a cost $c_{h}$ such that the following IC constraint is satisfied:

$$d_{h,0}(b_{h} - c_{h}) \geq d_{h,-1}((1 + \rho_{-1})b_{h} - c_{h}) \iff (1 + \rho_{-1})d_{h,-1} - d_{h,0} \leq (d_{h,-1} - d_{h,0})\frac{c_{h}}{b_{h}}.$$

Since $d_{h,-1} \geq d_{h,0}$, this IC condition is most easily satisfied if $c_{h}/b_{h} = 1/(1 + m)$. In that case, constraints (IC), (F), and ($\hat{CR}$) define a convex set and, for tolerance $K$ small enough, can be satisfied for all histories $h \in H$ if and only if

$$(1 + \rho_{-1})\hat{D}(\rho_{-1}|H) - \hat{D}(0|H) = \frac{1}{1 + m} \left[ \hat{D}(\rho_{-1}|H) - \hat{D}(0|H) \right]. \quad (8)$$

Since $\rho_{-1} < 0$, it follows that condition (8) always holds if $m$ is close to zero. This is intuitive: for a sufficiently small margin, say $m < 2\%$, reducing bids by 2% results in losses for each auction ($0.98 \times 1.02 - 1 < 0$). In contrast, if $m > 1/(1 + \rho_{-1}) - 1$ and demand increase $\hat{D}(\rho_{-1}|H) - \hat{D}(0|H)$ is sufficiently large, then (8) does not hold and some share of histories must be considered non competitive.

The solid line in Figure 5 plots our estimate of the 95% confidence bound as a function of $m$. In this data, a 2% drop in prices leads to a 44 percentage-point increase in the probability of winning the auction, from $\hat{D}(0|H) = 22.6\%$ to $\hat{D}(\rho_{-1}|H) = 66.2\%$, almost tripling demand. Hence, as minimum markup $m$ increases from 0, inequality (8) fails, implying that
(IC), (F), and (CR) cannot be solved together for all histories \( h \in H \). Correspondingly, the bound in Figure 5 is equal to 1 for low values of \( m \), and becomes less than 1 as \( m \) increases.\(^\text{25}\)

**Complementary upward and downward deviations.** Conditions (7) and (8) highlight that individual upward and downward deviations are rationalized as competitive by different costs. An upward deviation is least attractive when cost \( c_h \) is low. A downward deviation is least attractive when cost \( c_h \) is large. Hence, upward and downward deviations are complementary from the perspective of inference. The dashed-dotted line in Figure 5 plots our bound on the share of competitive histories using all three deviations (\( \rho \in \{-0.02, 0, 0.0008\} \)). For low values of \( m \), considering both upward and downward deviations leads to a tighter bound for the share of competitive histories than either upward or downward deviations alone (our estimate is 0.39 when \( m = 0.025 \)). The high costs needed to ensure that a downward deviation is not attractive also make upward deviations more attractive.

### 6.2 Findings from aggregate data

We now apply our tests to the full set of auctions in each of our datasets, taking \( H \) as the set of histories corresponding to all municipal or national auctions. Clearly, \( H \) is adapted. Going forward, when applying the results of Section 5.3, we set \( M = 0.5 \) and use the fixed set of deviations \( \{-0.02, 0, 0.001\} \) for all datasets. Deviation \( \rho = 0 \) is an infinitesimal downward deviation that breaks ties. Using a fixed set of deviations for all datasets is likely suboptimal for statistical power, but limits the scope for selection bias.\(^\text{26}\)

Figure 6 shows our estimates of the 95\% confidence bound on the share of competitive histories as a function of minimum markup \( m \), for city and national auctions. The dotted line corresponds to the estimated bound when we set \( \rho \in \{0.001, 0\} \). The dashed and the solid

\(^{25}\)The markups we consider contain prior estimates in the literature. Krasnokutskaya (2011) estimates markups ranging from 0.1\% to 24\% in Michigan highway procurement auctions. Bajari et al. (2014) estimate markups from 2.9\% to 26.1\% in California highway procurement auctions.

\(^{26}\)The magnitude of deviations was set using residual demand plotted in Figures OD.1(a) and OD.1(b) to achieve large gains in demand for downward deviations, and small drops in demand for upward deviations.
lines correspond to estimates from $\rho \in \{-0.02, 0\}$ and $\rho \in \{-0.02, 0, 0.001\}$, respectively. In the case of national auctions we use penalized incentive compatibility conditions accounting for rebidding detailed in Appendix OB. We note that upward deviations alone allow us to detect only a very small number of non-competitive histories in city-level auctions ($\hat{s} = 0.99$), and none in national-level auctions. One explanation for this is that looking at the full set of auctions causes us to mix competitive and non-competitive histories, thereby weakening our ability to detect non-competitive behavior. Correspondingly, the bound from Proposition 3 does not have much bite when $H$ is taken to be the entire sample of auctions. Note that this does not mean that upward deviations are uninformative: especially in the case of national data, considering both upward and downward deviations can yield significantly tighter bounds on the share of competitive histories than either deviation alone.

![Figure 6: Share of competitive histories, city and national level data.](image)

(a) city data  
(b) national data

Figure 6: Share of competitive histories, city and national level data.

Deviations $\{-0.02, 0, 0.001\}$; maximum markup 0.5.

---

27 Online Appendix OD shows that our estimates are not very sensitive to the value of maximum markup $M$, or assumptions about continuation payoffs upon re-bidding.

28 Even though the missing mass at $\Delta = 0$ is very noticeable in Figure 1, the density at $\Delta = 0$ is not quite low enough at the aggregate level for upward deviations alone to have statistical power.

29 When $m$ is small, downward deviations alone place no restrictions on the share of competitive histories.
6.3 Zeroing-in on specific firms

We now consider applying our tests to individual firms. As we highlight in Ortner et al. (2020), detecting non-competitive behavior at the firm level helps reduce the potential side-effects of regulatory oversight. Specifically, it ensures that a cartel cannot use the threat of regulatory crackdown to discipline bidders.

For both city and national samples, we consider the thirty firms that participate in the most auctions in each dataset. For each firm, we estimate a bound on the share of competitive histories taking $H$ to be the set of histories corresponding to all auctions in which the firm participates (set $H$ is clearly adapted to the firm’s information).

Panel (a) of Table 2 reports the results for firms active in the city sample. We order firms according to the number of auctions in which they participate. We report this number in Column 2. Column 3 reports the share of auctions each firm wins as a fraction of the number of auctions in which the firm participates. Column 4 shows estimates obtained using the bound from Proposition 3 in which we consider a single deviation with $\rho = 0.001$ instead of all possible upward deviations $\rho > 0$.

We use the sample revenue, $\hat{R}(\rho|H)$, to estimate $R(\rho|H)$. Column 5 shows estimates obtained using the bound from Proposition 4 using deviations $\{-0.02, 0, 0.001\}$, minimum markup $m = 0.025$ and maximum markup $M = 0.5$. Panel (b) of Table 2 reports the corresponding results for firms in the national sample. The bound from Proposition 3 is less than 1 for seventeen out of thirty firms in the city sample and six out of thirty firms in the national sample. The bound from Proposition 4 is less than 1 for nineteen firms for the city sample and twenty-four for the national sample.\[^{31}\]

\[^{30}\]Our inference also takes the set of deviations as fixed.\[^{31}\]We note that, for some specific firms, the bound we obtain from Proposition 3 is tighter than the bound we obtain from Proposition 4. As we explain in Appendix A, in finite samples, adding more deviations does not necessarily lead to tighter bounds on the share of competitive histories. The same margin of error (here 5%) is allocated across a greater number of estimates of demand.
<table>
<thead>
<tr>
<th>Rank</th>
<th>Participation</th>
<th>Share won</th>
<th>Share comp (Prop 3)</th>
<th>Share comp (Prop 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>347</td>
<td>0.19</td>
<td>0.86</td>
<td>0.88</td>
</tr>
<tr>
<td>2</td>
<td>336</td>
<td>0.21</td>
<td>0.84</td>
<td>0.86</td>
</tr>
<tr>
<td>3</td>
<td>299</td>
<td>0.08</td>
<td>1.00</td>
<td>0.98</td>
</tr>
<tr>
<td>4</td>
<td>293</td>
<td>0.05</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>5</td>
<td>290</td>
<td>0.14</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>6</td>
<td>287</td>
<td>0.20</td>
<td><strong>0.91</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>7</td>
<td>269</td>
<td>0.14</td>
<td>1.00</td>
<td><strong>0.93</strong></td>
</tr>
<tr>
<td>8</td>
<td>268</td>
<td>0.09</td>
<td>1.00</td>
<td><strong>0.97</strong></td>
</tr>
<tr>
<td>9</td>
<td>262</td>
<td>0.12</td>
<td><strong>0.87</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>10</td>
<td>259</td>
<td>0.18</td>
<td>1.00</td>
<td><strong>0.89</strong></td>
</tr>
<tr>
<td>11</td>
<td>252</td>
<td>0.12</td>
<td>1.00</td>
<td><strong>0.96</strong></td>
</tr>
<tr>
<td>12</td>
<td>241</td>
<td>0.12</td>
<td><strong>0.89</strong></td>
<td><strong>0.95</strong></td>
</tr>
<tr>
<td>13</td>
<td>239</td>
<td>0.16</td>
<td>1.00</td>
<td><strong>0.92</strong></td>
</tr>
<tr>
<td>14</td>
<td>238</td>
<td>0.09</td>
<td>1.00</td>
<td><strong>0.98</strong></td>
</tr>
<tr>
<td>15</td>
<td>227</td>
<td>0.11</td>
<td><strong>0.93</strong></td>
<td><strong>0.97</strong></td>
</tr>
<tr>
<td>16</td>
<td>226</td>
<td>0.12</td>
<td><strong>0.96</strong></td>
<td><strong>0.99</strong></td>
</tr>
<tr>
<td>17</td>
<td>225</td>
<td>0.08</td>
<td><strong>0.92</strong></td>
<td><strong>0.96</strong></td>
</tr>
<tr>
<td>18</td>
<td>223</td>
<td>0.12</td>
<td><strong>0.96</strong></td>
<td><strong>0.97</strong></td>
</tr>
<tr>
<td>19</td>
<td>220</td>
<td>0.07</td>
<td><strong>0.96</strong></td>
<td><strong>0.99</strong></td>
</tr>
<tr>
<td>20</td>
<td>218</td>
<td>0.08</td>
<td><strong>0.91</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>21</td>
<td>211</td>
<td>0.07</td>
<td><strong>0.89</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>22</td>
<td>210</td>
<td>0.14</td>
<td>1.00</td>
<td><strong>0.95</strong></td>
</tr>
<tr>
<td>23</td>
<td>209</td>
<td>0.17</td>
<td><strong>0.94</strong></td>
<td><strong>0.92</strong></td>
</tr>
<tr>
<td>24</td>
<td>204</td>
<td>0.15</td>
<td>1.00</td>
<td><strong>0.98</strong></td>
</tr>
<tr>
<td>25</td>
<td>203</td>
<td>0.11</td>
<td>1.00</td>
<td><strong>0.98</strong></td>
</tr>
<tr>
<td>26</td>
<td>199</td>
<td>0.06</td>
<td><strong>0.89</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>27</td>
<td>190</td>
<td>0.12</td>
<td>1.00</td>
<td><strong>0.96</strong></td>
</tr>
<tr>
<td>28</td>
<td>189</td>
<td>0.06</td>
<td><strong>0.96</strong></td>
<td>1.00</td>
</tr>
<tr>
<td>29</td>
<td>188</td>
<td>0.16</td>
<td><strong>0.86</strong></td>
<td><strong>0.93</strong></td>
</tr>
<tr>
<td>30</td>
<td>187</td>
<td>0.08</td>
<td><strong>0.86</strong></td>
<td>1.00</td>
</tr>
</tbody>
</table>

(a) City Data

<table>
<thead>
<tr>
<th>Rank</th>
<th>Participation</th>
<th>Share won</th>
<th>Share comp (Prop 3)</th>
<th>Share comp (Prop 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4,044</td>
<td>0.17</td>
<td>1.00</td>
<td><strong>0.84</strong></td>
</tr>
<tr>
<td>2</td>
<td>3,854</td>
<td>0.07</td>
<td>1.00</td>
<td><strong>0.91</strong></td>
</tr>
<tr>
<td>3</td>
<td>3,621</td>
<td>0.12</td>
<td>1.00</td>
<td><strong>0.84</strong></td>
</tr>
<tr>
<td>4</td>
<td>2,998</td>
<td>0.15</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>5</td>
<td>2,919</td>
<td>0.06</td>
<td><strong>0.93</strong></td>
<td><strong>0.92</strong></td>
</tr>
<tr>
<td>6</td>
<td>2,547</td>
<td>0.08</td>
<td>1.00</td>
<td><strong>0.71</strong></td>
</tr>
<tr>
<td>7</td>
<td>2,338</td>
<td>0.07</td>
<td>1.00</td>
<td><strong>0.71</strong></td>
</tr>
<tr>
<td>8</td>
<td>2,333</td>
<td>0.07</td>
<td>1.00</td>
<td><strong>0.71</strong></td>
</tr>
<tr>
<td>9</td>
<td>2,328</td>
<td>0.04</td>
<td>1.00</td>
<td><strong>0.93</strong></td>
</tr>
<tr>
<td>10</td>
<td>2,292</td>
<td>0.06</td>
<td>1.00</td>
<td><strong>0.72</strong></td>
</tr>
<tr>
<td>11</td>
<td>2,237</td>
<td>0.08</td>
<td><strong>0.93</strong></td>
<td><strong>0.90</strong></td>
</tr>
<tr>
<td>12</td>
<td>2,211</td>
<td>0.03</td>
<td>1.00</td>
<td><strong>0.95</strong></td>
</tr>
<tr>
<td>13</td>
<td>2,015</td>
<td>0.09</td>
<td>1.00</td>
<td><strong>0.73</strong></td>
</tr>
<tr>
<td>14</td>
<td>1,984</td>
<td>0.08</td>
<td>1.00</td>
<td><strong>0.72</strong></td>
</tr>
<tr>
<td>15</td>
<td>1,727</td>
<td>0.07</td>
<td>1.00</td>
<td><strong>0.80</strong></td>
</tr>
<tr>
<td>16</td>
<td>1,674</td>
<td>0.05</td>
<td>1.00</td>
<td><strong>0.80</strong></td>
</tr>
<tr>
<td>17</td>
<td>1,661</td>
<td>0.03</td>
<td>1.00</td>
<td><strong>0.94</strong></td>
</tr>
<tr>
<td>18</td>
<td>1,660</td>
<td>0.08</td>
<td>1.00</td>
<td><strong>0.71</strong></td>
</tr>
<tr>
<td>19</td>
<td>1,589</td>
<td>0.07</td>
<td>1.00</td>
<td><strong>0.75</strong></td>
</tr>
<tr>
<td>20</td>
<td>1,427</td>
<td>0.10</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>21</td>
<td>1,393</td>
<td>0.06</td>
<td>1.00</td>
<td><strong>0.81</strong></td>
</tr>
<tr>
<td>22</td>
<td>1,392</td>
<td>0.07</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>23</td>
<td>1,370</td>
<td>0.04</td>
<td><strong>0.96</strong></td>
<td><strong>0.87</strong></td>
</tr>
<tr>
<td>24</td>
<td>1,368</td>
<td>0.14</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>25</td>
<td>1,353</td>
<td>0.05</td>
<td>1.00</td>
<td><strong>0.75</strong></td>
</tr>
<tr>
<td>26</td>
<td>1,342</td>
<td>0.09</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>27</td>
<td>1,337</td>
<td>0.04</td>
<td><strong>0.96</strong></td>
<td><strong>0.82</strong></td>
</tr>
<tr>
<td>28</td>
<td>1,326</td>
<td>0.08</td>
<td>1.00</td>
<td><strong>0.89</strong></td>
</tr>
<tr>
<td>29</td>
<td>1,291</td>
<td>0.06</td>
<td><strong>0.95</strong></td>
<td><strong>0.81</strong></td>
</tr>
<tr>
<td>30</td>
<td>1,260</td>
<td>0.06</td>
<td><strong>0.95</strong></td>
<td><strong>0.90</strong></td>
</tr>
</tbody>
</table>

(b) National Data

95% confidence bound on the share of competitive auctions for top thirty most active firms. The first column corresponds to the ranking of the firms and the second column corresponds to the number of auctions in which each firm participates. Column 3 shows the fraction of auctions that each of these firms wins. Columns 4 and 5 present our 95% confidence bound on the share of competitive histories for each firm based on Proposition 3 and Proposition 4, respectively. For our estimates of column 5, we use deviations \{-0.02, 0, 0.001\}, minimum markup \(m = 0.025\) and maximum markup \(M = 0.5\).

Table 2: Share of competitive histories, individual firms
6.4 Consistency with proxies for collusion

We now show that our bounds on the share of competitive histories are consistent with proxies of collusive behavior.

**Before and after prosecution.** As we noted in Section 2, the JFTC investigated firms bidding in four groups of national auctions during the period for which we have data: auctions labeled Bridges, Electric, Floods, and Pre-Stressed Concrete. We now compute bounds on the set of competitive histories before and after investigation. We exclude the Bridge category because there are too few observations in the post-investigation sample to obtain a reliable confidence set (58 bids, vs more than 560 in the other industries).

Figure 7 shows our estimates of the 95% confidence bound on the share of competitive histories as a function of \( m \), for the three remaining groups of firms. The estimates are based on Proposition 4 with deviations \( \{-0.02, 0, .001\} \) and \( M = .5 \). For all three industries, we find that the share of competitive histories is higher after the investigation than before, consistent with the interpretation that collusion was more rampant before the investigation and less severe afterwards.

In Online Appendix OD.2 we perform sensitivity analysis of the share of competitive histories in these three industries to changes in (a) the confidence level used for inference, and (b) the set of deviations considered. We show that, for less conservative choices of tolerance parameter \( K \), our estimates on the share of competitive histories for Pre-Stressed Concrete are less than 1 in the after period, consistent with Figure 3.

**High vs. low bids.** We now compare the estimates that we obtain when focusing on histories with low bids relative to high bids. Because collusion typically elevates prices, we expect the share of competitive histories to be higher for histories with low bids and vice versa. Specifically, we divide the city-level data into histories with high bids relative to the reserve price (i.e., \( \frac{b}{r} \geq 0.9 \)) and those with low bids (i.e., \( \frac{b}{r} < 0.9 \)). Since the reserve price is
known to bidders in the city auctions, these two sets of histories are adapted.

Figure 8 plots our estimates of the bound on the share of competitive histories for histories with high bids (solid line) and low bids (dotted line) for the city auctions. As before, the estimates are based on Proposition 4 with deviations $\{-0.02, 0, 0.001\}$ and $M = .5$. As the figure shows, the fraction of competitive histories is lower for histories with high bids. This is consistent with the fact that collusion increases bids.

Figure 7: Share of competitive histories, before and after JFTC investigation.

Deviations $\{-0.02, 0, 0.001\}$; maximum markup 0.5.
Figure 8: Share of competitive histories by bid level, city data.

Deviations \{-0.02, 0.001\}; maximum markup 0.5.

7 Discussion

This paper develops tests of competitive bidding valid under minimal assumptions. In addition to the motivating observation that winning bids are isolated, we identify another suspicious pattern in the data: a 2% reduction in bids causes a large increase in demand.

Our tests are conservative: they can be passed by any firm that is bidding competitively under some information structure. Such caution is justified by the high direct and collateral costs of launching a formal investigation against non-collusive firms (Imhof et al., 2018). In a companion paper, (Ortner et al., 2020), we identify another important property of such tests: antitrust investigation based on tests that are robust to the information structure does not generate new collusive equilibria. This addresses the concern that data-driven regulation may inadvertently enhance a cartel’s ability to collude (Cyrenne, 1999, Harrington, 2004). For these two reasons, we believe that our tests provide a sensible starting point for data-driven antitrust in public procurement.

The Online Appendices collect important extensions of our baseline framework: how to
deal with secret reserve prices and re-bidding; how to deal with common values; and how to construct money denominated metrics of non-competitive behavior.

We conclude with a discussion of practical aspects of our tests: (i) firms’ responses to antitrust oversight, (ii) the relation between the rejection of the test and collusion, and (iii) non-collusive explanations for missing bids.

**What if firms adapt?** Non-competitive bidders may adapt to the screens used by antitrust authorities, reducing their efficacy. We show in a companion paper (Ortner et al., 2020) that antitrust oversight based on tests that are robust to the information structure always reduces the set of enforceable collusive schemes available to cartels. That is, screens based on robust tests always make cartels worse off, even if firms know they are being monitored and adapt their play accordingly.

Moreover, we note that simple adaptive responses to our tests may themselves lead to suspicious patterns. For instance, collusive firms may adapt their play to our tests by “filling the gap” in the distribution of $\Delta$ to avoid generating the suspicious patterns in Figure 1. While such adaptive response would reduce the profitability of upward deviations by winning bidders, it would also make downward deviations by losing bidders more profitable, potentially also leading to patterns that would fail our tests.\(^{32}\)

**Rejection of the test and collusion.** Our tests, which are aimed at detecting failures of competitive behavior, cannot distinguish among the various reasons why a given dataset may be inconsistent with competition. Failure to pass our test does not necessarily imply bidder collusion.\(^{33}\) However, findings from Section 6 show that rejection of our tests is in fact correlated with different markers of collusion. Indeed, in industries that were investigated for bid rigging, the fraction of competitive histories is lower before the investigation than

\(^{32}\)In our companion paper Kawai et al. (2020) we propose screens of non-competitive behavior that have statistical power precisely when the distribution of bid differences $\Delta$ has sufficient mass around $\Delta = 0$.

\(^{33}\)In addition, our tests cannot distinguish between tacit and explicit collusion.
after. The fraction of competitive histories is lower at histories at which bids are high relative to the reserve price. Altogether, this suggests that our test are sufficiently powered to flag cartels in practice.

**Non-collusive explanations.** It is instructive to evaluate potential non-collusive explanations for the bidding patterns in our data. The first is that bidders are committing errors, say playing an $\epsilon$-equilibrium of the game. This explanation is not entirely satisfactory for two reasons. First, the potential gains from downward deviations are not small, and bidders have many opportunities to learn. Second, natural models of erroneous play do not generate the patterns we see in the data. For instance, by adding noise on bids, quantal response equilibrium (McKelvey and Palfrey, 1995) would smooth out rather than enhance the pattern of missing bids.

Another possible direction to explain the anomalous bidding patterns we note in the data is to take into account dynamic payoff consequences of winning an auction, through either capacity constraints, or learning by doing. A rapid analysis suggests that such dynamic considerations are unlikely to explain the data. Capacity constraints essentially correspond to an increase in the bidder’s cost reflecting reduced continuation values. This increases the attractiveness of upward deviations. Similarly, learning-by-doing reduces the cost of accepting a project. This increases the attractiveness of downward deviations.

**Appendix**

**A Empirical Implementation**

This appendix clarifies how we compute estimates of competitiveness derived in Section 5.
A.1 Bounds for Section 5.1

Column 3 of Table 2 reports 95% confidence upper bounds on the share of competitive histories for individual firms using Proposition 3:

$$s_{\text{comp}} \leq 1 - \sup_{\rho > 0} \frac{\overline{R}(\rho|H) - \overline{R}(0|H)}{\rho}.$$  

Instead of considering all deviations $\rho > 0$, we take a single deviation with $\rho = .1\%$ and fix this in our asymptotics. We estimate the residual revenue using its empirical analog.\(^{34}\) We use Proposition OA.1 in Online Appendix OA to obtain a 95% confidence bound.

A.2 Bounds for Section 5.3

First, when estimating demand (i.e., equations (1) and (2)), we re-weight histories in order to minimize standard errors (see Online Appendix OA for details). Second, instead of using a consistency requirement ($\widehat{CR}$) for aggregate demand taking the form

$$\forall n \in \mathcal{M}, \ D_n(\omega_{H}, H) \in [\widehat{D}(\rho_n|H) - K, \widehat{D}(\rho_n|H) + K],$$

we impose linear constraints of the form

$$\langle Q, (\widehat{D}(\rho_n|H) - D_n(\omega_{H}, H))_{n \in \mathcal{M}} \rangle \leq X$$  \hspace{1cm} (9)

where $Q$ is a $K \times |\mathcal{M}|$ real matrix, and $X$ is a $K \times 1$ vector of thresholds such that (9) holds with sufficiently high probability under an appropriate concentration result (i.e. Proposition 1 or OA.1). This allows us to focus on binding constraints.

Third, we note that costs ($c_h$) do not affect either the objective function in (P) or consistency requirement ($\widehat{CR}$). This means that it is sufficient to know demands ($d_{h,n}$) to determine whether (IC), (F), (MKP) hold. This means that it is sufficient to look for solutions of the form $\omega_{H} = (d_{h,n})_{h \in H, n \in \mathcal{M}}$.

A solution $\omega_H$ of the adjusted program (P) is now a real vector of dimension $|H| \times |\mathcal{M}|$. Because the dimension of $\omega_H$ grows with sample size and the objective function is not concave we study a convex relaxation of (P). Observe that $|H_{\text{comp}}| = \mathbb{E}_{\mu_{\omega_H}}[\nu(\omega_h)]$ where $\mu_{\omega_H} \in \Delta([0,1]^{|\mathcal{M}|})$ is the sample distribution over demand profiles $\omega_h = (d_{h,n})_{n \in \mathcal{M}}$ induced by a candidate solution $\omega_H$, and $\nu(\omega_h) = 1_{\omega_h \text{ satisfies (IC),(F),(MKP)}}$. In addition, $D(\omega_H|H) = \ldots$  

\(^{34}\)We also estimated a smoothed revenue using a kernel CDF estimate. The results are similar.
\( \mathbb{E}_{\mu_H} d_{h,n} \). Hence, it follows that

\[
\hat{s} \leq \max_{\mu \in \Delta([0,1]^{|M|})} \mathbb{E}_{\mu} v(\omega_h)
\]

under the constraint that

\[
\left\langle Q, (\hat{D}(\rho_n|H) - \mathbb{E}_{\mu} d_{h,n})_{n \in M} \right\rangle \leq X.
\]

The convexified right-hand side problem is linear, and its complexity is no longer related to sample size \(|H|\). While \(\Delta([0,1]^{|M|})\) is infinite dimensional, the fact that \([0,1]^{|M|}\) is compact and finite dimensional means that it is covered by finitely many balls of radius \(r\) for any \(r > 0\). Hence for all \(\epsilon > 0\), there exists a finite set \(O \subset [0,1]^{|M|}\) such that

\[
\hat{s} - \epsilon \leq \max_{\mu \in \Delta(O)} \mathbb{E}_{\mu} v(\omega_h) \tag{CVX-P}
\]

under the constraint that

\[
\left\langle Q, (\hat{D}(\rho_n|H) - \mathbb{E}_{\mu} d_{h,n})_{n \in M} \right\rangle \leq X.
\]

The right-hand-side problem is now a well behaved finite dimensional linear problem.

In practice, for deviations \([-0.02, 0.001]\), we set \(Q = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}\) and use the following parallelized algorithm.

1. Draw 100 samples of 1000 tuples in \([0,1]^3\) using a seeded uniform distribution and sort each tuple in decreasing order.

2. For each sample \(j \in \{1, \ldots, 100\}\) of 1000 points (generating a set \(O_j\)) compute the solution \(\mu_j^*\) to the associated (CVX-P) problem. Let \(O_j\) denote the support of \(\mu_j^*\), truncated to cover 99\% of the mass under \(\mu_j^*\).

3. Set \(O = \bigcup_{j \in \{1, \ldots, 100\}} O_j\), and solve the associated (CVX-P) problem.

4. Assess convergence by comparing solution to that obtained starting from different random seeds.
B Proofs

Proof of Proposition 1. Let \( H \) be an adapted set of histories, and fix \( \rho \in (-1, \infty) \). Assume for simplicity that one auction happens at each time \( t \in \{0, 1, ..., T\} \). Bidding outcomes are revealed in real time. For each time \( t \), define

\[
\varepsilon_t \equiv \sum_{h_{i,t} \in H} \mathbb{E}_\sigma [1_{\sigma_{b_{i,t}}>b_{i,t}(1+\rho)}|h_{i,t}] - 1_{\sigma_{b_{i,t}}>b_{i,t}(1+\rho)}.
\]

Note that \( \hat{D}(\rho|H) - D(\rho|H) = \frac{1}{|H|} \sum_{t=0}^T \varepsilon_t \). Note further that, by the law of iterated expectations, for all public histories \( h_{0:t-s} \in H \) with \( s \geq 0 \), \( \mathbb{E}_\sigma [\varepsilon_t|h_{0:t-s}] = 0 \). Hence, \( S_T \equiv \sum_{t=0}^T \varepsilon_t \) is a Martingale with respect to public histories \( (h_{0:t})_{t \geq 0} \). Since the absolute value of its increments \( \varepsilon_t \) is bounded above by \( N_t \), the number of bidders participating at time \( t \) with histories in \( H \), the Azuma-Hoeffding Inequality implies that for every \( \nu > 0 \),

\[
\text{prob}(|S_T| \geq \nu|H|) \leq 2 \exp \left( -\frac{\nu^2|H|^2}{2 \sum_{t=0}^T N_t^2} \right).
\]

Observing that \( \sum_{t=0}^T N_t^2 \leq \sum_{t=0}^T N_t N_{\max} = N_{\max}|H| \), this implies that

\[
\text{prob}(|S_T| \geq \nu|H|) \leq 2 \exp \left( -\frac{\nu^2|H|}{2 N_{\max}} \right).
\]

This concludes the proof. ■

Proof of Proposition 4. By Proposition 1, we have that

\[
\text{prob} \left( |\hat{D}(\rho_n|H) - D_n(\omega_H, H)| \geq K \right) \leq 2 \exp(-K^2|H|/2N_{\max})
\]

for each deviation \( n \), with \( \omega_H \) denoting the true environment. The Boole-Frechet inequality implies that

\[
\text{prob} \left( \exists n \in \mathcal{M} \text{ s.t. } |\hat{D}(\rho_n|H) - D_n(\omega_H, H)| \geq K \right) \leq 2|\mathcal{M}| \exp(-K^2|H|/2N_{\max}).
\]

This implies that, with probability at least \( 1 - 2|\mathcal{M}| \exp(-K^2|H|/2N_{\max}) \), the constraints in Program (P) are satisfied when we set the environment equal to the true environment \( \omega_H \). Hence, with probability at least \( 1 - 2|\mathcal{M}| \exp(-K^2|H|/2N_{\max}) \), \( \hat{s} \) is weakly larger than the share of competitive histories \( s_{\text{comp}} \) under the true environment \( \omega_H \). ■

42
References


Online Appendix – Not for Publication

OA Asymptotics of Sample Residual Demand

In this appendix, we establish asymptotic analogues of Proposition 1 providing tighter controls over subjective beliefs. We also emphasize one-sided linear inequalities.

Take as given an adapted set of histories \( H \) corresponding to auctions taking place at times \( \{0, \cdots, T\} \). Consider deviations \( \rho \in \mathcal{R} \equiv \{\rho_n, \ n \in \mathcal{M}\} \). For each time \( t \), we denote by \( \{h_{i,t} \in H\} \) the set of individual histories corresponding to bidders \( i \) active at time \( t \). We denote by \( |\{h_{i,t} \in H\}| \) the number of such histories. Instead of using equations (1) and (2), for each \( \rho \in \mathcal{R} \) we now define average residual demand \( \overline{D}(\rho) \) and its sample analog \( \hat{D}(\rho) \) as follows:

\[
\overline{D}(\rho) \equiv \frac{1}{T+1} \sum_{t=0}^{T} \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} D_i((1 + \rho)b_{i,t}|h_{i,t}), \tag{O1}
\]

\[
\hat{D}(\rho) \equiv \frac{1}{T+1} \sum_{t=0}^{T} \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} 1_{b_{i,t} > (1+\rho)b_{i,t}}. \tag{O2}
\]

The weighting scheme in (O1) and (O2) reduces the standard error of estimates. It reflects the fact that expectation errors across time are uncorrelated since beliefs over time satisfy the martingale property. In contrast, expectation errors across players in the same time period may be correlated. As a result, convergence between true and sample residual demand occurs at rate \( \sqrt{1/T} \) rather than \( \sqrt{1/|H|} \).

Let \( \overline{D} \equiv (\overline{D}(\rho))_{\rho \in \mathcal{R}} \) and \( \hat{D} \equiv (\hat{D}(\rho))_{\rho \in \mathcal{R}}, \) where \( \hat{D} \) and \( \overline{D} \) are defined in (O1) and (O2). For any vector \( \lambda \in \mathbb{R}^{|\mathcal{R}|} \), we are interested in bounding the tails of

\[
\langle \lambda, \overline{D} - \hat{D} \rangle.
\]

Define \( 1_{h_{i,t}} \equiv (1_{b_{i,t} \leq (1+\rho)b_{i,t}})_{\rho \in \mathcal{R}}, \) and \( d_{h_{i,t}} \equiv (d_{h_{i,t},n})_{n \in \mathcal{M}}. \) We assume throughout that

\[
\lim_{T \to +\infty} \sum_{t=0}^{T} \text{var} \left( \langle \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \rangle | h^0_t \right) = +\infty.
\]

The Central Limit Theorem for renormalized martingale increments will imply that

\[
\langle \lambda, \overline{D} - \hat{D} \rangle
\]
is asymptotically Gaussian with a variance proportional to

\[
\sum_{t=0}^{T} \text{var} \left( \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \langle \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \rangle | h_{i,t}^{(0)} \right) \]

However, we cannot define a computable consistent sample estimator of this variance since it depends on the unobserved expected demand \(d_{h_{i,t}}\). Instead, we define a computable upper-bound to this variance exploiting the fact that belief \(d_{h_{i,t}}\) minimizes the conditional mean-squared expectation error among predictors measurable with respect to the players’ information. Specifically, we replace \(d_{h_{i,t}}\) with progressive sample means:

\[
\hat{D}_t(\rho) \equiv \frac{1}{T+1} \sum_{t=0}^{T} \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} 1_{\lambda_{b_{-i,s}} > (1+\rho) b_{i,s}}.
\]

Let \(\hat{D}_t \equiv (\hat{D}_t(\rho))_{\rho \in \mathbb{R}}\). Note that it is adapted to the information available to bidders at time \(t\) before bidding. Given \(\lambda\), define

\[
\hat{\sigma}_\lambda \equiv \sqrt{\frac{1}{T+1} \sum_{t=0}^{T} \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \langle \lambda, 1_{h_{i,t}} - \hat{D}_t \rangle^2}.
\]

Note that \(\hat{\sigma}_\lambda\) is computable from data observed by the econometrician.

The following asymptotic bound holds.

**Proposition OA.1.** Take \(x > 0\). As \(T\) becomes arbitrarily large,

\[
\limsup \text{prob} \left( \frac{\sqrt{T+1}}{\hat{\sigma}_\lambda} \langle \lambda, \hat{D} - \hat{D} \rangle \geq x \right) \leq 1 - \Phi(x)
\]

where \(\Phi\) is the c.d.f. of the standard normal \(N(0,1)\).

Note that bounds on lower tails follow from applying Proposition OA.1 to direction \(-\lambda\).

**Proof.** Let

\[
S_T \equiv (T+1) \langle \lambda, \hat{D} - \hat{D} \rangle = \sum_{t=0}^{T} X_t
\]

with

\[
X_t = \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \langle \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \rangle.
\]
The fact that beliefs are in equilibrium implies that $\mathbb{E}[X_t|h^0_t] = 0$: $X_t$ is a martingale increment from the perspective of filtration $(h^t_t)_{t \geq 0}$. Since $X_t$ is uniformly bounded by $||\lambda||_{\text{sup}}$, the Central Limit Theorem for martingale increments holds (Billingsley (1995), Theorem 35.11). This implies that,

$$\lim_{T \to \infty} \text{prob} \left( \frac{\sqrt{T+1}}{\sigma_\lambda} \left< \lambda, D_t - \hat{D}_t \right> \geq x \right) = 1 - \Phi(x)$$

with

$$\sigma_\lambda \equiv \sqrt{\frac{1}{T+1} \sum_{t=0}^{T} \text{var} \left( \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \left< \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \right| h^0_t \right) \right)}$$

We cannot directly exploit this result to get explicit bounds on the distribution of $\left< \lambda, D_t - \hat{D}_t \right>$ because $d_{h_{i,t}}$ is not directly observed, so that we can’t form a consistent estimate of $\sigma_\lambda$. Instead we use a consistent estimator $\hat{\sigma}_\lambda$ of an upper bound to $\sigma_\lambda$.

Jensen’s inequality implies that

$$\text{var} \left( \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \left< \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \right| h^0_t \right) \leq \frac{1}{|\{h_{i,t} \in H\}|} \sum_{h_{i,t} \in H} \text{var} \left( \left< \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \right| h^0_t \right) \right).$$

Furthermore, since $d_{h_{i,t}} = \mathbb{E}[1_{h_{i,t}}|h^0_t]$, and since $h_{i,t}$ includes the information provided in history $h^0_t$, it follows that for any $h^0_t$-measurable random variable $Z_t \in \mathbb{R}^{|H|}$,

$$\text{var} \left( \left< \lambda, 1_{h_{i,t}} - d_{h_{i,t}} \right| h^0_t \right) \leq \mathbb{E} \left( \left< \lambda, 1_{h_{i,t}} - Z_t \right| h^0_t \right)^2.$$

The Law of Large Number for martingale increments implies that almost surely,

$$\lim_{T \to \infty} \frac{1}{T+1} \sum_{t=0}^{T} \mathbb{E} \left[ \left< \lambda, 1_{h_{i,t}} - Z_t \right| h^0_t \right)^2 - \left< \lambda, 1_{h_{i,t}} - Z_t \right| h^0_t \right)^2 = 0.$$

Hence, for any $\epsilon > 0$, almost surely as $T$ becomes large, $(1 + \epsilon)\hat{\sigma}_\lambda \geq \sigma_\lambda$. Since $x > 0$, setting $Z_t = \hat{D}_t$ this implies that

$$\limsup \text{prob} \left( \frac{\sqrt{T+1}}{\sigma_\lambda} \left< \lambda, D_t - \hat{D}_t \right> \geq x \right) \leq \limsup \text{prob} \left( \frac{\sqrt{T+1}}{\sigma_\lambda} \left< \lambda, D_t - \hat{D}_t \right> \geq x(1 + \epsilon) \right) = 1 - \Phi(x(1 + \epsilon))$$
Proposition OA.1 follows from the fact that the result holds for any $\epsilon > 0$, and $\Phi$ is continuous.

The Boole-Fréchet inequality implies the following multidimensional corollary. Let $Q$ denote a $K \times |\mathcal{M}|$ matrix of the form $Q = (\frac{1}{\sigma_{k}}\lambda_{k})_{k \in \{1, \ldots, K\}}$ with each $\lambda_{k}$ a $1 \times |\mathcal{M}|$ vector. Let $X = (x_{k})_{k \in \{1, \ldots, K\}}$ denote a $K \times 1$ vector of positive real numbers.

$$\lim \inf \text{prob} \left( \sqrt{T+1} \langle Q, \mathcal{D} - \hat{\mathcal{D}} \rangle \leq X \right) \geq \sum_{k=1}^{K} \Phi(x_{k}) - K + 1.$$ 

### OB Multistage Bidding

National level auctions in our data follow a first-price auction format with a secret reserve price. This means that the auction is a multistage game, with stages $k \in \{1, \cdots, K\}$. The auctioneer picks a secret reserve price $r$. At each stage $k$, bidders submit bids $b_{i,k}$. A winner is declared if and only if $\min_{i} b_{i,k} \leq r$. In this case, the winner is paid her bid. If instead $\min_{i} b_{i,k} > r$ the game continues to an additional stage. At the end of each stage without a winner the lowest bid is revealed. The reserve price is constant across stages.

In this Appendix we extend the revealed preference inequalities of Section 5 to multistage first-price auctions.

In a multistage auction, a bidder’s continuation strategy after her first bid is a contingent plan dependent on the information revealed at each stage. We denote by $b_{i,1}$ bidder $i$’s first bid, and by $\beta_{i}$ her continuation play, mapping future information to bids.

Given an equilibrium strategy $\sigma_{i} = (b_{i,1}, \beta_{i})$ by player $i$ we consider first-stage-only deviations $\sigma'_{i} = (b'_{i,1}, \beta_{i})$ such that player $i$’s initial bid is different, but her continuation contingent plan, as a function of her own private signals, and the play of others, is unchanged.

Let $\text{win}_{i,k}$ denote the event that bidder $i$ wins in round $k$. Expected profits under $\sigma_{i}$ and $\sigma'_{i}$ take the form

$$\mathbb{E}_{\sigma_{i}}[\pi_{i}] = (b_{i,1} - c_{i})\text{prob}_{\sigma_{i},\sigma_{-i}}(\text{win}_{i,1}) + \mathbb{E}_{\sigma_{i},\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_{i}) 1_{\text{win}_{i,k}} \right]$$

$$\mathbb{E}_{\sigma'_{i}}[\pi_{i}] = (b'_{i,1} - c_{i})\text{prob}_{\sigma'_{i},\sigma_{-i}}(\text{win}_{i,1}) + \mathbb{E}_{\sigma'_{i},\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_{i}) 1_{\text{win}_{i,k}} \right]$$

We now introduce a classification of histories following upward and downward deviations.
in the first round as a function of how they affect the continuation play. We say that a deviation is marginal for continuation, if it changes whether the auction continues after period 1. When a deviation is marginal for information, it changes the information available to participants in future periods. If a deviation is non-marginal, it does not affect continuation play. This corresponds to the following formal definition.

**Definition OB.1.** Consider an upward deviation \( b'_{i,1} > b_{i,1} \). It is marginal for continuation (MC) if and only if \( b_{i,1} \leq r < \land b_{-i,1} \), and \( b'_{i,1} > r \). It is marginal for information (MI) if and only if \( r < b_{i,1} < \land b_{-i,1} \). It is non-marginal (NM) otherwise.

Consider a downward deviation \( b'_{i,1} < b_{i,1} \). It is marginal for continuation (MC) if and only if \( b'_{i,1} \leq r < \land b_{-i,1} \), and \( b_{i,1} > r \). It is marginal for information (MI) if and only if \( r < b'_{i,1} < \land b_{-i,1} \). It is non-marginal (NM) otherwise.

Note that we can assess the marginality of deviations using data, since it only relies on observed period 1 bids. Note also that the probability a given deviation is marginal for continuation or information only depends on the agent’s beliefs about bids \( b_{-i,1} \).

For bids \( b_{i,1} \) and \( b'_{i,1} \), we have that

\[
\begin{align*}
\mathbb{E}_{\sigma_i}[\pi_i] &= (b_{i,1} - c_i)\text{prob}_{\sigma_i,\sigma_{-i}}(\text{win}_{i,1}) + \mathbb{E}_{\sigma_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{MC}} \text{prob}_{\sigma_{-i}}(\text{MC}) \\
&\quad + \mathbb{E}_{\sigma_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{MI}} \text{prob}_{\sigma_{-i}}(\text{MI}) + \mathbb{E}_{\sigma_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{NM}} \text{prob}_{\sigma_{-i}}(\text{NM}) \\
\mathbb{E}_{\sigma'_i}[\pi_i] &= (b'_{i,1} - c_i)\text{prob}_{\sigma'_i,\sigma_{-i}}(\text{win}_{i,1}) + \mathbb{E}_{\sigma'_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{MC}} \text{prob}_{\sigma_{-i}}(\text{MC}) \\
&\quad + \mathbb{E}_{\sigma'_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{MI}} \text{prob}_{\sigma_{-i}}(\text{MI}) + \mathbb{E}_{\sigma'_i,\sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i)1_{\text{win}_{i,k}} \right]_{\text{NM}} \text{prob}_{\sigma_{-i}}(\text{NM}) 
\end{align*}
\]

Equilibrium implies that under player \( i \)'s beliefs \( \mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma'_i}[\pi_i] \). We now establish implications of this equilibrium condition that can be taken to the data.

For all deviations, the following hold:

- Bids must decrease with the stage of the game: \( b_{i,k} > b_{i,k+1} \); indeed, since the reserve price is constant, any bid submitted in period \( k \) wins with probability 0 in period \( k+1 \) if the auction continues.

- Continuation payoffs under \( \sigma_i \) and \( \sigma'_i \) are equal conditional on the deviation being non-marginal.

50
If the deviation is *an upward deviation* then,

- Player $i$’s continuation value under $\sigma_i$ is equal to zero when the deviation is marginal for continuation.
- If continuation strategies $\beta_i, \beta_{-i}$ are monotonic in observed bids, then

$$
\mathbb{E}_{\sigma_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MI} \right] \geq \mathbb{E}_{\sigma_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MI} \right].
$$

It follows from this that $\mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma'_i}[\pi_i]$ implies

$$
(b_{i,1} - c_i) \text{prob}_{\sigma_i, \sigma_{-i}}(\text{win}_{i,1}) \geq (b'_{i,1} - c_i) \text{prob}_{\sigma'_i, \sigma_{-i}}(\text{win}_{i,1}). \quad (O1)
$$

This coincides with the IC constraint for upward deviations used in Sections 5.1 and 5.3.

If the deviation is *a downward deviation* then player $i$’s continuation value under $\sigma'_i$ is equal to zero when the deviation is marginal for continuation. Furthermore we assume that for some $\alpha \in (0, 1)$

$$
\mathbb{E}_{\sigma'_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MI} \right] \geq (1 - \alpha) \mathbb{E}_{\sigma_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MI} \right]. \quad (O2)
$$

In words, following a downward deviation that is marginal for information (meaning that the bid is in fact above the reserve price, which it would have to beat to win at a later stage) the change in the information provided in the continuation stage does not destroy all the continuation value of the bidder. Note that if at the end of each stage the auctioneer revealed an exogenous signal of the reserve price, rather than the endogenous minimum bid, then condition (O2) would hold with $\alpha = 0$. In our empirical investigation, we use $\alpha = .5$.

Finally, we observe that the following bounds hold

$$
\mathbb{E}_{\sigma_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MI} \right] \leq \mathbb{E}[(r - c_i)^+],
$$

$$
\mathbb{E}_{\sigma_i, \sigma_{-i}} \left[ \sum_{k>1} (b_{i,k} - c_i) 1_{\text{win}_{i,k}} \middle| \text{MC} \right] \leq \mathbb{E}[(r - c_i)^+].
$$

Altogether, with optimality condition $\mathbb{E}_{\sigma_i}[\pi_i] \geq \mathbb{E}_{\sigma'_i}[\pi_i]$ this implies that
Equations (O1) and (O3) replace (IC) in the inference problem defined Section 5.3. In addition to disciplining residual demand under \( \omega_H \), expanded consistency requirement (\( \widehat{CR} \)) must ensure that the probability of events \( \text{MI} \) and \( \text{MC} \) under \( \omega_H \) must also be close to their sample probability for downward deviations. For any \( \rho < 0 \), denoting by \( m_i h \) and \( m_c h \) the probability that downward deviation \( \rho \) is marginal for information or continuation at \( h \) we must have

\[
\frac{1}{|H|} \sum_{h \in H} m_i h \in \left[ \frac{1}{|H|} \sum_{h \in H} 1_{r<(1+\rho)b_{i1}<\land b_{-i1}} \pm K \right]
\]

\[
\frac{1}{|H|} \sum_{h \in H} m_c h \in \left[ \frac{1}{|H|} \sum_{h \in H} 1_{(1+\rho)b_{i1} \leq r<\land b_{-i1}} \pm K \right]
\]

where \( K \) is a tolerance parameter chosen to ensure adequate coverage. Alternative coverage sets centered around the same sample probabilities of being marginal for information or continuation can be used along the lines described in Appendix A.2.

## OC Further Theoretical Results

### OC.1 Connection with Bayes Correlated Equilibrium

In this section we further extend the estimator introduced in Section 5.3 and clarify what would need to be added so that asymptotically, it exploits all implications from equilibrium. This allows us to connect with the work of Bergemann and Morris (2016).

For simplicity we assume that player identities \( i \), bids \( b \) and costs \( c \) take a fixed finite number of values \( (i, b, c) \in I \times B \times C \) that does not grow with sample size \( |H| \). Ties between bids are resolved with uniform probability. Deviations \( \rho_n \in (-1, \infty) \) correspond to the ratios of different bids on finite grid \( B \).

We extend problem (P) as follows. For any environment \( \omega_H \) and \( (i, b, c) \in I \times B \times C \), let us define \( H_{i,b,c}(\omega_H) \equiv \{ h \in H | (i_h, b_h, c_h) = (i, b, c) \} \), histories at which bidder \( i \) experiences a cost \( c \) and bids \( b \). Note that \( H_{i,b,c} \) is adapted to the information of player \( i \). For any
tolerance function $K : \mathbb{N} \to \mathbb{R}^+$ such that
\[
\lim_{k \to \infty} K(k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \exp \left( -K(k)^2 k / 2N_{\max} \right) = 0
\]
we consider inference problem (P')
\[
\hat{s} = \max_{\omega_H} \frac{|H_{\text{comp}}(\omega_H)|}{|H|} \quad \text{(P')}
\]
s.t. $\forall (i, b, c), \forall n, \quad D_n(\omega_H, H_{i,b,c}(\omega_H)) \in \left[ \hat{D}(\rho_n|H_{i,b,c}(\omega_H)) - K(|H_{i,b,c}(\omega_H)|), \right.$
\[
\left. \hat{D}(\rho_n|H_{i,b,c}(\omega_H)) + K(|H_{i,b,c}(\omega_H)|) \right].
\]

Problem (P') differs from (P) by imposing demand consistency requirements conditional on all triples $(i, b, c)$. Proposition 4 continues to hold with an identical proof: with probability approaching 1 as $|H|$ goes to $\infty$, $\hat{s}$ is an upper bound to the share of competitive histories.

Imposing consistency requirements conditional on bids and costs lets us establish a converse: if data passes our extended tests, then the joint distribution of bids and costs is an $\epsilon$-Bayes correlated equilibrium in the sense of Hart and Mas-Colell (2000).

Consider an $\omega_H$ solving (P'). Let $\hat{\mu} \in \Delta \left( [B \times C]^I \right)$ denote the sample distribution over bids and costs implied by $(H, \omega_H)$.

**Proposition OC.1.** For any $\epsilon > 0$, for $|H|$ large enough, $\hat{s} = 1$ implies that $\hat{\mu}$ is an $\epsilon$-Bayes correlated equilibrium.

**Proof.** Consider demand and costs $(d_{h,n}, c_h)_{h \in H}$ solving Problem (P'), and $\hat{\mu}$ the corresponding sample distribution over profiles of bids $b$ and costs $c$.

In order to deal with ties, we denote by $\land b_{-i} \succ b_i$ the event “$\land b_{-i} > b_i$, or $\land b_{-i} = b_i$ and the tie is broken in favor of bidder $i$.”

For $|H|$ large enough, we have that for all $(i, b, c)$ and all $n$,
\[
\frac{1}{|H|} \left| \sum_{h \in H_{i,b,c}} d_{n,h} - \text{prob}_{\hat{\mu}}(\land b_{-i} \succ (1 + \rho_n)b_i|i, b, c) \right| \leq \epsilon. \quad \text{(O1)}
\]

In addition, $\hat{s} = 1$ implies that (IC) holds at all histories: for all $h, n$,
\[
d_{h,n}(1 + \rho_n)b_h - c_h \leq d_{h,0}(b_h - c_h).
\]
Summing over histories $h \in H_{i,b,c}$ yields

$$\frac{1}{|H|} \sum_{h \in H_{i,b,c}} d_{h,n}((1 + \rho_n)b_h - c_h) - d_{h,0}(b_h - c_h) \leq 0.$$ 

Hence for $|H|$ large enough, for all $(b_i, c_i)$,

$$\sum_{b_{-i}, c_{-i}} \hat{\mu}(b_i, c_i, b_{-i}, c_{-i}) \left(1_{\hat{D}(0\mid H)} - \hat{D}(0\mid H)\right) \leq \epsilon.$$ 

It follows that $\hat{\mu}$ is an $\epsilon$-Bayes correlated equilibrium in the sense of Hart and Mas-Colell (2000). □

**OC.2 Complementarities between upward and downward deviations**

In this appendix we clarify complementarities between downward and upward deviations and establish a possibility result in a stylized setting. Even if neither individual deviation implies that a positive share of auctions is non competitive, the joint restrictions imposed by upward and downward deviations can imply that a positive share of auctions is non competitive. For simplicity we set tolerance $K = 0$.

As we discussed in Section 6, individual upward and downward deviations respectively imply strict bounds on the share of competitive histories if and only if

$$\hat{D}(0\mid H) - (1 + \rho_1)\hat{D}(\rho_1\mid H) < \frac{1}{1+M} \left[\hat{D}(0\mid H) - \hat{D}(\rho_1\mid H)\right],$$

$$(1 + \rho_{-1})\hat{D}(\rho_{-1}\mid H) - \hat{D}(0\mid H) > \frac{1}{1+m} \left[\hat{D}(\rho_{-1}\mid H) - \hat{D}(0\mid H)\right].$$

To clarify the existence of complementarities between upward and downward deviations, we

---

35Checking whether these constrains hold can be performed rapidly, and suggests a rough rationale by which one could pick $\rho_{-1}$ and $\rho_1$: obtain a smooth estimate of the true demand, and pick $\rho_{-1}$ and $\rho_1$ so that the conditions above hold with a reliable margin.
now consider the special case in which
\[
\hat{D}(0|H) - (1 + \rho_1)\hat{D}(\rho_1|H) = \frac{1}{1 + M} \left[ \hat{D}(0|H) - \hat{D}(\rho_1|H) \right],
\]
\[
(1 + \rho_{-1})\hat{D}(\rho_{-1}|H) - \hat{D}(0|H) = \frac{1}{1 + m} \left[ \hat{D}(\rho_{-1}|H) - \hat{D}(0|H) \right].
\]

Individual upward and downward deviations imply no restrictions on the set of competitive histories. However, different deviations are potentially rationalized by using different costs at the same history. We show this is indeed the case, and that jointly considering upward and downward deviations can yield strict constraints on the share of competitive histories. The following lemma clarifies that markup constraints will play a role in our argument.

**Lemma OC.1.** Under (O2) and if \( m = 0 \) and \( M = +\infty \), then all histories can be rationalized as competitive.

**Proof.** The following demand and costs rationalize the observed bidding behavior while satisfying consistency requirement \((\hat{C}\hat{R})\). At every history \( h \) such that the bidder wins, we set \( d_{h,0} = 1, d_{h,-1} = 1, d_{h,1} = \hat{D}(\rho_1|H)/\hat{D}(0|H) \) and \( c_h = 0 \). Since \( \rho_1 > 0 \), \( d_{h,1} \leq 1 \).

At every history \( h \) such that the bidder loses, but would win after reducing its bids by \( \rho_{-1} \), we set \( d_{h,0} = d_{h,1} = 0, d_{h,-1} = 1 \) and \( c_h = b_h \).

At every history such that the bidder loses even after deviation \( \rho_{-1} \), we set \( d_{h,-1} = d_{h,0} = d_{h,1} = 0 \), and \( c_h = b_h \).

It is immediate that these demand and costs satisfy (IC), (F) and \((\hat{C}\hat{R})\).

We return now to the case where (O2) and (O3) hold for \( m > 0 \). Assume that there exists an environment \((\omega_H)\) satisfying (IC), (F), (MKP) and \((\hat{C}\hat{R})\) with \( K = 0 \). We establish lower bounds for the number of histories at which \( c_h/b_h \) must be equal to \( \frac{1}{1+m} \) and \( \frac{1}{1+M} \). Whenever these two lower bounds are mutually incompatible, the share of competitive histories is strictly less than one.

**Histories such that** \( c_h/b_h = 1/(1 + M) \). (IC) for upward deviation \( \rho_1 \) implies that for all histories \( h \),
\[
d_{h,0} - (1 + \rho_1)d_{h,1} \geq (d_{h,0} - d_{h,1}) \frac{c_h}{b_h}.
\]
Summing over all histories, conditions (\(\widehat{CR}\)) and (O2) imply that
\[
\frac{1}{|H|} \sum_{h \in H} (d_{h,0} - d_{h,1}) \frac{c_h}{b_h} \leq \frac{1}{|H|} \sum_{h \in H} d_{h,0} - (1 + \rho_1)d_{h,1} = \frac{1}{1 + M} (\hat{D}(0|H) - \hat{D}(\rho_1|H)) = \frac{1}{|H|} \sum_{h \in H} (d_{h,0} - d_{h,1}) \frac{1}{1 + M}.
\]
Since \(d_{h,0} - d_{h,1} \geq 0\) and \(c_h/b_h \geq \frac{1}{1+M}\), this implies that whenever \(d_{h,0} - d_{h,1} > 0\), \(c_h/b_h = 1/(1 + M)\).

Note that if \(d_{h,0} = d_{h,1} > 0\) then \((1 + \rho_1)d_{h,1} < 0\) so that (IC) cannot hold. Hence \(d_{h,0} - d_{h,1} = 0\) implies \(d_{h,0} = d_{h,1} = 0\). This implies that
\[
\frac{1}{|H|} \sum_{h \in H} \mathbf{1}_{d_{h,0} - d_{h,1} > 0} \geq \frac{1}{|H|} \sum_{h \in H} \mathbf{1}_{d_{h,0} > 0} \geq \frac{1}{|H|} \sum_{h \in H} d_{h,0} = \hat{D}(0|H).
\]
Hence the share of histories such that \(c_h/b_h = \frac{1}{1 + M}\) is at least equal to \(\hat{D}(0|H)\).

**Histories such that** \(c_h/b_h = 1/(1 + m)\). (IC) for downward deviation \(\rho_{-1}\) implies that for all histories \(h\),
\[
(1 + \rho_{-1})d_{h,-1} - d_{h,0} \leq (d_{h,-1} - d_{h,0}) \frac{c_h}{b_h}.
\]
Summing over all histories in \(H\), conditions (\(\widehat{CR}\)) and (O3) imply that
\[
\frac{1}{|H|} \sum_{h \in H} (d_{h,-1} - d_{h,0}) \frac{c_h}{b_h} \geq \frac{1}{|H|} \sum_{h \in H} (1 + \rho_{-1})d_{h,-1} - d_{h,0} = \frac{1}{1 + m} (\hat{D}(\rho_{-1}|H) - \hat{D}(0|H)) = \frac{1}{|H|} \sum_{h \in H} (d_{h,-1} - d_{h,0}) \frac{1}{1 + m}.
\]
Since \(d_{h,-1} - d_{h,0} \geq 0\) and \(c_h/b_h \leq \frac{1}{1+m}\), this implies that whenever \(d_{h,-1} - d_{h,0} > 0\), then \(c_h/b_h = 1/(1 + m)\). In addition, for all \(h\), we have that
\[
(1 + \rho_{-1})d_{h,1} - d_{h,0} = (d_{h,0} - d_{h,1}) \frac{1}{1 + m} \Rightarrow d_{h,0} = \frac{1 + \rho_{-1} - \frac{1}{1+m}}{1 - \frac{1}{1+m}} d_{h,-1} = (1 - \nu) d_{h,-1}
\]
with \( \nu \equiv -\rho/(1 - \frac{1}{1+m}) > 0 \). Hence, we have that

\[
\frac{1}{|H|} \sum_{h \in H} d_{h,-1} - d_{h,0} \leq \frac{1}{|H|} \sum_{h \in H} (d_{h,-1} - d_{h,0}) \mathbf{1}_{d_{h,-1} - d_{h,0} > 0} \leq \frac{1}{|H|} \sum_{h \in H} \nu d_{h,-1} \mathbf{1}_{d_{h,-1} - d_{h,0} > 0} \leq \frac{1}{|H|} \sum_{h \in H} \nu \mathbf{1}_{d_{h,-1} - d_{h,0} > 0}.
\]

This implies that the share of histories such that \( c_h/b_h = 1/(1+m) \) is greater than \( \frac{1}{\nu}(\hat{D}(\rho_{-1}|H) - \hat{D}(0|H)) \).

Hence, if \( \hat{D}(0|H) + \frac{1}{\nu}(\hat{D}(\rho_{-1}|H) - \hat{D}(0|H)) > 1 \), then joint upward and downward deviations imply strict constraints on the share of competitive histories. For example, if \( m = 3\% \), \( \rho_{-1} = -1.5\% \), \( \hat{D}(\rho_{-1}|H) = 65\% \) and \( \hat{D}(0|H) = 25\% \), then \( \frac{1}{\nu} \approx 1.94 \), and \( \hat{D}(0|H) + \frac{1}{\nu}(\hat{D}(\rho_{-1}|H) - \hat{D}(0|H)) \approx 1.025 \).

**OC.3 Common Values**

We now show how to extend the analysis in Section 5.3 to allow for common values. Because expected costs conditional on winning now depend on a bidder’s bid, costs and demand associated with history \( h \in H \) now take the form \( \omega_h = (d_{h,n}, c_{h,n})_{n \in M} \), where for each \( n \in M \), \( c_{h,n} = \mathbb{E}[c|h, \wedge b_{-i,h} > (1 + \rho_n)b_h] \) is the bidder’s expected cost at history \( h \) conditional on winning at bid \( (1 + \rho_n)b_h \).

We make the following monotonicity assumption.

**Assumption C.1.** For all histories \( h \) and all bids \( b, b', b'' \) with \( b < b' < b'' \), \( \mathbb{E}[c|h, \wedge b_{-i,h} \in (b,b')] \leq \mathbb{E}[c|h, \wedge b_{-i,h} \in (b',b'')] \).

In words, bidders’ expected costs are increasing in opponents’ bids. This implies that expected costs \( c_{h,n} \) conditional on winning are weakly increasing in the deviation \( \rho_n \). This condition on costs follows from affiliation when bidders’ signals are one-dimensional and bidders use monotone bidding strategies. We now show that, under these conditions, allowing for common values does not relax the constraints in Program (P).

Note first that, for each deviation \( n \), expected costs conditional on winning \( (c_{h,n})_{n \in M} \) satisfy:

\[
\forall n \in M, \quad d_{h,n}c_{h,n} = d_{h,0}c_{h,0} + (d_{h,n} - d_{h,0})\hat{c}_{h,n}, \tag{O4}
\]
where $\hat{c}_{h,n} = \mathbb{E}[c|h, \wedge b_i \in (b_n, (1 + \rho_n)b_n)]$. Our assumptions on costs imply that $\hat{c}_{h,n}$ is weakly increasing in $n$.

Consider first downward deviations $\rho_n < 0$ (i.e., $n < 0$). For such deviations, incentive compatibility constraint (IC) holds if and only if

$$\frac{d_{h,n}(1 + \rho_n)b_h - d_{h,0}b_h}{d_{h,n} - d_{h,0}} \leq \hat{c}_{h,n}.$$ 

Consider next upward deviations $\rho_n > 0$ (i.e., $n > 0$). For any such deviation, constraint (IC) becomes

$$\hat{c}_{h,n} \leq \frac{d_{h,0}b_h - d_{h,n}(1 + \rho_n)b_h}{d_{h,0} - d_{h,n}}.$$ 

Since $\hat{c}_{h,n}$ is weakly increasing in $n$, $\hat{c}_{h,n} \geq \hat{c}_{h,n'}$ for all $n > 0$ and $n' < 0$. Hence there exist costs $(c_{h,n})_{n \in \mathcal{M}}$ satisfying (IC) if and only if

$$\max_{n < 0} \frac{d_{h,n}(1 + \rho_n)b_h - d_{h,0}b_h}{d_{h,n} - d_{h,0}} \leq \min_{n > 0} \frac{d_{h,0}b_h - d_{h,n}(1 + \rho_n)b_h}{d_{h,0} - d_{h,n}}.$$  

(O5)

Condition (O5) implies that there also exists a constant profile of costs $c_{h,n} = c_h$ (i.e. a private value cost), that satisfies (IC).

## OD Further Empirical Findings

### OD.1 Additional Figures for Section 2

Figure OD.1 illustrates the clustering of bids we highlight in Section 2 more directly. The two panels of Figure OD.1 plot the sample demand function, $\hat{D}(\cdot)$, for city-level and national-level auctions. We define the sample demand as follows: $\hat{D}(\rho) = \frac{1}{|H|} \sum_{i,a} 1_{h,a(1+\rho) < \wedge b_{-i,a}}$, where $|H|$ denotes the number of all bids in the dataset. In words, $\hat{D}(\rho)$ is the sample probability with which bidders win an auction if the bids are changed by a factor of $(1 + \rho)$. For panel (a), we find that a drop in bids of 2% increases the likelihood of winning by more than 3-fold from 16.3% to 56.9%. For panel (b), we find that a 2% drop in bids also increases the likelihood of winning by about 3-fold from 10.8% to 33.2%.

Figure OD.2 plots the distribution of differences $\Delta^2$ between bids after the lowest bid is

---

36We replace $(b, b')$ by $(b', b)$ in the event that $b' < b$. 

58
excluded. Formally, let $NW(a)$ denote the set of non-winning bidders in auction $a$. Then,

$$\forall i \in NW(a), \quad \Delta^2_{i,a} = \frac{b_{i,a} - \min_{j \in NW(a)} b_{j,a}}{r}.$$  

Although there is some bunching at exactly zero making the distributions of $\Delta^2$ somewhat irregular, the kernel density estimates show that there is no corresponding missing mass.

Figure OD.2: Distribution of bid-difference $\Delta^2$ excluding winning bids.

The dotted curves correspond to local (6th order) polynomial density estimates with bandwidth set to 0.0075.

Figure OD.3 plots the distribution of bid differences $\Delta$ for the set of auctions whose prices
were not renegotiated up (about 15.3% of the sample). As the figure shows, the missing mass at $\Delta = 0$ is just as visible when we focus on this set of auctions.

![Figure OD.3: Distribution of bid-difference $\Delta$.](image)

National data, auctions with no renegotiation. The dotted curves correspond to local polynomial density estimates with bandwidth set to 0.0075.

**OD.2 Additional Findings Related to Section 6.4**

This appendix provides sensitivity analysis of the share of competitive histories for the industries that were investigated for bid rigging by the JFTC to changes in (a) the choice of confidence level, and (b) the set of deviations considered.

Figure OD.4 presents estimates of the 90% confidence bound on the share of competitive histories for the three industries we studied in Section 6.4. Relative to the bounds in Figure 7 in Section 6.4, these bounds are obtained using a smaller tolerance parameter $K$ in program (P). Under this less conservative bound, firms labeled Floods and Pre-Stressed Concrete appear to have continued colluding in the period after the investigation.

Figure OD.5 again presents estimates of the 90% confidence bound on the share of competitive histories for the three industries we studied in Section 6.4, but with the set of deviations $\{-0.02, 0, 0.002\}$. Relative to the bounds in Figure OD.4, we now only find evidence of non-competitive behavior for firms labeled Pre-Stressed Concrete in the after period.
The difference between our estimates in Figure OD.4 and Figure OD.5 can be explained as follows. For Flood auctions occurring after investigation, a .1% upward deviation causes no change in demand. As a result, our estimate on the share of competitive histories in the after period are strictly less than 1 when we consider such a small upward deviation. However, we are worried that this insensitivity of demand to a small upward deviation might be a mechanical consequence of the small number of observations we have in the after period for this industry. Indeed, a .2% upward deviation causes a small drop in demand, and our estimate on the share of competitive histories in the after period is exactly 1 in Figure OD.5.
Figure OD.5: 90% confidence bound on share of competitive histories.

Deviations \{-0.02, 0, 0.002\}; maximum markup 0.5.

OD.3 Bounds on Other Moments

This appendix shows how to adapt the approach of Section 5.3 to obtain robust bounds on other moments of interest: (i) the share of competitive auctions, and (ii) the total deviation temptation.

Maximum share of competitive auctions. The bound on the share of competitive histories provided by Proposition 4 allows some histories in the same auctions to have different competitive vs. non-competitive status. This may underestimate the prevalence of non-competition in a given dataset. In particular, if one player is non-competitive, she must expect other players to be non-competitive in the future. Otherwise, if all of her opponents
played competitively, her stage-game best reply would be a profitable dynamic deviation.

For this reason, one might be interested in providing an upper bound on the share of competitive auctions, where an auction is considered to be competitive if and only if every player is competitive at their respective histories.

Take as given an adapted set of histories $H$, corresponding to a set $A$ of auctions. For every demand and costs $\omega_H$, let

$$A_{\text{comp}}(\omega_H) \equiv \{ A' \subset A \text{ s.t. } \forall a \in A', \forall h \in a, (d_h, c_h) \text{ satisfy } (F), (IC) \text{ and } (MKP) \}.$$

be the set of competitive auctions under $\omega_H$. Program (P) then becomes

$$\tilde{s}_{\text{auc}} = \max_{\omega_H} \frac{|A_{\text{comp}}(\omega_H)|}{|A|}$$

s.t. $\forall n$, $D_n(\omega_H, H) \in \left[ \tilde{D}(\rho_n|H) - K, \tilde{D}(\rho_n|H) + K \right]$. 

$\tilde{s}_{\text{auc}}$ provides an upper bound to the fraction of competitive auctions.

**Total deviation temptation.** Regulators may want to investigate an industry only if firms fail to optimize in a significant way. Our methods can be used to derive a lower bound on the bidders’ deviation temptation.

Given demand and costs $\omega_H$, define

$$U(\omega_H) \equiv \frac{1}{|H|} \sum_{h \in H} \left[ (b_h - c_h)d_{h,0} - \max_{n \in \{-\rho_n, \ldots, \rho_n\}} [(1 + \rho_n)b_h - c_h]d_{h,n} \right].$$

Our inference problem now becomes:

$$\tilde{D}T = \max_{\omega_H} U(\omega_H)$$

s.t. $\forall n$, $D_n(\omega_H, H) \in \left[ \tilde{D}(\rho_n|H) - K, \tilde{D}(\rho_n|H) + K \right]$. 

In this case, with probability approaching 1 as $|H|$ gets large, $-\tilde{D}T$ is a lower bound for the average total deviation-temptation per auction. This lets a regulator assess the extent of firms’ failure to optimize before launching a costly audit. In addition, since the sum of deviation temptations must be compensated by a share of the cartel’s future excess profits (along the lines of Levin (2003)), $\tilde{D}T$ provides an indirect measure of the excess profits generated by the cartel.
Figure OD.6 reports estimates for firms in the city of Tsuchiura, as a function of minimum markup $m$.

![Graph](image)

Figure OD.6: Total deviation temptation as a fraction of profits, Tsuchiura. Deviations \{-0.02, 0, 0.001\}. Maximum markup 0.5.

### OD.4 Sensitivity to Economic Plausibility Constraints

Figure OD.7 shows that, for our city-level data, our estimates on the share of competitive histories are insensitive to changes in maximum markup $M$. Figure OD.8 illustrates the sensitivity of our estimates to parameter $\alpha \in [0, 1]$ in downward deviation IC constraint (O3) for auctions with re-bidding. Recall that parameter $\alpha$ measures the extent to which a deviation by a firm in round 1 affects her continuation profits in the following rounds when the deviation changes the information bidders have in the following rounds.
Figure OD.7: Share of competitive histories for different maximum markups, city data, deviations \{-0.02, 0, 0.001\}.

Figure OD.8: Share of competitive histories, national-level data. Deviations \{-0.02, 0, 0.001\}, \(M = 0.5\).