Online Appendix

Collusion in Auctions with Constrained Bids:
Theory and Evidence from Public Procurement

Sylvain Chassang          Juan Ortner*
New York University       Boston University

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Abstract

This Online Appendix to “Collusion in Auctions with Constrained Bids: Theory and Evidence from Procurement Auctions” provides several extensions. We analyze variants of our baseline model allowing for endogenous participation by cartel members, as well as non-performing bidders. A back-of-the-envelope calibration of the model described in Section 4 lets us get a sense of potential treatments effects as the level of the minimum price varies. Finally we collect remaining proofs and present additional results for the model of Section 4.

Keywords: collusion, cartel enforcement, minimum prices, entry deterrence, procurement.

*Chassang: chassang@nyu.edu, Ortner: jortner@bu.edu.
OA  Participation by non-performing bidders

The official rationale for introducing minimum prices is that it reduces the incidence of non-performing bidders, i.e. bidders unable to execute the tasks described in the procurement contract. In addition to reducing the cost of procured services, the auctioneer is also interested in reducing the likelihood that a contract is assigned to a non-performing bidder.

The effect of minimum prices can be captured in the framework of Section 4. Non-performing bidders can be modeled as entrants whose cost of production is set to 0. To simplify the analysis, we further assume that the cost of entry of non-performing bidders is equal to 0, and that other bidders are informed of the non-performing status of the entrant. We denote by $q$ the likelihood that a non-performing entrant is present.

It is immediate that the characterization of equilibrium bids given by Proposition OE.1 and the results in Proposition 5 and Proposition 6 continue to hold: they rely only on the bidder-side of the market. Hence the possibility of non-performance does not affect our analysis. We now clarify the effect of minimum bids on non-performance.

Lemma OA.1 (likelihood of non-performance). Under both competition and collusion, the likelihood that the contract is awarded to a non-performing entrant is equal to $q \times \mathbb{E} \left[ \frac{1}{\sum_{i \in \hat{N}_t} 1_{c_{i,t} \leq p}} \right]$. It is decreasing in minimum price $p$.

Proof. Since costs are public information across participants, the only equilibrium under competition is such that the equilibrium bid is equal to $\max\{p, \hat{c}_{(2)}\}$, the maximum between the minimum price and the second lowest cost. Hence the non-performing bidder wins: with probability 1 when all other bidders have a cost of production above $p$; by tie-breaking when several other bidders have a cost of production below $p$.

Under collusion, the assumption that non-performing entrants have a cost of entry of 0, and the assumption that their non-performing status is known to other bidders, imply that the cartel is unable to deter entry by non-performing entrants. As a result, when a non-performing entrant is present, cartel members do not bid below their cost of production. Hence, the non-performing entrant wins the contract for the same configuration of costs as in the case of competition.  

\[\square\]
OB  Endogenous participation

OB.1 Model

We extend the model in the main text to allow for endogenous participation by cartel members. The main point of the extension is to show that, in an optimal equilibrium, the cartel will actively manage the number of firms that participate at each auction. This allows a cartel to sustain high prices even if it’s composed of a large number of firms. We also show that firms can implement the optimal equilibrium by dividing themselves into different sub-cartels.

At each period $t \in \mathbb{N}$, firms in $N = \{1, ..., n\}$ simultaneously choose whether or not to participate in the auction. We let $E_{i,t} \in \{0, 1\}$ denote the entry decision of firm $i \in N$, with $E_{i,t} = 1$ denoting entry.\(^1\) For simplicity, we assume that procurements costs of those firms that enter the market are independently drawn from c.d.f. $F$ with support $[\underline{c}, \overline{c}]$ and density $f$. We denote by $\tilde{N}_t = \{i \in N : E_{i,t} = 1\}$ the set of firms that participate at period $t$, and by $c_t = (c_{i,t})_{i \in \tilde{N}_t}$ the cost realization of all firms in $\tilde{N}_t$. Note that cost vector $c_t$ contains information about the set participants $\tilde{N}_t$ at period $t$.

The timing of information and decisions within period $t$ is as follows.

1. Firms $i \in N$ simultaneously make entry decisions $E_{i,t} \in \{0, 1\}$. Entry decisions are publicly observed.
2. Production costs $c_t = (c_{i,t})_{i \in \tilde{N}_t}$ of participating firms are drawn and publicly observed.
3. Participating firms submit public bids $b_t = (b_{i,t})_{i \in \tilde{N}_t}$ and numbers $\gamma_t = (\gamma_{i,t})_{i \in \tilde{N}_t}$, resulting in allocation $x_t = (x_{i,t})_{i \in \tilde{N}_t}$.
4. Firms make transfers $T_{i,t}$.

The history among cartel members at the beginning of time $t$ is

$$h_t = \{c_s, b_s, \gamma_s, x_s, T_s\}_{s=0}^{t-1}.$$ 

Let $\mathcal{H}^t$ denote the set of period $t$ public histories and $\mathcal{H} = \bigcup_{t \geq 0} \mathcal{H}^t$ denote the set of all histories (note that, for all $s$, cost vector $c_s = (c_{i,s})_{i \in \tilde{N}_s}$ contains information about the firms

\(^1\)Note that we assume that all firms in $N$ can participate at every period. The model can be easily extended to allow the set of potential participants to be randomly drawn at each period.

\(^2\)The allocation is determined in the same way as in the main text.
that participate at time $s$). Our solution concept is subgame perfect equilibrium (SPE), with strategies

$$
\sigma_i : h_t \mapsto (E_{i,t}, b_{i,t}(c_t), \gamma_{i,t}(c_t), T_{i,t}(c_t, b_t, \gamma_t, x_t))
$$
such that entry decisions $E_{i,t}$, bids $(b_{i,t}(c_t), \gamma_{i,t}(c_t))$ and transfers $T_{i,t}(c_t, b_t, \gamma_t, x_t)$ can depend on all public data available at the time of decision-making.

**OB.2 Optimal collusion**

For any SPE $\sigma$ and any history $h_t$, we denote by $V(\sigma, h_t)$ the surplus generated by $\sigma$ under history $h_t$. As in the main text, we denote by $V_p$ the highest surplus that firms can sustain in a SPE. Given a history $h_t$ and a strategy profile $\sigma$, we denote by $E(h_t, \sigma)$ and by $\beta(c_t|h_t, \sigma)$ the entry profile and bidding profile induced by strategy profile $\sigma$ at history $h_t$.

**Lemma OB.1 (stationarity).** Consider a subgame perfect equilibrium $\sigma$ that attains $V_p$. Equilibrium $\sigma$ delivers surplus $V(\sigma, h_t) = V_p$ after all on-path histories $h_t$.

There exists an integer $\tilde{n} \leq n$ and a bidding profile $\beta^*$ such that, in an equilibrium that attains $V_p$, $\tilde{n}$ firms enter and bid according to $\beta(c_t|h_t, \sigma) = \beta^*(c_t)$ after all on-path histories $h_t$.

**Proof.** The proof is identical to the proof of Lemma 1 and hence omitted. □

We denote by $V_p$ the lowest possible equilibrium payoff for a given firm. Similarly, for any $\tilde{N} \subset N$, we denote $V_p^{[\tilde{N}]}$ the lowest equilibrium payoff for a firm starting at a history at which $|\tilde{N}|$ firms chose to participate in the current auction (and before their procurement costs are drawn). Since firms are assumed to be symmetric, $V_p$ and $V_p^{[\tilde{N}]}$ are the same across firms.

Given a bidding profile $(\beta, \gamma)$, let us denote by $\beta^W(c)$ and $x(c)$ the induced winning bid and allocation profile for realized costs $c = (c_i)_{i \in \tilde{N}}$. Recall that, for each firm $i$,

$$
\rho_i(\beta^W, \gamma, x)(c) \equiv 1_{\beta^W(c) > p} + \frac{1_{\beta^W(c) = p}}{1 + \sum_{j \in \tilde{N} \setminus \{i\} : x_j(c) > 0} \gamma_j(c)}.
$$

is a deviator's highest possible chance of winning the contract by attempting to undercut the equilibrium winning bid.

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3Recall that the cost vector $c = (c_i)_{i \in \tilde{N}}$ contains information about the set of entrants. Hence, $\beta^W(c)$ and $x(c)$ are allowed to depend on the set of entrants.
Lemma OB.2 (enforceable bidding and participation). Entry profile $E \in \{0,1\}^N$ leading to set of participants $\tilde{N} = \{i \in N : E_i = 1\}$, winning bid profile $\beta^W(c)$ and allocation $x(c)$ are sustainable in SPE if and only if for all $c = (c_i)_{i \in \tilde{N}}$,

$$\sum_{i \in \tilde{N}} (\rho_i(\beta^W, \gamma, x)(c) - x_i(c)) [\beta^W(c) - c_i]_+ + x_i(c) [\beta^W(c) - c_i]_- \leq \delta(V_p - |\tilde{N}| V_p) - (n - |\tilde{N}|) V_{\text{irr}}^{(\tilde{N})} + 1. \tag{O1}$$

The second term on the right-hand side of (O1) captures the cost of keeping potential participants out of the auction. Indeed, when the set of participants $\tilde{N}$ is a strict subset of $N$, the cartel has to promise firms that stay out of the auction a payoff at least as large as $V_{\text{irr}}^{(\tilde{N})}$. Note that when all firms enter the auction (i.e., when $\tilde{N} = N$), obedience constraint (O1) is the same as the obedience constraint in our baseline model (under the assumption of symmetry; i.e., $V_{i,p} = V_p$ for all $i$).

Proof. We start with some preliminary observations. Fix an SPE $\sigma$ and a history $h_t$. Let $E$, $\beta(c)$, $\gamma(c)$ and $T(c, b, \gamma, x)$ be the entry, bidding and transfer profile that firms use in this equilibrium after history $h_t$. Let $\beta^W(c)$ and $x(c)$ be, respectively, the winning bid and the allocation induced by bidding profile $(\beta(c), \gamma(c))$. Let $h_{t+1} = h_t \cup (c, b, \gamma, x, T)$ be the concatenated history composed of $h_t$ followed by $(c, b, \gamma, x, T)$, and let $\{V(h_{t+1})\}_{i \in N}$ be the vector of continuation payoffs after history $h_{t+1}$. We let $h_{t+1}(c) = h_t \cup (c, \beta(c), \gamma(c), x(c), T(c, \beta(c), \gamma(c), x(c)))$ denote the on-path history that follows $h_t$ when current costs are $c$. Note that the following inequalities must hold:

(i) for all $i \in N$ such that $E_i = 1$ and $c_i \leq \beta^W(c)$,

$$x_i(c)(\beta^W(c) - c_i) + T_i(c, \beta(c), \gamma(c), x(c)) + V_i(h_{t+1}(c)) \geq \rho_i(\beta^W, \gamma, x)(c)(\beta^W(c) - c_i) + \delta V_p. \tag{O2}$$

(ii) for all $i \in N$ such that $E_i = 1$ and $c_i > \beta^W(c)$,

$$x_i(c)(\beta^W(c) - c_i) + T_i(c, \beta(c), \gamma(c), x(c)) + \delta V_i(h_{t+1}(c)) \geq \delta V_p. \tag{O3}$$

(iii) for all $i \in N$ such that $E_i = 0$,

$$T_i(c, \beta(c), \gamma(c), x(c)) + \delta V_i(h_{t+1}(c)) \geq V_{\text{irr}}^{(\tilde{N})} + 1. \tag{O4}$$
(iv) for all $i \in N$,
\[
T_i(c, \beta(c), \gamma(c), x(c)) + \delta V_i(h_{t+1}(c)) \geq \delta V_p.
\] (O5)

Relative to our baseline model, the new constraint is (O4). This inequality must hold since bidder $i \in N$ with $E_i = 0$ can obtain at least $V^{i\hat{N}}_{t+1}$ by participating in the current auction rather than staying out.

Conversely, suppose there exists an entry profile $E$, a winning bid profile $\beta^W(c)$, an allocation $x(c)$, a transfer profile $T$ and equilibrium continuation payoffs $\{V(i(h_{t+1}(c))\}_{i \in N}$ that satisfy inequalities (O2)-(O5) for some $\gamma(c)$ that is consistent with $x(c)$ (i.e., $\gamma(c)$ is such that $x_i(c) = \gamma_i(c)/\sum_{j:x_j(c) > 0} \gamma_j(c)$ for all $i$ with $x_i(c) > 0$). Then, $(E, \beta^W, x, T)$ can be supported in an SPE as follows. Firms in $N$ adopt entry decisions given by $E$. Let $\tilde{N} = \{i \in N : E_i = 1\}$. For all $c = (c_i)_{i \in \tilde{N}}$, firms $i \in \tilde{N}$ bid $\beta^W(c)$. Firms $i \in \tilde{N}$ with $x_i(c) = 0$ choose $\tilde{\gamma}_i(c) = 0$, and firms $i \in \tilde{N}$ with $x_i(c) > 0$ choose $\tilde{\gamma}_i(c) = \gamma_i(c)$. Note that, for all $i \in \tilde{N}$, $x_i(c) = \tilde{\gamma}_i(c)/\sum_j \tilde{\gamma}_j(c)$ and $p_i(\beta^W, \gamma_i(c)) = \rho_i(\beta^W, \gamma, x)(c)$. If no firm deviates at the entry and bidding stages, firms make transfers $T_i(c, \beta(c), \gamma(c), x(c))$. If no firm deviates at the transfer stage, in the next period firms play an SPE that gives payoff vector $\{V(h_{t+1}(c))\}_{i \in N}$. If firm $i \notin \tilde{N}$ enters, the cartel reverts to an equilibrium that gives firm $i$ a payoff of $V^{i\hat{N}}_{t+1}$; if firm $i \in \tilde{N}$ does not participate, the cartel reverts to an equilibrium that gives bidder $i$ a continuation payoff of $V_p$; if a firm $i \in \tilde{N}$ deviates at the bidding stage, there are no transfers and the cartel reverts to an equilibrium that gives firm $i$ a continuation payoff of $V_p$; if firm $i \in N$ deviates at the transfer stage, the cartel reverts to an equilibrium that gives firm $i$ a continuation payoff of $V_p$ (deviations by more than one firm go unpunished). Since (O2) holds, under this strategy profile no participating firm has an incentive to undercut the winning bid $\beta^W(c)$. Since (O3) holds, no participating firm with $c_i > \beta^W(c)$ and $x_i(c) > 0$ has an incentive to bid above $\beta^W(c)$ and lose. Moreover, (O2) and (O3) also guarantee that firms $i \in \tilde{N}$ have an incentive to participate. Upward deviations by a firm $i \in \tilde{N}$ with $c_i < \beta^W(c)$ who wins the auction are not profitable since the firm would lose the auction by bidding $b > \beta^W(c)$. Since (O4) holds, firms $i \notin \tilde{N}$ have no incentive to participate. Finally, since (O5) holds, all firms have an incentive to make their required transfers.

We now turn to the proof of Lemma OB.2. Suppose there is an SPE $\sigma$ and a history $h_t$ at which firms bid according to a bidding profile $(\beta, \gamma)$ that induces winning bid $\beta^W(c)$ and
allocation \( x(c) \). Since the equilibrium must satisfy (O2)-(O5) for all \( c \),

\[
\sum_{i \in \bar{N}} \left\{ (\rho_i(\beta^W, \gamma, x(c)) - x_i(c)) [\beta^W(c) - c_i]^+ + x_i(c) [\beta^W(c) - c_i]^- \right\} \\
\leq \sum_{i \in N} T_i(c, \beta^W, c) + \delta \sum_{i \in N} V_i(h_{t+1}(c)) - \delta |\bar{N}|V_p - (n - |\bar{N}|)V_{pBar}^{\bar{N}+1} \\
\leq \delta(V_p - |\bar{N}|V_p) - (n - |\bar{N}|)V_{pBar}^{\bar{N}+1},
\]

where we used \( \sum_i T_i(c, \beta^W, c, x(c)) = 0 \) and \( \sum_i V_i(h_{t+1}(c)) \leq V_p \).

Next, consider an entry profile \( E \), a winning bid profile \( \beta^W(c) \) and an allocation \( x(c) \) that satisfy (O1) for all \( c = (c_i)_{i \in \bar{N}} \) for some \( \gamma(c) \) consistent with \( x(c) \) (i.e., such that \( x_i(c) = \gamma_i(c)/\sum_{j \in \bar{N}} \gamma_{ij}(c) \) for all \( i \in \bar{N} \) with \( x_i(c) > 0 \)). We now construct an SPE that supports \( E, \beta^W(\cdot) \) and \( x(\cdot) \) in the first period. Let \( \{V_i\}_{i \in \bar{N}} \) be an equilibrium payoff vector with \( \sum_i V_i = V_p \). For each \( c = (c_i)_{i \in \bar{N}} \) and each \( i \in N \), we construct transfers \( T_i(c) \) as follows:

\[
T_i(c) = \begin{cases} 
\delta(V_i - V_p) + (\rho_i(\beta^W, \gamma, x(c)) - x_i(c))(\beta^W(c) - c_i) + \epsilon(c) & \text{if } i \in \bar{N}, c_i \leq \beta^W(c), \\
\delta(V_i - V_p) - x_i(c)\beta^W(c) - c_i) + \epsilon(c) & \text{if } i \in \bar{N}, c_i > \beta^W(c), \\
\delta V_i + V_p^{\bar{N}+1} + \epsilon(c) & \text{if } i \notin \bar{N},
\end{cases}
\]

where \( \epsilon(c) \geq 0 \) is a constant to be determined below. Note that, for all \( c \),

\[
\sum_{i \in \bar{N}} T_i(c) - n\epsilon(c) = -\delta(V_p - |\bar{N}|V_p) + (n - |\bar{N}|)V_{pBar}^{\bar{N}+1} \\
+ \sum_{i \in N} \left\{ (\rho_i(\beta^W, \gamma, x(c)) - x_i(c)) [\beta^W(c) - c_i]^+ + x_i(c) [\beta^W(c) - c_i]^- \right\} \leq 0,
\]

where the inequality follows since \( \beta^W \) and \( x \) satisfy (O1). We set \( \epsilon(c) \geq 0 \) such that transfers are budget balance; i.e., such that \( \sum_{i \in \bar{N}} T_i(c) = 0 \).

The SPE we construct is as follows. At \( t = 0 \), for each \( c \) all participating firms bid \( \beta^W(c) \). Firms \( i \in \bar{N} \) with \( x_i(c) = 0 \) choose \( \tilde{\gamma}_i(c) = 0 \), and firms \( i \in \bar{N} \) with \( x_i(c) > 0 \) choose \( \tilde{\gamma}_i(c) = \gamma_i(c) \). Note that, for all \( i \in \bar{N} \), \( x_i(c) = \tilde{\gamma}_i(c)/\sum_j \tilde{\gamma}_j(c) \) and \( \rho_i(\beta^W, \tilde{\gamma}, x(c)) = \rho_i(\beta^W, \gamma, x(c)) \). If no firm deviates at the entry stage nor at the bidding stage, firms exchange transfers \( T_i(c) \). If no firm deviates at the transfer stage, from \( t = 1 \) onwards they play an SPE that supports payoff vector \( \{V_i\} \). If firm \( i \in N \) deviates either at the bidding stage or at the transfer stage, from \( t = 1 \) onwards firms play an SPE that gives firm \( i \) a payoff \( V_p \) (if more than one firm deviates, firms punish the lowest indexed
firm that deviated). If firm $i \notin \tilde{N}$ deviates at the entry stage and enters, firms revert to an equilibrium that gives firm $i$ a payoff of $V|\tilde{N}|+1$. If firm $i \in \tilde{N}$ does not enter, firms revert to an equilibrium that gives firm $i$ a payoff of $V_p$ starting at $t=1$. This strategy profile satisfies (O2)-(O5), and so $\beta^W$ and $x$ are sustainable in SPE.

For each $\tilde{N}$ and each $c = (c_i)_{i \in \tilde{N}}$, we define

$$b_p^*(c; \tilde{N}) \equiv \sup \left\{ b \leq r : \sum_{i \in \tilde{N}} (1 - x_i^*(c)) [b - c_i]^+ \leq \delta(V_p - |\tilde{N}|V_p) - (n - |\tilde{N}|)\sum_{i \in \tilde{N}} x_i(c) \right\},$$

where $x^*(c)$ is the efficient allocation (ties broken randomly). Let $\beta_p(c; \tilde{N}) = \max\{p, b_p^*(c; \tilde{N})\}$, and let $x_p(c) = (x_{i,p})_{i \in \tilde{N}}$ be the most efficient allocation that is consistent with (O1) given $c$ and the winning bid $\beta_p(c; \tilde{N})$. Finally, let $\tilde{N}_p^* \in \arg \max_{\tilde{N} \in 2^N} \mathbb{E}[\beta_p^*(c; \tilde{N}) - \sum_{i \in \tilde{N}} x_{i,p}(c)c_i]$.

**Proposition OB.1.** In any efficient equilibrium, on the equilibrium path, $|\tilde{N}_p^*|$ bidders enter the auction at every period and the winning bid is set equal to $\beta_p^*(c; \tilde{N}_p^*)$. Moreover, the allocation is conditionally efficient: whenever $\beta_p^*(c; \tilde{N}_p^*) > p$, the contract is allocated to the bidder with the lowest procurement cost.

**Proof.** By Lemma OB.1, there exists an optimal equilibrium in which, at every on-path history, the same number of firms participate and participating firms use the same bidding profile $(\beta, \gamma)$. For each cost vector $c = (c_i)_{i \in \tilde{N}}$, let $\beta^W(c)$ and $x(c)$ denote the winning bid and the allocation induced by this bidding profile under cost vector $c$.

We next show that, if an optimal equilibrium is such that $|\tilde{N}|$ firms participate in the auction at each period along the equilibrium path, then the winning bid must be equal to $\beta_p(c; \tilde{N})$ for all cost vectors $c = (c_i)_{i \in \tilde{N}}$.

Consider first cost vectors $c$ such that $b_p^*(c; \tilde{N}) > p$. Towards a contradiction, suppose there exists $c$ with $\beta^W(c) \neq b_p^*(c; \tilde{N}) > p$. Since $x^*(c)$ is the efficient allocation, the procurement cost under allocation $x(c)$ is at least as large as the procurement cost under allocation $x^*(c)$. Since bidding profile $(\beta, \gamma)$ is optimal, it must be that $\beta^W(c) > b_p^*(c; \tilde{N}) > p$. Indeed, if $\beta^W(c) < b_p^*(c; \tilde{N})$, then the cartel would strictly prefer to use a bidding profile that allocates the contract efficiently and has winning bid $b_p^*(c; \tilde{N})$ under cost vector $c$ than to use
bidding profile \((\beta(c), \gamma(c))\). By Lemma OB.2, \(\beta^W(c)\) and \(x(c)\) must satisfy

\[
\delta(V_p - |N|V_{p,N}) - (n - |N|)V_{p,N}^{\bar{N}+1} \geq \sum_{i \in N} \left\{ (1 - x_i(c)) \left[ \beta^W(c) - c_i \right]^+ + x_i(c) \left[ \beta^W(c) - c_i \right]^+ \right\} \\
\geq \sum_{i \in N} (1 - x_i^*(c)) \left[ \beta^W(c) - c_i \right]^+,
\]

which contradicts \(\beta^W(c) > b^*_p(c; \bar{N}) > p\). Therefore, \(\beta^W(c) = b^*_p(c; \bar{N})\) for all \(c\) such that \(b^*_p(c; \bar{N}) > p\).

Next, we show that \(\beta^W(c) = p\) for all \(c\) such that \(b^*_p(c; \bar{N}) \leq p\). Towards a contradiction, suppose there exists \(c\) with \(b^*_p(c; \bar{N}) \leq p\) and \(\beta^W(c) > p\). By Lemma OB.2, \(\beta^W(c)\) and \(x(c)\) satisfy

\[
\delta(V_p - |N|V_{p,N}) - (n - |N|)V_{p,N}^{\bar{N}+1} \geq \sum_{i \in N} \left\{ (1 - x_i(c)) \left[ \beta^W(c) - c_i \right]^+ + x_i(c) \left[ \beta^W(c) - c_i \right]^+ \right\} \\
\geq \sum_{i \in N} (1 - x_i^*(c)) \left[ \beta^W(c) - c_i \right]^+,
\]

which contradicts \(\beta^W(c) > p \geq b^*_p(c; \bar{N})\). Therefore, \(\beta^W(c) = p\) for all \(c\) such that \(b^*_p(c; \bar{N}) \leq p\). Combining this with the arguments above, \(\beta^W(c) = \beta^*_p(c; \bar{N}) = \max\{p, b^*_p(c; \bar{N})\}\).

The results above show that if in an optimal equilibrium \(|\bar{N}|\) firms participate in the auction at each period along the equilibrium path, then the winning bid is equal to \(\beta_p(c; \bar{N})\) for all cost vectors \(c = (c_i)_{i \in \bar{N}}\). For any \(\bar{N} \subset N\), winning with \(\beta_p(c; \bar{N})\) and allocation \(x_p(c)\) are sustainable in a SPE. Therefore, in an optimal equilibrium, the number of firms that participate must be equal to \(|\bar{N}^*_p|\) for some \(\bar{N}^*_p \in \arg \max_{\bar{N} \subset N} \mathbb{E}[\beta^*_p(c; \bar{N}) - \sum_{i \in \bar{N}} x_{i,p}(c) c_i]\).

Proposition OB.1 characterizes entry and bidding behavior of firms in an efficient equilibrium. We note that a large group of firms can achieve the highest surplus \(V_p\) by dividing themselves into sub-cartels of size \(|\bar{N}^*_p|\). Under such equilibria, firms would coordinate on the auctions at which each subcartel will be active. We note that this type of bidding arrangement is broadly consistent with our data. Indeed, as we show in Appendix A, the firms that participate frequently in Tsuchiura appear to be organized in smaller subgroups of firms that interact frequently among each other.

\(^4\)Recall that \(x_p(c)\) is the most efficient allocation that is consistent with (O1) when the winning bid is \(\beta_p(c; \bar{N})\).
Our next result clarifies how minimum prices affect the set of payoffs that firms can sustain in SPE.

**Proposition OB.2** (worst case punishment).  
(i) \( V_0 = 0 \), and \( V_p > 0 \) whenever \( p > \zeta \);  
\( \forall \tilde{N} \subset N, V_{\tilde{N}}[\tilde{N}] = 0 \), and \( V_{\tilde{N}}[\tilde{N}] > \delta V_p \) whenever \( p > \zeta \);  
(ii) there exists \( \overline{p} > \zeta \) such that for all \( p \in [\zeta, \overline{p}] \),  
\[ \delta (V_p - |N^*_p|V_p) - (n - |N^*_p|)V_p^{[|N^*_p|]+1} < \delta (V_0 - |N^*_0|V_0) - (n - |N^*_0|)V_0^{[|N^*_0|]+1}. \]

**Proof.** We first establish part (i). Suppose that \( p = 0 \). Consider the following entry and bidding profile. All firms in \( N \) enter the auction. Then, for all cost realizations \( c = (c_i)_{i \in N} \), all firms \( i \in N \) bid \( c(1) = \min_{k \in N} c_k \). Firm \( i \in N \) chooses \( \gamma_i = 1 \) if \( c_i = c(1) \) and chooses \( \gamma_i = 0 \) otherwise. Note that this entry and bidding profile constitute an equilibrium of the stage game, and so the infinite repetition of this strategy profile constitutes an SPE. Moreover, this strategy profile gives all players a payoff of 0, so \( V_0 = 0 \).

Consider next a subgame at which \( \tilde{N} \subset N \) entered the auction. Consider the following bidding profile: for all \( c = (c_i)_{i \in \tilde{N}} \), all firms \( i \in \tilde{N} \) bid \( c(1) = \min_{k \in \tilde{N}} c_k \). Firm \( i \in \tilde{N} \) chooses \( \gamma_i = 1 \) if \( c_i = c(1) \) and chooses \( \gamma_i = 0 \) otherwise. Then, regardless of how firms behave, starting from the next period firms play an equilibrium that gives all bidders a payoff of \( V_0 = 0 \). One can check that no firm has an incentive to deviate in the initial period, so this strategy profile constitutes an SPE. Moreover, this strategy profile gives all players a payoff of 0, so \( V_{\tilde{N}}[\tilde{N}] = 0 \).

Suppose next that \( p > \zeta \), and note that  
\[ V_p \geq v_p = \frac{1}{1 - \delta} \mathbb{E} \left[ \frac{1}{n} 1_{c_i \leq p}(p - c_i) \right] > 0, \]
where the first inequality follows since \( v_p \) is the minimax payoff for a firm in an auction with minimum price \( p \). Similarly, note that for all \( \tilde{N} \),  
\[ V_{\tilde{N}}[\tilde{N}] \geq \mathbb{E} \left[ \frac{1}{|\tilde{N}|} 1_{c_i \leq p}(p - c_i) \right] + \delta V_p. \]

Indeed, firm \( i \) can obtain at least \( \mathbb{E} \left[ \frac{1}{|\tilde{N}|} 1_{c_i \leq p}(p - c_i) \right] \) in an auction in which \( |\tilde{N}| \) firms participate; and its continuation value starting the next period must be at least as large as \( \delta V_p \). Finally, since \( \mathbb{E} \left[ \frac{1}{|\tilde{N}|} 1_{c_i \leq p}(p - c_i) \right] > 0 \) for all \( p > \zeta \), it follows that \( V_{\tilde{N}}[\tilde{N}] > \delta V_p \). This
establishes part (i).

We now turn to the proof of part (ii). Fix $p > c$, and let $|N^*_p|$, $x^p(\cdot)$ and $\beta^*_p(\cdot) = \max\{p, b^*_p(\cdot)\}$ be, respectively, the number of participants, the allocation, and the winning bid in an optimal equilibrium with minimum price $p$. The surplus that the cartel generates in an optimal equilibrium under minimum price $p$ is

$$V_p = \frac{1}{1-\delta} \mathbb{E} \left[ \beta^*_p(c) - \sum x^p_i(c) c_i \right] \quad \text{for} \quad |N^*_p| \text{ bidders participate},$$

Consider next a setting without minimum price, and consider the following strategy profile for the cartel. For all on-path histories, $|N^*_p|$ firms participate in the auction. All participating bidders bid $\beta(c) = b^*_p(c)$; participating bidder $i$ chooses $\gamma_i(c) = 1$ if $c_i$ is the lowest cost in $c$, and $\gamma_i(c) = 0$ otherwise. Note that the allocation induced by this bidding profile is the efficient allocation $x^*$. Let $\hat{V}_p$ be the total payoff that the cartel generates under this entry and bidding profile:

$$\hat{V}_p = \frac{1}{1-\delta} \mathbb{E} \left[ b^*_p(c) - \sum x^*_i(c) c_i \right] \quad \text{for} \quad |N^*_p| \text{ bidders participate}.$$ 

If no firm deviates at the entry and bidding stages, firms make transfers $T_i(c)$ to be determined below. If no firm deviates at the transfer stage, in the next period firms continue playing the same entry and bidding profile. If a firm who was not suppose to participate in the auction enters, the cartel reverts to an equilibrium that gives firm $i$ a payoff of $V_{0|N^*_0+1} = 0$; if firm $i$ who was supposed to enter does not participate, the cartel reverts to an equilibrium that gives bidder $i$ a continuation payoff of $V_{0} = 0$; if a firm $i$ that participates in the auction deviates at the bidding stage, there are no transfers and the cartel reverts to an equilibrium that gives firm $i$ a continuation payoff of $V_{0} = 0$; if firm $i \in N$ deviates at the transfer stage, the cartel reverts to an equilibrium that gives firm $i$ a continuation payoff of $V_{0} = 0$ (deviations by more than one firm go unpunished).

Before constructing the transfers $T(c)$, note that

$$V_p - \hat{V}_p = \frac{1}{1-\delta} \mathbb{E} \left[ (p - b^*_p(c)) - \sum (x^p_i(c) - x^*_i(c)) c_i \right] \mathbf{1}_{b^*_p(c) < p} \quad \text{for} \quad |N^*_p| \text{ bidders participate},$$ 

$$\leq \frac{1}{1-\delta} \mathbb{E} \left[ (p - b^*_p(c)) \mathbf{1}_{b^*_p(c) < p} \right] \quad \text{for} \quad |N^*_p| \text{ bidders participate},$$

where the first equality follows since $x^p(c) = x^*(c)$ whenever $\beta^*_p(c) = b^*_p(c) > p$, and the inequality follows since $x^*$ is the efficient allocation. Note that $b^*_p(c) \geq c + \Delta$ for some
\[ \Delta > 0. \]

Let \( \overline{c} \equiv \underline{c} + \Delta \). Then, for all \( p \in (\underline{c}, \overline{c}) \), \( b_p^*(c) \geq p \), and so \( \delta \hat{V}_p \geq \delta \hat{V}_p > \delta(V_p - |N_p^*|V_p) - (n - |N_p^*|V_p^{[N_p^*]+1}) \) (where the last inequality follows from part (i) of the Lemma).

Set \( p \in (\underline{c}, \overline{c}) \). The transfers we construct are as follows. Let \( N_p^* \subset N \) be the set of firms that participate. Then, for all \( i \in N \),

\[
T_i(c) = \begin{cases} 
-\delta \hat{V}_p + (1 - x_i^*(c))(b_p^*(c) - c_i) + \epsilon(c) & \text{if } i \in N_p^*, c_i \leq \beta(c), \\
-\delta \hat{V}_p + \epsilon(c) & \text{if } i \not\in N_p^*,
\end{cases}
\]

where \( \epsilon(c) \geq 0 \) is a constant to be determined below. Note that, for all \( c \),

\[
\sum_{i \in N} T_i(c) - n\epsilon(c) = -\delta \hat{V}_p + \sum_{i \in N_p^*} (1 - x_i^*(c))(b_p^*(c) - c_i) + \epsilon(c) \leq 0,
\]

where the first inequality follows since \( \delta \hat{V}_p > \delta(V_p - |N_p^*|V_p) - (n - |N_p^*|V_p^{[N_p^*]+1}) \), and the last one follows from the definition of \( b_p^*(c) \).

One can check that, under this strategy profile, no firm has an incentive to deviate at any stage. Hence, this strategy profile is a SPE, and so \( V_0 \geq \hat{V}_p \). Since \( \delta \hat{V}_p > \delta(V_p - |N_p^*|V_p) - (n - |N_p^*|V_p^{[N_p^*]+1}) \), it follows that \( \delta \hat{V}_0 > \delta(V_p - |N_p^*|V_p) - (n - |N_p^*|V_p^{[N_p^*]+1}) \).

Proposition OB.2 shows that, when entry is endogenous, minimum prices limit the cartel’s surplus in two ways. First, as in our baseline model, minimum prices limit the cartel’s ability to punish firms that deviate at the bidding stage, thereby reducing the bids that can be sustained in a SPE. Second, minimum prices increase the cost of keeping potential participants out of the auction.

**OB.3 Large cartel limit**

We now discuss the cartel’s ability to sustain high prices at the large cartel limit, i.e. when the number \( n \) of cartel members grows large. We first consider the case where minimum

\[ \overline{c} \equiv \underline{c} + \Delta. \]

Indeed, \( b_p^*(c) \) attains its lowest value equal to when all participating firms have cost \( \underline{c} \): this lowest value is \( \underline{c} + \frac{1}{|N_p^*| - 1}(\delta(V_p - |N_p^*|V_p) - (n - |N_p^*|V_p^{[N_p^*]+1})). \)
prices $p$ are set to 0.

We first consider the case of exogenous participation described in the main text. In this case we assume that $|\hat{N}_t| \geq \rho n$ for some $\rho \in (0, 1)$. The highest sustainable price is determined by condition (1) in the main text. Since pledgeable surplus is bounded above by $\frac{1}{1-\delta}(r - \xi)$ (since production costs are bounded below by $\xi$), it must be that the highest sustainable price converges to $c$ almost surely as the cartel size $n$ becomes large. As a result expected cartel profits must go to zero as the cartel grows large.

In contrast, when the number of participants is endogenous as in the previous subsection, expected profits are weakly increasing in cartel size. This follows from the fact that when minimum price $p$ is equal to zero the cartel can costlessly control the number of participants in each auction. Since costs are public, any non-equilibrium entrant can be deprived of surplus by setting prices to her cost of production. In formal terms, $V_{|\hat{N}|+1} | p = 0 = 0$ (see Proposition OB.2).

This implies that in the absence of minimum prices, the fact that the number of cartel members in our data is large does not hinder the cartel’s ability to sustain high prices. What matters isn’t the total size of the cartel, but the number of cartel members participating in each auction. This finding is consistent with our data. While the number of high-frequency participants in our data ranges from 0 to 13 across years, the median number of participants in a given auction is equal to 3. We also note that large cartels are not unheard off in the field of construction. A 2008 press release by the UK’s Office of Fair Trading noted that it had filed a case against 112 firms in the construction sector.\(^6\) Reportedly, at least 80 of these firms have admitted engaging in bid-rigging.\(^7\) We also note that firms in this cartel used monetary transfers. Another example of large scale collusion is the Dutch construction cartel, which included approximately 650.\(^8\)

Interestingly, minimum prices also make sustaining cartels with endogenous participation more difficult. It is no longer costless to keep potential participants from entering since $V_{|\hat{N}|+1} > 0$ whenever $p > c$. As a result, the introduction of minimum prices increases participation by cartel members, making it more difficult to sustain high prices. Table A.2 shows that this is true in our data. Following the introduction of minimum prices the number of both cartel participants and entrants increases.

\(^7\)https://en.wikipedia.org/wiki/Price_fixing_cases#Construction.
OC  Measurement Error and Ommited Variable Bias

OC.1 Measurement Error

Proposition 6 requires conditioning on the entrant vs. long-run player status of the winning bidder. In this Appendix we show that Proposition 6 is robust to some forms of measurement error. The main requirement is that no long-run player be wrongly classified as an entrant. This motivates our choice to err on the side of inclusiveness when classifying firms as long-run players in our empirical analysis.

Let $E_W \in \{0, 1\}$ denote the entrant ($E_W = 1$) or long-run player ($E_W = 0$) status of the winning bidder. Proposition 6 establishes that under collusion, there exists $\eta > 0$ such that, for all $p \in [\beta_0^*(), \beta_0^*() + \eta]$ and all $q > p$:

(i) $\text{prob}(\beta_p^* \geq q | \beta_p^* \geq p, E_W = 0) \leq \text{prob}(\beta_0^* \geq q | \beta_0^* \geq p, E_W = 0);$  
(ii) $\text{prob}(\beta_p^* \geq q | \beta_p^* > p, E_W = 1) = \text{prob}(\beta_0^* \geq q | \beta_0^* > p, E_W = 1).$

Now assume that we only observe a signal $\hat{E}_W \in \{0, 1\}$ of $E_W$. In our empirical analysis, $\hat{E}_W = 0$ if the auction winner is a sufficiently frequent participant. By adjusting the participation-threshold above which a bidder is declared a long-run player, we can trade-off the misclassification of entrants as long-run players, and the misclassification of long-run players as entrants.

Assumption OC.1. Assume that

(i) $\text{prob}(E_W | \hat{E}_W, \beta, p) = \text{prob}(E_W | \hat{E}_W);$  
(ii) $\text{prob}(E_W = 0 | \hat{E}_W = 1) = 0.$

Assumption OC.1 states that measurement error is independent of winning bids, and that true long-run players are never classified as entrants.

Proposition OC.1. If Assumption OC.1 holds, then there exists $\eta > 0$ such that, for all $p \in [\beta_0^*(), \beta_0^*(()) + \eta]$ and all $q > p$:

(i) $\text{prob}(\beta_p^* \geq q | \beta_p^* \geq p, \hat{E}_W = 0) \leq \text{prob}(\beta_0^* \geq q | \beta_0^* \geq p, \hat{E}_W = 0);$  
(ii) $\text{prob}(\beta_p^* \geq q | \beta_p^* > p, \hat{E}_W = 1) = \text{prob}(\beta_0^* \geq q | \beta_0^* > p, \hat{E}_W = 1).$
**Proof.** Let $p$ be such that Proposition 6 holds. We first establish point (ii). Assume firms are collusive. We have that

\[
\text{prob}(\beta^*_p \geq q|\beta^*_p > p, \hat{E}_W = 1) = \text{prob}(\beta^*_p \geq q|\beta^*_p > p, E_W = 0) \text{prob}(E_W = 0|\hat{E}_W = 1) \\
+ \text{prob}(\beta^*_p \geq q|\beta^*_p > p, E_W = 1) \text{prob}(E_W = 1|\hat{E}_W = 1) \\
= \text{prob}(\beta^*_p \geq q|\beta^*_p > p, E_W = 1) \text{prob}(E_W = 1|\hat{E}_W = 1) \\
= \text{prob}(\beta^*_0 \geq q|\beta^*_0 > p, E_W = 1) \text{prob}(E_W = 1|\hat{E}_W = 1) \\
= \text{prob}(\beta^*_0 \geq q|\beta^*_0 > p, \hat{E}_W = 1)
\]

where we used the assumption that $\text{prob}(E_W = 0|\hat{E}_W = 1)$ and Proposition 6 (ii).

Point (i) follows from a similar line of reasoning. We have that

\[
\text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, \hat{E}_W = 0) = \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E_W = 0) \text{prob}(E_W = 0|\hat{E}_W = 0) \\
+ \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E_W = 1) \text{prob}(E_W = 1|\hat{E}_W = 0) \\
\leq \text{prob}(\beta^*_0 \geq q|\beta^*_0 \geq p, E_W = 0) \text{prob}(E_W = 0|\hat{E}_W = 0) \\
+ \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E_W = 1) \text{prob}(E_W = 1|\hat{E}_W = 0)
\]

(O6)

Observe that if $E_W = 1$, then $\beta^*_p > p$ and $\beta^*_0 > p$ both imply that $\beta^*_p = \beta^*_0$. Hence

\[
\text{prob}(\beta^*_p \geq q|\beta^*_p > p, E_W = 1) = \text{prob}(\beta^*_p \geq q|\beta^*_p > p, E_W = 1) \text{prob}(\beta^*_p > p|\beta^*_p \geq p, E_W = 1) \\
= \text{prob}(\beta^*_p \geq q|\beta^*_0 > p, E_W = 1) \text{prob}(\beta^*_p > p|\beta^*_0 \geq p, E_W = 1)
\]

Since $\beta^*_0 = p$ implies $\beta^*_p = p$ it follows that $\text{prob}(\beta^*_p > p|\beta^*_p \geq p, E_W = 1) \leq \text{prob}(\beta^*_0 > p|\beta^*_0 \geq p, E_W = 1)$. Substituting into (O6), and we get that indeed

\[
\text{prob}(\beta^*_p \geq q|\beta^*_p > p, \hat{E}_W = 0) \leq \text{prob}(\beta^*_0 \geq q|\beta^*_0 \geq p, \hat{E}_W = 0).
\]

**OC.2 Omitted variable bias**

If participation is correlated with both unobserved auction characteristics and the introduction of minimum prices, OLS estimates of the impact of minimum prices on winning bids controlling for the number of auction participants will be biased.
Consider the simple linear model of centered winning bids $\beta_W$

$$
\beta_W = \langle X, \alpha \rangle + \gamma Z + \varepsilon
$$

where: centered observable characteristics $X = (\text{min\_price}, N)$ include minimum-price-status and participation; $Z$ is an unobserved auction characteristic correlated with participation. Then the OLS estimator $\hat{\alpha}$ takes the form

$$
\hat{\alpha} = (X'X)^{-1}X'\beta_W = \alpha + \gamma(X'X)^{-1}X'Z + (X'X)^{-1}X'\varepsilon.
$$

Note that we can always change the sign of the omitted variable so that $\gamma > 0$. The free variable is then the correlation between the omitted variable and participation. We assume the omitted variable is uncorrelated to minimum-price-status.

We address the possibility of omitted variable bias in two ways. First, we formulate a simple instrumentation strategy using recent past participation for similar auctions as an instrument. Second, in case it cannot be successfully resolved by instrumentation, we discuss the potential sign of this bias.

**Instrumentation.** One omitted variable of prominent interest that could be taken care of by this strategy is erroneously high reserve prices: if city engineers sometimes overestimate maximum costs, this may jointly lead to more entry and higher prices.

To address this type of bias, we propose to use the number of bidders in previous comparable auctions as an instrument for current participation. This variable is strongly correlated with the current number of bidders and uncorrelated with auction-specific omitted variables – plausibly including erroneously high reserve prices.

Our empirical findings are reported in Table A.3 of the main Appendix. Our main empirical results continue to hold when we instrument the number of bidders with its lagged value:

- the introduction of a minimum price has a negative effect on winning bids, and
- the effect of the policy change is concentrated on the auctions won by bidders who participate frequently.

**Likely sign of the bias.** It is useful to evaluate the sign of potential bias absent instrument-ation, in the event that the assumptions needed for successful instrumentation do not hold.
Denote \(\begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\) the coefficients of \((X'X)\). Matrix \((X'X)^{-1}\) takes the form

\[
\frac{1}{\sigma_1^2 \sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}.
\]

Since \(\sigma_1^2 \sigma_2^2 - \sigma_{12}^2 > 0\) (by Cauchy-Schwarz) and \(X'Z = \begin{pmatrix} 0 \\ NZ \end{pmatrix}\), it follows that the bias has the sign of \(\begin{pmatrix} -\sigma_{12}NZ \\ \sigma_1^2 NZ \end{pmatrix}\).

We are specifically interested in \(-\sigma_{12}NZ\): this is the bias in our estimate of the impact of minimum prices on winning bids. Note that the covariance \(\sigma_{12}\) between minimum price status and participation is positive. Hence the bias in our estimate of the impact of minimum prices is: positive if participation is negatively correlated with the omitted variable; negative if participation is positively correlated with the omitted variable.

Subjectively, it seems more plausible that entry will be positively correlated with omitted variables that also increase winning bids. This would be the case if the omitted variable is erroneously high reserve prices. In this case, omitted variable bias would go against our findings.

### OD Calibration

Our calibration exercise seeks to gauge the range of plausible treatment effects one may have expected from a model such as ours. As a result we do not seek to estimate costs from bids. Instead, we consider distribution of costs obtained by deflating winning bids with a fixed markup. This rough assumption lets us get back-of-the-envelope estimates of average and conditional treatment effects.

#### Modeling choices and degrees of freedom.

We implement directly the model of Section 4. Our key modeling choices and degrees of freedom are the following:

- We fix the number of cartel bidders to three in each auction. An entrant participates with probability \(q\) in the range \([.6, .7]\). In data from Tsuchiura, on average three cartel members participate in each auction, and bidders labelled as entrants are present in 66% of auctions.
• We keep the firms’ yearly discount factor $\delta_Y$ as a free parameter in the range $[.7, .9]$. We note that auctions are not regularly spread out within the year, but rather occur in batches. This generates an effective discount factor $\delta = \delta_Y^{\frac{D}{365}}$, where $D$ is the average number of days between batches. The mean delay is 19 days.

• We do not estimate a cost distribution from winning bids but investigate treatment effects for not-implausible cost-distributions obtained in the following back of the envelope manner. Given the empirical distribution of winning bids $b$, we draw 4 independent values $\tilde{c}_i, i \in \{1, \ldots, 4\}$ according to distribution $c_i \sim \frac{1}{1+M} b$, where $M$ is a fixed markup taking values in the range $[.2,.6]$. We then set as costs

$$c_i = \lambda \frac{\sum_{i=1}^{3} \tilde{c}_i}{3} + (1 - \lambda) \bar{c}_i$$

$$c_4 = \lambda \frac{\sum_{i=1}^{3} \tilde{c}_i}{3} + (1 - \lambda) \bar{c}_4$$

where $\lambda$ parametrizes the correlation between the costs of participating cartel members. Given $\lambda$, the correlation between the costs of two cartel members is $\lambda^2 + \frac{2}{3} \lambda (1 - \lambda)$. Cost $c_4$ is the entrant’s cost if an entrant enters. In our data, correlation between bids is above 99%. We consider values of $\lambda$ in the range $[.95,.99]$.

The reserve price $r$ is set at

$$r = (1 + m) \times \frac{\sum_{i=1}^{3} c_i}{3}$$

where $m$ is in the range $[.4,.6]$.

• Minimum prices are a constant ratio of the reserve price. Consistent with our data we set this minimum price ratio in the range $[.75,.8]$. 

• We assume that cartel members follow the equilibrium strategies of the model in Section 4. We describe these strategies in detail in Appendix OE.2.

Findings. For each configuration of the parameters above, we simulate 1000 auctions with and without a minimum price. We compute the percentage change in average winning bids following the introduction of minimum prices for the unconditional distribution of winning bids, and for the conditional distribution of winning bids above the minimum price. We refer
to these percentage changes in average procurement costs as the average and conditional treatment effects.

Figure OD.1 reports the conditional treatment effects for each of the configurations of parameters above. As anticipated, conditional treatment effects are negative. Their range, goes from $-28\%$ to $-3\%$ and includes conditional treatment effects of the magnitude we find in our data.

Figure OD.2 reports the unconditional treatment effects for each of the configuration of parameters above. Treatment effects can be negative or positive. Their range, goes from $-11\%$ to $+11\%$ and includes unconditional treatment effects of the magnitude we find in our data. As Figure OD.3 shows, a key factor in explaining whether the average treatment effect is negative is the minimum price ratio. When it is relatively low, the truncation of the left tail of winning bids does not affect average winning bids much. When it is high, the truncation of the left tail of winning bids cannot be compensated by a drop in the right tail of winning bids.

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10Therefore the distribution of treatment effects is the one induced by placing a uniform distribution over the product set of parameters we consider.
Figure OD.2: Unconditional treatment effects.

Figure OD.3: Unconditional treatment effect increase with the minimum price ratio.
OE  Proofs

OE.1 Proofs of Section 3

Proof of Proposition 4. We first show that there exists a symmetric equilibrium as described in the statement of the proposition, and then we show uniqueness.

Consider first a minimum price \( p \leq b^A_0(\bar{c}) \). Clearly, in this case all firms using the bidding function \( b^A_0(\cdot) \) is a symmetric equilibrium of the auction with minimum price \( p \).

Consider next the case in which \( b^A_0(\bar{c}) < p \). For any \( c \in [\underline{c}, \bar{c}] \), define

\[
P(c) \equiv \sum_{j=0}^{\tilde{N}-1} \left( \frac{\tilde{N} - 1}{j} \right) \frac{1}{j+1} F(c)^j (1 - F(c))^{\tilde{N}-j-1}.
\]

\( P(c) \) is the probability with which a firm with cost \( c' \leq c \) wins the auction if all firms use a bidding function \( \beta(\cdot) \) with \( \beta(c') = b \geq p \) for all \( c' \leq c \) and \( \beta(c') > b \) for all \( c' > c \).

Let \( \hat{c} \in (\underline{c}, \bar{c}) \) be the unique solution to \( P(\hat{c})(p - \hat{c}) = (1 - F(\hat{c}))^{\tilde{N}-1}(b^A_0(\hat{c}) - \hat{c}) \).\(^{11}\)

Let \( b^A_p(\cdot) \) be given by

\[
b^A_p(c) = \begin{cases} 
b^A_0(c) & \text{if } c \geq \hat{c}, \\
p & \text{if } c < \hat{c}.
\end{cases}
\]

Note that if all firms bid according to bidding function \( b^A_p(\cdot) \), the probability with which a firm with cost \( c < \hat{c} \) wins the auction is \( P(\hat{c}) \). We now show that all firms bidding according to \( b^A_p(\cdot) \) is an equilibrium.

Suppose that all firms \( j \neq i \) bid according to \( b^A_p(\cdot) \). Note first that it is never optimal for firm \( i \) to bid \( b \in (p, b^A_p(\hat{c})) \). Indeed, if \( c_i < b^A_p(\hat{c}) \), bidding \( b \in (p, b^A_p(\hat{c})) \) gives firm \( i \) a strictly lower payoff than bidding \( b^A_p(\hat{c}) \): in both cases firm \( i \) wins with probability \( (1 - F(\hat{c}))^{\tilde{N}-1} \), but by bidding \( b^A_p(\hat{c}) \) the firm gets a strictly larger payoff in case of winning.

If \( c_i > b^A_p(\hat{c}) \), bidding \( b \in (p, b^A_p(\hat{c})) \) gives firm \( i \) a strictly lower payoff than bidding \( b^A_p(c_i) \).

Suppose that \( c_i \geq \hat{c} \). Since \( b^A_p(x) = b^A_0(x) \) for all \( x \geq \hat{c} \), firm \( i \) with cost \( c_i \) gets a larger payoff bidding \( b^A_p(c_i) \) than bidding \( b^A_p(x) \) with \( x \in [\hat{c}, \bar{c}] \). If \( c_i = \hat{c} \), firm \( i \) is by construction

\(^{11}\)Note first that such a \( \hat{c} \) always exists whenever \( b^A(\bar{c}) < p \). Indeed, in this case \( P(\bar{c})(p - \bar{c}) = p - \bar{c} > b^A_0(\bar{c}) - \bar{c} \), while \( P(p)(p - p) = 0 < (1 - F(p))^{\tilde{N}-1}(b^A_0(p) - \bar{c}) \). By the Intermediate value Theorem, there exists \( \hat{c} \in [\underline{c}, p] \) such that \( P(\hat{c})(p - \hat{c}) = (1 - F(\hat{c}))^{\tilde{N}-1}(b^A_0(\hat{c}) - \hat{c}) \). Moreover, for all \( c \leq p \), \( \frac{\partial}{\partial c} P(c)(p - c) = -P(c) + P'(c)(p - c) \leq -P(c) < -(1 - F(c))^{\tilde{N}-1} = \frac{\partial}{\partial c}(1 - F(c))^{\tilde{N}-1}(b^A_0(c) - c) \), so \( \hat{c} \) is unique.
indifferent between bidding \( p \) and bidding \( b_p^{AI}(\hat{c}) \). Moreover, for all \( c_i > \hat{c} \),

\[
(1 - F(c_i))\tilde{N}^{-1}(b_p^{AI}(c_i) - c_i) \geq (1 - F(\hat{c}))\tilde{N}^{-1}(b_p^{AI}(\hat{c}) - \hat{c}) + (1 - F(\hat{c}))\tilde{N}^{-1}(\hat{c} - c_i) \\
= P(\hat{c})(p - \hat{c}) + (1 - F(\hat{c}))\tilde{N}^{-1}(\hat{c} - c_i) \\
> P(\hat{c})(p - \hat{c}) + P(\hat{c})(\hat{c} - c_i),
\]

where the strict inequality follows since \( P(\hat{c}) > (1 - F(\hat{c}))\tilde{N}^{-1} \) and \( c_i > \hat{c} \). Hence, firm \( i \) strictly prefers to bid \( b_p^{AI}(c_i) \) when her cost is \( c_i > \hat{c} \) than to bid \( p \). Combining all these arguments, a firm with cost \( c_i \geq \hat{c} \) finds it optimal to bid \( b_p^{AI}(c_i) \) when her cost is \( c_i \geq \hat{c} \).

Finally, suppose that \( c_i < \hat{c} \). Firm \( i \)'s payoff from bidding \( b_p^{AI}(c_i) = p \) is \( P(\hat{c})(p - c_i) \). Note that, for all \( c \geq \hat{c} \),

\[
P(\hat{c})(p - c_i) = P(\hat{c})(p - \hat{c}) + P(\hat{c})(\hat{c} - c_i) \\
\geq (1 - F(c))\tilde{N}^{-1}(b_p^{AI}(c) - \hat{c}) + P(\hat{c})(\hat{c} - c_i) \\
> (1 - F(c))\tilde{N}^{-1}(b_p^{AI}(c) - c_i),
\]

where the first inequality follows since \( P(\hat{c})(p - \hat{c}) = (1 - F(\hat{c}))\tilde{N}^{-1}(b_p^{AI}(\hat{c}) - \hat{c}) \geq (1 - F(c))\tilde{N}^{-1}(b_p^{AI}(c) - \hat{c}) \) for all \( c \geq \hat{c} \), and the second inequality follows since \( P(\hat{c}) > (1 - F(c))\tilde{N}^{-1} \) for all \( c \geq \hat{c} \) and since \( c_i < \hat{c} \). Therefore, firm \( i \) finds it optimal to bid \( b_p^{AI}(c_i) = p \) when her cost is \( c_i < \hat{c} \).

Next we establish uniqueness. We start with a few preliminary observations. Fix an auction with minimum price \( p > 0 \) and let \( b_p \) be the bidding function in a symmetric equilibrium. By standard arguments (see, for instance, Maskin and Riley (1984)), \( b_p \) must be weakly increasing; and it must be strictly increasing and differentiable at all points \( c \) such that \( b_p(c) > p \). Lastly, \( b_p \) must be such that \( b_p(\bar{c}) = \bar{c} \).\(^{12}\)

Consider a bidder with cost \( c \) such that \( b_p(c) > p \), and suppose all of her opponents bid according to \( b_p \). The expected payoff that this bidder gets from bidding \( b_p(\bar{c}) > p \) is \( (1 - F(\bar{c}))\tilde{N}^{-1}(b_p(\bar{c}) - c) \). Since bidding \( b_p(c) > p \) is optimal, the first-order conditions imply that \( b_p \) solves

\[
b_p'(c) = \frac{f(c)}{1 - F(c)}(\tilde{N} - 1)(b_p(c) - c),
\]

with boundary condition \( b_p(\bar{c}) = \bar{c} \). Note that bidding function \( b_0^{AI} \) solves the same differential equation with the same boundary condition, and so \( b_p(c) = b_0^{AI}(c) \) for all \( c \) such that

\(^{12}\)This condition holds for the case in which \( r \geq \bar{c} \). If \( r < \bar{c} \), then \( b_p \) must be such that \( b_p(r) = r \).
such that \( b_p(c) > p \).

Consider the case in which \( p < b_0^{AI}(c) \), and suppose that there exists a symmetric equilibrium \( b_p \neq b_0^{AI} \). By the previous paragraph, \( b_p(c) = b_0^{AI}(c) \) for all \( c \) such that \( b_p(c) > p \). Therefore, if \( b_p \neq b_0^{AI} \) is an equilibrium, there must exist \( c > c \) such that \( b_p(c) = p \) for all \( c < c \), and \( b_p(c) = b_0^{AI}(c) \) for all \( c \geq c \). For this to be an equilibrium, a bidder with cost \( c \) must be indifferent between bidding \( b_0^{AI}(c) = b_p(c) \) or bidding \( p \): \( P(\hat{c})(p - \hat{c}) = (1 - F(\hat{c}))^{1 - \hat{b}}(b_0^{AI}(\hat{c}) - \hat{c}) \).

But this can never happen when \( p < b_0^{AI}(c) \) since \( P(c)(p - c) = p - c < b_0^{AI}(c) - c \), and for all \( c \in [c, p] \), \( \frac{\partial}{\partial c} P(c)(p - c) = -P(c) + P'(c)(p - c) \leq -P(c) < -(1 - F(c))^{1 - \hat{b}} = \frac{\partial}{\partial c}(1 - F(c))^{1 - \hat{b}}(b_0^{AI}(c) - c) \). Therefore, in this case the unique symmetric equilibrium is \( b_0^{AI} \).

Consider next the case with \( p > b_0^{AI}(c) \). By the arguments above, any symmetric equilibrium \( b_p \) must be such that \( b_p(c) = b_0^{AI}(c) \) for all \( c \) with \( b_p(c) > p \). Therefore, in any symmetric equilibrium, there exists \( c > c \) such that \( b_p(c) = p \) for all \( c < c \), and \( b_p(c) = b_0^{AI}(c) \) for all \( c \geq c \). Moreover, \( c \) satisfies \( P(c)(p - c) = (1 - F(c))^{1 - \hat{b}}(b_0^{AI}(c) - \hat{c}) \). When \( p > b_0^{AI}(c) \), there exists a unique such \( c \) (see footnote 11). Therefore, in this case the unique symmetric equilibrium is \( b_0^{AI} \).

\[ \hat{b} \]

**Proof of Corollary 4.** Suppose first that \( p \leq b_0^{AI}(c) \). Then, \( \text{prob}(\beta_p^{AI} \geq q|\beta_p^{AI} > p) = \text{prob}(\beta_0^{AI} \geq q|\beta_0^{AI} > p) \) for all \( q > p \).

Consider next the case in which \( p > b_0^{AI}(c) \). For all \( b \in [b_0^{AI}(c), b_0^{AI}(\tau)] \), let \( c(b) \) be such that \( b_0^{AI}(c(b)) = b \). Since \( \hat{c} \) is such that \( b_0^{AI}(\hat{c}) > p \), it follows that \( \hat{c} > c(p) \). Note then that, for all \( q \geq b_0^{AI}(\hat{c}) \), \( \text{prob}(\beta_p^{AI} \geq q|\beta_p^{AI} > p) = \frac{(1 - F(c(q)))^{\hat{b}}}{(1 - F(c(p)))^{\hat{b}}} > \frac{(1 - F(c(q)))^{\hat{b}}}{(1 - F(c(p)))^{\hat{b}}} = \text{prob}(\beta_0^{AI} \geq q|\beta_0^{AI} > p) \).

For \( q \in (p, b_0^{AI}(\hat{c})) \), \( \text{prob}(\beta_p^{AI} \geq q|\beta_p^{AI} > p) = 1 > \frac{(1 - F(c(q)))^{\hat{b}}}{(1 - F(c(p)))^{\hat{b}}} = \text{prob}(\beta_0^{AI} \geq q|\beta_0^{AI} > p) \).

\[ \hat{b} \]

**OE.2 Additional results and Proofs for Section 4**

This appendix analyzes the model with entry in Section 4. We let \( \hat{N}_e \) denote the set of all participants in the auction; i.e., \( \hat{N}_e = \hat{N} \) when \( E = 0 \), and \( \hat{N}_e = \hat{N} \cup \{e\} \) when \( E = 1 \). Given a history \( h_t \) and an equilibrium \( \sigma \), we let \( \beta(c|h_t, \sigma) \) be the bidding profile of cartel members and short-lived firm induced by \( \sigma \) at history \( h_t \) as a function of procurement costs \( c = (c_i)_{i \in \hat{N}_e} \).\(^{13}\) Our first result generalizes Lemma 1 to the current setting.

\(^{13}\)Since the vector of costs \( c \) includes the cost of the short-lived firm in case of entry, the cartel’s bidding profile can be different depending on whether the short-lived firm enters the auction or not.
Lemma OE.1 (stationarity – entry). Consider a subgame perfect equilibrium $\sigma$ that attains $V_p$. Equilibrium $\sigma$ delivers surplus $V(\sigma, h_t) = V_p$ after all on-path histories $h_t$.

There exists a fixed bidding profile $\beta^*$ such that, in a Pareto efficient equilibrium, firms bid $\beta(c_i| h_t, \sigma) = \beta^*(c_i)$ after all on-path histories $h_t$.

Proof. The proof is identical to the proof of Lemma 1, and hence omitted. ■

Given a bidding profile $(\beta, \gamma)$, we let $\beta^W(c)$ be the winning bid and $x(c) = (x_i(c))_{i \in \tilde{N}_c}$ be the induced allocation when realized costs are $c = (c_i)_{i \in \tilde{N}_c}$. As in Section 2, for all $i \in \tilde{N}_c$ we let

$$\rho_i(\beta^W, \gamma, x)(c) \equiv 1_{\beta^W(c) > p} + \frac{1_{\beta^W(c) = p}}{\sum_{j \in \tilde{N}_c \setminus \{i\}, x_j(c) > 0} \gamma_j(c) + 1}.$$

Lemma OE.2 (enforceable bidding – entry). A winning bid profile $\beta^W(c)$ and an allocation $x(c)$ are sustainable in SPE if and only if, for $E \in \{0, 1\}$ and for all $c$,

$$\sum_{i \in \tilde{N}} \{(\rho_i(\beta^W, \gamma, x)(c) - x_i(c))|\beta^W(c) - c_i|^+ + x_i(c)|\beta^W(c) - c_i|^- \} \leq \delta(V_p - \sum_{i \in \tilde{N}} V_{i,p}). \quad (O7)$$

$$E \times \{(\rho_e(\beta^W, \gamma, x)(c) - x_e(c))|\beta^W(c) - c_e|^+ + x_e(c)|\beta^W(c) - c_e|^- \} \leq 0. \quad (O8)$$

Proof. We start with a few preliminary observations. Fix an SPE $\sigma$ and a history $h_t$, and suppose that the entry decision of the short-lived firm at time $t$ is $E$. For each $c$, let $\beta(c)$, $\gamma(c)$ and $T(c, b, \gamma, x)$ be the bidding profile of cartel members and short-lived firm and the transfer profile of cartel members in this equilibrium after history $h_t \cup (E, c)$. For each $c$, let $\beta^W(c)$ and $x(c)$ be winning bid and the allocation induced by bidding profile $(\beta(c), \gamma(c))$. For each $h_{t+1} = h_t \cup (E, c, b, \gamma, x, T)$, let $\{V(h_{t+1})\}_{i \in \tilde{N}}$ be the vector of continuation payoffs of cartel members after history $h_{t+1}$. We let $h_{t+1}(c) = h_t \cup (E, c, \beta(c), \gamma(c), x(c), T(c, \beta(c), \gamma(c), x(c)))$ denote the on-path history that follows $h_t \cup (E, c)$. With this notation, the inequalities (7)-(9) in Appendix B must also hold in this setting. Moreover, if $E = 1$, it must also be that

$$x_e(c)[\beta^W(c) - c_e]^+ \geq \rho_e(\beta^W, \gamma, x)(c)[\beta^W(c) - c_e]^+ \text{ and } x_e(c)[\beta^W(c) - c_e]^\leq 0. \quad (O9)$$

Conversely, suppose there exists a winning bid profile $\beta^W(c)$, an allocation $x(c)$, a transfer profile $T$ and equilibrium continuation payoffs $\{V_i(h_{t+1}(c))\}_{i \in \tilde{N}}$ that satisfy inequalities (7)-(9) in Appendix B for some $\gamma(c)$ that is consistent with $x(c)$ (i.e., $x_i(c) = \gamma_i(c)/\sum_{j : x_j(c) > 0} \gamma_j(c)$).
for all $i \in \hat{N}_e$ with $x_i(c) > 0$ and satisfy (O9) if $E = 1$. Then, $(\beta^W, x, T)$ can be supported in an SPE as follows. For all $c$, all firms $i \in \hat{N}_e$ bid $\beta^W(c)$. Firms $i \in \hat{N}_e$ with $x_i(c) = 0$ choose $\tilde{\gamma}_i(c) = 0$ and firms $i \in \hat{N}_e$ with $x_i(c) > 0$ choose $\tilde{\gamma}_i(c) = \gamma_i(c)$. Note that, for all $i \in \hat{N}_e$, $x_i(c) = \tilde{\gamma}_i(c)/\sum_j \tilde{\gamma}_j(c)$ and $\rho_i(\beta^W, \tilde{\gamma}, x)(c) = \rho_i(\beta^W, \gamma, x)(c)$. If no firm $i \in \hat{N}$ deviates at the bidding stage, cartel members make transfers $T_i(c, \beta(c), \gamma(c), x(c))$. If no firm $i \in \hat{N}$ deviates at the transfer stage, in the next period cartel members play an SPE that gives payoff vector $\{V(h_{t+1}(c))\}_{i \in N}$. If firm $i \in \hat{N}$ deviates at the bidding stage, there are no transfers and the cartel reverts to an equilibrium that gives firm $i$ a payoff of $V_{i,p}$; if firm $i \in N$ deviates at the transfer stage, the cartel reverts to an equilibrium that gives firm $i$ a payoff of $V_{i,p}$ (deviations by more than one firm go unpunished). Since (7) holds, under this strategy profile no firm $i \in \hat{N}$ has an incentive to undercut the winning bid $\beta^W(c)$. Since (8) holds, no firm $i \in \hat{N}$ with $c_i > \beta^W(c)$ and $x_i(c) > 0$ has an incentive to bid above $\beta^W(c)$ and lose. Upward deviations by a firm $i \in \hat{N}_e$ with $c_i < \beta^W(c)$ who bids $\beta^W(c)$ are not profitable since the firm would lose the auction by bidding $b > \beta^W(c)$. Since (O9) holds, the short-lived firm does not have an incentive to deviate when $E = 1$. Finally, since (9) holds, all firms $i \in N$ have an incentive to make their required transfers.

We now turn to the proof of the Lemma. The proof that (O7) must hold in any equilibrium uses the same arguments used in the proof of Lemma 2, and hence we omit it. Since (O9) must hold for $E = 1$, it follows that

$$E \times \{(\rho_e(\beta^W, \gamma, x)(c) - x_e(c))[\beta^W(c) - c_e]^+ + x_e(c)[\beta^W(c) - c_e]^- \} \leq 0.$$ 

Next, consider a winning bid profile $\beta^W(c)$ and an allocation $x(c)$ that satisfy (O7) and (O8) for all $c$ for some $\gamma(c)$ consistent with $x(c)$ (i.e., such that $x_i(c) = \gamma_i(c)/\sum_{j,x_j(c)>0} \gamma_j(c)$ for all $i$ with $x_i(c) > 0$). We construct an SPE that supports $\beta^W(c)$ and $x(c)$ in the first period. Let $\{V_i\}_{i \in N}$ be an equilibrium payoff vector with $\sum_i V_i = \bar{V}_p$. For each $c = (c_i)_{i \in \hat{N}_e}$ and $i \in N$, we construct transfers $T_i(c)$ as follows:

$$T_i(c) = \begin{cases} 
-\delta(V_i - V_{i,p}) + (\rho_i(\beta^W, \gamma, x)(c) - x_i(c))(\beta^W(c) - c_i) + \epsilon(c) & \text{if } i \in \hat{N}, c_i \leq \beta^W(c), \\
-\delta(V_i - V_{i,p}) - x_i(c)(\beta^W(c) - c_i) + \epsilon(c) & \text{if } i \in \hat{N}, c_i > \beta^W(c), \\
-\delta(V_i - V_{i,p}) + \epsilon(c) & \text{if } i \notin \hat{N},
\end{cases}$$

where $\epsilon(c) \geq 0$ is a constant to be determined below. Since $\beta^W(c)$ and $x(c)$ satisfy (O7), it
follows that for all $c$,

$$
\sum_{i \in N} T_i(c) - n \epsilon(c) = -\delta(V_p - \sum_{i \in N} V_{i,p}) + \sum_{i \in N} \left\{ (\rho_i(\beta^W, \gamma, x)(c) - x_i) [\beta^W(c) - c_i]^+ + x_i [\beta^W(c) - c_i]^+ \right\} \leq 0.
$$

We set $\epsilon(c) \geq 0$ such that transfers are budget balance; i.e., such that $\sum_{i \in N} T_i(c) = 0$.

The SPE we construct is as follows. At $t = 0$, for each $c = (c_i)_{i \in \hat{N}_e}$ all firms $i \in \hat{N}_e$ bid $\beta^W(c)$. Firms $i \in \hat{N}_e$ with $x_i(c) = 0$ choose $\tilde{\gamma}_i(c) = 0$, and firms $i \in \hat{N}_e$ with $x_i(c) > 0$ choose $\tilde{\gamma}_i(c) = \gamma_i(c)$. Note that, for all $i \in \hat{N}_e$, $x_i(c) = \tilde{\gamma}_i(c)/\sum_j \tilde{\gamma}_j(c)$ and $\rho_i(\beta^W, \tilde{\gamma}, x)(c) = \rho_i(\beta^W, \gamma, x)(c)$. If no firm $i \in \hat{N}$ deviates at the bidding stage, cartel members exchange transfers $T_i(c)$. If no firm $i \in N$ deviates at the transfer stage, from $t = 1$ onwards firms play an SPE that supports payoff vector $\{V_i\}$. If firm $i \in N$ deviates either at the bidding stage or at the transfer stage, from $t = 1$ onwards firms play an SPE that gives firm $i$ a payoff $V_{i,p}$ (if more than one firm deviates, then firms punish the lowest indexed firm that deviated). One can check that this strategy profile satisfies (7)-(9) in Appendix B and (O9). Hence, winning bid profile $\beta^W$ and allocation $x$ are implementable.

Recall that

$$
b^*_p(c) = \sup \left\{ b \leq r : \sum_{i \in \hat{N}} (1 - x^*_i(c)) [b - c_i]^+ \leq \delta(V_p - \sum_{i \in \hat{N}} V_{i,p}) \right\}.
$$

**Proposition OE.1.** In an optimal equilibrium, the on-path bidding profile is such that:

(i) if $E = 0$, the cartel sets winning bid $\beta^*_p(c) = \max\{b^*_p(c), p\}$;

(ii) if $E = 1$, the winning bid is $\beta^*_p(c) = \max\{p, \min\{c_e, b^*_p(c)\}\}$ when a cartel wins the auction, and is $\beta^*_p(c) = \max\{c_e, p\}$ when the entrant wins the auction.

**Proof.** The proof of part (i) is identical to the proof of Proposition 1, and hence omitted.

We now turn to part (ii). Note first that, by Lemma OE.2, entry by the short-lived firm reduces the set of sustainable bidding profiles and thus the profits that the cartel can obtain in an auction. Therefore, in an optimal equilibrium the cartel seeks to maximize its payoff and minimize the short-lived firm’s payoff from entry.
Suppose \( E = 1 \). For any \( c \), let \( \beta^W(c) \) and \( x(c) \) be, respectively, the winning bid and allocation in an optimal equilibrium. We let \( c_{(1)} = \min_{i \in \tilde{N}} c_i \) be the lowest cost among participating cartel members. Consider first cost realizations \( c \) such that \( c_{(1)} > c_e \geq p \). In this case, \( x_e(c) = 1 \) in an optimal bidding profile. Indeed, by equation (O8), \( \beta^W(c) \leq c_e \) if \( x_e(c) < 1 \). Hence, the cartel is better-off letting the short-lived firm win whenever \( c_{(1)} > c_e \geq p \). Moreover, by setting \( \beta^W(c) = c_e \), the cartel guarantees that the short-lived firm earns zero payoff.\(^{14}\)

Consider next \( c \) such that \( c_{(1)} > p > c_e \). By (O8), it must be that \( x_e(c) > 0 \). In this case, in an optimal equilibrium the cartel sets winning bid equal to \( \beta^W(c) = p \), as this minimizes the short-lived firm’s payoff from winning.

Consider next \( c \) such that \( c_{(1)} < c_e \) and \( c_e \geq p \). Clearly, an optimal bidding profile for the cartel must be such that \( x_e(c) = 0 \). Equation (O8) then implies that \( \beta^W(c) \leq c_e \). We now show that, in this case, \( \beta^W(c) = \max \{ p, \min \{ c_e, b^*_p(c) \} \} \). There are two cases to consider:

(a) \( b^*_p(c) > c_e \), and 
(b) \( b^*_p(c) \leq c_e \). Consider case (a), so \( b^*_p(c) > c_e \geq p \). It follows that

\[
\sum_{i \in \tilde{N}} (1 - x^*_i(c))[c_e - c_i]^+ < \sum_{i \in \tilde{N}} (1 - x^*_i(c))[b^*_p(c) - c_i]^+ \leq \delta(V_p - \sum_{i \in N} V_{i,p}).
\]

Therefore, a bidding profile that induces winning bid \( c_e \) and allocation \( x^*(c) \) satisfies (O7) and (O8). Since such a bidding profile is optimal for the cartel among all bidding profiles with winning bid lower than \( c_e \), it must be that \( \beta^W(c) = c_e \).

Consider next case (b). Note that for all \( b > \max \{ b^*_p(c), p \} \) and any allocation \( x(c) \),

\[
\sum_{i \in \tilde{N}} \left\{ (1 - x_i(c))[b - c_i]^+ + x_i(c)[b - c_i]^+ \right\} \geq \sum_{i \in \tilde{N}} (1 - x^*_i(c))[b - c_i]^+ > \delta(V_p - \sum_{i \in N} V_{i,p}),
\]

so \( \max \{ b^*_p(c), p \} \) is the largest winning bid that can be supported in an equilibrium. Therefore, in an optimal equilibrium cartel members must use a bidding profile inducing winning bid \( \max \{ b^*_p(c), p \} \).

Finally, consider \( c \) such that \( c_{(1)} < p \) and \( c_e < p \). We now show that, in an optimal equilibrium, \( \beta^W(c) = p \). Indeed, by (O8), a winning bid \( \beta^W(c) > p > c_e \) can only be implemented if \( x_e(c) = 1 \). But this is clearly suboptimal for the cartel. Indeed, the cartel could make strictly positive profits by having a firm with cost \( c_{(1)} \) bidding \( p \); and doing this would also strictly reduce the short-lived firm’s expected payoff from entering. Therefore, in

\(^{14}\)This is achieved by having all participating cartel members bidding \( \beta^W(c) = c_e \) and \( \gamma_i(c) = 0 \), and having the entrant bidding \( \beta^W(c) = c_e \) and \( \gamma_e(c) = 1 \).
an optimal equilibrium it must be that $\beta^W(c) = p$. ■

Proposition OE.1 characterizes bidding behavior under an optimal equilibrium. In periods in which the short-lived firm does not participate, the cartel’s bidding behavior is the same as in Section 2. Entry by a short-lived firm reduces the cartels profits in two ways: (i) the cartel losses the auction whenever the entrant’s procurement cost is low enough, and (ii) entry leads to weakly lower winning bids when the cartel wins the auction.

By Proposition OE.1, the winning bid when the entrant wins the auction is $\beta^*_p(c) = \max\{c(e), p\}$. For $p \leq c$, the entrant earns zero payoff from participating in the auction. Therefore, for $p \leq c$ the entrant participates in the auction if and only if its entry cost is equal to zero. For $p > c$, the entrant’s payoff from participating in the auction is strictly positive. From now on we assume that the distribution of entry costs $F_k$ has a mass point at zero, so that there is positive probability of entry for all minimum prices $p$.

Our last result in this section extends Proposition 2 to the current setting. Recall that $\beta^*_0(c)$ is the lowest bid under minimum price $p = 0$.

**Proposition OE.2** (worse case punishment – entry). (i) $V_{i,0} = 0$, and $V_{i,p} > 0$ whenever $p > c$;

(ii) there exists $\eta > 0$ such that, for all $p \in [\beta^*_0(c), \beta^*_0(c) + \eta]$, $V_p - \sum_{i \in N} V_{i,p} \leq V_0 - \sum_{i \in N} V_{i,0}$. The inequality is strict if $p \in (\beta^*_0(c), \beta^*_0(c) + \eta]$.

**Proof.** We first establish part (i). Suppose that $p \leq c$ and fix equilibrium payoffs $\{V_i\}_{i \in N}$. Fix $j \in N$ and consider the following strategy profile. At $t = 0$, firms’ behavior depends on whether $j \in \hat{N}$ or $j \notin \hat{N}$. If $j \in \hat{N}$, all firms $i \in \hat{N}_e$ bid $\min\{c_j, \hat{c}_{(2)}\}$ (where $\hat{c}_{(2)}$ is the second lowest procurement cost among firms in $\hat{N}_e$). Firm $i \in \hat{N}_e$ chooses $\gamma_i = 1$ if $c_i = \min_{k \in \hat{N}_e} c_k$, and chooses $\gamma_i = 0$ otherwise. Note that this bidding profile constitutes an equilibrium of the stage game. If $j \notin \hat{N}$, at $t = 0$ participating firms play according to some equilibrium of the stage game. If all firms bid according to this profile, firm $j$’s transfer is $T_j = -\delta V_j$ at the end of the period regardless of whether $j \in \hat{N}$ or $j \notin \hat{N}$. The transfer of firm $i \in N \setminus \{j\}$ is $T_i = \frac{1}{n-1} \delta V_j$ at the end of the period, so $\sum_i T_i = 0$. If no firm deviates at the bidding or transfer stage, at $t = 1$ firms play according to an equilibrium that delivers payoffs $\{V_i\}$. If firm $i$ deviates at the bidding stage, there are no transfers and at $t = 1$ firms play the strategy just described with $i$ in place of $j$. If no firm deviates at the bidding stage and firm $i$ deviates at the transfer stage, at $t = 1$ firms play the strategy just described with $i$ in place of $j$.

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15 We assume that the short-lived firm participates in the auction whenever its indifferent.
of j (if more than one firm deviates at the bidding or transfer stage, from $t = 1$ firms play according to an equilibrium that delivers payoffs $\{V_i\}_{i \in N}$. Note that this strategy profile gives player $j$ a payoff of 0. Moreover, no firm has an incentive to deviate at $t = 0$, and so $V_{i,p} = 0$ for all $p \leq \zeta$.

Suppose next that $p > \zeta$, and note that $V_{i,p} \geq u_{i,p} \equiv 1 - \delta \text{prob}(i \in \tilde{N})E_{F_i} \left[ \frac{1}{N+1} \mathbb{1}_{c_i \leq p}(p - c_i) | i \in \tilde{N} \right] > 0,$

where the inequality follows since firm $i$ can always guarantee a payoff at least as large as $u_{i,p}$ by bidding $p$ whenever $c_i \leq p$ and bidding $b \geq c_i$ otherwise. This establishes part (i).

We now turn to part (ii). Note that $\beta_0^*(\zeta) = \zeta$.\textsuperscript{16} Fix $\eta > 0$ and $p \in [\zeta, \zeta + \eta]$. For $E = 0, 1$, let $(\beta^E, \gamma^E)$ be the bidding profile that firms use on the equilibrium path at periods in which the short-lived firm's entry decision is $E$ under an equilibrium that attains $V_p$ when the minimum price is $p$. Let $\beta_p^*(c)$ and $x_p^*(c)$ denote, respectively, the winning bid and the allocation under this optimal equilibrium. The cartel’s expected payoff under this optimal equilibrium satisfies

\[
(1 - \delta)V_p = \text{prob}(E = 0|p)E \left[ \sum_{i \in \tilde{N}} x_i^p(c)(\beta_p^*(c) - c_i) | E = 0 \right] \\
+ \text{prob}(E = 1|p)E \left[ \sum_{i \in \tilde{N}} x_i^p(c)(\beta_p^*(c) - c_i) | E = 1 \right].
\]

Suppose there is no minimum price and consider the following bidding profile for cartel members. For $E = 0, 1$ and all $c$ such that $\beta_p^*(c) > p$, participating firms bid according to $(\beta^E, \gamma^E)$. For $E = 0$ and all $c$ such that $\beta_p^*(c) = p$, all participating cartel members bid $c(2)$; firm $i \in \tilde{N}$ with $c_i = c(1) = \min_{j \in \tilde{N}} c_j$ sets $\gamma_i = 1$, and firm $i \in \tilde{N}$ with $c_i > c(1)$ sets $\gamma_i = 0$. For $E = 1$ and all $c$ such that $\beta_p^*(c) = p$, all participating firms bid $\min\{c(2), c_e\}$; firm $i \in \tilde{N}_e$ sets $\gamma_i = 1$ if $c_i = \min_{k \in \tilde{N}_e} c_k$ and sets $\gamma_i = 0$ otherwise. Note that, for $c$ such that $\beta_p^*(c) = p$, the bidding profile that firms use constitutes an equilibrium of the stage game when there is no minimum price. Note further that the entrant earns a lower expected payoff under this bidding profile than under the optimal equilibrium for minimum price $p \in [\zeta, \zeta + \eta]$; indeed, under this bidding profile, the entrant earns the same payoff than under the optimal equilibrium whenever $\beta_p^*(c) > p$, and earns a payoff of zero whenever

\textsuperscript{16}Indeed, by Proposition OE.1, $\beta_0^*(\zeta) = \zeta$ whenever $E = 1$ and $c_e = \zeta$.  

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\( \beta_p^*(c) = p \). Therefore, the probability of entry under this strategy profile is lower than under the optimal equilibrium when minimum price is \( p \). Let \( \beta(c) \) and \( x(c) \) denote the winning bid and the allocation that this bidding profile induces. Let \( \hat{\beta} \) be the cartel’s total surplus under this strategy profile, and note that

\[
(1 - \delta) \hat{V}_p = \text{prob}(E = 0 | \text{no min price}) \mathbb{E} \left[ \sum_{i \in \mathcal{N}} x_i(c)(\beta(c) - c_i) | E = 0 \right] \\
+ \text{prob}(E = 1 | \text{no min price}) \mathbb{E} \left[ \sum_{i \in \mathcal{N}} x_i(c)(\beta(c) - c_i) | E = 1 \right] \\
\geq \text{prob}(E = 1 | p) \mathbb{E} \left[ \sum_{i \in \mathcal{N}} x_i^p(c)(\beta_p^*(c) - c_i) \mathbf{1}_{\beta_p^*(c) > p} | E = 0 \right] \\
+ \text{prob}(E = 1 | p) \mathbb{E} \left[ \sum_{i \in \mathcal{N}} x_i^p(c)(\beta_p^*(c) - c_i) \mathbf{1}_{\beta_p^*(c) > p} | E = 1 \right],
\]

where we used the fact that the \( \text{prob}(E = 0 | p) \leq \text{prob}(E = 0 | \text{no min price}) \) and that the cartel’s payoff conditional on \( E = 0 \) is weakly larger than its payoff conditional on \( E = 1 \).

Note that \( b_p^*(c) \geq c + \frac{\delta(V_p - \sum_{i \in \mathcal{N}} V_{i,p})}{n-1} > c \). By Proposition OE.1, \( \beta_p^*(c) = \max\{p, b_p^*(c)\} \) whenever \( E = 0 \). Therefore, for \( \eta > 0 \) small enough and for \( E = 0 \), \( \beta_p^*(c) > p \) for all \( c \) and all \( p \in [c, c + \eta] \). For all such \( \eta > 0 \) and for all \( p \in [c, c + \eta] \), \( \text{prob}(\beta_p^*(c) = p | E = 0) = 0 \). Moreover, Proposition OE.1 also implies that \( \text{prob}(\beta_p^*(c) = p | E = 1) = F_e(p) \) for all \( p \in [c, c + \eta] \). Therefore, for \( \eta > 0 \) small enough and for \( p \in [c, c + \eta] \),

\[
(1 - \delta)(\overline{V}_p - \hat{V}_p) \leq \text{prob}(E = 1 | p) \mathbb{E} \left[ \sum_{i \in \mathcal{N}} x_i^p(c)(\beta_p^*(c) - c_i) \mathbf{1}_{\beta_p^*(c) = p} | E = 1 \right] \\
\leq \text{prob}(E = 1 | p) \frac{n}{n+1} F_e(p) \mathbb{E} [(p - c_{(1)}) \mathbf{1}_{c_{(1)} \leq p}],
\]

where the second inequality follows since the probability with which the cartel wins the auction when the entrant’s cost is below \( p \) is bounded above by \( \frac{n}{n+1} \), and since the cartel’s payoff from winning the auction at price \( p \) is bounded above by \( (p - c_{(1)}) \mathbf{1}_{c_{(1)} \leq p} \). Let \( F \) be a

\footnote{Indeed, \( \inf p \beta_p^*(c) \) is attained when all cartel members participate and they all have a cost equal to \( c \). In this case, \( \beta_p^*(c) = c + \frac{\delta(V_p - \sum_{i \in \mathcal{N}} V_{i,p})}{n-1} \).

\footnote{Indeed, \( b_p^*(c) > p \) for all \( c \) and all \( p \in [c, c + \eta] \). Therefore, by Proposition OE.1, for all \( p \in [c, c + \eta] \) and for \( E = 1 \), the winning bid \( \beta_p^*(c) \) is equal to \( p \) only when the entrant’s cost is below \( p \).}
distribution with support $[\xi, \tau]$ such that $\mathbb{E}[(p-c_1)1_{c_1 \leq p}] \leq \int_\xi^p (p-c)n(1-F(c))^{-1} f(c)\,dc.$

Note then that

$$
(1-\delta)(\bar{V}_p - \check{V}_p) \leq \text{prob}(E = 1|p) \frac{n}{n+1} F_e(p) \int_\xi^p (p-c)n(1-F(c))^{-1} f(c)\,dc.
$$

On the other hand, for each $i \in N$,

$$(1-\delta)V_{i,p} \geq u_{i,p} \geq \frac{n}{n+1} \text{prob}(i \in \hat{N}) \mathbb{E}_{F_i}[(p-c_i)1_{c_i \leq p}] = \frac{n}{n+1} \text{prob}(i \in \hat{N}) \int_\xi^p (p-c)f_i(c)\,dc.
$$

Note that, for $p = c$, $\hat{V}_p \geq V_p - \sum_{i \in N} V_{i,p} = \bar{V}_p$. Note further that

$$
\left. \frac{\partial}{\partial p} \right|_{p=\xi} F_e(p) \int_\xi^p (p-c)n(1-F(c))^{-1} f(c)\,dc = 0
$$

$$
\left. \frac{\partial^2}{\partial p^2} \right|_{p=\xi} F_e(p) \int_\xi^p (p-c)n(1-F(c))^{-1} f(c)\,dc = 0
$$

$$
\left. \frac{\partial}{\partial p} \right|_{p=\xi} \int_\xi^p (p-c)f_i(c)\,dc = 0
$$

$$
\left. \frac{\partial^2}{\partial p^2} \right|_{p=\xi} \int_\xi^p (p-c)f_i(c)\,dc = f_i(\xi) > 0.
$$

Therefore, there exists $\eta > 0$ small enough such that $\hat{V}_p \geq V_p - \sum_{i \in N} V_{i,p}$ for all $p \in [\xi, \xi + \eta]$, with strict inequality if $p > c$. To establish part (ii) of the Lemma, we show that $V_0 \geq \hat{V}_p$ for all $p \in [\xi, \xi + \eta]$.

Suppose there is no minimum price, and consider the following strategy profile. Along the equilibrium path, bidders bid according to the bidding profile described above, which generates surplus $\hat{V}_p$ for the cartel. If firm $i \in \hat{N}$ deviates at the bidding stage, there are no transfers and in the next period cartel members play an equilibrium that gives firm $i$ a payoff of $V_{i,0} = 0$ (if more than one firm deviates, cartel members punish the lowest indexed firm that deviated). If no firm deviates at the bidding stage, each firm $i \in N$ makes transfer $T_i(c)$ to be determined below. If a firm $i \in N$ deviates at the transfer stage, in the next period firms play an equilibrium that gives firm $i$ a payoff of $V_{i,0} = 0$ (if more than one firm deviates, cartel members again punish the lowest indexed firm that deviated). Otherwise, if no firm deviates at the bidding and transfer stages, in the next period firms continue playing

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\textsuperscript{19}For instance, choose $F$ such that for all $i \in N$ and all $c \in [\xi, \tau]$, $F_i(c) \geq F_i(c)$. 

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the same strategies as above.

Let \( \{V_i\}_{i \in N} \) be a payoff profile with \( \sum_i V_i = \hat{V}_p \) and \( V_i \geq V_{i,0} = 0 \) for all \( i \). The transfers \( T_i(c) \) are determined as follows. For all \( c \) such that \( \beta^*_p(c) = p, T_i(c) = 0 \) for all \( i \in N \). Otherwise,

\[
T_i(c) = \begin{cases} 
-\delta V_i + (1 - x_i^p(c)) (\beta^*_p(c) - c_i) + \epsilon(c) & \text{if } i \in \tilde{N}, c_i \leq \beta^*_p(c), \\
-\delta V_i + \epsilon(c) & \text{otherwise}, 
\end{cases}
\]

where \( \epsilon(c) \geq 0 \) is a constant to be determined.\(^{20}\) Note that

\[
\sum_i T_i(c) - n \epsilon(c) = -\delta \hat{V}_p + \sum_i (1 - x_i^p(c)) [\beta^*_p(c) - c_i]^+ \leq 0,
\]

where the inequality follows since \( \beta^*_p(c) \) is implementable with minimum price \( p \), and since \( \hat{V}_p \geq \nabla p - \sum_{i \in N} V_{i,p} \). We set \( \epsilon(c) \geq 0 \) such that \( \sum_i T_i(c) = 0 \). This strategy profile generates total surplus \( \hat{V}_p \) for the cartel. One can check that no firm has an incentive to deviate at any stage, and so this strategy profile constitutes an equilibrium. Hence, it must be that \( \nabla_0 \geq \hat{V}_p \geq \nabla_p - \sum_{i \in N} V_{i,p} \) for all \( p \in [\underline{c}, \underline{c} + \eta] \), and the second inequality is strict if \( p > \underline{c} \). \( \blacksquare \)

**Proof of Proposition 5.** Consider first a collusive environment and suppose that \( E \in \{0, 1\} \). By Propositions OE.1 and OE.2, for all \( p \in [\beta^*_0(c), \beta^*_0(c) + \eta] \), \( \beta^*_p(c) \leq \beta^*_0(c) \) for all \( c \) such that \( \beta^*_0(c) \geq p \). Therefore, for all \( p \in [\beta^*_0(c), \beta^*_0(c) + \eta] \) and all \( q > p \), \( \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E) \leq \text{prob}(\beta^*_0 \geq q|\beta^*_0 \geq p, E) \). This completes the proof of part (i).

Consider next a competitive environment. Let \( \hat{c}(2) \) be the second lowest cost among all participating firms (including the entrant if \( E = 1 \)). Then, for all \( p > 0 \) and all \( q > p \), \( \text{prob}(\beta^*_p^{\text{comp}} \geq q|\beta^*_p^{\text{comp}} > p, E) = \text{prob}(\hat{c}(2) \geq q|\hat{c}(2) > p, E) = \text{prob}(\beta^*_0^{\text{comp}} \geq q|\beta^*_0^{\text{comp}} > p, E) \). This completes the proof of part (ii). \( \blacksquare \)

**Proof of Proposition 6.** We start with part (i). As a first step, we show that for \( E = 0, 1 \), \( \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E, \text{cartel wins}) \leq \text{prob}(\beta^*_0 \geq q|\beta^*_0 \geq p, E, \text{cartel wins}) \). In the case of \( E = 0 \), the result follows from Proposition 5(i). Suppose next that \( E = 1 \), and consider cost realizations \( c \) such that the cartel wins. By Propositions OE.1 and OE.2, for all \( p \in [\beta^*_0(c), \beta^*_0(c) + \eta] \), \( \beta^*_p(c) \leq \beta^*_0(c) \) whenever \( \beta^*_0(c) \geq p \). Therefore, for all \( p \in [\beta^*_0(c), \beta^*_0(c) + \eta] \) and all \( q > p \), \( \text{prob}(\beta^*_p \geq q|\beta^*_p \geq p, E = 1, \text{cartel wins}) \leq \text{prob}(\beta^*_0 \geq q|\beta^*_0 \geq p \).\(^{20}\)

\(^{20}\)Recall that \( x^p(c) \) is the allocation under an optimal equilibrium when the minimum price is \( p \). Therefore, \( x^p(c) \) is such that \( x_i^p(c) = 0 \) for all \( i \) with \( c_i > \beta^*_p(c) \).
\( p, E = 1, \text{cartel wins} \).

It then follows that, for any \( p \in [\beta_0^*(c), \beta_0^*(c) + \eta] \) and any \( q > p \),

\[
\begin{align*}
\operatorname{prob}(\beta_p^* \geq q | \beta_p^* \geq p, \text{cartel wins}) &= \operatorname{prob}(E = 0 | p > 0) \operatorname{prob}(\beta_p^* \geq q | \beta_p^* \geq p, E = 0, \text{cartel wins}) \\
&\quad + \operatorname{prob}(E = 1 | p > 0) \operatorname{prob}(\beta_p^* \geq q | \beta_p^* \geq p, E = 1, \text{cartel wins}) \\
&\leq \operatorname{prob}(E = 0 | p > 0) \operatorname{prob}(\beta_0^* \geq q | \beta_0^* \geq p, E = 0, \text{cartel wins}) \\
&\quad + \operatorname{prob}(E = 1 | p > 0) \operatorname{prob}(\beta_0^* \geq q | \beta_0^* \geq p, E = 1, \text{cartel wins}) \\
&\leq \operatorname{prob}(E = 0 | p = 0) \operatorname{prob}(\beta_0^* \geq q | \beta_0^* \geq p, E = 0, \text{cartel wins}) \\
&\quad + \operatorname{prob}(E = 1 | p = 0) \operatorname{prob}(\beta_0^* \geq q | \beta_0^* \geq p, E = 1, \text{cartel wins}) \\
&= \operatorname{prob}(\beta_0^* \geq q | \beta_0^* \geq p, \text{cartel wins}),
\end{align*}
\]

where the first inequality follows from the arguments in the previous paragraph, and the second inequality follows since \( \operatorname{prob}(E = 1 | p = 0) \leq \operatorname{prob}(E = 1 | p > 0) \) (i.e., the probability of entry increases with the minimum price) and since the cartel’s winning bid is lower when the entrant participates.

We now turn to part (ii). Consider cost realizations \( c \) such that the entrant wins. By Proposition OE.1, \( \beta_0^*(c) = c(e) \) and \( \beta_p^*(c) = \max\{c(e), p\} \). Therefore, for all \( p > 0 \) and all \( q > p \), \( \operatorname{prob}(\beta_p^* \geq q | \beta_p^* > p, \text{entrant wins}) = \operatorname{prob}(\beta_0^* \geq q | \beta_0^* > p, \text{entrant wins}) \). This completes the proof of part (ii). ■

**References**