Dynamic Contracting with Limited Liability Constraints

Edoardo Grillo                Juan Ortner*
Collegio Carlo Alberto        Boston University

March 15, 2018

Abstract

We study a dynamic mechanism design problem in which a buyer seeks to procure an item from a single seller in multiple periods. The seller is privately informed about her procurement cost at each period, and this cost may be serially correlated over time. We restrict the buyer to use mechanisms satisfying a limited liability constraint: the seller’s flow payoffs must be non-negative at each period. Limited liability constraints give rise to new dynamic distortions and inefficiencies. The optimal mechanism is path dependent, favoring sellers who had low cost realizations in the past.

KEYWORDS: dynamic mechanism design, dynamic contracting, limited liability, cash-constraints, path dependence, dynamic distortions.

*Grillo: edoardo.grillo@carloalberto.org, Ortner: jortner@bu.edu.
1 Introduction

We study a dynamic mechanism design problem in which a buyer seeks to procure an item from a single seller in two periods. The seller’s procurement cost is continuously distributed at each period, and is her private information. We allow the seller’s costs to be serially correlated over time, and assume that the buyer has full commitment power.

The novel feature of our analysis is that we restrict the buyer to use mechanisms that give non-negative flow payoffs to the seller at each period. Such limited liability constraints are highly relevant in procurement settings, where firms may be cash constrained and face imperfect credit markets. Our analysis highlights the dynamic distortions that these constraints generate, as well as their implications in terms of payoffs and the frequency with which the principal and agent contract.

In the absence of limited liability constraints, the optimal mechanism pays the seller a low fixed amount in the first period, and a high price in the second period. A seller with a high cost realization in the first period obtains negative flow profits, but is willing to accept the contract because she expects to earn high profits in the second period. As in Baron and Besanko (1984) and Eső and Szentes (2017), under the optimal mechanism the seller only gets informational rents for her first-period private information. Moreover, as Eső and Szentes (2017) highlight, the dynamic nature of the problem is “irrelevant.” under the optimal contract the buyer obtains the same expected profits she would obtain if she could observe and contract on the seller’s “orthogonalized” second period cost.

The mechanism described above, which gives negative flow payoff in the first period to sellers with high costs, is not feasible in the presence of limited liability constraints. Instead, the optimal mechanism takes the following form. A seller with a low cost in the first period gets the same payments and allocation across all periods as in the optimal mechanism without limited liability constraints. In contrast, a seller with a high cost in the first period gets zero flow profits at the initial date: the payment she receives exactly covers her procurement cost. To provide incentives for truthful reporting to sellers with high cost realizations in the first period, the optimal mechanism adjusts the payment and probability of contracting in the second period as a function of the first period report. In particular, firms that report a lower cost in the first period get a higher compensation in the second period, and procure with higher probability.

Limited liability constraints give rise to the following implications. First, limited liability constraints generate new distortions and inefficiencies. Indeed, the probability with which the seller procures in the second period is lower and this decreases social
surplus. The buyer is always worse-off relative to the benchmark setting without limited liability constraints. Perhaps surprisingly, we show that whether the seller is worse-off or not depends on the correlation between her first period and second period cost. When costs are independently distributed over time, the seller obtains the same expected payoff with limited liability constraints than without these constraints. Instead, if costs are positively serially correlated, limited liability constraints make the seller worse-off.

Second, the optimal contract under limited liability constraints displays a form of path dependence: sellers that performed well in the first period (i.e., sellers with a low cost realization) are given a preferential treatment in the second period. This result holds regardless of the correlation structure of costs across periods. This resonates with practices of several firms. For instance, Toyota has been known for favoring suppliers which had performed well in the past (Roberts, 2007).

Third, our results show that the dynamic irrelevance result in Eső and Szentes (2017) no longer holds when the principal is restricted to offer mechanisms satisfying limited liability constraints: the principal obtains lower payoffs relative to the case in which she can observe and contract on the agent’s “orthogonalized” second period cost. Intuitively, under the optimal contract with limited liability, the contractual terms in the second period must be distorted to provide incentives for truthful reporting in the first period.

Related literature. Our paper is primarily related to the large and growing literature on dynamic mechanism design (Courty and Hao (2000), Battaglini (2005), Eső and Szentes (2007), Pavan et al. (2014), Garrett and Pavan (2012), Battaglini and Lamba (2015) and Board and Skrzypacz (2016)). We add to this literature by studying the effects that limit liability constraints have on dynamic contracts. Recent work by Krishna et al. (2013), Krasikov and Lamba (2016) and Ashlagi et al. (2016) also study dynamic contracting environments with limited liability constraints. Krishna et al. (2013) and Krasikov and Lamba (2016) study settings in which the agent’s private information can take two values: Krishna et al. (2013) consider i.i.d. types, while Krasikov and Lamba (2016) allow for serially correlated types. Ashlagi et al. (2016) consider a model in which the agent’s type is continuously distributed and is drawn i.i.d. over time. Our model allows the agent’s private information to be continuously distributed and serially correlated. Moreover, we provide an explicit characterization of the optimal contract in this setting, and discuss several of its economic implications. Lastly, our proof techniques are broadly different from Krishna et al. (2013), Krasikov and Lamba (2016) and Ashlagi et al. (2016).
Finally, Board (2011) shows that the practices that firms like Toyota use of favoring suppliers who have performed well in the past can be rationalized with relational incentives. Our arguments provide an alternative rationalization to such practices based on limited liability constraints.

The paper proceeds as follows. Section 2 introduces our framework. Section 3 characterizes the optimal contract in the presence of limited liability constraints, and compares it to the optimal contract without such constraints. Section 4 discusses the key economic implications of our model. All proofs are in the Appendix.

2 Model

We consider a buyer who wants a seller to perform a job in two subsequent periods, $t = 1, 2$. The buyer gets a value $v > 0$ for each period in which the job is done and 0 otherwise.

The seller incurs a cost whenever she performs the job. This cost is the seller’s private information and changes over time. In particular, the cost the seller incurs from performing the job in period $t = 1, 2$ is random and equal to $\theta_t$. Cost $\theta_1$ is distributed according to cdf $F_1(\cdot)$, with support $[\bar{\theta}, \tilde{\theta}]$ and density $f_1(\cdot)$ satisfying $f_1(\theta) > 0$ for all $\theta \in [\bar{\theta}, \tilde{\theta}]$. Similarly, given first period cost realization $\theta_1$, cost $\theta_2$ is distributed according to $F_2(\cdot | \theta_1)$ with support $[\bar{\theta}, \tilde{\theta}]$ and density $f_2(\cdot | \theta_1)$ satisfying $f_2(\theta | \theta_1) > 0$ for all $\theta \in [\bar{\theta}, \tilde{\theta}]$. Distribution $F_2(\cdot | \theta_1)$ is such that $\frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1}$ and $\frac{\partial f_2(\theta_2 | \theta_1)}{\partial \theta_1}$ exist and are bounded for all $\theta_1, \theta_2 \in [\bar{\theta}, \tilde{\theta}]^2$. We make the following assumptions on the cost distributions.\(^1\)

**Assumption 1.** $F_1(\theta)/f_1(\theta)$ is increasing in $\theta$, and $F_2(\theta | \theta_1)/f_2(\theta | \theta_1)$ is increasing in $\theta$ for all $\theta_1$.

**Assumption 2.** $\theta_2 - \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} \frac{\alpha}{f_1(\theta_1)f_2(\theta_2 | \theta_1)}$ is increasing in $\theta_2$ for all $\theta_1$ and for all $\alpha \in [-1, 1]$.

**Assumption 3.** $v - \bar{\theta} - F_1(\bar{\theta})/f_1(\bar{\theta}) \geq 0$ and $\forall \theta_1, \theta_2, v \geq \theta_2 - \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} \frac{F_1(\theta_1)}{f_1(\theta_1)f_2(\theta_2 | \theta_1)}$.

\(^1\)Assumptions 1 and 2 guarantee that the allocation that maximizes pointwise the principal’s payoffs is monotone, and hence incentive compatible. Assumption 3 guarantees that, in the absence of limited liability constraints, the buyer procures with probability 1 in both periods. Assumption 4 implies that costs are positively serially correlated over time.
Assumption 4. \( \partial F_2(\theta_2 \mid \theta_1) / \partial \theta_1 \leq 0. \) Thus, if \( \theta_1 > \theta_1' \), then \( F(\theta_2 \mid \theta_1) \) first-order stochastically dominates \( F(\theta_2 \mid \theta_1') \).

In order to procure the job, the buyer offers a mechanism to the seller. The mechanism is represented by (i) a set of messages \( M_t \) the seller can send in period \( t = 1, 2 \), (ii) a profile of allocation rules representing the probability with which the job is allocated to the seller as a function of her reports, \( x_1(m_1), x_2(m_1, m_2) \), and (iii) monetary transfers from the buyer to the seller as a function of the seller’s reports, \( P_1(m_1), P_2(m_1, m_2) \).

Denote by \( m^t \) the history of messages sent by the seller up to period \( t \). Thus, the flow utility that the buyer and the seller get in period \( t \) when the report is \( m^t \), previous reports are \( m^{t-1} \) and the actual type of the seller is \( \theta' \in [\underline{\theta}, \bar{\theta}] \) are respectively

\[
\begin{align*}
v x_t(m^t, m^{t-1}) &- P_t(m^t, m^{t-1}), \\
P_t(m^t, m^{t-1}) &- x_t(m^t, m^{t-1})\theta',
\end{align*}
\]

where \( P_1(m_1, m^0) \) and \( x_1(m_1, m^0) \) denote \( P_1(m_1), x_1(m_1) \). Both players share a common discount factor \( \delta = 1 \). By the revelation principle (Sugaya and Wolitzky, 2017), we restrict attention to direct mechanisms; i.e., mechanisms with \( M_1 = M_2 = [\underline{\theta}, \bar{\theta}] \).

For any \( \theta_1, \theta_2 \) and \( \theta'_2 \), let

\[
\hat{u}(\theta_2, \theta'_2|\theta_1) = P_2(\theta'_2, \theta_1) - x_2(\theta'_2, \theta_1)\theta_2,
\]

denote the expected payoff of a seller who reported cost \( \theta_1 \) in the first period, and who has cost \( \theta_2 \) in the second period but reports cost \( \theta'_2 \). Define \( u(\theta_2|\theta_1) \equiv \hat{u}(\theta_2, \theta_2|\theta_1) \) to be the payoff of a seller who reports truthfully at period 2 and who reported \( \theta_1 \) at period 1. For any \( \theta_1 \) and \( \theta'_1 \), let

\[
\hat{U}(\theta_1, \theta'_1) = P_1(\theta'_1) - x_1(\theta'_1)\theta_1 + \int_{\bar{\theta}}^{\underline{\theta}} u(\theta_2|\theta'_1)f_2(\theta_2 \mid \theta_1)d\theta_2
\]

denote the expected payoff of a seller who has cost \( \theta_1 \) in the first period but reports cost \( \theta'_1 \), and who expects to report truthfully at period 2. Define \( U(\theta_1) \equiv \hat{U}(\theta_1, \theta_1) \) to be the payoff of a seller who reports truthfully at period 1.
To satisfy incentive compatibility, the mechanism must be such that:

\[ \forall \theta_1, \theta'_1 \quad U(\theta_1) \geq \hat{U}(\theta_1, \theta'_1), \quad (1) \]
\[ \forall \theta_1, \theta_2, \theta'_2 \quad u(\theta_2|\theta_1) \geq \hat{u}(\theta_2, \theta'_2|\theta_1). \quad (2) \]

We restrict attention to mechanisms under which the seller earns positive flow payoffs every period. Formally, we look for mechanisms that satisfy the following constraints:

\[ \forall \theta_1 \quad P_1(\theta_1) - x_1(\theta_1)\theta_1 \geq 0 \quad (3) \]
\[ \forall \theta_1, \theta_2 \quad P_2(\theta_1, \theta_2) - x_2(\theta_1, \theta_2)\theta_2 \geq 0 \quad (4) \]

These constraints are relevant when the seller does not have cash reserves and has no access to credit markets. Note that any mechanism satisfying (3) and (4) satisfies individual rationality for the seller; indeed, (3) and (4) imply that \( U_1(\theta) \geq 0 \) for all \( \theta \in [\bar{\theta}, \overline{\theta}] \).

The problem of the buyer is to find the mechanism that maximizes her expected payoffs, subject to incentive compatibility constraints and limited liability constraints. Formally, the buyer solves:

\[
\max_{x_1(\cdot), P_1(\cdot), x_2(\cdot), P_2(\cdot)} \int_{\bar{\theta}}^{\overline{\theta}} \left[ v x_1(\theta_1) - P_1(\theta_1) + \int_{\theta}^{\overline{\theta}} (v x_2(\theta_1, \theta_2) - P_2(\theta_1, \theta_2)) dF_2(\theta_2|\theta_1) \right] dF_1(\theta_1) \quad (5)
\]
subject to:
\[ (1), (2), (3) \text{ and } (4) \]

3 Results

Before presenting our results, it is useful to characterize the buyer’s optimal mechanism in the benchmark case in which the mechanism need not satisfy constraints (3) and (4). To guarantee that the seller is willing to participate, in this benchmark case we require

---

2Note that, if constraints (1) and (2) are satisfied, then it is optimal for the seller to report her type truthfully at period 2 regardless of whether or not she misreported at period 2.

3Our main qualitative results continue to hold if we assume that the seller can suffer a per period loss of at most \(-L\) with \(L < \overline{L}\) for some \(\overline{L} > 0\).
the mechanism to be individually rational:

\[ \forall \theta_1, \quad U(\theta_1) \geq 0. \]  \hfill (6)

**Proposition 0.** Suppose Assumptions 1-4 hold. Then, the mechanism that maximizes (5) subject to (1), (2) and (6) satisfies:

\[ \forall \theta_1 \in [\theta, \bar{\theta}], \quad x_{1b}(\theta_1) = 1; \quad P_{1b}(\theta_1) = \mathbb{E}[\theta_2 | \bar{\theta}], \]

\[ \forall \theta_1, \theta_2 \in [\theta, \bar{\theta}], \quad x_{2b}(\theta_1, \theta_2) = 1; \quad P_{2b}(\theta_1, \theta_2) = \bar{\theta}. \]

Under the optimal mechanism in Proposition 0, the buyer implements the optimal allocation at time \( t = 2 \) and pays a transfer of \( \bar{\theta} \), leaving the seller with significant expected rents. At time \( t = 1 \), however, the buyer extracts the expected value of these future rents by paying the seller a small amount \( \mathbb{E}[\theta_2 | \bar{\theta}] \in (\theta, \bar{\theta}) \). As in Baron and Besanko (1984) and Eső and Szentes (2017), under this optimal mechanism the seller only gets rents for his private information at \( t = 1 \). Moreover, as emphasized by Eső and Szentes (2017), the dynamic nature of the problem is “irrelevant.” Indeed, under the optimal mechanism the buyer obtains the same expected profits she would obtain if at the time of contracting she knew the seller’s “orthogonalized” private information at time \( t = 2 \), but not the one at \( t = 1 \).

We now compare the optimal mechanism in Proposition 0 to the optimal mechanism when limited liability constraints (3) and (4) must be satisfied. Define \( \theta^\dagger \equiv \mathbb{E}[\theta_2 | \bar{\theta}] \), and note that \( \theta^\dagger \in (\theta, \bar{\theta}) \). Let \( \vartheta : [\theta^\dagger, \bar{\theta}] \rightarrow [\theta, \bar{\theta}] \) be the solution to differential equation

\[ \frac{d\vartheta(\theta_1)}{d\theta_1} = -\frac{1}{F_2(\vartheta(\theta_1) | \theta_1)} \]  \hfill (7)

with initial condition \( \vartheta(\theta^\dagger) = \bar{\theta} \). Note that \( \frac{d\vartheta(\theta_1)}{d\theta_1} < 0 \), so \( \vartheta(\theta_1) < \bar{\theta} \) for all \( \theta_1 > \theta^\dagger \). We extend \( \vartheta \) to \([\theta, \bar{\theta}]\) by setting \( \vartheta(\theta) = \bar{\theta} \) for all \( \theta < \theta^\dagger \). We have the following result.
Proposition 1. Suppose Assumptions 1-4 hold. Then, the mechanism that maximizes (5) subject to (1), (2), (3) and (4) is given by:

∀θ₁, \( x^*_1(\theta_1) = 1 \); 
\[
P^*_1(\theta_1) = \begin{cases} 
\theta_1 & \text{if } \theta_1 > \theta^t = \mathbb{E}[\theta_2 | \theta], \\
\mathbb{E}[\theta_2 | \theta] & \text{if } \theta_1 \leq \theta^t.
\end{cases}
\]

∀θ₁, θ₂, \( x^*_2(\theta_1, \theta_2) = \begin{cases} 
1 & \text{if } \theta_2 \leq \vartheta(\theta_1), \\
0 & \text{if } \theta_2 > \vartheta(\theta_1),
\end{cases} \)
\[
P^*_2(\theta_1, \theta_2) = \begin{cases} 
\vartheta(\theta_1) & \text{if } \theta_2 \leq \vartheta(\theta_1), \\
0 & \text{if } \theta_2 > \vartheta(\theta_1).
\end{cases}
\]

The optimal mechanism described in Proposition 1 takes a simple structure. In the first period, the allocation is the same as in Proposition 0. However, because of limited liability constraints, the report the seller makes in period 1 now affects the payment she receives at \( t = 1 \), as well as the payment and allocation in period 2. In particular, if the period 1 report is above \( \theta^t = \mathbb{E}[\theta_2|\theta] \), the seller receives a transfer at \( t = 1 \) that exactly covers her cost. Moreover, the seller procures at \( t = 2 \) if and only if her period 2 cost is lower than \( \vartheta(\theta_1) < \bar{\theta} \).

To gain intuition for the solution in Proposition 1, recall that in the absence of limited liability constraints, the transfer that the buyer pays to a seller of type \( \theta_1 \) in period 1 is given by \( P^b_1(\theta_1) = \mathbb{E}[\theta_2|\theta] = \theta^t \). When the seller’s first period cost is below \( \mathbb{E}[\theta_2|\theta] \), the prices and allocations in Proposition 0 satisfy the limited liability constraints (3) and (4), and hence are implementable. In contrast, when the seller’s first period cost is larger than \( \mathbb{E}[\theta_2|\theta] \), the prices and allocations in Proposition 0 do not satisfy constraint (3). The optimal mechanism in this case maintains the period 1 allocation as in the benchmark case \( (x^*_1(\cdot) = x^b_1(\cdot)) \), and sets the highest possible price \( P^*_1(\theta_1) = \theta_1 \) for all \( \theta_1 > \mathbb{E}[\theta_2|\theta] \). To provide incentives for truthful reporting, the optimal mechanism adjusts the continuation payoff of the seller by changing the transfer she will receive in the second period, and hence also the probability with which she procures in the second period – this is the role of function \( \vartheta(\theta_1) \).

Lastly, we explain why function \( \vartheta(\theta_1) \) solves (7). Consider for simplicity the case in which costs are independently distributed, with \( f_2(\theta_2|\theta_1) = f_2(\theta_2) \). Note that,

\[
U(\theta_1) = \int_{\theta_1}^{\bar{\theta}} x^*_1(\theta) d\bar{\theta} - \theta_1 = P^*_1(\theta_1) - x^*_1(\theta_1)\theta_1 + \int_{\theta}^{\bar{\theta}} (P^*_2(\theta_1, \theta_2) - x^*_2(\theta_1, \theta_2)\theta_2) f_2(\theta_2) d\theta_2
\]
\[
= \int_{\theta}^{\vartheta(\theta_1)} (\vartheta(\theta_1) - \theta_2) f_2(\theta_2) d\theta_2, \quad (8)
\]
where the first equality follows from incentive compatibility at $t = 1$. Differentiating both sides of (8) with respect to $\theta_1$ and rearranging, we obtain $\vartheta' (\theta_1) = \frac{-1}{F_2 (\vartheta (\theta_1))}$.

We end this section with an example in which costs are iid and uniformly distributed.

**Example 1.** Suppose $\theta_1$ and $\theta_2$ are iid and uniformly distributed on interval $[0, 1]$, so that $\theta_1 = 1/2$. In this case, the solution to differential equation (7) plus boundary condition $\vartheta (\theta_1) = \vartheta = 1$ is $\vartheta (\theta_1) = \sqrt{2(1 - \theta_1)}$ for all $\theta_1 \in [1/2, 1]$. The optimal allocation rule in period 1 is thus given by $x_1^*(\theta_1) = 1$ for all $\theta_1$, and $x_2^*(\theta_1, \theta_2) = 1_{\theta_2 \leq \sqrt{2(1 - \theta_1)}}$.

## 4 Economic implications

This section highlights the key economic implications of the optimal mechanism in Proposition 1.

**Inefficiencies and payoffs.** As Proposition 1 shows, limited liability constraints give rise to new distortions: the probability that the seller procures in period 2 is lower than in the benchmark case of Proposition 0. As a result, total surplus falls.

Who bears the cost of these new distortions? Because of the added constraints, the buyer’s payoff must fall relative to the benchmark in Proposition 0. Whether or not the seller’s payoff is lower under the solution in Proposition 1 depends on the correlation structure of costs across periods. By the envelope formula (and using the fact that the a seller with cost $\theta$ earns zero expected payoff), under both mechanisms (i.e., the one in Proposition 0 and the one in Proposition 1), the expected payoff of a seller with first period cost $\theta_1$ is\(^4\)

$$U (\theta_1) = \int_{\theta_1}^{\vartheta} x_1 (\hat{\theta}) d\hat{\theta} - \int_{\theta_1}^{\vartheta} \int_{\theta_1}^{\vartheta} u (\theta_2 | \hat{\theta}) \frac{\partial f (\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 d\hat{\theta}$$

$$= \int_{\theta_1}^{\vartheta} x_1 (\hat{\theta}) d\hat{\theta} - \int_{\theta_1}^{\vartheta} \int_{\theta_1}^{\vartheta} x_2 (\hat{\theta}, \theta_2) \frac{\partial F (\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 d\hat{\theta}$$

(9)

From equation (9), it follows that when costs are independent across periods, the seller obtains the same expected payoff under the mechanism in Proposition 0 than under the mechanism in Proposition 1: indeed, in this case $\frac{\partial F (\theta_2 | \hat{\theta})}{\partial \theta} = 0$ and $x_1^*(\theta_1) = x_1^{sb}(\theta_1) = 1$ for all $\theta_1$. Therefore, when costs are independent the buyer bears all the cost of the added distortions that limited liability constraints introduce.

\(^4\)See Appendix A for a derivation of equation (9).
Consider next the case in which costs are positively correlated, with $\frac{\partial F(\theta_2|\theta_1)}{\partial \theta_1} \leq 0$ for all $\theta_1, \theta_2$, and recall from Propositions 0 and 1 that the allocation is less efficient in the presence of limited liability constraints (i.e, $x^*_2(\theta_1, \theta_2) \geq x^*_2(\theta_1, \theta_2)$ for all $(\theta_1, \theta_2)$). As a result, the seller obtains a lower payoff under the mechanism in Proposition 1 relative to her payoff under the mechanism in Proposition 0.

**Dynamic relevance.** Our analysis shows that, in the presence of limited liability constraints, the dynamic irrelevance result in Eső and Szentes (2017) no longer holds.

To see why, suppose for simplicity that the seller’s costs are independently distributed across periods. Consider, as Eső and Szentes (2017) do, a setting in which the buyer can observe the seller’s second-period cost at the time of contracting. In this setting, the buyer can implement the same allocation and obtain the same profits as in the mechanism in Proposition 0, while satisfying incentive compatibility at $t=1$ and limited liability constraints (3) and (4). Indeed, this can be achieved by offering a mechanism with $x_1(\theta_1) = x_2(\theta_1, \theta_2) = 1$ for all $\theta_1, \theta_2$, and with payments $P_1(\theta_1) = \overline{\theta}$ for all $\theta_1$, and $P_2(\theta_1, \theta_2) = \theta_2$ for all $\theta_1, \theta_2$. Note that the buyer obtains a strictly larger payoff under this mechanism than under the mechanism in Proposition 1.

**Path dependence.** A notable feature of the mechanism in Proposition 1 is that it exhibits path dependence: the probability that the seller procures in period 2 (conditional on $\theta_2$) depends on her cost in the first period. Indeed, low reports in period 1 are rewarded by increasing the posted price in period 2, hence expanding the set of reports for which the job is procured. Note that this is true even when the firm’s costs are independent over time.

The model therefore generates the prediction that buyers will give preferential treatment to suppliers that had good past performance. This is consistent with procurement practices of many firms. For instance, Toyota has been known for favoring suppliers that performed well in the past (Roberts, 2007). Prior papers, like Board (2011), have shown that these practices can be rationalized using relational incentives. Our arguments give an alternative rationalization driven by limited liability constraints.

---

5In Appendix B we show that the result in Eső and Szentes (2017) also fails to hold when costs are positively correlated over time.

6Since costs are assumed to be independent across periods, this is equivalent to having the buyer observing the seller’s orthogonalized second period cost.
Appendix

A Proofs of Propositions 0 and 1

We start with a few preliminary observations. By performing standard manipulations of incentive compatibility constraints (1) (or applying the envelope formula, Milgrom and Segal (2002)), we obtain:

\[ \forall \theta_1, \quad U(\theta_1) = U(\theta) + \int_{\theta_1}^{\theta} x_1(\theta)d\theta - \int_{\theta_1}^{\theta} \int_{\theta_1}^{\theta} u(\theta_2|\theta) \frac{\partial f(\theta_2|\theta)}{\partial \theta} d\theta_2 d\theta. \]  

(10)

Similarly, (2) implies

\[ \forall \theta_1, \theta_2, \quad u(\theta_2|\theta_1) = u(\theta_2|\theta_1) + \int_{\theta_2}^{\theta} x_2(\theta_1, \theta)d\theta \]  

(11)

Moreover, the allocations \( x_1(\cdot) \) and \( x_2(\cdot, \cdot) \) must be monotone decreasing. We note that in the special case in which \( \theta_2 \) is independent of \( \theta_1 \), the last term in (10) disappears and we retrieve the usual envelope condition.

Using the definitions of \( U(\theta_1) \) and \( u(\theta_2|\theta_1) \) as well as (10) and (11), we can rewrite the payments functions as a function of the allocations:

\[ P(\theta_1) = x_1(\theta_1)\theta_1 - \int_{\theta}^{\theta} u(\theta_2|\theta_1) f_2(\theta_2|\theta_1) d\theta_2 \]

\[ + U(\theta) + \int_{\theta_1}^{\theta} x_1(\theta)d\theta - \int_{\theta_1}^{\theta} \int_{\theta_1}^{\theta} u(\theta_2|\theta) \frac{\partial f(\theta_2|\theta)}{\partial \theta} d\theta_2 d\theta \]  

(12)

\[ P_2(\theta_1, \theta_2) = x_2(\theta_1, \theta_2)\theta_2 + u(\theta_2|\theta_1) + \int_{\theta_2}^{\theta} x_2(\theta_1, \theta)d\theta \]  

(13)

Using (11) and integrating by parts, we can show that

\[ \int_{\theta_1}^{\theta} \int_{\theta}^{\theta} u(\theta_2|\theta) \frac{\partial f_2(\theta_2|\theta)}{\partial \theta} d\theta_2 d\theta = \int_{\theta_1}^{\theta} \int_{\theta}^{\theta} \left( u(\theta_2|\theta) + \int_{\theta_1}^{\theta} x_2(\theta_2, \theta)d\theta \right) \frac{\partial f_2(\theta_2|\theta)}{\partial \theta} d\theta_2 d\theta \]

\[ = \int_{\theta_1}^{\theta} \int_{\theta}^{\theta} x_2(\theta_2, \theta) \frac{\partial F_2(\theta_2|\theta)}{\partial \theta} d\theta_2 d\theta \]  

(14)
where we used the fact that \( \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} u(\theta_1) \frac{\partial f_2(\theta_2)}{\partial \theta_1} d\theta_2 d\theta_1 = 0 \) since
\[
\int_{\theta_1}^{\theta_2} \frac{\partial f_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 = \frac{\partial}{\partial \hat{\theta}} \int_{\theta_1}^{\theta_2} f(\theta_2 | \hat{\theta}) d\theta_2 = 0
\]

Moreover:
\[
\int_{\theta_1}^{\theta_2} u(\theta_2 | \theta_1) f_2(\theta_2 | \theta_1) d\theta_2 = \int_{\theta_1}^{\theta_2} \left[ u(\theta_1) + \int_{\theta_1}^{\theta_2} x_2(\theta_1, \hat{\theta}) d\hat{\theta} \right] f_2(\theta_2 | \theta_1) d\theta_2 = \\
= \int_{\theta_1}^{\theta_2} \left[ u(\theta_1) f_2(\theta_2 | \theta_1) + x_2(\theta_1, \theta_2) F_2(\theta_2 | \theta_1) \right] d\theta_2
\]

Thus, from (12) and (13), we can write constraints (3) and (4) as:
\[
\int_{\theta_1}^{\theta_2} x_1(\hat{\theta}) d\hat{\theta} + U(\theta_1) \geq \int_{\theta_1}^{\theta_2} \left[ u(\theta_1) f_2(\theta_2 | \theta_1) + x_2(\theta_1, \theta_2) F_2(\theta_2 | \theta_1) \right] d\theta_2 \\
+ \int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} \left[ x_2(\hat{\theta}, \theta_2) \frac{\partial F_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} \right] d\theta_2 d\hat{\theta}
(15)
\]
\[
\int_{\theta_1}^{\theta_2} x_2(\theta_1, \hat{\theta}) d\hat{\theta} + u(\theta_1) \geq 0
(16)
\]

Now consider the objective function of the buyer. Substituting for (12), simplifying the terms that depend on \( P_2(\theta_1, \theta_2) \) and integrating by parts,\(^7\) the buyer’s objective function becomes:
\[
E_{\theta_1, \theta_2}[W] = \int_{\theta_1}^{\theta_2} \left\{ \left[ v - \theta_1 - \frac{F(\theta_1)}{f(\theta_1)} \right] x_1(\theta_1) + \int_{\theta_1}^{\theta_2} (v - \theta_2) x(\theta_1, \theta_2) f_2(\theta_2 | \theta_1) d\theta_2 - \right. \\
U(\theta_1) + \left. \frac{F(\theta_1)}{f(\theta_1)} \int_{\theta_1}^{\theta_2} \left[ x(\theta_1, \theta_2) \frac{\partial F(\theta_2 | \theta_1)}{\partial \theta_1} \right] d\theta_2 \right\} f_1(\theta_1) d\theta_1
(17)
\]

**Proof of Proposition 0.** The optimal mechanism in the absence of limited liability constraints can be found by maximizing (17) subject to the constraints that \( x_1(\cdot), x_2(\cdot, \cdot) \) are decreasing. By Assumptions 1-3, the solution to that problem is to set \( x_{\text{sb}}^1(\theta_1) = \)

\(^7\)In particular, the last term of (17) follows since
\[
\int_{\theta_1}^{\theta_2} \int_{\theta_1}^{\theta_2} x_2(\theta_1, \theta_2) \frac{\partial F_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 d\theta_1 f_1(\theta_1) d\theta_1 = \int_{\theta_1}^{\theta_2} \frac{F_1(\theta_1)}{f_1(\theta_1)} \int_{\theta_1}^{\theta_2} x_2(\theta_1, \theta_2) \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} d\theta_2 d\theta_1 f_1(\theta_1) d\theta_1.
\]
We have $x_{sb}^{th}(\theta_1, \theta_2) = 1$ for all $\theta_1, \theta_2 \in [\theta, \bar{\theta}]$ and to set $U(\bar{\theta}) = 0$. Using (13) and $u(\bar{\theta} | \theta_1) = 0$, the second period transfer is given by $P_{sb}(\theta_1, \theta_2) = \bar{\theta}$. Lastly, using (12) and $u(\theta_2 | \theta_1) = P_{sb}(\theta_1, \theta_2) - x_{sb}^{th}(\theta_1, \theta_2) = \bar{\theta} - \theta_2$, the first period transfer is given by

$$P_{sb}^{th}(\theta_1) = \bar{\theta} - \int_{\theta}^{\bar{\theta}} (\bar{\theta} - \theta_2) f_1(\theta_2 | \theta_1) d\theta_2 - \int_{\theta}^{\bar{\theta}} f_1(\theta_2 | \theta_1) d\theta_2 - \int_{\theta}^{\bar{\theta}} (\bar{\theta} - \theta_2) (f_2(\theta_2 | \theta_1) - f(\theta_2 | \theta_1)) d\theta_2$$

Finally, Assumption 4 implies that, with these transfers and allocation, IR constraint (6) holds for all $\theta_1 \in [\theta, \bar{\theta}]$. Indeed, for all $\theta_1 \in [\theta, \bar{\theta}]$,

$$U(\theta_1) = P(\theta_1) - x_{1}^{th}(\theta_1) \theta_1 + \int_{\theta}^{} u(\theta_2 | \theta_1) f_2(\theta_2 | \theta_1) d\theta_2$$

$$= \mathbb{E}[\theta_2 | \bar{\theta}] - \theta_1 + \bar{\theta} - \mathbb{E}[\theta_2 | \theta_1] \geq 0,$$

where the last inequality follows since Assumption 4 implies that $\mathbb{E}[\theta_2 | \bar{\theta}] - \mathbb{E}[\theta_2 | \theta_1] \geq 0$ for all $\theta_1 \in [\theta, \bar{\theta}]$. 

**Proof of Proposition 1.** To prove Proposition 1, we consider the solution of the following relaxed problem:

$$\max_{x_1(\cdot), x_2(\cdot, \cdot)} \mathbb{E}[\theta_1, \theta_2][W] \text{ subject to } (15), (16) \text{ and } x_1(\cdot), x_2(\cdot, \cdot) \text{ decreasing} \quad (RP)$$

Recall that we are assuming $\frac{F_1(\theta)}{F_1(\theta)}$ to be increasing in $\theta$ and that $v$ is sufficiently high to guarantee that $v - \bar{\theta} - \frac{F_1(\theta)}{F_1(\bar{\theta})} > 0$. Thus the optimal allocation rule in the first period is $x_1(\theta_1) = 1$ for all $\theta_1$.

Consider limited liability constraint (15). Exploiting the fact that $x_1(\theta_1) = 1$ for all $\theta_1$, this constraint can be written as:

$$\bar{\theta} - \theta_1 \geq -U(\bar{\theta}) + u(\bar{\theta} | \theta_1) + \int_{\theta}^{\bar{\theta}} x_2(\theta_1, \theta_2) F_2(\theta_2 | \theta_1) d\theta_2 + \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} x_2(\theta, \theta_2) \frac{\partial F_2(\theta_2 | \theta)}{\partial \theta} d\theta_2 d\theta$$

(18)
Notice that \( u(\theta_1) \) enters on the right-hand side of (18) and does not enter on the objective function. Thus, under an optimal mechanism, \( u(\theta_1) = 0 \) for all \( \theta_1 \). Now set \( x_2(\theta_1, \theta_2) = 1 \) for all \((\theta_1, \theta_2)\), which is the allocation that maximizes the principal’s payoff in the absence of limited liability constraints (see Proposition 0). Then, the right-hand side of (18) is equal to:

\[
-U(\theta) + \int_{\theta}^{\theta_1} F_2(\theta_2 | \theta_1) d\theta_2 + \int_{\theta}^{\theta_1} \frac{\partial F_2(\theta_2 | \theta)}{\partial \theta_1} d\theta_2 d\theta
\]

Taking the derivative with respect to \( \theta_1 \) we get

\[
\int_{\theta}^{\theta_1} \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} d\theta_2 - \int_{\theta}^{\theta_1} \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} d\theta_2 = 0
\]

Thus, under this allocation, the right-hand side of (18) is constant in \( \theta_1 \). Instead the left-hand side of (18) is decreasing in \( \theta_1 \). We conclude that there must exist \( \theta^* \in [\theta, \bar{\theta}] \) such that (18) binds only if \( \theta_1 > \theta^* \).

For \( \theta_1 \leq \theta^* \), the optimal allocation rule is given by \( x_1^*(\theta_1) = x_2^*(\theta_1) = 1 \) for all \( \theta_1 \) and \( x_2^*(\theta_1, \theta_2) = x_2^*(\theta_1, \theta_2) = 1 \) for all \((\theta_1, \theta_2)\). For \( \theta_1 < \theta^* \), transfers are given by \( P_1(\theta_1) = \mathbb{E}[\theta_2 | \theta] \) and \( P_2(\theta_1, \theta_2) = \bar{\theta} \) for all \( \theta_2 \).

Thus, from now on, we will focus on the case in which \( \theta_1 > \theta^* \), so that (18) binds. For each \( \theta_1 \), let \( \lambda(\theta_1) \) be the Lagrange multiplier associated with constraint (18). By our discussion above, \( \lambda(\theta_1) = 0 \) for all \( \theta \leq \theta^* \). Consider the Lagrangean of Program (RP). Differentiating this Lagrangean with respect to \( x_2(\theta_1, \theta_2) \), we get the following first-order condition:

\[
(v - \theta_2) f_2(\theta_2 | \theta_1) f_1(\theta_1) + F_1(\theta_1) \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} - \lambda(\theta_1) F_2(\theta_2 | \theta_1) - \int_{\theta_1}^{\theta} \lambda(\hat{\theta}) d\hat{\theta}
\]

Following Jullien (2000), term \( \int_{\theta_1}^{\theta} \lambda(\hat{\theta}) d\hat{\theta} \) in the equation above corresponds to the shadow value a uniform relaxation of the limited liability constraints (18) for all types in the interval \([\theta, \theta_1]\).

If we differentiate the Lagrangean of Program (RP) with respect to \( U(\theta) \), we obtain

\[
-1 + \int_{\theta_1}^{\theta} \lambda(\hat{\theta}) d\hat{\theta}
\]

(where we used the fact that \( \lambda(\hat{\theta}) = 0 \) for all \( \hat{\theta} \leq \theta^* \)). Note that this derivative cannot be positive: if it where, it would be optimal to set \( U(\theta) = +\infty \), which can be satisfied only if at least one of the payments is set to \( +\infty \). Clearly, this cannot
be true at an optimal mechanism. It thus follows $-1 + \int_{\theta^*}^{\theta} \lambda(\hat{\theta}) d\hat{\theta} \leq 0$, and so $U(\bar{\theta}) = 0$ is optimal.

Dividing (19) by $f_2(\theta_2 | \theta_1)f_1(\theta_1)$ and rearranging, we get that the first-order condition is proportional to

$$(v - \theta_2) + \frac{1}{f_2(\theta_2 | \theta_1)} \frac{1}{f_1(\theta_1)} \frac{\partial F_2(\theta_2 | \theta_1)}{\partial \theta_1} \left[ F_1(\theta_1) - \int_{\theta}^{\theta_1} \lambda(\hat{\theta}) d\hat{\theta} \right] - \frac{\lambda(\theta_1)}{f_1(\theta_1)} \frac{F_2(\theta_2 | \theta_1)}{f_2(\theta_2 | \theta_1)} = 0,$$  \hspace{1cm} (20)

By the arguments above, we know that $\int_{\theta}^{\theta_1} \lambda(\hat{\theta}) d\hat{\theta} \in [0, 1]$ for all $\theta_1$. It follows from Assumptions 1-3 that, for each $\theta_1 > \theta^*$, the first-order condition (19) is decreasing in $\theta_2$. Moreover, first-order condition (19) is strictly positive at $\theta_2 = \theta^*$. For each $\theta_1 \geq \theta^*$, let $\vartheta(\theta_1)$ be the value of $\theta_2$ at which the first-order condition (19) is equal to zero (if such a $\theta_2$ does not exist, set $\vartheta(\theta_1) = 0$). Therefore, for all $\theta_1 > \theta^*$, the optimal allocation rule in the second period is given by:

$$x^*_2(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_2 \leq \vartheta(\theta_1), \\ 0 & \text{if } \theta_2 > \vartheta(\theta_1). \end{cases}$$

Note that since $\lambda(\theta_1) = 0$ for all $\theta_1 \leq \theta^*$, Assumption 3 implies that $\vartheta(\theta^*) = 0$.

Substituting for this allocation rule in (18), and using the fact that this constraint binds for all $\theta_1 > \theta^*$, it follows that for all $\theta_1 > \theta^*$

$$\bar{\theta} - \theta_1 = \int_{\theta}^{\vartheta(\theta_1)} F_2(\theta_2 | \theta_1) d\theta_2 + \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\vartheta(\hat{\theta})} \frac{\partial F_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 d\hat{\theta},$$

where we used $U(\bar{\theta}) = 0$. Differentiating both sides of this equation with respect to $\theta_1$, we get that $\vartheta(\cdot)$ satisfies differential equation

$$\frac{d\vartheta(\theta_1)}{d\theta_1} = - \frac{1}{F_2(\vartheta(\theta_1) | \theta_1)}.$$

\hspace{1cm} (20)

Indeed, at $\theta_2 = \bar{\theta}$, equation (20) is equal to

$$(v - \bar{\theta}) + \frac{1}{f_2(\bar{\theta} | \theta_1)} \frac{1}{f_1(\theta_1)} \frac{\partial F_2(\bar{\theta} | \theta_1)}{\partial \theta_1} \left[ F_1(\theta_1) - \int_{\theta}^{\theta_1} \lambda(\hat{\theta}) d\hat{\theta} \right] - \frac{\lambda(\theta_1)}{f_1(\theta_1)} \frac{F_2(\bar{\theta} | \theta_1)}{f_2(\bar{\theta} | \theta_1)} = v - \bar{\theta} > 0,$$

where the equality follows since $F_2(\bar{\theta} | \theta_1) = 0$ and $\frac{\partial F_2(\bar{\theta} | \theta_1)}{\partial \theta_1} = 0$.  

15
with boundary condition \( \vartheta(\theta^i) = \bar{\theta} \). Computing the equation above at \( \theta_1 = \theta^i \) and using \( \vartheta(\theta^i) = \bar{\theta} \) yields

\[
\bar{\theta} - \theta^i = \int_{\theta}^{\bar{\theta}} F_2(\theta_2 | \theta^i) d\theta_2 + \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} \frac{\partial F_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\theta_2 d\hat{\theta}
\]

\[
= \int_{\theta}^{\bar{\theta}} F_2(\theta_2 | \theta^i) d\theta_2 + \int_{\theta}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} \frac{\partial F_2(\theta_2 | \hat{\theta})}{\partial \hat{\theta}} d\hat{\theta} d\theta_2
\]

\[
= \int_{\theta}^{\bar{\theta}} F_2(\theta_2 | \theta^i) d\theta_2 + \int_{\theta}^{\bar{\theta}} [F_2(\theta_2 | \bar{\theta}) - F_2(\theta_2 | \theta^i)] d\theta_2 = \int_{\theta}^{\bar{\theta}} F_2(\theta_2 | \bar{\theta}) d\theta_2.
\]

Since \( \int_{\theta}^{\bar{\theta}} F_2(\theta_2 | \bar{\theta}) d\theta_2 = \bar{\theta} - E[\theta_2 | \bar{\theta}] \), it follows that \( \theta^i = E[\theta_2 | \bar{\theta}] \). Finally, it is immediate to verify that the allocations in the first and second period are monotone, and so (1) and (2) are satisfied. ■

**B Orthogonalized types and dynamic relevance**

Following Eső and Szentes (2017), consider the following “orthogonalization” of the seller’s private information. At time \( t = 1 \), the seller observes \( \epsilon_1 = F_1(\theta_1) \). At time \( t = 2 \), the seller observes \( \epsilon_2 = F_2(\theta_2 | \theta_1) \). Note that \( \epsilon_1 \) and \( \epsilon_2 \) are iid uniformly distributed on \([0, 1] \). Moreover, this model is strategically equivalent to our original model in which the seller observes \( \theta_1 \) and \( \theta_2 \) (provided that the seller remembers at time \( t = 2 \) the realization of \( \theta_1 \) ). Indeed, \( \theta_1 = F_1^{-1}(\epsilon_1) \) and \( \theta_2 = F_2^{-1}(\epsilon_2 | \theta_1) \).

Within this model, consider (as Eső and Szentes (2017) do) a benchmark setting in which, at time \( t = 1 \), the buyer observes the realization of \( \epsilon_2 \), and can contract on this information. Consider the following direct mechanism.

The allocation is \( x_1(\epsilon_1) = x_2(\epsilon_1, \epsilon_2) = 1 \) for all \( \epsilon_1, \epsilon_2 \); that is, the job is procured with probability one in both periods independently of the reports. Note that this allocation is equivalent to the one induced by the second best mechanism in Proposition 0.

Transfers are instead given by \( P_1(\epsilon_1) = \bar{\theta} \) for all announced \( \epsilon_1 \in [0, 1] \). For all \( \epsilon_1, \epsilon_2 \in [0, 1]^2 \), \( P_2(\epsilon_1, \epsilon_2) = F_2^{-1}(\epsilon_2 | \theta_1) \). Note that transfer \( P_2(\epsilon_1, \epsilon_2) \) is equal to what the realized second period cost, \( \theta_2 \), would be given the orthogonalized type \( \epsilon_2 \) if the seller’s first period cost was \( \bar{\theta} \). Since \( \partial F_2(\theta_2 | \theta_1)/\partial \theta_1 \leq 0 \) for all \( \theta_1, \theta_2 \in [\theta, \bar{\theta}] \) (Assumption 4), it follows that \( P_2(\epsilon_1, \epsilon_2) \geq F_2^{-1}(\epsilon_2 | \theta_1) = \theta_2 \); i.e., transfer \( P_2(\epsilon_1, \epsilon_2) \) is weakly higher
than the seller’s second period cost. Hence, this mechanism satisfies limited liability constraints (3) and (4).

Note that for all $\epsilon_1 \in [0,1]$, a seller with such orthogonalized type would have a first-period cost equal to $\theta_1 = F_1^{-1}(\epsilon_1)$. His expected payoff would be given by

$$P_1(\epsilon_1) - F_1^{-1}(\epsilon_1) + \mathbb{E}[P_2(\epsilon_1, \epsilon_2) - F_2^{-1}(\epsilon_2|\theta_1)] = \bar{\theta} - \theta_1 + \mathbb{E}[\theta_2|\bar{\theta}] - \mathbb{E}[\theta_2|\theta_1],$$

which is equal to what the seller gets under the mechanism in Proposition 0. Hence, this mechanism also satisfies incentive compatibility at $t = 1$, and is thus implementable in this benchmark environment in which the buyer observes the realization of $\epsilon_2$ at the time of contracting. Furthermore, this contract gives the buyer the same expected payoff as the contract in Proposition 0 – and so it gives the buyer a larger payoff than the contract in Proposition 1.

References


Sugaya, T. and A. Wolitzky (2017): “Revelation Principles in Multistage Games,”

18