Pooling and Tranching under Belief Disagreement

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Abstract

We study optimal security design when issuer and market participants disagree about the characteristics of the underlying asset. We show that pooling and tranching assets can be preferable to selling optimal securities backed by individual assets: pooling mitigates belief disagreement between issuer and investors, and tranching allows the issuer to exploit belief disagreement among investors. Interestingly, differences in beliefs can make pooling and tranching complements. Pooling and tranching can be optimal even when the number of securities in the pool is small, and remain optimal even when the issuer sells all tranches, including the most junior ones.

Keywords: Disagreement, Security Design, Optimism, Overconfidence, Pooling, Tranching, Behavioral Finance

JEL classification codes: G30, G32, D84, D86

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1 Introduction

Which security does a firm optimally issue when firm and market participants agree to disagree (Aumann, 1976) about the firm’s cash-flow distribution? Existing literature, reviewed below, has used belief disagreement to explain capital structure choices, investment decisions, the choice of debt maturities, or the emergence of intermediaries. Complementing this literature, we study how differences in beliefs between issuer and markets, and between market participants influence a firm’s optimal security design in the sense of Allen and Gale (1988) – i.e., in a model that imposes only minimal restrictions on the shape of the contract. We formally show that disagreement in beliefs can generate various commonly observed financial contracts, and in particular previously unexplained aspects of pooling and tranching.

We consider an issuer who owns an asset that will pay uncertain cash-flows at a future date. To raise capital, the issuer designs a security which is backed by the cash flows of one or multiple assets. Following DeMarzo and Duffie (1999), we assume the issuer discounts future cash-flows more than the market.\textsuperscript{1} We allow for different types of investors in the market, who may differ in their beliefs about the assets’ cash-flow distribution. Our main assumption is that the issuer is more optimistic than market participants.\textsuperscript{2} The issuer’s problem is to design a set of monotonic securities (one for each type of investor) backed by the underlying assets that maximizes her expected payoff, which is given by the sum of the market prices of the securities she sells and the expected discounted value of retained cash flows.

We provide three sets of main results. First, we provide conditions under which it is optimal for the issuer to sell different tranches to the different types of investors. Second,

\textsuperscript{1}The assumption that the issuer discounts future cash-flows at a higher rate than the market is a metaphor, for example for a situation in which the issuer has some profitable investment opportunity. Also, the assumption will hold if the issuer faces credit constraints or, as in the case of financial entities, minimum-capital requirements.

\textsuperscript{2}This assumption is different from assuming difference in the success probability of a project with binary payoff, as has been employed in the previous literature. The difference is meaningful as it leads to a distinction between outside equity and debt, among others.
we show that selling a security backed by a pool of several underlying assets can be strictly preferred to selling individual asset-backed securities. Third, we provide conditions under which pooling and tranching are complements.

The intuition behind the optimality of tranching is simple, and related to Garmaise (2001): when there are differences in beliefs among investors, it is optimal for the issuer to design different securities, targeted to the different investor types. We show that, under certain conditions, the issuer finds it optimal to retain the most junior tranche.\(^3\) In the special case in which all investors in the market share the same beliefs, we show that, under standard conditions, the optimal security is debt. The intuition for this result is closely related to earlier studies on capital structure choice and investment amid disagreement, including De Meza and Southey (1996); Heaton (2002); Hackbarth (2008): the issuer finds it optimal to only sell cash-flows in the left tail of the cash-flow distribution, which the market values relatively more, and to retain the right tail of the distribution, which the issuer values relatively more.

For our results on pooling, we consider an issuer who owns two underlying assets.\(^4\) We start by assuming that there is a single type of investor in the market. In this setting, we show that an optimistic issuer may strictly prefer to sell a security backed by the pool of assets than to sell individual asset-backed securities. Intuitively, while outside investors might be very relatively pessimistic about the probability of an individual asset delivering high profits, they will typically be less pessimistic about the event that at least one of several assets pays off a high return. As a result, an issuer who owns multiple assets may find it strictly optimal to sell a “senior” security backed by the pool of assets. The following example illustrates the

\(^3\)The latter prediction depends on the assumption that the issuer is more optimistic than the market. The prediction would be reversed in a (perhaps trivial) twist on the model that involves allowing some investors to be more optimistic than the issuer.

\(^4\)For simplicity we focus on the case of two assets, although the results extend to the case in which the issuer owns several assets. The fact that disagreement makes pooling and tranching optimal even for a small number of assets is one dimension of distinction from asymmetric information theories of security design.
logic of this result.

**Example 1.** Consider first an issuer who owns a single asset, which can either pay a return of 1 or a return of 0. The market believes that the probability of the asset paying off is \( \frac{1}{3} \); the issuer believes in an upside probability of \( \frac{2}{3} \). The issuer discounts future cash-flows with a factor of 0.6, whereas the market does not discount. The market is therefore willing to pay \( \frac{1}{3} \) for the asset. Since the asset is worth \( \frac{2}{3} \cdot 0.6 = 0.4 \) to the issuer, she retains it.

Consider now an issuer who owns two of these assets with iid returns. The issuer’s payoff from retaining the two assets is 0.8, which is strictly larger than her payoff from selling two individual securities, each backed by an asset. Suppose instead that the issuer sells a “senior” security backed by the pool of assets that pays 1 if at least one asset pays off and zero otherwise. Investors are willing to pay \( 1 - \left( \frac{2}{3} \right)^2 = \frac{5}{9} \) for the security, while the issuer assigns to it a value of \( \left( 1 - \left( \frac{1}{3} \right)^2 \right) \cdot 0.6 = \frac{8}{15} < \frac{5}{9} \). Because the issuer retains a cash-flow of 1 in the event that both assets pay off, her expected payoff from selling this security is \( \frac{5}{9} + \left( \frac{2}{3} \right)^2 \cdot 0.6 \approx 0.822 \).

We stress that differences in beliefs between the issuer and the market are crucial for pooling to be optimal in this setting. Indeed, because the issuer discounts future cash-flows more than the market, with homogenous beliefs it would always be optimal for the issuer to sell the entire firm. As a result, when issuer and market share the same beliefs, the issuer is indifferent between pooling her assets or selling them as separate concerns.

Lastly, we consider the case of an issuer who owns two assets but faces different types of investors in the market. We impose restrictions on the primitives such that an issuer who does not pool the assets will find it optimal to not tranche the asset, i.e. only sell a single senior tranche (designed for one type of investor), and under which an issuer who does not tranche (i.e., sells to only one type of investors) will find it optimal not to pool. We show that, in this setting, it can still be strictly optimal for the issuer to pool and tranche – in
other words, we provide conditions under which pooling and tranching are complements. Importantly, the ability to sell multiple tranches to the market has a positive feedback to the attractiveness of pooling.

These results complement asymmetric information theories of pooling and tranching (De-Marzo, 2005). Despite a fundamentally different mechanism, some predictions are similar, whereas others differ. For example, our model shares the feature with asymmetric information theories that pooling a set of securities and selling a single senior tranche backed by the pool can be preferable to selling a set of senior claims on the individual securities, and that the ability to design a non-linear security (a single senior tranche) can make pooling attractive in the first place. Also, in both theories, pooling becomes relatively less attractive when the correlation of the underlying assets increases.\(^5\) Our model’s predictions add to the results of asymmetric information models by explaining also why issuers sell multiple tranches, and showing how the ability to sell multiple tranches increases the attractiveness of pooling. Moreover, pooling is attractive in our setting even when the number of the securities in the pool is small, and selling all tranches (rather than just the senior tranche) can be optimal.

The paper proceeds as follows. We discuss the related literature in Section 2. Section 3 introduces the basic model and derives the optimality of tranching. Sections 4 presents the results on pooling and its interaction with tranching. Section 5 concludes. All proofs are in the Appendix.

\(^5\)Our model may help explain in ways consistent with the empirical evidence on issuers’ relatively optimistic beliefs (Cheng et al., 2014) and the pro-cyclical nature of belief disagreement (see e.g. Chen et al., 2002; Scheinkman and Xiong, 2003; Hong and Stein, 2007) why pooling and tranching appear to coincide in the time series and cross-section (Fender and Mitchell, 2005; Coval et al., 2009; Stein, 2010; Chernenko et al., 2014; Fuster and Vickery, 2014).
2 Related Literature

The idea that belief disagreement can shape security design goes back at least to Modigliani and Miller (1958), who write “Grounds for preferring one type of financial structure to another still exist within the framework of our model. If the owners of a firm discovered a major investment opportunity which they felt would yield much more than [the market’s discount rate], they might well prefer not to finance it via common stock. A better course would be to finance the project initially with debt. Still another possibility might be to [issue] a convertible debenture.” (excerpts from p. 292) Our paper offers a formal investigation into the role of disagreement in optimal design of securities.

Many papers have invoked differences in beliefs to explain stylized facts of entrepreneurship as well as corporate investment, financing, payout and capital structure choices. By contrast, we allow for a less restrictive state space (\(N\) states instead of two) and/or contracting space (all monotonic securities, rather than a choice between equity and debt, debt of various maturities, or similar) and study the question of which security is optimal under these general conditions.

Our paper also relates to Garmaise (2001), who shows that tranching can be optimal in a model in which there is disagreement among investors and in which the prices of securities are determined through a first price auction, and to Coval and Thakor (2005), who show that rational actors can arise to intermediate between optimistic entrepreneurs and pessimistic investors, issuing safe debt and retaining a mezzanine tranche of the projects they finance (see also Gennaioli et al., 2013).

\textsuperscript{6}De Meza and Southey (1996); Boot et al. (2006, 2008); Landier and Thesmar (2009); Malmendier et al. (2011); Boot and Thakor (2011); Bayar et al. (2011); Thakor and Whited (2011); Huang and Thakor (2013); Adam et al. (2014); Bayar et al. (forthcoming). See also Simsek (2013), who studies how differences in beliefs among investors affect asset prices in the presence of collateral constraints.

\textsuperscript{7}Yang (2013) shows that limited channel capacity can render debt and pooling optimal.

\textsuperscript{8}Our paper is also related to a literature on corporate financial choices amid an ambiguity-averse pool of investors (e.g. Dicks and Fulghieri, 2015), because ambiguity aversion on behalf of the market collapses to disagreement between issuer and market. Lee and Rajan (2017) study optimal security design when both
Whereas some intuitions and predictions of our model are similar to previous models exploring asymmetric information as a friction to rationalize particular securities – including debt, convertible debt, and pooling –, the mechanism that our paper highlights is sharply different from those earlier contributions.\footnote{See, for instance, Myers and Majluf (1984); Noe (1988); Innes (1990); Nachman and Noe (1990); Gorton and Pennacchi (1990); Stein (1992); Nachman and Noe (1994); Manove and Padilla (1999); Inderst and Mueller (2006); Axelsson (2007); Fulghieri et al. (2013). We discuss differences in predictions throughout the paper.} In particular, similarities and differences to the most closely related paper by DeMarzo (2005) have been discussed above.

Our theory also makes no use of moral hazard as a driver of the optimal security as in Admati and Pfleiderer (1994); Bergemann and Hege (1998); Schmidt (2003); Winton and Yerramilli (2008); Antic (2014); Hébert (2014).

3 Basic Model

3.1 Payoffs, Beliefs, and Objectives

At date $t = 0$, an issuer owns a risky asset yielding state-contingent payoffs at date $t = 1$. For now we treat this as a single asset, but this could be a pool of several assets. There is a finite set of possible states of nature $S = \{1, \ldots, K\}$ at $t = 1$, and the asset pays an amount $X_s \in \mathbb{R}_+$ in state $s \in S$, with $X_s \neq X_{s'}$ for all $s, s' \in S$.\footnote{The assumption that states are finite is for simplicity. Our main results are robust to having a continuum of states.} We order the states so that $X_1 < X_2 < \ldots < X_K$. Let $X_0 = 0$. With little loss of generality, we assume that there exists $\Delta > 0$ such that $X_s - X_{s-1} = \Delta$ for all $s \geq 1$.

We let $\pi^I$ be the probability distribution over $S$ that represents the issuer’s beliefs. Market participants have different beliefs about the cash-flow distribution of the underlying asset than the issuer. In particular, we assume that there are two types of investors in the market,
The two types of investors differ in their beliefs about the cash-flow distribution of the asset that the issuer owns. For \( j = 1, 2 \), let \( \pi^j \) be the probability distribution over \( S \) representing the beliefs of investors of type \( \tau_j \). We assume that the issuer is more optimistic than both types of investors: for \( j = 1, 2 \), \( \pi^I \) first-order stochastically dominates \( \pi^j \).

The issuer has to design securities \( (F^1, F^2) \in \mathbb{R}^K_+ \) backed by the cash-flows \( X = (X_s)_{s \in S} \) to sell in the market. Thus, securities \( (F^1, F^2) \) must be such that \( 0 \leq F^1_s + F^2_s \leq X_s \) for all \( s \in S \). Following DeMarzo and Duffie (1999) we assume that the issuer discounts retained cash-flows at a rate that is higher than the market rate (which is normalized to 1). Thus, the issuer attaches a value of \( \delta \sum_{s \in S} \pi^I_s (X_s - F^1_s - F^2_s) \) to retained cash-flows, where \( \delta \in (0, 1) \) is the issuer’s discount rate. The payoff of an issuer who sells to the market securities \( (F^1, F^2) \) at a prices \( p^1 \) and \( p^2 \) is then given by \( p^1 + p^2 + \delta \sum_{s \in S} \pi^I_s (X_s - F^1_s - F^2_s) \).

The price that investors of type \( t_j \) are willing to pay for security \( F \) is \( p^j(F) := \sum_s \pi^j_s F_s \). For any security \( F \), let \( p(F) = \max \{ p^1(F), p^2(F) \} \) be the highest price that market participants are willing to pay for \( F \). Overall, the issuer’s payoff from selling securities \( (F^1, F^2) \) is

\[
U(F^1, F^2) := p(F^1) + p(F^2) + \delta \sum_{s \in S} \pi^I_s (X_s - F^1_s - F^2_s). \tag{1}
\]

As is standard in the literature of optimal security design (e.g. DeMarzo and Duffie, 1999), we assume the issuer is restricted to sell securities that are monotonic.\(^1\)

**Definition 1.** Say that securities \( F^1 \) and \( F^2 \) are monotonic if \( F^1_s \) and \( F^2_s \) are increasing in \( s \) and if \( X_s - F^1_s - F^2_s \) is increasing in \( s \).

\(^{11}\)The assumption that there are two types of investors is for simplicity; all of our results extend to the case with \( n \geq 1 \) types of investors. In particular, the number of tranches will depend on the number of investors with different beliefs.

\(^{12}\)As is well known, this assumption can be microfounded with a moral hazard problem. To avoid high payments implied by a non-increasing security, the issuer could easily inflate cash-flows, e.g., by borrowing privately, and thus decrease payments to the investor.
Let \( F \) be the set of feasible securities

\[
F := \{ F_1, F_2 \in \mathbb{R}_+^K : 0 \leq F_1^s + F_2^s \leq X_s \forall s \in S \text{ and } F_1 \text{ and } F_2 \text{ are monotonic} \}.
\] (2)

The issuer’s problem is to find the securities \((F^1, F^2) \in F\) that solve

\[
\sup_{(F^1, F^2) \in F} U(F^1, F^2).
\] (3)

### 3.2 Optimal Security Design with Divergent Beliefs

In this section we present the solution to problem (3). We introduce additional notation before presenting our results. For any \( s \in S \), let \( A_s := \{s, s+1, ..., K\} \) be the event that the asset yields cash-flows weakly larger than \( X_s \). For all \( s \in S \), let \( \pi^I(A_s) := \sum_{s' \geq s} \pi^I_{s'} \) and \( \pi^j(A_s) := \sum_{s' \geq s} \pi^j_{s'} \) be, respectively, the probability that the issuer and investors of type \( j \) assign to event \( A_s \). The assumption that \( \pi^I \) first-order stochastically dominates \( \pi^j \) implies that \( \pi^I(A_s) \geq \pi^j(A_s) \) for all \( s \in S \) and for \( j = 1, 2 \).

**Lemma 1.** Let \((F^1, F^2)\) be a solution to (3). Then, there exists \((\hat{F}^1, \hat{F}^2) \in F\) such that, for \( j = 1, 2 \),

\[
p(\hat{F}^j) = p^j(\hat{F}^j).
\]

By Lemma 1, it is without loss of optimality to consider solutions \((F^1, F^2)\) to (3) such that, for \( j = 1, 2 \), security \( F^j \) is bought by investors of type \( t_j \).

The following result characterizes the optimal security. In what follows, for \( j = 1, 2 \), we use \(-j\) to denote the investors of type \( t_i \neq t_j \).

**Proposition 1.** The optimal securities \((F^1, F^2)\) satisfy: \( F_1^1 + F_1^2 = X_1 \), and for \( j = 1, 2 \) and
for all \( s \in S \setminus \{1\} \),

\[
F^j_s = \begin{cases} 
F^j_{s-1} + X_s - X_{s-1} & \text{if } \pi^j(A_s) > \max \{ \pi^{1-j}(A_s), \delta \pi^j(A_s) \} ; \\
F^j_{s-1} & \text{if } \pi^j(A_s) \leq \max \{ \pi^{1-j}(A_s), \delta \pi^j(A_s) \} .
\end{cases}
\] (4)

The key value that determines the shape of the optimal securities \((F^1, F^2)\) at each state \( s \) is the difference between \( \delta \pi^j(A_s) \) and \( \max \{ \pi^1(A_s), \pi^2(A_s) \} \); i.e., the difference between the probability that the issuer and market assign to profits being larger than \( X_s \). If \( \delta \pi^j(A_s) < \max \{ \pi^1(A_s), \pi^2(A_s) \} \), the optimal securities \((F^1, F^2)\) pay the largest possible amount (subject to monotonicity constraints) in state \( s \); i.e., \( F^1_s + F^2_s = F^1_{s-1} + F^2_{s-1} + X_s - X_{s-1} \). In contrast, if \( \delta \pi^j(A_s) \geq \max \{ \pi^1(A_s), \pi^2(A_s) \} \), the optimal securities pay the least possible amount (again, subject to monotonicity constraints) at state \( s \); i.e., \( F^1_s + F^2_s = F^1_{s-1} + F^2_{s-1} \).

The following corollaries immediately follow.

**Corollary 1.** Suppose that there exists \( s_1, s_2 \in S, s_1 < s_2 \), such that

(i) \( \pi^1(A_s) > \max \{ \pi^2(A_s), \delta \pi^j(A_s) \} \) if and only if \( s \leq s_1 \), and

(ii) \( \pi^2(A_s) > \max \{ \pi^1(A_s), \delta \pi^j(A_s) \} \) if and only if \( s \in (s_1, s_2] \).

Then, the optimal securities \((F^1, F^2)\) are \( F^1_s = \min \{ X_s, X_{s_1} \} \) and

\[
F^2_s = \begin{cases} 
0 & \text{if } s \leq s_1, \\
X_s - X_{s_1} & \text{if } s \in (s_1, s_2], \\
X_{s_2} - X_{s_1} & \text{if } s > s_2.
\end{cases}
\]

Under the conditions in Corollary 1, the issuer sells a senior tranche \( F^1 \), which is bought by investors of type \( t_1 \), and a mezzanine tranche \( F^2 \), which is bought by investors of type
Finally, the issuer only retains the most junior cash-flows \( X_s - X_{s_{t_2}} \) at states \( s > s_s \). The mezzanine tranche can be interpreted as preferred equity or junior debt.

### 3.3 Single Investor

A special case of the model is one in which there is effectively a single investor in the market. To formalize this, suppose that \( \pi^1 = \pi^2 \), so all investors share the same beliefs. We use the convention that \( F_0 = X_0 = 0 \) for any security \( F \).

**Corollary 2.** Suppose that \( \pi^1 = \pi^2 \). Then, it is optimal to sell \((F^1, F^2)\) with \( F^2_s = 0 \) for all \( s \), and

\[
\forall s \in S, \quad F^1_s = \begin{cases} 
F^1_{s-1} + X_s - X_{s-1} & \text{if } \pi^1(A_s) > \delta \pi^I(A_s), \\
F^1_{s-1} & \text{if } \pi^1(A_s) \leq \delta \pi^I(A_s).
\end{cases}
\]

Corollary 2 characterizes the optimal security in the case in which all investors share the same beliefs. By Corollary 2, when the ratio \( \frac{\pi^1(A_s)}{\delta \pi^I(A_s)} \) is decreasing in \( s \), the optimal security is given by a debt contract with face value \( s^* = \min \{ s \in S : \pi^1(A_s) > \delta \pi^I(A_s) \} \). Holding \( \delta \) fixed, the face value of debt \( s^* \) depends on how different the beliefs of the issuer and market are. When the market is extremely pessimistic, the firm issues only risk-free debt. (Once that option is exhausted, it stops issuance altogether, as we show in Section 5.) By contrast, the issuer sells the whole firm when belief disagreement is small.

This prediction is strongly consistent with the timing of securities issuances to meet market sentiment (e.g., Marsh (1982); Baker and Wurgler (2002), and in particular Dittmar and Thakor (2007)). Moreover, this prediction is in stark contrast to several theories of security design based on asymmetric information. Most prominently, the traditional “pecking order” hypothesis holds that firms issue equity only as a “last resort” (e.g., Myers, 1984) – hence, only the worst firms that have run out of other options issue equity. We stress that
there is substantial empirical evidence in line with our model’s predictions. For instance, Frank and Goyal (2003); Fama and French (2005) show that firms issue equity predominantly when not in financial distress. Farre-Mensa (2015) analyses firms that are hit with negative cash-flow shocks and thus face a need to issue securities (a decrease in $\delta$ in our model), and shows that firms whose stock is overvalued issue equity, whereas undervalued firms issue debt. Similar in spirit, Erel et al. (2011) and McLean and Zhao (2014) find that equity issuance is cyclical and higher amid positive investor sentiment, whereas firms turn to issuing safer securities during market downturns.

4 Pooling and Tranching

In this section, we consider an issuer who owns several assets. We establish two main results. First, we show that an issuer who has more optimistic beliefs than the market can strictly benefit from pooling different assets and designing a security backed by the cash-flows generated by the pool. Second, we show that when there are different types of investors in the market, pooling and tranching can be complements. To formalize this second result, we consider settings in which the issuer (i) would not find it profitable to pool her assets if restricted to sell a single tranche, and (ii) would not find it profitable to tranch if restricted to not pool her assets. We show that, in such settings, it can be strictly profitable for the issuer to pool and tranche her assets.

4.1 General Framework

Consider an issuer who owns two assets, $X^1$ and $X^2$, with iid returns.\textsuperscript{13} Let $S = \{1, ..., K\}$ and let $\{X_s\}_{s \in S}$ be the possible cash-flow realizations of asset $X^a, a = 1, 2$. We continue to

\textsuperscript{13}We focus on the case of two assets for simplicity. The results can be extended to the case of $n > 2$ assets.
assume that $X_1 < X_2 < \ldots < X_K$, and that $X_s - X_{s-1} = \Delta > 0$ for all $s \geq 1$ (recall that $X_0 = 0$).

As in Section 3, we assume that there are two types of investors. Let $\pi^I$ be the probability distributions over $S$ representing the beliefs of the issuer; let $\pi^1$ and $\pi^2$ be the probability distributions over $S$ representing, respectively, the beliefs of investors of type $t_1$ and $t_2$. The issuer is more optimistic than the market, so $\pi^I$ first-order stochastically dominates $\pi^1$ and $\pi^2$. The issuer discounts future profits at rate $\delta < 1$, whereas the market discounts future profits at rate $1$.

The timing of events is as follows. First, the issuer decides whether to pool the assets or not. Then, she designs securities optimally.

**Separate assets**

Suppose that the issuer chooses not to pool her assets. For each asset $X^a$, $a = 1, 2$, let $\mathcal{F}_{X^a}$ be the set of securities $(F^1, F^2)$ backed by $X^a$ that are monotonic; i.e., that satisfy the conditions in Definition 1. For any $(F^1, F^2) \in \mathcal{F}_{X^a}$, we let $U_{X^a}(F^1, F^2)$ be the profits that the issuer obtains from selling securities $(F^1, F^2)$ (calculated as in Section 3). Then, an issuer who doesn’t pool the assets solves the following problem for each asset $a = 1, 2$:

$$\sup_{(F^1, F^2) \in \mathcal{F}_{X^a}} U_{X^a}(F^1, F^2). \quad (6)$$

The solution to this problem is characterized by Proposition 1.

**Pooled assets**

If the issuer pools the asset, she designs securities $(F^1, F^2)$ backed by the asset pool $Y = X^1 + X^2$. For $j = 1, 2$, let $F^j_{s,s'}$ be the payoff of security $F^j$ when asset 1’s realized return is $X_s$ and asset 2’s realized return is $X_{s'}$. We restrict the issuer to sell securities $(F^1, F^2)$ that
satisfy the following monotonicity requirements:

**Definition 2.** Say that securities $F^1$ and $F^2$ backed by asset $Y = X^1 + X^2$ are $X^1 X^2$-monotonic if:

(i) for $j = 1, 2$, $F^j_{s,s'}$ is increasing in $s$ and $s'$;

(ii) $X_s + X_{s'} - (F^1_{s,s'} + F^2_{s,s'})$ is increasing in $s$ and $s'$.

These monotonicity restrictions assume it is difficult for the issuer to manipulate profits across assets. For example, the issuer may face legal constraints that make it difficult for her to transfer profits from one asset to another.

Let $\mathcal{F}_Y$ be the set of feasible securities:

$$\mathcal{F}_Y := \left\{ F^1, F^2 \in \mathbb{R}^{[S]\times[S]} : 0 \leq F^1_{s,s'} + F^2_{s,s'} \leq X_s + X_{s'} \forall (s,s') \in S^2 \text{ and } (F^1, F^2) \text{ are } X^1 X^2\text{-monotonic} \right\}.$$

Note that the price that market participants of type $j$ are willing to pay for security $F$ is $p^j_Y(F) := \sum_{s \in S} \sum_{s' \in S} \pi^j_s \pi^j_{s'} F_{s,s'}$. The issuer’s profits from pooling the assets and selling securities $(F^1, F^2) \in \mathcal{F}(Y)$ are

$$U_Y(F^1, F^2) = p_Y(F^1) + p_Y(F^2) + \delta \sum_{s \in S} \pi^1_s \pi^2_{s'} (X_s + X_{s'} - F^1_s - F^2_s), \quad (7)$$

where, for any security $F$, $p_Y(F) = \max_{j=1,2} p^j_Y(F)$ is the highest price that investors are willing to pay for $F$. The problem of an issuer who pools the asset is then

$$\sup_{(F^1, F^2) \in \mathcal{F}_Y} U_Y(F^1, F^2). \quad (8)$$

By the same arguments as in Lemma 1, it is without loss of generality to restriction attention to securities $(F^1, F^2)$ such that, for $j = 1, 2$, investors of type $j$ buy security $F^j$. 

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4.2 Single investor

We start by considering the case in which $\pi^1 = \pi^2$, so there is one type of investor in the market. For notational simplicity, we will assume that the issuer sells the security to investor 1. We start with a simple example:

**Example 2.** Suppose the issuer has two assets, $X^1$ and $X^2$. Each of the assets can produce cash-flows in $\{X_1, X_2\}$, with $X_2 > X_1 > 0$. Let $\pi^I \in (0, 1)$ and $\pi \in (0, 1)$ be, respectively, the probability the issuer and market assigns to the asset yielding cash-flows $X_1$. The issuer is more optimistic than the market, so $\pi^I < \pi$. We further assume that $\delta(1 - \pi^I) > 1 - \pi$; that is, the issuer values cash-flows at state 2 more than the market does.

Consider first the problem of an issuer who does not pool the assets. By Proposition 1, for each asset $X^a$, an optimal security $F$ has $F_1 = X_1$ and $F_2 = X_1$ (since $\delta(1 - \pi^I) > 1 - \pi$). The issuer’s profits from selling the securities separately are $2X_1 + 2\delta(1 - \pi^I)(X_2 - X_1)$.

Suppose instead that the issuer pools the two assets and sells a single security backed by the pool. Let $Y = X^1 + X^2$ and $F_Y = \min\{Y, X_1 + X_2\}$; that is, security $F_Y$ pays $2X_1$ if both assets yield a return of $X_1$, and pays $X_1 + X_2$ if one of the two assets yields a return of $X_2$. The market-price of security $F_Y$ is $p(F_Y) = \pi^22X_1 + (1 - \pi^2)(X_1 + X_2)$, and the issuer’s payoff from selling $F_Y$ is $2X_1 + (1 - \pi^2 + \delta(1 - \pi^I)^2)(X_2 - X_1)$. The issuer strictly prefers to pool the assets and sell security $F_Y$ if $\pi < \sqrt{1 - \delta(1 - (\pi^I)^2)}$. Therefore, for $\pi \in \left(1 - \delta(1 - \pi^I), \sqrt{1 - \delta(1 - (\pi^I)^2)}\right)$, pooling is strictly optimal.

Intuitively, the market is relatively less pessimistic about the event that one of the two assets yields a cash-flow of $X_2$. By pooling the two assets, the issuer is able to design a security that pays off a high return precisely when this event occurs. The example illustrates why changes in belief divergence between issuers and the market should relate to the time-series variation in the issuance of asset-backed securities, as documented by Chernenko et al. (2013).
We now present a general result for the case in which there is a single type of investor in the market. As before, for each \( s \in S \) let \( A_s = \{ s, s + 1, \ldots, K \} \) be the event that an asset pays weakly more than \( X_s \). Recall that \( \pi^1 \) are the beliefs of the investors in the market. We make the following assumption:

**Assumption 1.** There exists \( k \in S \setminus \{ K \} \) such that \( \pi^1(A_s) \geq \delta \pi^I(A_s) \) if and only if \( s \leq k \).

Under Assumption 1, if the issuer sells the assets as separate concerns it is optimal for her to issue two securities \( \hat{F}_s = \min \{ X_s, X_K \} \). The issuer’s profits from doing so are \( 2U_{X_s}(\hat{F}) \).

Let

\[
\mathcal{F}^1_Y := \left\{ (F^1, F^2) \in \mathcal{F}_Y : F^2_s = 0 \text{ for all } s \in S \right\},
\]

(9)
denote the set of feasible securities when the issuer pools the assets and designs a single security to investors with beliefs \( \pi^1 \). Note that the seller’s profits from pooling in this case are \( \sup_{F \in \mathcal{F}^1_Y} U_Y(F) \). Our next result derives conditions under which pooling is strictly optimal for the issuer.

**Proposition 2.** Suppose that Assumption 1 holds, and that there exists \( \hat{s}, \hat{s}' \in S \) with \( \hat{s}' > k \) and \( \hat{s}' \geq \hat{s} \), such that

\[
\pi^1(A_{\hat{s}'}) (2\pi^1(A_{\hat{s}}) - \pi^1(A_{\hat{s}'}) > \delta \pi^I(A_{\hat{s}'}) (2\pi^I(A_{\hat{s}}) - \pi^I(A_{\hat{s}'})). \tag{10}
\]

Then, pooling is strictly optimal: \( \sup_{F \in \mathcal{F}^1_Y} U_Y(F) > 2U_{X_s}(\hat{F}) \).

To gain intuition for Proposition 2, let \( F^S = F^1 \) denote the security that corresponds to selling \( \hat{F}_s = \min \{ X_s, X_K \} \) separately on each individual asset:

\[
F^S_{s,s'} = \min \{ X_s, X_k \} + \min \{ X_{s'}, X_k \}. \tag{11}
\]
Suppose there exists states $\hat{s}$ and $\hat{s}'$ satisfying inequality (10), and let

\[
\hat{S} := \{(s, s') \in S^2 : s \geq \hat{s} \text{ and } s' \geq \hat{s}' \text{ or } s \geq \hat{s}' \text{ and } s' \geq \hat{s}\}.
\]  

(12)

Note that for $i = 1, I$,

\[
Pr_i(s, s' \in \hat{S}) = Pr_i(X^a \geq X_{\hat{s}'}, \text{ and } X^{a'} \geq X_{\hat{s}} \text{ for } a, a' = 1, 2)
= \pi^i(A_{\hat{s}'})(2\pi^i(A_{\hat{s}}) - \pi^i(A_{\hat{s}'})
\]

is the probability that, under beliefs $\pi^i$, one asset yields returns larger than $X_{\hat{s}'}$ and the other asset yields returns larger than $X_{\hat{s}}$. The inequality in (10) implies that the market values returns in $\hat{S}$ more than the issuer does. Therefore, the issuer strictly benefits from pooling the assets and selling security $F^P$ that pays

\[
F^{P}_{s, s'} = \begin{cases} 
F^S_{s, s'} & \text{if } s, s' \notin \hat{S}, \\
F^S_{s, s'} + X_{\hat{s}} - X_{\hat{s}-1} & \text{otherwise}.
\end{cases}
\]

(13)

The following result shows the counter-part of Proposition 2: when inequality (10) does not hold, there are no gains from pooling.

**Proposition 3.** Suppose Assumption 1 holds, and that

\[
\pi^1(A_{s'})(2\pi^1(A_{s}) - \pi^1(A_{s'})) < \delta \pi^I(A_{s'})(2\pi^I(A_{s}) - \pi^I(A_{s'}))
\]

(14)

for all $s, s'$ with $s' > k$ and $s' \geq s$. Then, there are no gains from pooling: $\sup_{F \in F^\hat{S}} U_Y(F) = 2U_{X^s}(\hat{F})$. 

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4.3 Multiple investors

We now consider a setting in which there are two types of investors. We show that belief disagreement among them can make pooling and tranching complements.

We make the following assumptions.

**Assumption 2.**  
(i) There exists \( k \in S \) such that \( \pi^1(A_{s'}) \geq \pi^2(A_{s'}) \) for all \( s' \leq k \) (with strict inequality for all \( s' \neq 1 \)) and \( \pi^1(A_{s'}) < \pi^2(A_{s'}) \) for \( s' > k \).

(ii) \( \pi^1(A_{s'}) > \delta \pi^I(A_{s'}) \) for all \( s' \leq k \) and \( \delta \pi^I(A_{s'}) \geq \pi^2(A_{s'}) > \pi^1(A_{s'}) \) for all \( s' > k \).

Assumption 2(i) implies that the c.d.f.'s of the investors' beliefs cross at exactly one point. When the two types of investors assign the same value to the underlying assets (i.e., when \( \sum_s \pi^1_s X^a_s = \sum_s \pi^2_s X^a_s \) ), Assumption 2(i) implies that \( \pi^1 \) second-order stochastically dominates \( \pi^2 \).

Assumption 2(ii) implies that an issuer who sells the two assets separately (i.e., an issuer who does not pool her assets) finds it optimal to not tranche the assets: for each asset \( a = 1, 2 \), she will sell a single security \( F^a \) targeted to investors with beliefs \( \pi^1 \). Indeed, by Proposition 1, under this assumption the optimal securities \((F^1, F^2)\) when selling assets \((X^a)_{a=1,2} \) separately are given by \( F^1_s = \min \{ X_k, X_s \} \) and \( F^2_s = 0 \) for all \( s \) and for \( a = 1, 2 \). The issuer’s profits from selling the two assets separately are then \( 2U_Y(F^1, F^2) \).

For \( j = 1, 2 \), let \( U^j_Y \) denote the issuers payoffs from pooling the assets and designing a single security for investors of type \( j \). We assume:

**Assumption 3.**  
(i) \( \pi^1(A_{s})(2\pi^1(A_{s'}) - \pi^1(A_{s})) < \delta \pi^I(A_{s})(2\pi^I(A_{s'}) - \pi^I(A_{s})) \) for all \( s, s' \) with \( s' > k \) and \( s' \geq s \).

(ii) \( U^1_Y > U^2_Y \).

Assumption 3 implies that if the issuer does not tranche, then there are no gains from pooling. Indeed, by Proposition 3, under Assumption 3(i) there are no gains from pooling.
if the issuer will only sell securities designed for investors of type $\pi^1$. Moreover, assumption 3(ii) says that doing this is more profitable than pooling and selling a security designed for investors of type $t_2$.\footnote{It can be shown that, under Assumption 2, $U^1_Y > U^2_Y$ whenever $k$ is large enough.}

Taken together, Assumption 2 and Assumption 3 imply that an issuer who does not tranche does not benefit from pooling, and an issuer who does not pool does not benefit from tranche.

The next proposition summarizes these results.

**Proposition 4.** Suppose Assumptions 2 and 3 holds. Then,

1. If the seller only sells securities designed to investors with beliefs $\pi^1$, there are no gains from pooling: $\sup_{F \in F^1_Y} U_Y(F) = 2U_{X^a}(F^{1,a}, F^{2,a})$.

2. If the seller does not pool the assets, there are no benefits from tranche.

Our next result shows that, in this environment, the issuer might still find it strictly optimal to pool the assets: by doing so, she can profit from selling an additional tranche to investors with beliefs $\pi^2$.

**Proposition 5.** Suppose Assumptions 2 and 3 holds, and that there exists $\hat{s}, \hat{s}' \in S$ with $\hat{s}' > k$ and $\hat{s}' \geq \hat{s}$, such that

$$\pi^2(A_{\hat{s}'}) (2\pi^2(A_{\hat{s}}) - \pi^2(A_{\hat{s}'}) > \delta \pi^t(A_{\hat{s}'})(2\pi^t(A_{\hat{s}}) - \pi^t(A_{\hat{s}'})).$$

(15)

Then, it is strictly optimal to pool and tranche: $\sup_{(F^1, F^2) \in F^T_Y} U_Y(F^1, F^2) > 2U_{X^a}(F^{1,a}, F^{2,a})$.

The results above show conditions under which pooling and tranche are complements: while neither pooling nor tranche are beneficial on their own, the issuer finds it strictly optimal to pool the assets and then tranche them compared to selling them as separate concerns. The intuition is similar to Proposition 2. Indeed, inequality (15) implies that investors with
beliefs $\pi^2$ value returns at states in $\hat{S} := \{(s, s') \in S^2 : s \geq \hat{s} \text{ and } s' \geq \hat{s}' \text{ or } s \geq \hat{s}' \text{ and } s' \geq \hat{s}\}$ more than the issuer does. Therefore, under these conditions, the issuer gains from pooling the assets, selling security $F^1_{s, s'} = \min \{X_s, X_k\} + \min \{X_{s'}, X_k\}$ to investors with beliefs $\pi^1$, and selling security $F^2$ with

$$F^2_{s, s'} = \begin{cases} 
0 & \text{if } s, s' \notin \hat{S}, \\
X_{\hat{s}} - X_{\hat{s}-1} & \text{otherwise}
\end{cases}$$

(16)

to investors with beliefs $\pi^2$.

5 Discussion

This paper offers a simple but broadly applicable theory of security design based on the premise that issuer and market openly disagree about the asset’s cash-flow distribution. Not surprisingly, debt of various types are an optimal security. More interestingly, we show conditions under which issuing securities backed by a pool of assets (instead of issuing one security for each asset) is optimal; in addition, when there is disagreement among investors, the issuer optimally sells different tranches to the market. The perhaps most interesting result is that differences in beliefs can make pooling and tranching complements.

Our model admits several natural extensions. We conclude the paper by briefly outlining a few of them.

Pre-existing Debt

Consider the problem of an issuer who has senior debt outstanding that is backed by the cash-flows that her asset will generate, and who is considering to issue a new security backed by the remaining cash-flows. For simplicity, we assume that there is a single investor type.
Suppose the issuer has debt outstanding with face value $D < X_K$. The issuer’s goal is to design a security $F \in \mathbb{R}^{K_+}$ to sell to the market, with $F$ backed by the remaining cash-flows; i.e., for all $s$, $F$ satisfies $0 \leq F_s \leq X_s - \min\{X_s, D\}$. As before, we restrict the issuer to design monotonic securities; that is, securities $F$ such that $F_s$ and $X_s - F_s - \min\{X_s, D\}$ are increasing in $s$. Let $\mathcal{F}_D$ denote the set of feasible securities: $\mathcal{F}_D := \{F \in \mathbb{R}^{K_+} : 0 \leq F_s \leq X_s - \min\{X_s, D\}\forall s \in S$ and $F_s$ and $X_s - F_s - \min\{X_s, D\}$ are increasing in $s\}$.

The issuer’s problem is $\sup_{F \in \mathcal{F}_D} U_D(F)$, where for any $F \in \mathcal{F}_D$,

$$U_D(F) := \sum_{s \in S} \pi^1_s F_s + \delta \sum_{s \in S} \pi^I_s (X_s - \min\{X_s, D\} - F_s).$$

Let $s_D = \max\{s \in S : X_s \leq D\}$.

**Proposition 6.** Suppose the issuer already has debt outstanding with face value $D$. Then, the optimal security is described by

$$\forall s \in S, \quad F_s = \begin{cases} 
0 & \text{if } s \leq s_D \\
F_{s-1} + X_s - X_{s-1} & \text{if } \pi^1(A_s) \geq \delta \pi^I(A_s) \text{ and } s > s_D, \\
F_{s-1} & \text{if } \pi^1(A_s) < \delta \pi^I(A_s) \text{ and } s > s_D.
\end{cases} \quad (17)$$

Corollary 6 shows that the firm in our model may stop the issuance of all securities when it becomes over-levered, and is thus similar to the underinvestment result in Heaton (2002). This prediction contrasts with that of informational theories of security design as well as with tradeoff models, in which the firm may start to issue equity instead of debt when it has preexisting debt. Our model’s prediction is supported by empirical evidence in Erel et al. (2011), who show that low market sentiment can lead firms not only to stop equity issuances but to not access credit markets at all.
Correlated Assets and Disagreement on Correlations

We conclude by briefly discussing the possibility of having assets with correlated returns, and of having disagreement about the correlation of these assets between issuer and market. For simplicity, we focus on the case in which there is a single investor.

Consider first the case in which the underlying assets’ returns are not iid. In the Appendix we consider a simple setting with two assets, each of which can yield two possible returns $X_1$ and $X_2$, as in Example 2 – the only difference is that we allow these returns to be correlated. Consistent with the time-series variation in the issuance of asset-backed securities discussed above, we show that pooling remains optimal as long as the correlation between the underlying assets is not too high relative to the disagreement in beliefs.

Second, our model assumes that issuer and market disagree about the return distribution of each of the underlying assets, but agree on the correlation between these assets (i.e., they agree that returns are iid). Disagreement about the correlation in the assets’ return can strengthen the investor’s incentives for pooling. To see this, consider again the setting in Example 2. Suppose that the market believes that the two assets are iid, while the issuer believes that the two assets are perfectly correlated. Assume again that $\pi > 1 - \delta(1 - \pi^I)$, so that the optimal security backed by asset $X^a$ has $F_s = X_1$ for $s = 1, 2$. The issuer’s profits from selling the securities separately are given by $2X_1 + 2\delta(1 - \pi^I)(X_2 - X_1)$, while her payoff from selling security $F_Y = \min\{Y, X_1 + X_2\}$ now is $2X_1 + (1 - \pi^2 + \delta(1 - \pi^I))(X_2 - X_1)$. Pooling is strictly optimal whenever $\pi \in (1 - \delta(1 - \pi^I), \sqrt{1 - \delta(1 - \pi^I)})$. 


A Proofs

Proofs of Section 3

Proof of Lemma 1. Let \((F^1, F^2) \in \mathcal{F}\) be a solution to the issuer’s problem. Note that the lemma clearly holds if the two securities \((F^1, F^2)\) are bought by different types of investors.

If the two securities \((F^1, F^2)\) are bought by investors of type \(t_i\), then the issuer’s payoff from selling security \((F^1, F^2)\) is

\[
U(F^1, F^2) = p^i(F^1) + p^i(F^2) + \delta \sum_{s \in S} \pi^i_s (X_s - F^1_s - F^2_s)
\]

\[
= \sum_{s \in S} p^i(F^1_s + F^2_s) + \delta \sum_{s \in S} \pi^i_s (X_s - F^1_s - F^2_s).
\]

Consider the pair of securities \((\tilde{F}^1, \tilde{F}^2)\) with \(\tilde{F}^i_s = F^1_s + F^2_s\) for all \(s\) and \(\tilde{F}^j_s = 0\) for all \(s\).

Since investors of type \(t_i\) buy the two securities \(F^1, F^2\), it must be that \(p^i(F^j) \geq p^{-i}(F^j)\) for \(j = 1, 2\). Note that \(p^i(\tilde{F}^i) = p^i(F^1) + p^i(F^2)\) and \(p^{-i}(\tilde{F}^i) = p^{-i}(F^1) + p^{-i}(F^2)\). Hence, \(p(\tilde{F}^i) = p^i(\tilde{F}^i)\). Moreover, \(p^j(\tilde{F}^{-i}) = 0\) for \(j = 1, 2\), and \(p(\tilde{F}^{-i}) = p^{-i}(\tilde{F}^{-i})\). Finally, note that

\[
U(\tilde{F}^1, \tilde{F}^2) = p^i(\tilde{F}^i) + \delta \sum_{s \in S} \pi^i_s (X_s - F^1_s - F^2_s)
\]

\[
= \sum_{s \in S} \pi^i_s (F^1_s + F^2_s) + \delta \sum_{s \in S} \pi^i_s (X_s - F^1_s - F^2_s) = U(F^1, F^2).
\]

\(\square\)

Proof of Proposition 1. Fix a pair of securities \((F^1, F^2) \in \mathcal{F}\) such that, for \(j = 1, 2\), security
$F^j$ is bought by investors of type $t_j$. The issuer’s payoff from selling this pair of securities is

$$U(F^1, F^2) = \sum_{s=1}^{K} \pi_s^1 F^1_s + \sum_{s=1}^{K} \pi_s^2 F^2_s + \delta \sum_{s=1}^{K} \pi_s^I (X_s - F^1_s - F^2_s)$$

$$= F^1_1 + F^2_1 + \sum_{s=2}^{K} (\pi^1(A_s) - \delta \pi^I(A_s)) (F^1_s - F^1_{s-1})$$

$$+ \sum_{s=2}^{K} (\pi^2(A_s) - \delta \pi^I(A_s)) (F^2_s - F^2_{s-1}) + \delta \sum_{s=1}^{K} \pi_s^I X_s. \quad (18)$$

Note that any pair of securities $(F^1, F^2) \in \mathcal{F}$ must be such that: (i) $F^1_1 + F^2_1 \in [0, X_1]$, (ii) for all $s > 1$, $F^1_s + F^2_s \in [F^1_{s-1} + F^2_{s-1}, F^1_{s-1} + F^2_{s-1} + X_s - X_{s-1}]$ and (iii) for $i = 1, 2$, $F^i_s \geq F^i_{s-1}$.

From equation (18), it is optimal for the issuer to set $F^1_1 + F^2_1 = X_1$. Moreover, for $s \in S \setminus \{1\}$ and $j = 1, 2$, it is optimal to set $F^j_s = F^j_{s-1} + X_s - X_{s-1}$ if $\pi^j(A_s) > \max \{\delta \pi^I(A_s), \pi^{-j}(A_s)\}$, and to set $F^j_s = F^j_{s-1}$ otherwise.

\[ Q.E.D. \]

### Proofs of Section 4

**Proof of Proposition 2.** By Proposition 1, under Assumption 1 the optimal security backed by a single asset $X^a$ is $F = \min\{X_s, X_k\}$. Note that selling two individual securities $F$, each backed by one of the assets, is the same as selling security $\tilde{F} \in \mathcal{F}_{Y^1}$ such that

$$\tilde{F}_{s, s'} = \begin{cases} 
X_s + X_{s'} & \text{if } s, s' \leq k, \\
X_k + X_{s'} & \text{if } s > k, s' \leq k, \\
X_s + X_k & \text{if } s \leq k, s' > k, \\
2X_k & \text{if } s > k, s' > k.
\end{cases} \quad (19)$$

Suppose there exists exists $\hat{s}, \hat{s}' \in S$ with $\hat{s}, \hat{s}' \in S$ with $\hat{s}' > k$ and $\hat{s}' \geq \hat{s}$, such that...
Define
\[ \hat{S} := \left\{ (s, s') \in S^2 : s \geq \hat{s} \text{ and } s' \geq \hat{s}' \text{ or } s \geq \hat{s}' \text{ and } s' \geq \hat{s} \right\}, \tag{20} \]
and consider the following alternative security \( F \in F_Y \):
\[
F_{s,s'} = \begin{cases} 
\tilde{F}_{s,s'} & \text{if } s, s' \in \hat{S}, \\
\tilde{F}_{s,s'} + \Delta = \tilde{F}_{s,s'} + X_{\hat{s}} - X_{\hat{s}-1} & \text{otherwise}. 
\end{cases} \tag{21}
\]
Note that security \( F \) satisfies the monotonicity requirements. Note further that, for any beliefs \( \pi \) over \( S \),
\[
\sum_s \sum_{s'} \pi_s \pi_{s'} (F_{s,s'} - \tilde{F}_{s,s'}) = \sum_{s=\hat{s}}^K \pi_s \sum_{s'=\hat{s}'}^K \pi_{s'} \Delta + \sum_{s=\hat{s}'}^{\hat{s}'-1} \pi_s \sum_{s'=\hat{s}}^{\hat{s}'} \pi_{s'} \Delta
\]
\[
= \Delta [\pi(A_{\hat{s}})\pi(A_{\hat{s}'}) + \pi(A_{\hat{s}'})\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'})]
\]
\[
= \Delta [\pi(A_{\hat{s}'})\pi(2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'}))].
\]
Therefore,
\[
U_Y(F) - 2U_{X_a}(F) = U_Y(F) - U_Y(\tilde{F})
= p_Y(F) - p_Y(\tilde{F}) + \delta \sum_s \sum_{s'} \pi_s^l \pi_{s'}^l (\tilde{F}_{s,s'} - F_{s,s'})
= \sum_s \sum_{s'} \pi_s^l \pi_{s'}^l (F_{s,s'} - \tilde{F}_{s,s'}) + \delta \sum_s \sum_{s'} \pi_s^l \pi_{s'}^l (\tilde{F}_{s,s'} - F_{s,s'})
= (\pi^1(A_{\hat{s}'}) (2\pi^1(A_{\hat{s}}) - \pi^1(A_{\hat{s}'}) - \delta \pi^l(A_{\hat{s}'}) (2\pi^l(A_{\hat{s}}) - \pi^l(A_{\hat{s}'}) ) \Delta > 0,
\]
where we used the equation above and the inequality in the statement of the proposition.
Proof of Proposition 3. Let $F^1$ be the optimal security when the issuer pools the two assets and sells a single security to market participants with beliefs $\pi^1$. Since the two assets have iid returns, it is without loss to assume that $F^1$ is symmetric: $F^1_{s,s'} = F^1_{s',s}$ for all $s, s' \in S$.

We show that $F^1_{s,s'} = F^S_{s,s'}$ for all $s, s' \in S$, where $F^S$ is the security that the issuer will effective sell if she were to sell two individual securities, each backed by its own asset:

$$F^S_{s,s'} = \begin{cases} 
X_s + X_{s'} & \text{if } s, s' \leq k, \\
X_k + X_{s'} & \text{if } s > k, s' \leq k, \\
X_s + X_k & \text{if } s \leq k, s' > k, \\
2X_k & \text{if } s > k, s' > k.
\end{cases} \quad (22)$$

We start by showing that $F^1_{s,s'} = F^S_{s,s'}$ for all $s, s'$ such that $s \leq k$ and $s' \leq k$. Towards a contradiction, suppose not, and let $\hat{s}, \hat{s}' \leq k$ be such that $F^1_{\hat{s},\hat{s}'} \neq F^S_{\hat{s},\hat{s}'} = X_{\hat{s}} + X_{\hat{s}'}$. Note that it must be that $F^1_{\hat{s},\hat{s}'} = X_{\hat{s}} + X_{\hat{s}'} - \epsilon$ for some $\epsilon > 0$. Consider the security $\tilde{F}$ such that

$$\tilde{F}_{s,s'} = \begin{cases} 
F^1_{s,s'} & s < \hat{s} \text{ and } s' < \hat{s}' \\
F^1_{s,s'} + \epsilon & \text{otherwise.}
\end{cases} \quad (23)$$

Note that, since $F^1$ is monotonic, then so is $\tilde{F}$.

For any beliefs $\pi$,

$$\sum_{s} \sum_{s'} \pi_s \pi_{s'} (\tilde{F}_{s,s'} - F^1_{s,s'}) = \sum_{s=1}^{\hat{s}-1} \pi_s \sum_{s'=s}^{K} \pi_{s'} \epsilon + \sum_{s=\hat{s}}^{K} \pi_s \sum_{s'=1}^{K} \pi_{s'} \epsilon$$

$$= \epsilon \left[ (1 - \pi(A_{\hat{s}})) \pi(A_{\hat{s}'}) + \pi(A_{\hat{s}}) \right]. \quad (24)$$

15To see why, suppose the seller finds it optimal to sell a security $F^1$ that is not symmetric. Let $\hat{F}^1$ be the security such that, for all $s, s' \in S$, $\hat{F}^1_{s,s'} = F^1_{s',s}$. Since the two assets have iid returns, securities $F^1$ and $\hat{F}^1$ yield the same profits to the issuer. Since $F^1$ satisfies the monotonicity requirements, so does $\hat{F}^1$. Let $G$ be a security such that, for all $s, s'$, $G_{s,s'} = \frac{1}{2}(F^1_{s,s'} + \hat{F}^1_{s,s'})$. Note that security $G$ is symmetric, satisfies the monotonicity requirements, and gives the same profits to the issuer as security $F^1$. 25
Note that

\[ U_Y(\hat{F}) - U_Y(F^1) = p_Y(\hat{F}) - p_Y(F^1) + \delta \sum_s \sum_{s'} \pi^1_s \pi^1_{s'} (F^1_{s,s'} - \hat{F}_{s,s'}) \]

\[ = \sum_s \sum_{s'} \pi^1_s \pi^1_{s'} (\hat{F}_{s,s'} - F^1_{s,s'}) + \delta \sum_s \sum_{s'} \pi^1_s \pi^1_{s'} (F^1_{s,s'} - \hat{F}_{s,s'}) \]

\[ = \epsilon \left[ (1 - \pi^1(\hat{s})) \pi^1(A_{\hat{s}}) + \pi^1(\hat{s}) - \delta (1 - \pi^1(\hat{s})) \pi^1(A_{\hat{s}}) - \delta \pi^1(\hat{s}) \right], \]

where we used equation (24). By Assumption 1, \( \pi^1(\hat{s}) > \delta \pi^1(\hat{s}) \) (since \( \hat{s} \leq k \)). Moreover, since \( \pi^1 \) is such that \( \pi^1(\hat{s}) \geq \pi^1(\hat{s}) \). Since \( \pi^1(\hat{s}) > \delta \pi^1(\hat{s}) \), it follows that \( (1 - \pi^1(\hat{s})) \pi^1(A_{\hat{s}}) > \delta (1 - \pi^1(\hat{s})) \pi^1(A_{\hat{s}}) \). Therefore, \( U_Y(\hat{F}) > U_Y(F^1) \), a contradiction to the fact that \( F^1 \) is optimal. Hence, \( F^1 \) is such that \( F^1_{s,s'} = X_s + X_{s'} \) for all \( s, s' \leq k \).

Next we show that \( F^1_{s,s'} = F^S_{s,s'} \) for all \( s, s' \) with \( s > k \) or \( s' > k \). Since \( F^1_{s,s'} = F^S_{s,s'} = X_s + X_{s'} \) for all \( s, s' \) with \( s \leq k \) and \( s' \leq k \), by monotonicity it must be that \( F^1_{s,s'} \geq \min\{X_s, X_k\} + \min\{X_{s'}, X_k\} = F^S_{s,s'} \) for all \( s, s' \) with \( s > k \) or \( s' > k \). Towards a contradiction, suppose that there is \( s, s' \) with \( s > k \) or \( s' > k \) such that \( F^1_{s,s'} > F^S_{s,s'} \). Let \( \hat{s} := \min\{s > k : F^1_{s,s'} > F^S_{s,s'} \text{ for some } s'\} \), and let \( \hat{s}' := \min\{s' : F^1_{\hat{s},s'} > F^S_{\hat{s},s'} \} \). By monotonicity, it must be that \( F^1_{s,s'} > F^S_{s,s'} \) for all \( s, s' \) with \( s \geq \hat{s} \) and \( s' \geq \hat{s}' \). Moreover, since \( F^1 \) and \( F^S \) are symmetric, it must also be that \( F^1_{s,s'} > F^S_{s,s'} \) for all \( s, s' \) with \( s \geq \hat{s}' \) and \( s' \geq \hat{s}' \). Lastly, by symmetry of \( F^1 \), it must be that \( \hat{s} \leq \hat{s}' \).

Let \( \hat{S} = \{s, s' \in S^2 : s \geq \hat{s} \text{ and } s' \geq \hat{s}' \} \). Moreover, since \( F^1 \) and \( F^S \) are symmetric, it must also be that \( F^1_{\hat{s},\hat{s}'} > F^S_{\hat{s},\hat{s}'} \) for all \( s, s' \) with \( s \geq \hat{s} \) and \( s' \geq \hat{s} \). Lastly, by symmetry of \( F^1 \), it must be that \( \hat{s} \leq \hat{s}' \).

Let \( \hat{F} \) be an alternative security with

\[ \hat{F}_{s,s'} = \begin{cases} F^1_{s,s'} & s, s' \notin \hat{S} \\ F^1_{s,s'} - \epsilon & s, s' \in \hat{S} \end{cases} \]  

(25)

\[ ^{16} \text{Indeed, suppose by contradiction that } \hat{s} > \hat{s}' \text{. Since } F^1_{\hat{s},\hat{s}'} > \hat{F}^1_{\hat{s},\hat{s}'}, \text{ by symmetry of } F^1 \text{ and } \hat{F} \text{ it must be that } F^1_{\hat{s},\hat{s}'} > \hat{F}^1_{\hat{s},\hat{s}'}. \text{ Since } \hat{s} > \hat{s}', \text{ this contradicts the fact that } \hat{s} := \min\{s > k : F^1_{s,s'} > \hat{F}^1_{s,s'} \text{ for some } s'\}. \text{ Therefore, it must be that } \hat{s}' \geq \hat{s}. \]
Note that, since $F^1$ satisfies the monotonicity requirements, so does $\tilde{F}$. Note further that, for any beliefs $\pi$,

$$
\sum_s \sum_{s'} \pi_s \pi_{s'} (\tilde{F}_{s,s'} - F^1_{s,s'}) = \sum_{s = \hat{s}}^K \pi_s \sum_{s' = \hat{s}'}^K \pi_{s'} (-\epsilon) + \sum_{s = \hat{s}'}^K \pi_s \sum_{s' = \hat{s}}^K \pi_{s'} (-\epsilon)
$$

$$
= -\epsilon [\pi(A_{\hat{s}}) \pi(A_{\hat{s}'}) + \pi(A_{\hat{s}'}) (\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'}))]
$$

$$
= -\epsilon [\pi(A_{\hat{s}'}) (2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'}))]. \tag{26}
$$

Note that

$$
U_Y(\tilde{F}) - U_Y(F^1) = p_Y(\tilde{F}) - p_Y(F^1) + \delta \sum_s \sum_{s'} \pi_s \pi_{s'} (F^1_{s,s'} - \tilde{F}_{s,s'})
$$

$$
= \sum_s \sum_{s'} \pi_{s'} \pi_{s'} (\tilde{F}_{s,s'} - F^1_{s,s'}) + \delta \sum_s \sum_{s'} \pi_s \pi_{s'} (F^1_{s,s'} - \tilde{F}_{s,s'})
$$

$$
= -\epsilon [\pi(\hat{s}') (2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'})) - \delta \pi(\hat{s}) (2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'}))],
$$

where we used equation (26). By the conditions in the Proposition, $\pi(\hat{s}') (2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'})) < \delta \pi(\hat{s}) (2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'}))$, and so $U_Y(\tilde{F}) > U_Y(F^1)$, which contradicts the fact that $F^1$ is optimal. Hence, it must be that $F^1_{\hat{s},\hat{s}'} = \tilde{F}^1_{\hat{s},\hat{s}'}$ for all $s, s'$, and so there are no gains from pooling.

Proof of Proposition 5. By the arguments in the main text, under Assumption 2 the optimal securities $(F^1, i, F^2, i)$ backed by a single asset $X^i$ are $F^1_s = \min\{X_s, X_k\}$ and $F^2_s = 0$. Note that selling two securities $F^1_s$, each backed by one of the assets, is the same as selling security

27
\( \tilde{F}^1 \in \mathcal{F}_Y \) such that

\[
\tilde{F}^1_{s,s'} = \begin{cases} 
X_s + X_{s'} & \text{if } s, s' \leq k, \\
X_k + X_{s'} & \text{if } s > k, s' \leq k, \\
X_s + X_k & \text{if } s \leq k, s' > k, \\
2X_k & \text{if } s > k, s' > k.
\end{cases}
\] (27)

Let \( \hat{S} := \{ (s, s') \in S^2 : s \geq \hat{s} \text{ and } s' \geq \hat{s}' \text{ or } s \geq \hat{s}' \text{ and } s' \geq \hat{s} \} \). Suppose the issuer pools the assets and sells securities \((\tilde{F}^1, \tilde{F}^2) \in \mathcal{F}_Y\), with \( \tilde{F}^1 \) as above, and

\[
\tilde{F}^2_{s,s'} = \begin{cases} 
0 & \text{if } s, s' \notin \hat{S}, \\
X_{\hat{s}} - X_{\hat{s}-1} = \Delta & \text{otherwise}
\end{cases}
\] (28)

Note that, for any beliefs \( \pi \),

\[
\sum_s \sum_{s'} \pi_s \pi_{s'} \tilde{F}^2_{s,s'} = \sum_{s=\hat{s}}^{K} \sum_{s'=\hat{s}'}^{K} \pi_s \pi_{s'} \Delta + \sum_{s=\hat{s}'}^{K} \sum_{s'=\hat{s}}^{s'-1} \pi_s \pi_{s'} \Delta \\
= \Delta \left[ \pi(A_{\hat{s}})\pi(A_{\hat{s}'}) + \pi(A_{\hat{s}'})\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'})\right] \\
= \Delta \left[ \pi(A_{\hat{s}'})\left(2\pi(A_{\hat{s}}) - \pi(A_{\hat{s}'})\right)\right].
\] (29)

The issuer’s payoff from selling the two assets as separate concerns, issuing for each asset \( X^a \) securities \((F^{1,a}, F^{2,a})\) with \( F^{1,a}_{s} = \min\{X_s, X_k\} \) and \( F^{2,a}_{s} = 0 \), is equal to

\[
2U_{X^a}(F^{1,a}, F^{2,a}) = \sum_s \sum_{s'} \pi^1_s \pi^1_{s'} F^{1}_{s,s'} + \delta \sum_s \sum_{s'} \pi^l_s \pi^l_{s'} (X_s + X_{s'} - \tilde{F}^{1}_{s,s'}). 
\] (30)

On the other hand, the payoff that the issuer gets from pooling the assets and selling securities
\((\tilde{F}^1, \tilde{F}^2) \in \mathcal{F}_Y\) is:

\[
U_Y(\tilde{F}^1, \tilde{F}^2) = \sum_s \sum_{s'} \pi_s^1 \pi_s^1 \tilde{F}^1_{s,s'} + \sum_s \sum_{s'} \pi_s^2 \pi_s^2 \tilde{F}^2_{s,s'} + \delta \sum_s \sum_{s'} \pi_s^I \pi_s^I (X_s + X_{s'} - \tilde{F}^1_{s,s'} - \tilde{F}^2_{s,s'}).
\]

Comparing (30) and (31), it follows that

\[
U_Y(\tilde{F}^1, \tilde{F}^2) - 2U_X(F^{1,a}, F^{2,a}) = \sum_s \sum_{s'} \pi_s^2 \pi_s^2 \tilde{F}^2_{s,s'} - \delta \sum_s \sum_{s'} \pi_s^I \pi_s^I \tilde{F}^2_{s,s'}
\]

\[
= (\pi^2(A_{s'}) (2\pi^2(A_s) - \pi^2(A_{s'})) - \delta \pi^I(A_{s'}) (2\pi^I(A_s) - \pi^I(A_{s'}))) \Delta > 0,
\]

where the second equality follows from (29) and the strict inequality follows from the assumption in the statement of the Proposition.

Proofs of Section 5

Proof of Proposition 6. The proof uses arguments similar to those in the proof of Proposition 1. For any security \(F \in \mathcal{F}_D\), the issuer’s payoff is

\[
U(F) = \sum_{s=1}^{K} \pi_s^M F_s + \delta \sum_{s=1}^{K} \pi_s^I (X_s - \min\{X_s, D\} - F_s)
\]

\[
= F_1 + \sum_{s=2}^{K} (\pi^M(A_s) - \delta \pi^I(A_s)) (F_s - F_{s-1}) + \delta \sum_{s=1}^{K} \pi_s^I (X_s - \min\{X_s, D\}),
\]

(32)

Note that any security \(F \in \mathcal{F}_D\) must be such that \(F_s = 0\) and for all \(s \leq s_D\), \(F_s \in [F_{s-1}, F_{s-1} + X_s - X_{s-1}]\) for all \(s > s_D\) and \(F_s \geq F_{s-1}\) for all \(s\). Moreover, any security that satisfies these conditions belongs to \(\mathcal{F}_D\). Mechanically, any optimal security \(F\) must have \(F_s = 0\) and for all \(s \leq s_D\). From equation (32), for any \(s > s_D\) it is optimal to set \(F_s = F_{s-1} + X_s - X_{s-1}\) if \(\pi^M(A_s) \geq \delta \pi^I(A_s)\), and to set \(F_s = F_{s-1}\) if \(\pi^M(A_s) < \delta \pi^I(A_s)\).
Lastly, we extend the example of section 4.2 to allow for non-zero correlation between the assets to be securitized. As in section 4.2, suppose the issuer owns two assets, $X^1$ and $X^2$, each of which can generate a return in $\{X_1, X_2\}$ (with $X_1 < X_2$). In contrast to section 4.2, suppose that the returns of assets $X^1$ and $X^2$ are correlated. Let $sk \in \hat{S} = \{11, 12, 21, 22\}$ denote the event that asset 1’s return is $X_s$ and asset 2’s return is $X_k$. The beliefs of the issuer and market over the set of possible return realizations are, respectively, $\hat{\pi}^I$ and $\hat{\pi}^M$. For $j = I, M$, $\hat{\pi}^j_{sk}$ denotes the probability that $j$ assigns to the event $sk$. We assume that the assets are symmetric, so that $\hat{\pi}^I_{12} = \hat{\pi}^I_{21}$ for $j = I, M$. The iid case of section 4.2 is the special case with $\hat{\pi}^j_{sk} = \pi^j_{s} \pi^j_{k}$ for $j = I, M$ and for all $sk \in \hat{S}$.

Suppose first that the issuer sells two individual securities, each backed by an asset. It can be shown that an optimal security $F$ has $F_1 = X_1$ and $F_2 \in [F_1, X_2]$. The price that the market is willing to pay for security $F$ is $p(F) = X_1(\hat{\pi}^M_{11} + \hat{\pi}^M_{12}) + F_2(\hat{\pi}^M_{21} + \hat{\pi}^M_{22})$; the issuer’s payoff from selling this security is

$$p(F) + \delta(X_2 - F_2)(\hat{\pi}^M_{21} + \hat{\pi}^M_{22}) = X_1(\hat{\pi}^M_{11} + \hat{\pi}^M_{12}) + F_2(\hat{\pi}^M_{21} + \hat{\pi}^M_{22}) + \delta(X_2 - F_2)(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}). \quad (33)$$

The issuer finds it optimal to set $F_2 = X_1$ if $\delta(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) > \hat{\pi}^M_{21} + \hat{\pi}^M_{22}$ and $F_2 = X_2$ if $\delta(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) \leq \hat{\pi}^M_{21} + \hat{\pi}^M_{22}$. In what follows we maintain the assumption that $\delta(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) > \hat{\pi}^M_{21} + \hat{\pi}^M_{22}$, so that an issuer who sells individual securities $F^1$ and $F^2$, each backed respectively by asset $X^1$ and $X^2$, finds it optimal to set $F^1_s = F^2_s = X_1$ for $s = 1, 2$.

Suppose next that the issuer pools the two assets and sells a single security backed by cash-flows $Y = X^1 + X^2$. Consider a security $F_Y = \min\{Y, X_1 + X_2\}$. The price that the market is willing to pay for security $F_Y$ is $p(F_Y) = \hat{\pi}^M_{11} 2X_1 + (1 - \hat{\pi}^M_{11})(X_1 + X_2)$, and the

$$p(F_Y) + \delta(Y - F_Y)(\hat{\pi}^M_{21} + \hat{\pi}^M_{22}) = \hat{\pi}^M_{11} 2X_1 + (1 - \hat{\pi}^M_{11})(X_1 + X_2) + \delta(Y - F_Y)(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}). \quad (33)$$

The issuer finds it optimal to set $F_Y = Y$ if $\delta(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) > \hat{\pi}^M_{21} + \hat{\pi}^M_{22}$ and $F_Y = X_1$ if $\delta(\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) \leq \hat{\pi}^M_{21} + \hat{\pi}^M_{22}$.
issuer’s payoff from selling this security is

\[ p(F_Y) + \delta \hat{\pi}^I_{22}(X_2 - X_1) = \hat{\pi}^M_{11} 2X_1 + (1 - \hat{\pi}^M_{11}) (X_1 + X_2) + \delta \hat{\pi}^I_{22}(X_2 - X_1). \]  

(34)

Comparing (33) and (34), the issuer strictly prefers selling security \( F_Y \) backed by the pool of assets than selling the two individual securities \( F^1_s = F^2_s = X_1 \) for \( s = 1, 2 \) if and only if

\[ 2\hat{\pi}^M_{21} + \hat{\pi}^M_{22} = 1 - \hat{\pi}^M_{11} > \delta (1 - \pi^I_{11}) = \delta (2\hat{\pi}^I_{21} + \hat{\pi}^I_{22}). \]

Combining this with \( \delta (\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) > \hat{\pi}^M_{21} + \hat{\pi}^M_{22} \), the issuer strictly prefers to pool the assets and sell security \( F_Y \) if

\[ \hat{\pi}^M_{11} \in (1 - \delta (\hat{\pi}^I_{21} + \hat{\pi}^I_{22}) - \hat{\pi}^M_{21}, 1 - \delta (2\hat{\pi}^I_{21} + \hat{\pi}^I_{22})). \]  

(35)

If the issuer and the market both perceive the asset to be perfectly correlated (so that \( \hat{\pi}^I_{21} = 0 \) for \( j = 1, 2 \)), the condition in (35) can never be satisfied, and hence pooling does not obtain.
References


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