Solutions
10 points possible

ENG EC/ME/SE 501:

Exercises (Set 1)

1pt 1. (a) Prove that matrix multiplication is associative: i.e. show that for any three matrices \( A, B, \) and \( C \) of compatible dimensions that \( A(BC) = (AB)C \).

(b) Prove that matrix multiplication is not commutative: i.e. it is not the case that \( AB = BA \) for any two square matrices \( A \) and \( B \).

1pt 2. (a) Prove that \( (AB)^T = B^TA^T \).

1pt (b) Prove that if \( A \) is invertible, then \( A^T \) is invertible.

1pt 3. (a) Suppose \( A(t) \) and \( B(t) \) are \( m \times n \) and \( n \times p \) matrices respectively. Find a formula for

\[
\frac{d}{dt}[A(t)B(t)]
\]

in terms of the derivatives of the individual matrices.

1pt (b) If \( A(t) \) is invertible, find a formula for the derivative (with respect to \( t \)) of its inverse.

4. Using Laplace’s expansion (i.e. expansion by cofactors), evaluate the determinants of the matrices

1pt

\[
\begin{pmatrix}
3 & 0 & 1 \\
2 & 4 & 3 \\
1 & 1 & 2
\end{pmatrix}
\quad\begin{pmatrix}
1 & x & x \\
a & ax + by & ax \\
0 & by & 1
\end{pmatrix}
\]

1pt

5. Find the inverses of the matrices in the previous problem. Assume that \( a, b, x, \) and \( y \) are integers. Under what further conditions on the symbolic entries \( a, b, x, y \) will the inverse of the second matrix have all its entries integers?

2pts 6. Let \( V \) be a finite dimensional vector space. A basis of \( V \) is any linearly independent set of vectors that span \( V \). (a) Show that any maximal linearly independent set of vectors in \( V \) is a basis for \( V \). (b) Show that any minimal spanning set of vectors is also a basis.

Definition: Given a square matrix \( A \), the determinant of the \((n-1)\times(n-1)\) submatrix \( M_{ij} \) obtained from \( A \) by deleting the \( i \)-th row and \( j \)-th column is called the \( ij \)-th minor of \( A \). Multiplying the \( ij \)-minor by \(-1^{i+j}\) yields the \( ij \)-th cofactor \( A_{ij} \). Among the list of determinant facts that were recalled in class, we have the following:

\[
\det A = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij}.
\]

(Please turn over.)
7. The *adjugate* of $A$ is the $n \times n$ matrix $\text{adj}(A)$ s.t. $\text{adj}(A)_{ij} = A_{ji}$. Using the properties of determinants reviewed in class, prove the following.

**Theorem:** If $\det A \neq 0$, then the matrix $A$ is invertible, and when this is the case

$$A^{-1} = \left(\frac{1}{\det A}\right) \text{adj}(A).$$
Exercise Set 1

(a) Assume the matrices of the problem, \(A\), \(B\), and \(C\), are of the dimensions \(m \times n\), \(n \times p\), and \(p \times q\), respectively. Allow \(D = BC\). Then,

\[
d_{ij} = \sum_{k=1}^{p} b_{ik} c_{kj}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, q.
\]

Allow \(E = AB\). Then,

\[
e_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, p
\]

Allow \(F = AD = A(BC)\). Then,

\[
f_{ij} = \sum_{k=1}^{n} a_{ik} d_{kj} = \sum_{k=1}^{n} a_{ik} \left\{ \sum_{h=1}^{p} b_{kh} c_{hj} \right\} = \sum_{k=1}^{n} \sum_{h=1}^{p} a_{ik} b_{kh} c_{hj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, q.
\]

Finally, allow \(G = EC = (AB)C\). Then,

\[
g_{ij} = \sum_{h=1}^{p} e_{ih} c_{hj} = \sum_{h=1}^{p} \left[ \sum_{k=1}^{n} \{ a_{ik} b_{kh} \} c_{hj} \right] = \sum_{k=1}^{n} \sum_{h=1}^{p} a_{ik} b_{kh} c_{hj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, q.
\]

Thus, \(F = G\), and \(A(BC) = (AB)C\).

(b) Allow

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

and

\[
B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}
\]

Then,

\[
AB = \begin{bmatrix} 8 & 5 \\ 20 & 13 \end{bmatrix}
\]

and

\[
BA = \begin{bmatrix} 13 & 20 \\ 5 & 8 \end{bmatrix}.
\]

Hence, \(AB \neq BA\) and matrix multiplication is in general not commutative.
(a) Assuming $A$ and $B$ are $n \times n$ matrices, allow $C = AB$. Then,

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}, i, j = 1, \ldots, n.$$ 

Allow $D = C^T$. Then,

$$d_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}, i, j = 1, \ldots, n.$$ 

Allow $E = B^T A^T$. Then,

$$e_{ij} = \sum_{k=1}^{n} b_{ki}a_{jk}, i, j = 1, \ldots, n.$$ 

Thus, $D = E$ and therefore $(AB)^T = B^T A^T$.

(b) If the matrix $A$ is invertible, there exist a matrix $B$ such that $AB = I$.

Taking the transpose of both sides of this equation gives $B^T A^T = I^T = I$ and hence, $A^T$ has an inverse.

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(a) Assigning $C(t) = A(t)B(t)$, yields

$$c_{ij}(t) = \sum_{k=1}^{n} a_{ik}(t)b_{kj}(t).$$ 

This implies that

$$\frac{d}{dt}(c_{ij}(t)) = \sum_{k=1}^{n} \left[ \frac{d}{dt}(a_{ik}(t))b_{kj}(t) + a_{ik}(t)\frac{d}{dt}(b_{kj}(t)) \right].$$

Simplification of the above yields

$$\frac{d}{dt}(c_{ij}(t)) = \sum_{k=1}^{n} \frac{d}{dt}(a_{ik}(t))b_{kj}(t) + \sum_{k=1}^{n} a_{ik}(t)\frac{d}{dt}(b_{kj}(t)).$$

From the above it is clear that

$$\frac{d}{dt}(C(t)) = \frac{d}{dt}(A(t)B(t)) = \frac{d}{dt}(A(t))B(t) + A(t)\frac{d}{dt}(B(t)).$$
(b) If $A(t)$ is invertible, there exist a matrix $B(t)$ such that

$$A(t)B(t) = I.$$ 

Taking the derivative of both sides gives

$$\frac{d}{dt}(A(t)B(t)) = \frac{d}{dt}(I) = 0,$$

where $0$ is the matrix of zeros.

From part (a), the expression becomes

$$\frac{d}{dt}(A(t))B(t) + A(t)\frac{d}{dt}(B(t)) = 0.$$ 

Moving the first term to the right hand side of the equation gives

$$A(t)\frac{d}{dt}(B(t)) = -\frac{d}{dt}(A(t))B(t).$$ 

After left-multiplying both sides of the equation by $B(t)$, the expression for the derivative of the inverse of $A(t)$ with respect to $t$ becomes

$$\frac{d}{dt}(B(t)) = -B(t)\frac{d}{dt}(A(t))B(t).$$

4

(a)

$$\begin{vmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$

$$= 3(8 - 3) + (2 - 4) = 3(5) - 2 = 13.$$ 

(b)

$$\begin{vmatrix} 1 & ax + by \\ by & 1 \end{vmatrix} - a \begin{vmatrix} x & x \\ by & 1 \end{vmatrix} = by$$

5

(a)

$$\begin{bmatrix} 3 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 3 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & 3 \\ 3 & 0 & 0 \end{bmatrix} \xrightarrow{R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & -3 & -5 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow R_2$$

$$R_3 - 3R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -2 \\ 0 & -3 & -5 \end{bmatrix} \xrightarrow{R_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -3 & -5 \end{bmatrix} \xrightarrow{R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 1 & 0 & -3 \end{bmatrix}$$
(a) Let \( \{p_1, \ldots, p_n\} \) be a maximal linearly independent set of vectors in \( V \), and let \( v \) be an arbitrarily chosen vector. The set \( \{p_1, \ldots, p_n, v\} \) must be linearly dependent. This means that there is a set of scalars \( \{a_1, \ldots, a_{n+1}\} \), not all of which are zero, such that

\[
a_1 p_1 + \cdots + a_n p_n + a_{n+1} v = 0.
\]

In this expression, it must be the case that \( a_{n+1} \neq 0 \), because otherwise there would be a nontrivial linear combination of \( p_1, \ldots, p_n \) that is equal to zero. This would contradict the linear independence of \( \{p_1, \ldots, p_n\} \).

Hence we may write

\[
v = -\frac{a_1}{a_{n+1}} p_1 - \cdots - \frac{a_n}{a_{n+1}} p_n.
\]

Since \( v \) was arbitrary, this proves that \( \{p_1, \ldots, p_n\} \) spans \( V \).

(b) Let \( \{p_1, \ldots, p_n\} \) be a minimal spanning set, and suppose there is a nontrivial linear combination

\[
a_1 p_1 + \cdots + a_n p_n = 0.
\]

We may assume without loss of generality that \( a_n \neq 0 \). Then we can write

\[
p_n = -\frac{a_1}{a_n} p_1 - \cdots - \frac{a_{n-1}}{a_n} p_{n-1}.
\]

Let \( v \) be an arbitrarily chosen vector. Since \( \{p_1, \ldots, p_n\} \) is a spanning set, there are scalars \( \beta_1, \ldots, \beta_n \) such that

\[
v = \beta_1 p_1 + \cdots + \beta_n p_n.
\]

But we can also write

\[
v = (\beta_1 - \frac{a_1 \beta_n}{a_n}) p_1 + \cdots + (\beta_{n-1} - \frac{a_{n-1} \beta_n}{a_n}) p_{n-1}.
\]
Since \( v \) was chosen arbitrarily, this means that any vector in \( V \) may be expressed as a linear combination of \( p_1, \ldots, p_{n-1} \) contradicting the assumption that \( \{p_1, \ldots, p_n\} \) is a minimal spanning set.

7. We compute the \( ij \)-th entry of \( A \cdot \text{adj}(A) \):

\[
\sum_{k=1}^{n} a_{ik} A_{jk}
\]

where \( A_{jk} \) is the \( jk \)-th cofactor of \( A \), which is \((-1)^{j+k} \cdot \det((n-1) \times (n-1)) \) submatrix of \( A \) obtained by deleting the \( j \)-th row and \( k \)-th column.

Case \( i=j \): \[
\sum_{k=1}^{n} a_{ik} A_{jk} = \sum_{k=1}^{n} a_{ik} A_{ik} = \det(A).
\]

Case \( i \neq j \): \[
\sum_{k=1}^{n} a_{ik} A_{jk} = \det(\hat{A}) \] where \( \hat{A} \) is the \( n \times n \) matrix whose \( j \)-th row is the \( i \)-th row of \( A \) with all other rows (including the \( i \)-th) being the same as \( A \). Since Property 4 of determinants (in the list of properties discussed in class) states that the determinant of a
matrix with two equal rows is zero, \( \det|\mathbf{A}| = 0 \). Thus we have

\[
\sum_{k=1}^{n} a_{ik} A_{jk} = \delta_{ij} \cdot \det|\mathbf{A}|
\]

where \( \delta_{ij} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{if } i \neq j
\end{cases} \). This proves the theorem.